Math 2300-013: Quiz 6

Name: Solution Score: ________

1. (2 points) Which of the following are true statements? Circle your answer(s).

   (i) If $a_n \to 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

   (ii) If $a_n \not\to 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

   (iii) If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

   (iv) If $\sum_{n=1}^{\infty} a_n$ diverges, then $a_n \not\to 0$.

   ] – This is the divergence test.

2. (2 points each) For each of the following series, determine if the series converges or diverges. If the series converges, find its sum. In all cases, justify your answers.

   (a) $\sum_{n=1}^{\infty} \sqrt[n]{4}$

   $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 4^{1/n} = 4^0 = 1$$

   Since the limit of the sequence $a_n = \sqrt[n]{4}$ is not 0, by the divergence test, the series $\sum_{n=1}^{\infty} \sqrt[n]{4}$ diverges.

   (b) $\sum_{n=2}^{\infty} \frac{(-3)^n}{7^{n+3}}$ $= \frac{(-3)^2}{7^5} + \frac{(-3)^3}{7^6} + \frac{(-3)^4}{7^7} + \frac{(-3)^5}{7^8} + \cdots$

   This is a geometric series with

   $$a = \frac{(-3)^2}{7^5} = \frac{9}{7^5}$$ ← first term

   $$r = \frac{-3}{7}$$ ← ratio you multiply each term by to get the next

   Since $|r| = \frac{3}{7} < 1$, the series converges to $\frac{a}{1-r} = \frac{\frac{9}{7^5}}{1+3/7} = \frac{\frac{9}{7^4}}{7+3} = \frac{9}{10 \cdot 7^4}$.

(c) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$

[For this series, I recommend the integral test. Let $f(x) = \frac{1}{x\sqrt{\ln x}}$ (the connect-the-dots function).]

Hypotheses of integral test

- $f(x) > 0$ for $x \geq 3$ ✓
- $f(x)$ decreasing for $x \geq 3$. (x, $\ln(x)$ are increas
- $f(x)$ continuous on $x \geq 3$.

$$\int_3^\infty \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{T\to\infty}\int_3^T \frac{1}{x\sqrt{\ln x}}\,dx$$

$$\left\{\begin{array}{l} u = \ln x \\ du = \frac{1}{x}dx \end{array}\right\} = \lim_{T\to\infty}\int_{\ln(3)}^{\ln(T)} \frac{1}{\sqrt{u}}\,du$$

$$= \lim_{T\to\infty} 2\sqrt{u}\,\Big|_{\ln(3)}^{\ln(T)}$$

$$= \lim_{T\to\infty}\left[2\sqrt{\ln(T)} - 2\sqrt{\ln(3)}\right]$$

$$= \infty$$

Conclusion: Since $\int_3^\infty \frac{1}{x\sqrt{\ln x}}\,dx$ diverges, the integral test says that the series $\sum_{n=3}^{\infty}\frac{1}{n\sqrt{\ln n}}$ diverges, too.

(d) $\displaystyle\sum_{n=1}^{\infty} \ln\left|\frac{\cos\left(\frac{1}{n}\right)}{\cos\left(\frac{1}{n+1}\right)}\right|$

[This series is telescoping:]

$$= \lim_{N\to\infty}\sum_{n=1}^{N}\left[\ln\left|\cos\left(\tfrac{1}{n}\right)\right| - \ln\left|\cos\left(\tfrac{1}{n+1}\right)\right|\right]$$

$$= \lim_{N\to\infty}\Big[\Big(\ln|\cos(1)| - \cancel{\ln|\cos(\tfrac{1}{2})|}\Big) + \Big(\cancel{\ln|\cos(\tfrac{1}{2})|} - \cancel{\ln|\cos(\tfrac{1}{3})|}\Big) + \Big(\cancel{\ln|\cos(\tfrac{1}{3})|} - \cancel{\ln|\cos(}$$

$$+ \cdots + \Big(\cancel{\ln|\cos(\tfrac{1}{N-1})|} - \cancel{\ln|\cos(\tfrac{1}{N})|}\Big) + \Big(\cancel{\ln|\cos(\tfrac{1}{N})|} - \ln|\cos(\tfrac{1}{N+1}$$

$$= \lim_{N\to\infty}\left[\ln|\cos(1)| - \ln\left|\cos\left(\tfrac{1}{N+1}\right)\right|\right]$$

$$= \ln(\cos(1)) - \ln|1|$$

$$= \ln(\cos(1)).$$

$\Longrightarrow$ The series $\displaystyle\sum_{n=1}^{\infty}\ln\left|\frac{\cos(\frac{1}{n})}{\cos(\frac{1}{n+1})}\right|$ converges to $\ln(\cos(1))$