Math 2300-013: Quiz 8

Name: Solution Score: ________

1. Consider the power series

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{\sqrt{n}}(5x-1)^n$$

(i) (4 points) Find the interval of convergence of the series.

Use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}(5x-1)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{3^n\cdot(5x-1)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{3\sqrt{n}(5x-1)}{\sqrt{n+1}}\right|$$

$$= |3(5x-1)| \quad \text{since } \frac{\sqrt{n}}{\sqrt{n+1}} \xrightarrow[n\to\infty]{} 0$$

By the ratio test, the series converges wherever

$$|3(5x-1)| < 1$$
$$-1 < 3(5x-1) < 1$$
$$-1/3 < 5x-1 < 1/3$$
$$2/3 < 5x < 4/3$$
$$2/15 < x < 4/15$$

Interval of Convergence: $\left(\frac{2}{15}, \frac{4}{15}\right]$

Endpoints: $x=\frac{2}{15}$; $\sum_{n=1}^{\infty}\frac{(-3)^n}{\sqrt{n}}\left(-\frac{1}{3}\right)^n = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges by p-test $(p=\frac{1}{2}\le 1)$

$x=4/15$; $\sum_{n=1}^{\infty}\frac{(-3)^n}{\sqrt{n}}\left(\frac{1}{3}\right)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges by alt. series test [$\frac{1}{\sqrt{n}}$ is decreasing + goes to 0].

(ii) (1 point) What is the radius of convergence of this series?

The interval $\left(\frac{2}{15}, \frac{4}{15}\right]$ is of length $\frac{4}{15}-\frac{2}{15} = \frac{2}{15}$, so the radius of convergence is $\frac{2}{15}\cdot\frac{1}{2} = \boxed{\frac{1}{15}}$

2. (a) (3 points) Write down a power series that converges to

$$f(x) = \frac{x^7 - 1}{1 - 3x^2}.$$

(Hint: Modify the power series for $\frac{1}{1-x}$.)

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \quad \text{for } x \text{ in } (-1,1) \text{ ie. } |x|<1$$

$$\frac{1}{1-3x^2} = 1 + (3x^2) + (3x^2)^2 + (3x^2)^3 + \cdots = \sum_{n=0}^{\infty} (3x^2)^n \quad \text{for } |3x^2|<1$$

$$\frac{x^7-1}{1-3x^2} = (1-3x^2)\left[1 + (3x^2) + (3x^2)^2 + \cdots\right] = (1-3x^2)\sum_{n=0}^{\infty}(3x^2)^n \quad \text{for } |3x^2|<1$$

$$\frac{x^7-1}{1-3x^2} = 1-3x^2 + (1-3x^2)(3x^2) + (1-3x^2)(3x^2)^2 + \cdots = \sum_{n=0}^{\infty}(1-3x^2)(3x^2)^n \quad \text{for } |3x^2|<1$$

* Note: As written, this is not in the form $\sum c_n x^n$, but it has all the right types of things so we could do it. It wouldn't have a nice pattern I think.

(b) (2 points) What is the radius of convergence of this series? (Hint: Modify the interval of convergence of the power series for $\frac{1}{1-x}$.)

The series for $\frac{x^7-1}{1-3x^2}$ we constructed above converges for

$$|3x^2| < 1$$

$$-1 < 3x^2 < 1$$

$$-\tfrac{1}{3} < x^2 < \tfrac{1}{3}$$

$\underbrace{-\tfrac{1}{3} < x^2}_{\text{always true}}$ and $x^2 < \tfrac{1}{3}$

$$\Downarrow$$

$$|x| < \sqrt{\tfrac{1}{3}}$$

$$-\sqrt{\tfrac{1}{3}} < x < \sqrt{\tfrac{1}{3}}$$

$\boxed{\text{radius of conv. is } \sqrt{\tfrac{1}{3}}}$