1. Recall the idea of linear approximation, which we will use to approximate the numbers $e^{.1}$ and e. Let $f(x) = e^x$. We want to find a linear function $L(x) = C_0 + C_1 x$ whose value and whose derivative at $x = 0$ matches $f(x)$ at $x = 0$.

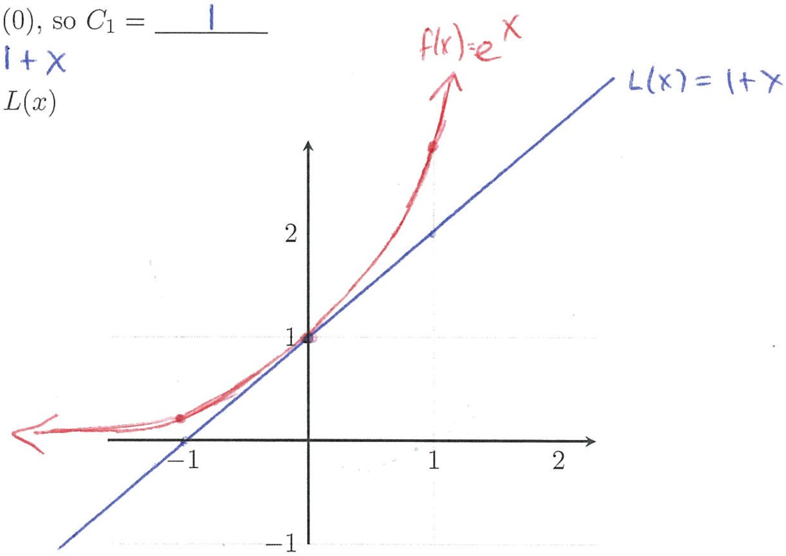
   (a) Fill in the table below:

$f(x) = e^x$	$f(0) = 1$	$L(x) = C_0 + C_1 x$	$L(0) = C_0$
$f'(x) = e^x$	$f'(0) = 1$	$L'(x) = C_1$	$L'(0) = C_1$

   (b) Now we want to match the values and derivatives of $f(x)$ and $L(x)$, in other words:
   - $f(0) = L(0)$, so $C_0 =$ 1
   - $f'(0) = L'(0)$, so $C_1 =$ 1

   (c) Thus $L(x) = 1 + x$

   (d) Graph e^x and $L(x)$



   (e) Use $L(x)$ to estimate $e^{0.1}$ and e. Compare them to the estimates your calculator gives you and comment on the accuracy. Explain from the graph why the error is so much larger for estimating the value of e.

$e^{0.1} \approx L(0.1) = 1.1$ Actual values: $e^{0.1} = 1.10517...$

$e^1 \approx L(1) = 2$ $e^1 = 2.718281....$

Error larger when approximating e because the tangent line L(x) was centered at x=0 and 1 is farther from 0 than 0.1 is.

Note: At home you can repeat this process using an arbitrary $f(x)$ to find that the general formulas for a linear approximation centered at $x = 0$ is

$$L(x) = f(0) + f'(0)x$$

This is the equation of the tangent line, and is called the 1st degree Taylor Polynomial for $f(x)$ centered at $x = 0$.

2. Now we'll repeat the ideas of the previous problem, but using a quadratic approximation instead of a linear approximation. Let $f(x) = e^x$. We want to find a quadratic function $Q(x) = C_0 + C_1x + C_2x^2$ whose value and whose derivative and whose second derivative at $x = 0$ matches $f(x)$ at $x = 0$.

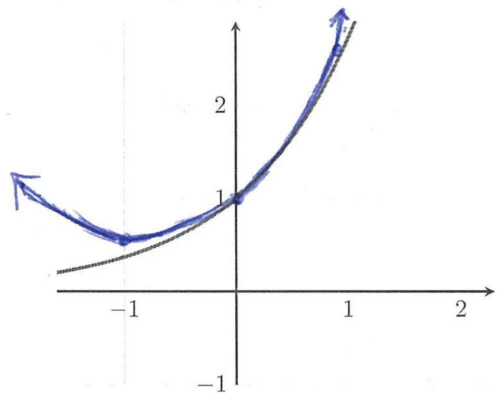
   (a) Fill in the table below:

$f(x) = e^x$	$f(0) = 1$	$Q(x) = C_0 + C_1x + C_2x^2$	$Q(0) = C_0$
$f'(x) = e^x$	$f'(0) = 1$	$Q'(x) = C_1 + 2C_2x$	$Q'(0) = C_1$
$f''(x) = e^x$	$f''(0) = 1$	$Q''(x) = 2C_2$	$Q''(0) = 2C_2$

   (b) Now we want to match the values and derivatives of $f(x)$ and $Q(x)$, in other words:
   - $f(0) = Q(0)$, so $C_0 =$ 1
   - $f'(0) = Q'(0)$, so $C_1 =$ 1
   - $f''(0) = Q''(0)$, so $C_2 = \frac{1}{2}$

   (c) Thus $Q(x) =$ $1 + x + \frac{x^2}{2}$

   (d) A graph of e^x is shown below. Graph $Q(x)$ on the same coordinate plane.



   (e) Use $Q(x)$ to estimate $e^{0.1}$ and e. Compare them to the estimates your calculator gives you and comment on the accuracy. Explain from the graph above and the graph from the previous problem why the errors are reduced.

