# § 5.6 : Integration by Parts

This is also known as "the backwards product rule":

Let $u(x)$ and $v(x)$ be two functions. By the product rule,

$$\frac{d}{dx}\Big(u(x)v(x)\Big) = u'(x)\,v(x) + u(x)\,v'(x) \qquad (1)$$

By (1),

subtract this

$$\int \frac{d}{dx}\big(u(x)\cdot v(x)\big)\,dx = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx$$

Hence, since

$$\int \frac{d}{dx}\big(u(x)v(x)\big)\,dx = \boxed{u(x)v(x)}$$

we get that

$$\boxed{\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx}$$

This is called the Integration by Parts formula. You may also see the shorthand:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

To apply integration by parts, the integrand must be a product of functions:

<u>Example</u> 1: Use IBP to compute

$$\int \underset{u}{x}\,\underset{v'}{e^x}\,dx \qquad u = x \qquad v' = e^x$$

$$u' = 1 \qquad v = e^x$$

By the formula,

$$\int x e^x\,dx = xe^x - \int 1 \cdot e^x\,dx$$

$$= xe^x - \int e^x\,dx$$

$$= xe^x - e^x + C$$

Note that our choice of $u = x$ and $v' = e^x$ is effective b/c the $\int u' v\,dx$ term is $\int e^x\,dx$ which we know how to integrate.

<u>Example</u> 2: Use IBP to find

$$\int \ln(x)\, dx$$

Note that $\int \ln(x)\,dx = \int \underset{\substack{\uparrow \\ v'}}{1} \cdot \underset{\substack{\uparrow \\ u}}{\ln(x)}\, dx$

$$u = \ln(x) \qquad v' = 1$$

$$u' = \frac{1}{x} \qquad v = x$$

Hence,

$$\int \ln(x)\,dx = \ln(x)\cdot x - \int x\left(\frac{1}{x}\right)dx$$

$$= x\ln(x) - \int 1\,dx$$

$$= x\ln(x) - x + C$$

Example 3: Use IBP twice to find

$$\int t^2 \sin(t)\, dt$$

$u = t^2 \qquad v' = \sin(t)$

$u' = 2t \qquad v = -\cos(t)$

Hence,

$$\int t^2 \sin(t)\, dt = t^2(-\cos t) - \int (-\cos(t)) \cdot 2t\, dt$$

$$= -t^2 \cos(t) + 2 \underbrace{\int t \cos t\, dt}_{(*)}$$

Do another integration by parts on (*):

$u = t \qquad v' = \cos t$

$u' = 1 \qquad v = \sin t$

$$(*) = t \sin t - \int \sin t\, dt$$

$$= t \sin t - (-\cos t)$$

$$= t \sin t + \cos t$$

So

$$\int t^2 \sin t \, dt = -t^2 \cos t + 2(t \sin t + \cos t) + C$$

$$= -t^2 \cos t + 2t \sin t + 2\cos t + C$$

Example 4: Definite integrals and IBP

Compute $\displaystyle\int_1^3 \frac{\ln(x)}{x^2}\, dx$

$u = \ln(x) \qquad v' = \frac{1}{x^2}$

$u' = \frac{1}{x} \qquad v = -\frac{1}{x}$

So
$$\int_1^3 \frac{\ln(x)}{x^2}\, dx = \ln(x)\left(-\frac{1}{x}\right)\Big]_1^3 - \int_1^3 \left(\frac{1}{x}\right)\left(-\frac{1}{x}\right) dx$$

$$= -\frac{\ln(x)}{x}\Big]_1^3 + \int_1^3 \frac{1}{x^2}\, dx$$

$$= -\frac{\ln(x)}{x}\Big]_1^3 + \left(-\frac{1}{x}\Big]_1^3\right)$$

$$= \frac{-\ln(3)}{3} - \frac{-\ln(1)}{1} + \left(-\frac{1}{3} - -\frac{1}{1}\right)$$

$$= \frac{-\ln(3)}{3} + \frac{2}{3}$$

More examples of IBP!

Example 5 Find $\int (\cos\theta)^2 \, d\theta$.

Note that $\int (\cos\theta)^2 \, d\theta = \int \cos\theta \cos\theta \, d\theta$

$u = \cos\theta \qquad v' = \cos\theta$

$u' = -\sin\theta \qquad v = \sin\theta$

So
$$\int \cos^2\theta \, d\theta = \cos\theta\sin\theta - \int (-\sin\theta)\sin\theta \, d\theta$$

$$= \cos\theta\sin\theta + \int \sin^2\theta \, d\theta$$

$$= \cos\theta\sin\theta + \int 1 - \cos^2\theta \, d\theta$$

$$= \cos\theta\sin\theta + \int 1 \, d\theta - \int \cos^2\theta \, d\theta$$

$$= \cos\theta\sin\theta + \theta - \int \cos^2\theta \, d\theta$$

Then

$$2\int \cos^2\theta \, d\theta = \cos\theta \sin\theta + \theta$$

Therefore,

$$\int \cos^2\theta \, d\theta = \frac{1}{2}(\cos\theta\sin\theta + \theta) + C.$$

Example 6: Indicate whether substitution or IBP is more appropriate.

(a) $\int x \sin x \, dx$ IBP $u = x$ $v' = \sin x$

(b) $\int \frac{x^2}{1+x^3} \, dx$ Sub $u = 1+x^3$

$\frac{1}{3}\ln|1+x^3|$

(c) $\int x e^{x^2} \, dx$ Sub $u = x^2$

$\frac{1}{2}e^{x^2}$

(d) $\int \arctan(x) \, dx$ IBP $u = \arctan(x)$ $v' = 1$

(e) $\int \frac{1}{\sqrt{3x+1}} \, dx$ Sub $u = 3x+1$

Example 7: Use IBP twice to find

$$\int e^x \sin(x)\,dx$$

$u = e^x \qquad v' = \sin x$

$u' = e^x \qquad v = -\cos x$

$$\int e^x \sin(x)\,dx = -e^x \cos x + \underbrace{\int e^x \cos x\,dx}_{(*)}$$

Do IBP on $(*)$:

$u = e^x \qquad v' = \cos x$

$u' = e^x \qquad v = \sin x$

$$(*) = e^x \sin(x) - \int e^x \sin x\,dx \,.$$

Hence,

$$\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx$$

So

$$2\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x$$

which implies that

$$\int e^x \sin x \, dx = \frac{1}{2}\left(-e^x \cos x + e^x \sin x\right) + C.$$