ALGEBRAIC SPACES

Contents

1. Introduction

Algebraic spaces were first introduced by Mike Artin, see [Art69c], [Art70], [Art73], [Art71a], [Art71b], [Art69a], [Art69b], and [Art74]. Some of the foundational material was developed jointly with Knutson, who produced the book [Knu71]. Artin defined (see [Art69b, Definition 1.3]) an algebraic space as a sheaf for the etale topology which is locally in the etale topology representable. In most of Artin's work the categories of schemes considered are schemes locally of finite type over a fixed excellent Noetherian base.

Our definition is slightly different. First of all we consider sheaves for the fppf topology. This is just a technical point and scarcely makes any difference. Second, we include the condition that the diagonal is representable.

After defining algebraic spaces we make some foundational observations. The main result in this chapter is that with our definitions an algebraic space is the same thing as an etale equivalence relation, see the discussion in Section 8 and Theorem 9.5. The analogue of this theorem in Artin's setting is [Art69b, Theorem 1.5], or [Knu71, Proposition II.1.7]. In other words, the sheaf defined by an etale equivalence relation has a representable diagonal. It follows that our definition agrees with Artin's original definition in a broad sense. It also means that one can give examples of algebraic spaces by simply writing down an etale equivalence relation.

In Section 13 we introduce various separation axioms on algebraic spaces that we have found in the literatur. Finally in Section 14 we give some weird and not so weird examples of algebraic spaces.

2. General remarks

We work in a suitable big fppf site Sch_{fppf} as in Topologies, Definition 5.6. So, if not explicitly stated otherwise all schemes will be objects of Sch_{fppf}. We will record elsewhere what changes if you change the big fppf site (insert future reference here).

We will always work relative to a base S contained in Sch_{fppf}. And we will then work with the big fppf site $(Sch/S)_{fppf}$, see Topologies, Definition 5.7. The absolute case can be recovered by taking $S = \mathrm{Spec}(\mathbf{Z})$.

If U, T are schemes over S, then we denote $U(T)$ for the set of T-valued points *over* S. In a formula: $U(T) = \mathrm{Mor}_S(T, U)$.

Note that any fpqc covering is a family of universally effective epimorphisms, see Descent, Lemma 5.2. Hence the topology on Sch_{fppf} is weaker than the canonical topology and all representable presheaves are sheaves.

3. Representable morphisms of presheaves

Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$ be a representable transformation of functors, see Categories, Definition 7.2. This means that for every $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $\xi \in G(U)$ the fiber product $h_U \times_{\xi, G} F$ is representable. Choose a representing object V_ξ and an isomorphism $h_{V_\xi} \to h_U \times_G F$. By the Yoneda lemma, see Categories, Lemma 3.5, the projection $h_{V_\xi} \to h_U \times_G F \to h_U$ comes from a unique morphism of schemes $a_\xi : V_\xi \to U$. Suggestively we could represent this by the diagram

$$\begin{tikzcd} V_\xi \arrow[r, rightsquigarrow] \arrow[d, "a_\xi"'] & h_{V_\xi} \arrow[r] \arrow[d] & F \arrow[d, "a"] \\ U \arrow[r, rightsquigarrow] & h_U \arrow[r, "\xi"] & G \end{tikzcd}$$

where the squiggly arrows represent the Yoneda embedding. Here are some lemmas about this notion that work in great generality.

Lemma 3.1. *Let S, X, Y be objects of Sch_{fppf}. Let $f : X \to Y$ be a morphism of schemes. Then*

$$h_f : h_X \longrightarrow h_Y$$

is a representable transformation of functors.

Proof. This is formal and relies only on the fact that the category $(Sch/S)_{fppf}$ has fibre products. $\square$

Lemma 3.2. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$, $b : G \to H$ be representable transformations of functors. Then*

$$b \circ a : F \longrightarrow H$$

is a representable transformation of functors.

