§7.5: The Logistic Equation Solutions

(Thanks to Faan Tone Liu)

Key Points:

- Review: Often, population growth can be modeled by $P' = kP$. The solution is $P(t) = P_0 e^{kt}$ (see example 1 below), and this situation is called exponential growth.

- Exponential growth is not realistic in the long run because $\lim_{t\to\infty} P(t) = \infty$, so we modify it to get the **Logistic Equation** (assuming $k>0$ (i.e. population grows not decays))

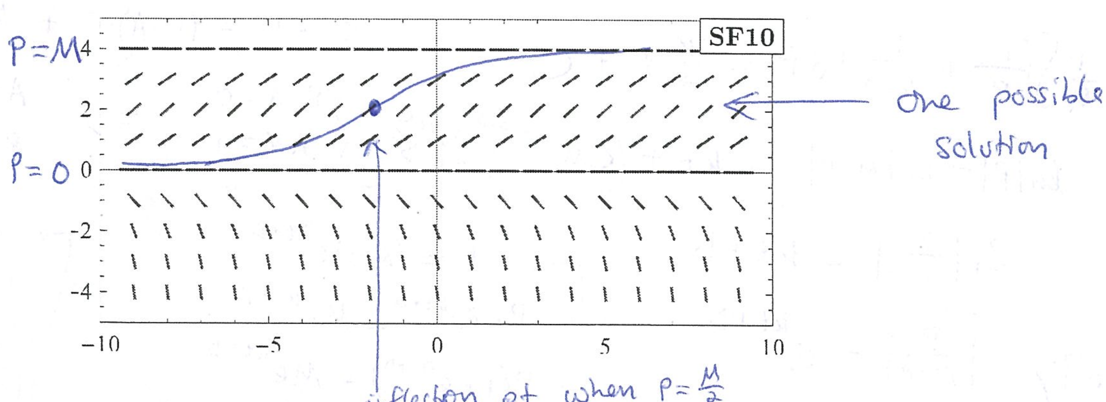
$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right),$$

  where M is a constant that represents the carrying capacity of the population. The solution is

$$\boxed{P(t) = \frac{M}{1+Ae^{kt}}, \quad A = \frac{M-P_0}{P_0}}$$

  * Note, to find $A = \frac{M-P_0}{P_0}$, use the initial condition $P(0) = P_0$

- The slope field for the logistic growth equation is



- Miscellaneous observations:
  - If P is small, then $\frac{dP}{dt} \approx$ kP (basically exponential growth).
  - If $P \approx M$, then $\frac{dP}{dt} \approx$ 0 (growth slows to 0).
  - Equilibrium (constant) solutions are: $P=0$, $P=M$.
  - If the population starts between 0 and M: $\lim_{t\to\infty} P(t) =$ M.
  - Using Calc I methods, we can show that $P(t)$ has an inflection point when $P = \frac{M}{2}$.

Examples:

1. (Review of exponential growth) Solve the differential equation $P' = kP$.

$$\frac{dP}{dt} = kP$$
$$\frac{1}{P}dP = k\,dt$$
$$\int \frac{1}{P}dP = \int k\,dt$$
$$\ln|P| = kt + C$$
$$|P| = e^{kt+C}$$
$$P = e^{C}\cdot e^{kt} \quad \text{(Population} \geq 0\text{)}$$
$$\boxed{P = P_0 \cdot e^{kt}} \quad (P_0 \text{ is population when } t=0)$$

2. Solve the logistic differential equation $P' = kP\left(1 - \frac{P}{M}\right)$.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$
$$\frac{1}{M}\cdot\frac{1}{P(1-\frac{P}{M})}dP = k\,dt\cdot\frac{1}{M}$$
$$\int \frac{1}{P(M-P)}dP = \int \frac{k}{M}dt$$

Use partial Fractions to simplify $\frac{1}{P(M-P)}$:

$$\frac{1}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}$$
$$1 = A(M-P) + BP$$
$$0P + 1 = (B-A)\cdot P + AM$$
$$\begin{cases} B - A = 0 \\ AM = 1 \end{cases} \implies \begin{matrix} A = \frac{1}{M} \\ B = \frac{1}{M} \end{matrix}$$

$$\frac{1}{M}\int\left(\frac{1}{P} + \frac{1}{M-P}\right)dP = \frac{k}{M}t + C$$
$$\ln|P| - \ln|M-P| = kt + CM$$

Just a bigger C, call it D

$$\ln\left|\frac{P}{M-P}\right| = kt + D$$
$$\left|\frac{P}{M-P}\right| = e^{kt+D}$$

$\begin{bmatrix} P \geq 0 \\ M \geq P \end{bmatrix}$

$$\frac{P}{M-P} = e^{kt+D}$$
$$P = (M-P)e^{kt+D}$$
$$P + Pe^{kt+D} = Me^{kt+D}$$
$$P(1+e^{kt+D}) = Me^{kt+D}$$
$$P = \frac{Me^{kt+D}}{1+e^{kt+D}}\cdot\frac{e^{-kt-D}}{e^{-kt-D}}$$
$$P = \frac{M}{1+e^{-kt-D}}$$
$$P = \frac{M}{1+e^{-D}e^{-kt}}$$
$$\boxed{P = \frac{M}{1+Ae^{-kt}}}$$

Using the initial condition $P(0) = P_0$, can find $A = \frac{M-P_0}{P_0}$

3. Suppose a population grows according to the logistic model with an initial population of 1000 and a carrying capacity of 10,000. If the population grows to 2500 after one year, what is the population after three more years?

$$P = \frac{M}{1+Ae^{-kt}}, \quad A = \frac{M-P_0}{P_0}, \quad M = 10000, \quad P_0 = 1000$$

Hence, $A = \frac{9000}{1000} = 9$,

So

$$P(t) = \frac{10000}{1+9e^{-kt}}$$

We know $P(1) = 2500$

$$2500 = \frac{10000}{1+9e^{-k}}$$
$$1 + 9e^{-k} = 4$$
$$9e^{-k} = 3$$
$$e^{-k} = \frac{1}{3}$$

Finally,

$$P(t) = \frac{10000}{1+9\cdot(\frac{1}{3})^t},$$

So

$$P(4) = \frac{10000}{1+9(\frac{1}{3})^4} = 9000.$$