Math 2300 – Calculus II – University of Colorado

Fall 2011 – Final exam review problems:

1. Find $f_x(1,0)$ for $f(x,y) = \dfrac{xe^{\sin(x^2y)}}{(x^2+y^2)^{3/2}}$.

2. Consider the solid region $\mathcal{W}$ situated above the region $0 \le x \le 2$, $0 \le y \le x$, and bounded above by the surface $z = e^{x^2}$.

   (a) Write an integral that evaluates the area of each cross-section of $\mathcal{W}$ with vertical planes $y = a$, where a is a constant in $[0,2]$. Can you evaluate this integral (as an expression depending on a)?

   (b) Write an integral that evaluates the area of each cross-section of $\mathcal{W}$ with vertical planes $x = b$, where b is a constant in $[0,2]$. Can you evaluate this integral (as an expression depending on b)?

   (c) Write a convenient double integral that evaluates the volume of the region $\mathcal{W}$, and evaluate it.

   (d) Let
   $$f(x,y) = \sqrt{x^2+y^2}.$$

   (a) Find $f_x(x,y)$.

   (b) Let $g(x) = f_x(x,0)$. Does $\lim_{x\to 0} g(x)$ exist? If so, evaluate it. If not, explain why not. Hint: Note that $f(x,0) = \sqrt{x^2} = |x|$.

3. Evaluate:

   (a)
   $$\int_1^2 \int_y^{y^2} (x^2+y)\,dx\,dy.$$

   (b)
   $$\int_0^1 \int_{e^y}^{e} \frac{1+e^x}{\ln x}\,dx\,dy.$$

4. (a) Set up a double integral that corresponds to the mass of the semidisk
   $$D = \{(x,y)\colon 0 \le y \le \sqrt{4-x^2}\},$$
   if the density at each of its points is given by the function $\delta(x,y) = x^2y$.

   (b) Find the mass of the above semidisk.

5. The power series $\sum C_n x^n$ diverges at $x = 7$ and converges at $x = -3$. At $x = -9$, the series is

   (a) Conditionally convergent
   (b) Absolutely convergent
   (c) Divergent
   (d) Cannot be determined.

6. The power series $\sum C_n (x-5)^n$ converges at $x = -5$ and diverges at $x = -10$. At $x = 11$, the series is

   (a) Conditionally convergent
   (b) Absolutely convergent
   (c) Divergent

(d) Cannot be determined.

7. The power series $\sum C_n x^n$ diverges at $x = 7$ and converges at $x = -3$. At $x = -4$, the series is

   (a) Conditionally convergent
   (b) Absolutely convergent
   (c) Divergent
   (d) Cannot be determined.

8. In order to determine if the series

$$\sum_{k=1}^{\infty} \frac{\sqrt{k+1}}{k^2+1}$$

   converges or diverges, the limit comparison test can be used. Decide which series provides the best comparison.

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$
   (b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$
   (c) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$

9. Decide whether the following statements are true or false. Give a brief justification for your answer.

   (a) If f is continuous for all x and $\displaystyle\int_0^{\infty} f(x)dx$ converges, then so does $\displaystyle\int_a^{\infty} f(x)dx$ for all positive a.
   (b) if $f(x)$ is continuous and positive for $x > 0$ and if $\displaystyle\lim_{x\to\infty} f(x) = 0$, then $\displaystyle\int_0^{\infty} f(x)dx$ converges.
   (c) If $\displaystyle\int_0^{\infty} f(x)dx$ and $\int_0^{\infty} g(x)dx$ both converge then $\displaystyle\int_0^{\infty} (f(x)+g(x))dx$ converges.
   (d) If $\displaystyle\int_0^{\infty} f(x)dx$ and $\displaystyle\int_0^{\infty} g(x)dx$ both diverge then $\displaystyle\int_0^{\infty} (f(x)+g(x))dx$ diverges.

10. For this problem, state which of the integration techniques you would use to evaluate the integral, but **DO NOT** evaluate the integrals. If your answer is **substitution**, also list u and du; if your answer is **integration by parts**, also list u, dv, du and v; if your answer is **partial fractions**, set up the partial fraction decomposition, but do not solve for the numerators; if your answer is **trigonometric substitution**, write which substitution you would use.

