§ Improper Integrals (Day 2)

Yesterday, we considered improper integrals of Type (I).

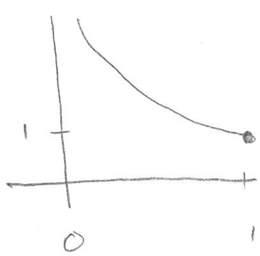
That is, integrals of form

$$\int_a^\infty f(x)\,dx$$

where one (or both) bounds of integration is infinite. Today, we'll consider Type (II) improper integrals (that is, when the integrand becomes infinite).

Example 1: Consider $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$

Note: This function is not defined at $x=0$, and looks like



$$\frac{1}{\sqrt{x}} \to \infty \quad \text{as} \quad x \to 0^+$$

One can compute $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$ in the following way:

First, let $0 < a < 1$. Then

$$\int_a^1 \frac{1}{\sqrt{x}}\,dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}}\Big]_a^1 = 2x^{\frac{1}{2}}\Big]_a^1 = 2 - 2a^{\frac{1}{2}}$$

Then,

$$\int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\left(\int_a^1 \frac{1}{\sqrt{x}}\,dx\right)$$

$$= \lim_{a\to 0^+}\left(2 - 2a^{\frac{1}{2}}\right)$$

$$= 2$$

Since this limit is finite, we say the improper integral converges.

Example 2: Find $\displaystyle\int_0^7 \frac{1}{\sqrt{49-x^2}}\,dx$

Notice, $f(x) = \dfrac{1}{\sqrt{49-x^2}}$ is not defined at $x = 7$.

Hence $\displaystyle\int_0^7 \frac{1}{\sqrt{49-x^2}}\,dx = \lim_{b\to 7^-}\left(\int_0^b \frac{1}{\sqrt{49-x^2}}\,dx\right)$

$$= \lim_{b \to 7^-} \left(\arcsin\left(\frac{x}{7}\right) \Big]_0^b \right) \qquad (\text{via } x = 7\sin\theta)$$

$$= \lim_{b \to 7^-} \left(\arcsin\left(\frac{b}{7}\right) - \arcsin\left(\frac{0}{7}\right) \right)$$

$$= \lim_{b \to 7^-} \left(\arcsin\left(\frac{b}{7}\right) \right)$$

$$= \arcsin(1) = \pi/2$$

A comparison test for improper integrals:

Suppose f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int_a^\infty f(x)\,dx$ is convergent, $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, $\int_a^\infty f(x)\,dx$ is divergent.

Example 3: Use the comparison thm to determine whether the following integrals are convergent or divergent.

(a) $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx$

(b) $\displaystyle\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx$

(a):
$$\int_0^\infty \frac{x}{x^3+1}\,dx = \underbrace{\int_0^1 \frac{x}{x^3+1}\,dx} + \underbrace{\int_1^\infty \frac{x}{x^3+1}\,dx}_{(*)}$$

finite b/c continuous on $[0,1]$

For $(*)$, note $\dfrac{x}{x^3+1} \le \dfrac{x}{x^3} = \dfrac{1}{x^2}$ so

$$\int_1^\infty \frac{x}{x^3+1}\,dx \le \int_1^\infty \frac{1}{x^2}\,dx < +\infty$$

since $p = 2 > 1$.. Hence, integral is convergent!

(b) : Note that

$$\frac{x+1}{\sqrt{x^4 - x}} \geq \frac{x}{\sqrt{x^4}} = \frac{x}{x^2} = \frac{1}{x} \quad \text{since } x \geq 1 > 0.$$

Hence,

$$\int_1^\infty \frac{x+1}{\sqrt{x^4 - x}}\, dx \geq \int_1^\infty \frac{1}{x}\, dx = +\infty$$

so the integral diverges!