Proof. This is entirely formal and works in any category. $\square$

Lemma 3.3. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$ be a representable transformations of functors. Let $b : H \to G$ be any transformation of functors. Consider the fibre product diagram*

$$\begin{tikzcd} H \times_{b,G,a} F \arrow[r, "b'"'] \arrow[d, "a'"'] & F \arrow[d, "a"] \\ H \arrow[r, "b"] & G \end{tikzcd}$$

Then the base change a' is a representable transformation of functors.

Proof. This is entirely formal and works in any category. $\square$

Lemma 3.4. *Let S be a scheme contained in Sch_{fppf}. Let $F_i, G_i : (Sch/S)_{fppf}^{opp} \to Sets$, $i = 1, 2$. Let $a_i : F_i \to G_i$, $i = 1, 2$ be representable transformations of functors. Then*

$$a_1 \times a_2 : F_1 \times F_2 \longrightarrow G_1 \times G_2$$

is a representable transformation of functors.

Proof. Write $a_1 \times a_2$ as the composition $F_1 \times F_2 \to G_1 \times F_2 \to G_1 \times G_2$. The first arrow is the base change of a_1 by the map $G_1 \times F_2 \to G_1$, and the second arrow is the base change of a_2 by the map $G_1 \times G_2 \to G_2$. Hence this lemma is a formal consequence of Lemmas 3.2 and 3.3. $\square$

Lemma 3.5. *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$ be a representable transformation of functors. If G is a sheaf, then so is F.*

Proof. Let $\{\varphi_i : T_i \to T\}$ be a covering of the site $(Sch/S)_{fppf}$. Let $s_i \in F(T_i)$ which satisfy the sheaf condition. Then $\sigma_i = a(s_i) \in G(T_i)$ satisfy the sheaf condition also. Hence there exists a unique $\sigma \in G(T)$ such that $\sigma_i = \sigma|_{T_i}$. By assumption $F' = h_T \times_{\sigma,G,a} F$ is a representable presheaf and hence (see remarks in Section 2) a sheaf. Note that $(\varphi_i, s_i) \in F'(T_i)$ satisfy the sheaf condition also, and hence come from some unique $(\mathrm{id}_T, s) \in F'(T)$. Clearly s is the section of F we are looking for. $\square$

4. Lists of useful properties of morphisms of schemes

For ease of reference we list in the following remarks the properties of morphisms which possess some of the properties required of them in later results.

Remark 4.1. Here is a list of properties/types of morphisms which are *stable under arbitrary base change*:

(1) closed, open, and locally closed immersions, see Schemes, Lemma 18.2,
(2) quasi-compact, see Schemes, Lemma 19.3,
(3) universally closed, see Schemes, Definition 20.1,
(4) (quasi-)separated, see Schemes, Lemma 21.13,
(5) monomorphism, see Schemes, Lemma 23.5
(6) surjective, see Morphisms, Lemma 10.3,
(7) radicial (or universally injective), see Morphisms, Lemma 11.2,
(8) affine, see Morphisms, Lemma 12.8,
(9) quasi-affine, see Morphisms, Lemma 13.5,
(10) (locally) of finite type, see Morphisms, Lemma 15.4,

(11) (locally) quasi-finite, see Morphisms, Lemma 19.12,
(12) (locally) of finite presentation, see Morphisms, Lemma 20.4,
(13) locally of finite type of relative dimension d, see Morphisms, Lemma 22.2,
(14) universally open, see Morphisms, Definition 24.1,
(15) flat, see Morphisms, Lemma 25.7,
(16) syntomic, see Morphisms, Lemma 26.4,
(17) smooth, see Morphisms, Lemma 28.5,
(18) unramified, see Morphisms, Lemma 29.5,
(19) etale, see Morphisms, Lemma 30.7,
(20) proper, see Morphisms, Lemma 35.5,
(21) H-projective, see Morphisms, Lemma 36.8,
(22) (locally) projective, see Morphisms, Lemma 36.9,
(23) finite or integral, see Morphisms, Lemma 37.6,
(24) finite locally free, see Morphisms, Lemma 38.4.