$e^{0.1} \approx Q(.1) = 1 + .1 + \frac{.01}{2} = 1.105$ Actual: $e^{0.1} = 1.1051709...$

$e^1 \approx Q(1) = 1 + 1 + \frac{1}{2} = 2.5$ $e^1 = 2.71828...$

Errors reduced because a parabola is more curvy and better approximates the graph of e^x

Note: At home you can repeat this process using an arbitrary $f(x)$ to find that the general formulas for a quadratic approximation centered at $x = 0$ is

$$Q(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

Notice that this is the linear approximation with an additional term added. It is called the 2nd degree Taylor Polynomial for $f(x)$ centered at $x = 0$.

3. Now we'll repeat the ideas of the previous problem, but using a cubic approximation. Let $f(x) = e^x$. We want to find a cubic function $T(x) = C_0 + C_1x + C_2x^2 + C_3x^3$ whose value and whose derivative and whose second derivative and whose third derivative at $x = 0$ matches $f(x)$ at $x = 0$.

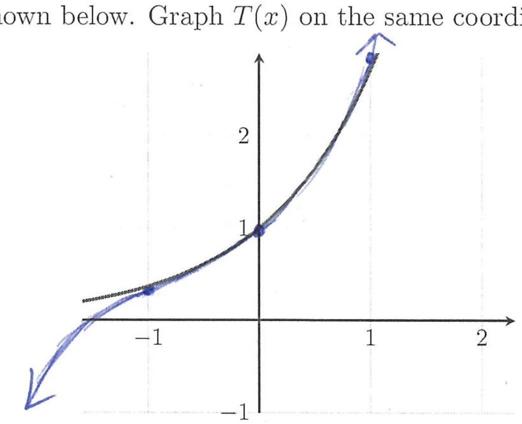
   (a) Fill in the table below:

$f(x) = e^x$	$f(0) = 1$	$T(x) = C_0 + C_1x + C_2x^2 + C_3x^3$	$T(0) = C_0 = 0!\,C_0$
$f'(x) = e^x$	$f'(0) = 1$	$T'(x) = C_1 + 2C_2x + 3C_3x^2$	$T'(0) = C_1 = 1!\,C_1$
$f''(x) = e^x$	$f''(0) = 1$	$T''(x) = 2C_2 + 6C_3x$	$T''(0) = 2C_2 = 2!\,C_2$
$f'''(x) = e^x$	$f'''(0) = 1$	$T'''(x) = 6C_3$	$T'''(0) = 6C_3 = 3!\,C_3$

   (b) Now we want to match the values and derivatives of $f(x)$ and $T(x)$, in other words:
   - $f(0) = T(0)$, so $C_0 =$ 1
   - $f'(0) = T'(0)$, so $C_1 = 1$
   - $f''(0) = T''(0)$, so $C_2 = \frac{1}{2}$
   - $f'''(0) = T'''(0)$, so $C_3 = \frac{1}{6}$

   (c) Thus $T(x) =$ ____ $\leftarrow$ $1 + x + \frac{x^2}{2} + \frac{x^3}{6}$

   (d) A graph of e^x is shown below. Graph $T(x)$ on the same coordinate plane.



   (e) Use $T(x)$ to estimate $e^{.1}$ and e. Compare them to the estimates your calculator gives you and comment on the accuracy. Compare the accuracy to that of the previous two problems.

   $e^{0.1} \approx T(.1) = 1 + 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} = 1.1051\overline{6}$

   $e^1 \approx T(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} = 2.\overline{6}$ Accuracy is Improving!

Note: The general formula for a cubic approximation centered at $x = 0$ is given below. Notice that the first three terms are the same as the quadratic approximation.

$$T(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

This is called the 3rd degree Taylor Polynomial for $f(x)$ centered at $x = 0$.

4. Fill in the table below, for $f(x) = e^x$. Note that $f^{(n)}(x)$ means the nth derivative of $f(x)$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}$	$\frac{f^{(n)}(0)}{n!}x^n$
0	e^x	1	1	1
1	e^x	1	1	x
2	e^x	1	$\frac{1}{2}$	$\frac{x^2}{2!}$
3	e^x	1	$\frac{1}{6}$	$\frac{x^3}{3!}$
4	e^x	1	$\frac{1}{24}$	$\frac{x^4}{4!}$
5	e^x	1	$\frac{1}{120}$	$\frac{x^5}{5!}$
6	e^x	1	$\frac{1}{720}$	$\frac{x^6}{6!}$

5. Add the terms in the last column to get the 6th degree Taylor Polynomial for $f(x) = e^x$ centered at $x = 0$.

$$P_6(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}$$

6. Use the Taylor Polynomial from Problem 5 to estimate e.

$$e \approx P_6(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} = 2.718055\ldots$$

$$\left(\frac{1957}{720}\right)$$

7. Write down the general formula for the nth degree Taylor polynomial for $f(x)$, centered at $x = 0$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

8. Give an explanation of where the factorials come from.

We need to divide by $n!$ to compensate for the power rule for derivatives, which "pulls down" n, then $n-1$, ... etc...

<u>Key point</u>: Two functions are "close" if (some of) their derivatives agree! They are "closer" if <u>more</u> of their derivates agree.