   (a) $\displaystyle\int \tan x dx$
   (b) $\displaystyle\int \frac{dx}{x^2-9}$
   (c) $\displaystyle\int e^x \cos x dx$
   (d) $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2} dx$
   (e) $\displaystyle\int \frac{\sin(\ln x)}{x} dx$

(f) $\int x^{3/2} \ln x dx$

(g) $\int \frac{1}{\sqrt{x^2+4}} dx$

(h) $\int \frac{e^x+1}{e^x+x} dx$

11. If
$$\sum_{n=1}^{\infty} a_n(x-1)^n = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{4} - \frac{(x-1)^4}{8} + \frac{(x-1)^5}{16} - \cdots,$$
then the correct formula for a_n is:

(a) $a_n = \frac{(-1)^n}{2n}$

(b) $a_n = \frac{(-1)^{n+1}}{2^n}$

(c) $a_n = \frac{(-1)^n}{2^n}$

(d) $a_n = \frac{(-1)^{n+1}2}{2^n}$

12. **True or False?** If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=0}^{\infty} a_n$ converges.

(a) True

(b) False

13. **True or False?** If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=0}^{\infty} a_n$ diverges.

(a) True

(b) False

14. True/False: If $\sum a_n$ is convergent, then the power series $\sum a_n x^n$ has convergence radius at least $R = 1$.

15. If one uses the Taylor polynomial $P_3(x)$ of degree $n = 3$ to approximate $\sin x = \sum_0^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ at $x = 0.1$, would one get an overestimate or an underestimate?

16. Suppose that x is positive but very small. Arrange the following in **increasing** order:
$$x,\ \sin x,\ \ln(1+x),\ 1-\cos(x),\ e^x - 1,\ x\sqrt{1-x}.$$

17. (a) Find the Taylor series about 0 for $f(x) = x^2 e^{x^2}$.

(b) Is this function even or odd? Justify your answer.

(c) Find $f^{(3)}(0)$ and $f^{(6)}(0)$.

18. Suppose that ibuprofen is taken in 200mg doses every six hours, and that all 200mg are delivered to the patient's body immediately when the pill is taken. After six hours, 12.5% of the ibuprofen remains. Find expressions for the amount of ibuprofen in the patient immediately before and after the n^{th} pill taken. Include work; without work, you may receive no credit.

19. Find an equation for a sphere if one of its diameters has endpoints $(2, 1, 4)$ and $(4, 3, 10)$.

20. A cube is located such that its top four corners have the coordinates $(-1,-2,2)$, $(-1,3,2)$, $(4,-2,2)$, and $(4,3,2)$. Give the coordinate of the center of the cube.

21. Find the equation of the largest sphere with center $(5,4,9)$ contained in the first octant.

22. Evaluate the double integral (using the most convenient method):

    (a) $\displaystyle\int_0^1 \int_{e^y}^{e} \frac{x}{\ln x}\, dx\, dy$

    (b) $\displaystyle\int_0^1 \int_x^1 \frac{y^2}{1+y^4}\, dy\, dx$

23. The following sum of double integrals describes the mass of a thin plate $\mathcal{R}$ in the xy-plane, of density $\delta(x,y) = x+y$:

$$\text{mass} = \int_{-4}^{0} \int_{0}^{2x+8} \delta(x,y)\ dy\ dx + \int_{0}^{4} \int_{0}^{-2x+8} \delta(x,y)\ dy\ dx$$

    (a) Describe the thin plate (shape, intersections with the coordinate axes).

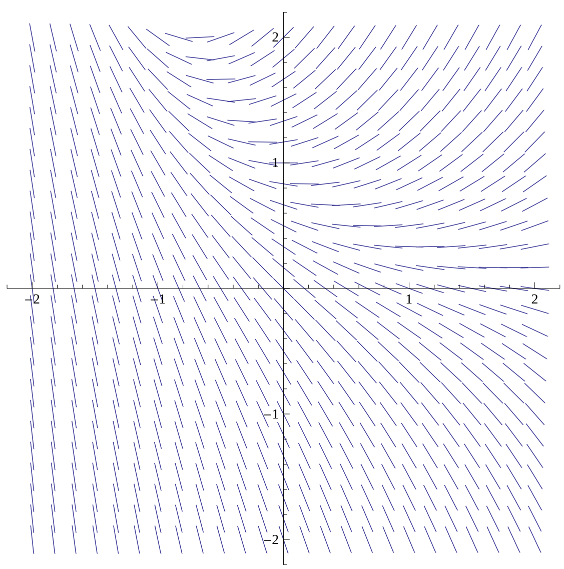
    (b) Write an expression for the mass of the plate as only one double integral.