Add more as needed.

Remark 4.2. Of the properties of morphisms which are stable under base change (as listed in Remark 4.1) the following are also *stable under compositions*:

(1) closed, open and locally closed immersions, see Schemes, Lemma 24.3,
(2) quasi-compact, see Schemes, Lemma 19.4,
(3) universally closed, see Schemes, Definition 35.4,
(4) (quasi-)separated, see Schemes, Lemma 21.13,
(5) monomorphism, see Schemes, Lemma 23.4,
(6) surjective, see Morphisms, Lemma 10.2,
(7) radicial (or universally injective), see Morphisms, Lemma 11.3,
(8) affine, see Morphisms, Lemma 12.7,
(9) quasi-affine, see Morphisms, Lemma 13.4,
(10) (locally) of finite type, see Morphisms, Lemma 15.3,
(11) (locally) quasi-finite, see Morphisms, Lemma 19.11,
(12) (locally) of finite presentation, see Morphisms, Lemma 20.3,
(13) universally open, see Morphisms, Definition 24.3,
(14) flat, see Morphisms, Lemma 25.5,
(15) syntomic, see Morphisms, Lemma 26.3,
(16) smooth, see Morphisms, Lemma 28.4,
(17) unramified, see Morphisms, Lemma 29.4,
(18) etale, see Morphisms, Lemma 30.6,
(19) proper, see Morphisms, Lemma 35.4,
(20) H-projective, see Morphisms, Lemma 36.7,
(21) finite or integral, see Morphisms, Lemma 37.5,
(22) finite locally free, see Morphisms, Lemma 38.3.

Add more as needed.

Remark 4.3. Of the properties mentioned which are stable under base change (as listed in Remark 4.1) the following are also *fpqc local on the base* (and a fortiori fppf local on the base):

(1) for immersions we have this for
    (a) closed immersions, see Descent, Lemma 12.17,
    (b) open immersions see Descent, Lemma 12.14, and
    (c) quasi-compact immersions, see Descent, Lemma 12.19,

(2) quasi-compact, see Descent, Lemma 12.1,
(3) universally closed, see Descent, Definition 12.3,
(4) (quasi-)separated, see Descent, Lemmas 12.2, and 12.5,
(5) monomorphism, see Descent, Lemma 12.29,
(6) surjective, see Descent, Lemma 12.6,
(7) radicial (or universally injective), see Descent, Lemma 12.7,
(8) affine, see Descent, Lemma 12.16,
(9) quasi-affine, see Descent, Lemma 12.18,
(10) (locally) of finite type, see Descent, Lemmas 12.8, and 12.10,
(11) (locally) quasi-finite, see Descent, Lemma 12.22,
(12) (locally) of finite presentation, see Descent, Lemmas 12.9, and 12.11,
(13) locally of finite type of relative dimension d, see Descent, Lemma 12.23,
(14) universally open, see Descent, Lemma 12.4,
(15) flat, see Descent, Lemma 12.13,
(16) syntomic, see Descent, Lemma 12.24,
(17) smooth, see Descent, Lemma 12.25,
(18) unramified, see Descent, Lemma 12.26,
(19) etale, see Descent, Lemma 12.27,
(20) proper, see Descent, Lemma 12.12,
(21) finite or integral, see Descent, Lemma 12.21,
(22) finite locally free, see Descent, Lemma 12.28.

Note that the property of being an "immersion" may not be fpqc local on the base, but in Descent, Lemma 13.1 we proved that it is fppf local on the base.

5. Properties of representable morphisms of presheaves

Here is the definition that makes this work.

Definition 5.1. With S, and $a : F \to G$ representable as above. Let P be a property of morphisms of schemes which

(1) is preserved under any base change, see Schemes, Definition 18.3, and
(2) is fppf local on the base, see Descent, Definition 11.1.

