Section 5.7: Integrals involving trig functions

Example 1: Evaluate $\int \sin^5 x \, dx$

Note: $\sin^5 x = (\sin^2 x)^2 \sin x$

$= (1 - \cos^2 x)^2 \sin x$

Now $\int \sin^5 x \, dx = \int (1-\cos^2 x)^2 \sin x \, dx$

$u = \cos x$ $\qquad = -\int (1-u^2)^2 \, du$

$\frac{du}{dx} = -\sin x$

$= -\int 1 - 2u^2 + u^4 \, du$

$= -\left(\frac{u^5}{5} - \frac{2u^3}{3} + u\right) + C$

$= -\frac{\cos^5 x}{5} + \frac{2\cos^3 x}{3} - \cos x + C$

Example 2 Evaluate

$$\int \sin^6 x \cos^3 x \, dx$$

Note that $\sin^6 x \cos^3 x = \sin^6 x \,(1 - \sin^2 x) \cos x$

so

$$\int \sin^6 x \cos^2 x \, dx = \int \sin^6 x \,(1 - \sin^2 x) \cos x \, dx$$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$$= \int u^6 (1 - u^2) \, du$$

$$= \frac{u^7}{7} - \frac{u^9}{9} + C$$

$$= \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + C$$

Remark: In general for

$$\int \sin^n x \cos^m x \, dx$$

if n is odd, can strip at one sine and convert rest to cosines and then use substitution $u = \cos x$. If

m is odd, can strip out one cosine and convert

rest to sines, and then use substitution $u = \sin x$.

If both n and m odd, we can use either method.

Q: What if both n and m are even?

Not a general approach to this case, but we'll look

at a simple case.

Example 3: Evaluate

$$\int \sin^2 x \cos^2 x \, dx$$

Recall that $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$

$$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$$

Hence,

$$\int \sin^2 x \cos^2 x \, dx = \int \frac{1}{2}(1 - \cos(2x)) \cdot \frac{1}{2}(1 + \cos(2x)) \, dx$$

$$= \frac{1}{4} \int 1 - \cos^2(2x)\, dx$$

$$= \frac{1}{4} \int 1\, dx - \frac{1}{4} \int \cos^2(2x)\, dx \qquad u = 2x$$

$$= \frac{1}{4} x - \frac{1}{8} \int \cos^2(u)\, du$$

$$= \frac{1}{4} x - \frac{1}{8} \left(\int \frac{1}{2} (1 + \cos(2u))\, du \right)$$

$$= \frac{1}{4} x - \frac{1}{16} \left(u + \frac{\sin(2u)}{2} \right) + C$$

$$= \frac{1}{4} x - \frac{1}{16} (2x) + \frac{1}{32} \sin(4x) + C$$

$$= \frac{1}{8} x + \frac{1}{32} \sin(4x) + C$$

Example 4: Evaluate

$$\int \cos(15x)\cos(4x)\,dx$$

To do this integral, useful to know identity

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$

Applying this to our example, we get

$$\int \cos(15x)\cos(4x)\,dx = \frac{1}{2}\int \cos(15x-4x)\,dx + \frac{1}{2}\int \cos(15x+4x)\,dx$$

$$= \frac{1}{2}\int \cos(11x)\,dx + \frac{1}{2}\int \cos(19x)\,dx = \frac{1}{2}\cdot\frac{\sin(11x)}{11} + \frac{1}{2}\,\frac{\sin(19x)}{19} + C$$

$$= \frac{1}{22}\sin(11x) + \frac{1}{38}\sin(19x) + C$$

What about integrals involving $\sec x$ and $\tan x$?

Recall the useful facts:

- $\tan^2 x + 1 = \sec^2 x$
- $\dfrac{d}{dx} \tan x = \sec^2 x$
- $\dfrac{d}{dx} \sec x = \sec x \tan x$

<u>Example</u> 5: Evaluate

$$\int \sec^9 x \cdot \tan^5 x \, dx$$

peel off $\tan x$ → peel off $\sec x$

Note: $\sec^9 x \tan^5 x = \sec^9 x \,(\underbrace{\sec^2 x - 1}_{(\tan^2 x)^2})^2 \tan x$

$= \sec^8 x \,(\sec^2 x - 1)^2 \sec x \tan x$

So $$\int \sec^9 x \cdot \tan^5 x \, dx = \int \sec^8 x \,(\sec^2 x - 1)^2 \sec x \tan x$$

dx

$u = \sec x$

$\dfrac{du}{dx} = \sec x \tan x$

$$= \int u^2 (u^2 - 1)^2 \, du \quad \ldots \quad \text{you can finish!}$$

In general, if the exponent on tangent is odd and we have at least one secant in the integrand we can follow above method.

<u>Example</u> 6: Evaluate

$$\int \sec^4 x \tan^6 x \, dx$$

peel off $\sec^2 x$

$$= \int \overbrace{(1 + \tan^2 x)}^{\sec^2 x} \tan^6 x \cdot \sec^2 x \, dx = \int (1 + u^2) u^6 \, du$$

$u = \tan x$

$$\frac{du}{dx} = \sec^2 x$$

$$= \int u^6 + u^8 \, du$$

$$= \frac{(\tan x)^7}{7} + \frac{(\tan x)^9}{9} + C$$

Please see link on course website for several more examples!

Powers of tan x:

Example 7

$$\int \tan x \, dx$$

$$\int \frac{\sin x}{\cos x} \, dx = \int \frac{\cancel{\sin x}}{u} \cdot \frac{du}{-\cancel{\sin x}} = -\ln|\cos x|$$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

Example 8:

$$\int \tan^3 x \, dx$$

$$\int \tan^3 x \, dx = \int \tan x \, (\sec^2 x - 1) \, dx$$

$$= \int \tan x \sec^2 x \, dx - \int \tan x \, dx$$

$$= \frac{1}{2}\tan^2 x + \ln|\cos x| + C$$