§5.7: Partial Fractions Decomposition

(Thanks to Faan Tone Liu)

Key Points:

- Use this method to integrate rational functions.
- Degree of numerator must be lower than degree of denominator. If needed, start with long division.
- Factor the denominator
- Solve the decomposition according to the right form:
  - Linear Factors: $\dfrac{5x+3}{(x+1)(x+4)} = \dfrac{A}{x+1} + \dfrac{B}{x+4}$
  - Repeated Linear Factors: $\dfrac{2x-4}{(x-2)(x+3)^2} = \dfrac{A}{x-2} + \dfrac{B}{x+3} + \dfrac{C}{(x+3)^2}$
  - Irreducible Quadratic Factors: $\dfrac{2x^2-3x-1}{(x-1)(x^2+9)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+9}$
  - Mixtures are possible!
- Make sure you remember how to integrate the "outputs" of the partial fractions decomposition. For example:

$$\frac{5}{2x+1},\ \frac{3}{(x+4)^2},\ \frac{2x}{x^2+9},\ \frac{4}{x^2+25},\ \text{or}\ \frac{3x+1}{x^2+16}.$$

- Other notes and tips:

u-sub works

$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$

Split into $\frac{3x}{x^2+16} + \frac{1}{x^2+16}$

And use the rules to the left

Examples:

1. $\displaystyle\int \frac{2x^2+3x-3}{x^2-x}\,dx$

First, use long division to re-write the integrand so the fraction involved has a numerator with a lower degree than the denominator.

$$\begin{array}{r} 2 \\ x^2 - x \,\big)\, \overline{2x^2+3x-3} \\ -(2x^2-2x) \\ \hline 5x-3 \end{array} \qquad\qquad \frac{2x^2+3x-3}{x^2-x} = 2 + \frac{5x-3}{x^2-x}$$

The form of the decomposition of $\frac{5x-3}{x(x-1)}$ is:

$$\frac{5x-3}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

Here are two methods to find A and B in the partial fractions decomposition:

Method I: Equate Coefficients	Method II: Judicious Substitution
$5x-3 = A(x-1) + Bx$	$5x-3 = A(x-1) + Bx$
$5x-3 = Ax - A + Bx$	$\underline{x=1}$: $5-3 = B$
$5x-3 = (A+B)x - A$	$B = 2$
$\begin{cases} 5 = A+B \\ -3 = -A \end{cases} \Rightarrow \begin{array}{l} A=3 \\ B=2 \end{array}$	$\underline{x=0}$: $-3 = A(-1)$
	$A = 3$

Now, $\displaystyle\int \frac{2x^2+3x-3}{x^2-x}\,dx = \int 2 + \frac{3}{x} + \frac{2}{x-1}\,dx$

$$= 2x + 3\ln|x| + 2\ln|x-1| + C$$

2. $\displaystyle\int \frac{x^2+x-5}{(x-2)(x-1)^2}\,dx$

$$\frac{x^2+x-5}{(x-2)(x-1)^2} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$x^2+x-5 = A(x-1)^2 + B(x-1)(x-2) + C(x-2)$$

Method II:

$\underline{x=1}$: $1+1-5 = C(-1)$

$-3 = -C$

$C = 3$

$\underline{x=2}$: $4+2-5 = A$

$A = 1$

Now, $x^2+x-5 = (x-1)^2 + B(x-1)(x-2) + 3(x-2)$

Try something like $x=0$ (or do the compare coefficients method)

$\underline{x=0}$:

$-5 = 1 + B(-1)(-2) + 3(-2)$

$-5 = 1 + 2B - 6$

$0 = 2B$

$B = 0$

Now, we have

$$\int \frac{x^2+x-5}{(x-2)(x-1)^2}\,dx = \int \frac{1}{x-2}\,dx + \int \frac{0}{x-1}\,dx + \int \frac{3}{(x-1)^2}\,dx$$

$$= \ln|x-2| + \int 0\,dx + \int 3(x-1)^{-2}\,dx$$

Could use u-sub with $u = x-1$, $du = dx$

$$= \ln|x-2| + \text{Const.} + -3(x-1)^{-1} + \text{Const.}$$

$$= \ln|x-2| + \frac{-3}{x-1} + C$$

3. $\displaystyle\int \frac{10}{(x+1)(x^2+9)}\,dx$ ⚠ Note x^2+9 cannot be factored!

Method I:

$$\frac{10}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$$

$$10 = A(x^2+9) + (Bx+C)(x+1)$$

$$10 = Ax^2 + 9A + Bx^2 + Bx + Cx + C$$

$$10 = (A+B)x^2 + (B+C)x + 9A + C$$

$$\begin{cases} A+B=0 \\ B+C=0 \\ 9A+C=10 \end{cases} \Rightarrow \begin{cases} A-C=0 \\ B=-C \\ 9A+C=10 \end{cases} \Rightarrow \begin{cases} A=C \\ B=-C \\ 9A+A=10 \end{cases} \Rightarrow \begin{cases} A=C=1 \\ B=-1 \end{cases}$$

Now, $\displaystyle\int \frac{10}{(x+1)(x^2+9)}\,dx = \int \frac{1}{x+1}\,dx + \int \frac{-x+1}{x^2+9}\,dx$

$$= \ln|x+1| + \int \frac{-x}{x^2+9}\,dx + \int \frac{1}{x^2+9}\,dx$$

$$= \ln|x+1| - \frac{1}{2}\int \frac{1}{u}\,du + \frac{1}{3}\arctan(x/3) + C$$

$$= \ln|x+1| - \frac{1}{2}\ln|x^2+9| + \frac{1}{3}\arctan(x/3) + C$$

u-sub!

$$\boxed{\begin{aligned} u &= x^2+9 \\ du &= 2x\,dx \\ \tfrac{1}{2}du &= x\,dx \end{aligned}}$$