In this case we say that a has *property* P if for every $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $\xi \in G(U)$ the resulting morphism of schemes $V_\xi \to U$ has property P.

It is important to note that we will only use this definition for properties of morphisms that are stable under base change, and local in the fppf topology on the base. This is not because the definition doesn't make sense otherwise; rather it is because we may want to give a different definition which is better suited to the property we have in mind.

Remark 5.2. Consider the property $\mathcal{P} =$"surjective". In this case there could be some ambiguity if we say "let $F \to G$ be a surjective map". Namely, we could mean the notion defined in Definition 5.1 above, or we could mean a surjective map of presheaves, see Sites, Definition 3.1. If not mentioned otherwise when discussing morphisms of algebraic spaces we will allways mean the first of the two.

Here is a sanity check.

Lemma 5.3. *Let S, X, Y be objects of Sch_{fppf}. Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{P}$ be as in Definition 5.1. Then $h_X \longrightarrow h_Y$ has propery $\mathcal{P}$ if and only if f has property $\mathcal{P}$.*

Proof. Note that the lemma makes sense by Lemma 3.1. Proof omitted. $\square$

Lemma 5.4. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $\mathcal{P}$ be a property as in Definition 5.1 which is stable under composition. Let $a : F \to G$, $b : G \to H$ be representable transformations of functors. If a and b have property $\mathcal{P}$ so does $b \circ a : F \longrightarrow H$.*

Proof. Note that the lemma makes sense by Lemma 3.2. Proof omitted. $\square$

Lemma 5.5. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $\mathcal{P}$ be a property as in Definition 5.1. Let $a : F \to G$ be a representable transformations of functors. Let $b : H \to G$ be any transformation of functors. Consider the fibre product diagram*

$$\begin{tikzcd} H \times_{b,G,a} F \arrow[r, "b'"] \arrow[d, "a'"'] & F \arrow[d, "a"] \\ H \arrow[r, "b"] & G \end{tikzcd}$$

If a has property $\mathcal{P}$ then also the base change a' has property $\mathcal{P}$.

Proof. Note that the lemma makes sense by Lemma 3.3. Proof omitted. $\square$

Lemma 5.6. *Let S be a scheme contained in Sch_{fppf}. Let $F_i, G_i : (Sch/S)_{fppf}^{opp} \to Sets$, $i = 1, 2$. Let $a_i : F_i \to G_i$, $i = 1, 2$ be representable transformations of functors. Let $\mathcal{P}$ be a property as in Definition 5.1 which is stable under composition. If a_1 and a_2 have property $\mathcal{P}$ so does $a_1 \times a_2 : F_1 \times F_2 \longrightarrow G_1 \times G_2$.*

Proof. Note that the lemma makes sense by Lemma 3.4. Proof omitted. $\square$

Lemma 5.7. *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$,. Let $a : F \to G$ be a representable transformation of functors. Let $\mathcal{P}$, $\mathcal{P}'$ be properties as in Definition 5.1. Suppose that for any morphism of schemes $f : X \to Y$ we have $\mathcal{P}(f) \Rightarrow \mathcal{P}'(f)$. If a has property $\mathcal{P}$ then a has property $\mathcal{P}'$.*

Proof. Formal. $\square$

Here is a characterization of those functors for which the diagonal is representable.

Lemma 5.8. *Let S be a scheme contained in Sch_{fppf}. Let F be a presheaf of sets on $(Sch/S)_{fppf}$. The following are equivalent:*

(1) *The diagonal $F \to F \times F$ is representable.*
(2) *For every scheme U over S, $U/S \in Ob((Sch/S)_{fppf})$ and any $\xi \in F(U)$ the map $\xi : h_U \to F$ is representable.*

Proof. This is completely formal, see Categories, Lemma 7.3. It depends only on the fact that the category $(Sch/S)_{fppf}$ has products of pairs of objects and fibre products, see Topologies, Lemma 5.9. $\square$

In the situation of the lemma, for any morphism $\xi : h_U \to F$ as in the lemma, it makes sense to say that ξ has property P, for any property as in Definition 5.1. In particular this holds for $P =$ "surjective" and $P =$ "etale", see Remark 4.3 above. We will use these in the definition of algebraic spaces below.

