Calculus of Parametric Eq. (§1.7, 3.4, 6.4)

Thanks to Faan Tone Liu

Key Points:

- Suppose $x(t)$ and $y(t)$ are parametric equations and t represents time. Then
  - $\frac{dx}{dt}$ represents ____.
  - $\frac{dy}{dt}$ represents ____.
  - $\frac{dy}{dx}$ represents ____.
- Formulae for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t are

$$\frac{dy}{dx} = \qquad\qquad\qquad\qquad\qquad ,$$

$$\frac{d^2y}{dx^2} = \qquad\qquad\qquad\qquad\qquad .$$

- The instantaneous speed (of the snail) along the curve as a function of t is:

- The arc length of the curve for t in $[a, b]$ is the integral of ____________ :

$$\text{Arc Length} = \qquad\qquad\qquad\qquad\qquad .$$

- Other Notes:

Examples:

1. Consider the parametric curve given by $\begin{cases} x = 5\cos(3t) \\ y = 5sin(3t). \end{cases}$

   At $t = \frac{\pi}{4}$ and $t = \frac{\pi}{2}$, find the slope of the tangent line and the speed.

2. A half-line is parameterized by $\begin{cases} x = 2 + 3t \\ y = -1 + 5t, \end{cases}$ where $t \geq 0$.

   (a) Does (5,4) lie on the ray?

   (b) Does (2,1) lie on the ray?

   (c) Does (-1,-6) lie on the ray?

   (d) When does the line hit the y-axis?

   (e) What is the speed of motion along the line?

   (f) What is the slope of the line?

3. Use technology to graph
$$\begin{cases} x = t^2 \\ y = t^3 - 3t \end{cases} \quad ; \ t \in \mathbb{R}.$$

   (a) Find equations for the tangent lines to the curve at (3,0).

   (b) At $t = 1$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Check the concavity.

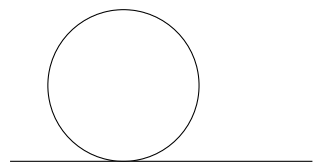
4. Consider the parametric curve given by $\begin{cases} x = t\cos t \\ y = t\sin t \end{cases}$, where $0 \leq t \leq 4\pi$.

   (a) At time $t = 3.5$, which direction is the particle moving? Find the speed of the particle at this time.

   (b) Find the average speed and the arc length of the particle on $[0, 4]$. What do you notice?

5. A **cycloid** is a path traced by a point on the edge of a wheel.

   (a) Find parametric equations for the cycloid generated by the wheel of radius 1 shown. Suppose t is measured in radians.



   (b) For what value(s) of t is the tangent line horizontal?

   (c) For what value(s) of t is the point stopped?