Solutions

§8.7: Taylor Series

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Key Points:

- The Taylor Series for $f(x)$ centered at $x = a$ is given by

$$f(x) = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- If $f(x)$ has a power series representation, then it is the Taylor Series.
- A Taylor Series centered at $x = 0$ is also called a MacLauren Series.

Examples:

1. Find the Taylor Series for $f(x) = e^x$ about $x = 0$

$f(x) = e^x \qquad f(0) = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$f''(x) = e^x \qquad f''(0) = 1$

$\vdots \qquad \vdots$

$f^{(n)}(x) = e^x \qquad f^{(n)}(0) = 1$

$\vdots \qquad \vdots$

$$f(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

2. Find the Taylor Series for $f(x) = \sin(x)$ about $x = 0$

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$

$f^{(5)}(x) = \cos x \qquad f^{(5)}(0) = 1$

$\vdots \qquad \vdots$

$$\sin(x) = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^{(2n+1)}}{(2n+1)!}(-1)^n$$

3. Find the Taylor Series for $f(x) = \cos(x)$ about $x = 0$

$f(x) = \cos x \qquad f(0) = 1$

$f'(x) = -\sin x \qquad f'(0) = 0$

$f''(x) = -\cos x \qquad f''(0) = -1$

$f'''(x) = \sin x \qquad f'''(0) = 0$

$f^{(4)}(x) = \cos x \qquad f^{(4)}(0) = 1$

$f^{(5)}(x) = -\sin x \qquad f^{(5)}(0) = 0$

$\vdots \qquad \vdots$

$$\cos(x) = 1 + \frac{0}{1!}x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 - \frac{0}{5!}x^5 - \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$$

⚠ Note: could also find this series by differentiating the one for sin x!

4. Find the Taylor Series for for $f(x) = \ln(1+x)$ about $x = 0$

$f(x) = \ln(1+x)$ $\quad f(0) = 0$

$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$ $\quad f'(0) = 1$

$f''(x) = -(1+x)^{-2} = \frac{-1}{(1+x)^2}$ $\quad f''(0) = -1$

$f'''(x) = 2(1+x)^{-3} = \frac{2}{(1+x)^3}$ $\quad f'''(0) = 2$

$f^{(4)}(x) = -6(1+x)^{-4} = \frac{-6}{(1+x)^4}$ $\quad f^{(4)}(0) = -6$

$f^{(5)}(x) = 24(1+x)^{-5} = \frac{24}{(1+x)^5}$ $\quad f^{(5)}(0) = 24$

$\vdots$

$f^{(n)}(x) = (n-1)!(-1)^{n+1}(1+x)^{-n} = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n}$ $\quad f^{(n)}(0) = (-1)^{n+1}(n-1)!$

$\vdots$

$f(x) = 0 + x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 - \frac{6}{4!}x^4 + \cdots$

$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!} x^n$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$

5. Evaluate $\int \frac{e^x - 1}{x}\, dx$ using infinite Series.

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n!}$

$\frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$

$\int \frac{e^x-1}{x}\, dx = \int 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots\, dx = \sum_{n=1}^{\infty} \int \frac{x^{n-1}}{n!}\, dx$

$\int \frac{e^x-1}{x}\, dx = x + \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} + \frac{x^4}{4 \cdot 4!} + \cdots + C = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!} + C$