    (c) Calculate the area and the mass of the plate.

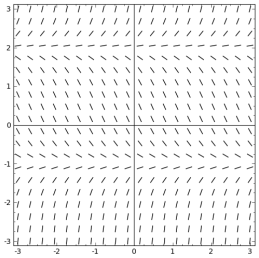
24. If one uses the Taylor polynomial $P_5(x)$ of degree $n=5$ to approximate $e^x = \displaystyle\sum_0^\infty \frac{x^n}{n!}$ at $x=-0.2$, would one get an overestimate or an underestimate?

25. A slope field for the differential equation $y' = y - e^{-x}$ is shown. Sketch the graphs of the solutions that satisfy the given initial conditions. Make sure to label each sketched graph.

    (a) $y(0)=0$ (b) $y(0)=1$ (c) $y(0)=-1$



26. For each differential equation, find the corresponding slope field. (*Not all slope fields will be used.*)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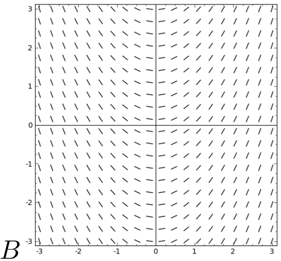
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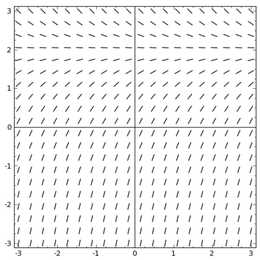
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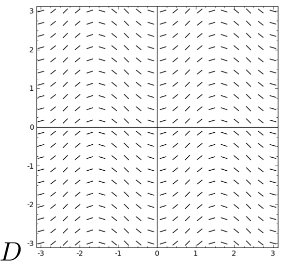
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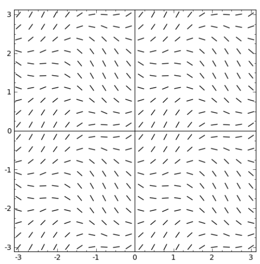
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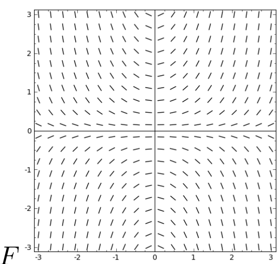
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Equation	**Slope Field**
$\dfrac{dy}{dx} = x$	____
$y' = xy$	____
$y' = \sin(2x)$	____
$\dfrac{dy}{dx} = 2 - y$	____

27. Let $f(x, y) = e^{x^2y+xy^2}$.

(a) Find $f_x(x, y)$.
(b) Find $f_y(x, y)$.
(c) Find $\dfrac{\partial}{\partial x} f_y(x, y)$.
(d) Find $\dfrac{\partial}{\partial y} f_x(x, y)$. What's curious about the relation of this answer to that of part (c) above?

28. Find the equation of the sphere that passes through the origin and whose center is $(1, 2, 3)$.

29. You want to estimate $\sin x$ using the first 3 nonzero terms in the Taylor series. What formula for the error bound would you use to get the best estimate for the error?

30. Find the arc length of the curve $y = x^{3/2}$ from $(1, 1)$ to $(2, 2\sqrt{2})$.

31. Find the arc length of the curve $y = (x^6 + 8)/(16x^2)$ from $x = 2$ to $x = 3$.

32. Let $f(x, y) = x^5 - 10x^3y^2 + 5xy^4$. Show that

$$\frac{\partial}{\partial x} f_x(x, y) + \frac{\partial}{\partial y} f_y(x, y) = 0.$$

33. Additional problems from your text:

(a) Section 8.1 (Areas and Volumes) Exercises 3, 11, 25
(b) Section 8.2 (Volumes by Revolution & Cross Sections) Exercises 9, 23, 35
(c) Section 8.3 (Arc Length and Parametric Curves) Exercises 11, 15, 17, 19
(d) Section 11.4 (Separation of Variables) Exercises 3, 9, 13, 21, 41