6. Algebraic spaces

Here is the definition.

Definition 6.1. Let S be a scheme contained in Sch_{fppf}. An *algebraic space over* S is a presheaf

$$F : (Sch/S)_{fppf}^{opp} \longrightarrow Sets$$

with the following properties

(1) The presheaf F is a sheaf.
(2) The diagonal morphism $F \to F \times F$ is representable.
(3) There exists a scheme $U \in \mathrm{Ob}(Sch_{fppf})$ and a map $h_U \to F$ which is surjective, and etale.

There are two differences with the "usual" definition, for example the definition in Knutson's book [Knu71].

The first is that we require F to be a sheaf in the fppf topology. One reason for doing this is that many natural examples of algebraic spaces satisfy the sheaf condition for the fppf coverings (and even for fpqc coverings). Also, one of the reasons that algebraic spaces have been so useful is via Mike Artin's results on algebraic spaces. Built into his method is a condition which garantees the result is locally of finite presentation over S. Combined it somehow seems to us that the fppf topology is the natural topology to work with. In the end the resulting category of algebraic spaces ends up being "the same". Namely, allthough the actual sheaves F being considered may be different, in the end the category of algebraic spaces defined using sheaves in the etale topology is equivalent the the category we define here. This will be clear later when we introduce presentations (insert future reference here).

The second is that we only require the diagonal map for F to be representable, whereas in [Knu71] it is required that it also be quasi-compact. If $F = h_U$ for some scheme U over S this corresponds to the condition that S be quasi-separated. Our point of view is to try to prove a certain number of the results that follow only assuming that the diagonal of F be representable, and simply add an addition hypothesis wherever this is necessary. In any case it has the pleasing consquence that the following lemma is true.

Lemma 6.2. *A scheme is an algebraic space. More precisely, given a scheme $T \in Ob((Sch/S)_{fppf})$ the representable functor h_T is an algebraic space.*

Proof. The functor h_T is a sheaf by our remarks in Section 2. The diagonal $h_T \to h_T \times h_T = h_{T\times T}$ is representable because $(Sch/S)_{fppf}$ has fibre products. The identity map $h_T \to h_T$ is surjective etale. $\square$

Definition 6.3. Let F, F' be algebraic spaces over S. A *morphism $f : F \to F'$ of algebraic spaces over* S is a transformation of functors from F to F'.

The category of algebraic spaces over S contains the category $(Sch/S)_{fppf}$ as a full subcategory via the Yoneda embedding $T/S \mapsto h_T$. From now on we no longer distinghuish between a scheme T/S and the algebraic space it represents. Thus when we say "Let $f : T \to F$ be a morphism from the scheme T to the algebraic space F", we mean that $T \in \mathrm{Ob}((Sch/S)_{fppf})$, that F is an algebraic space over S, and that $f : h_T \to F$ is a morphism of algebraic spaces over S.

7. Glueing algebraic spaces

In this section we really start abusing notation and not distinguish between schemes and the spaces they represent.

Lemma 7.1. *Let $S \in Ob(Sch_{fppf})$. Let $U \in Ob(Sch/S)_{fppf}$. Given a set I and sheaves F_i on $Ob(Sch/S)_{fppf}$, if $U \cong \coprod_{i \in I} F_i$ as sheaves, then each F_i is representable by an open and closed subscheme U_i and $U \cong \coprod U_i$ as schemes.*

Proof. By assumption this means there exists an fppf covering $\{U_j \to U\}_{j \in J}$ such that each $U_j \to U$ factors through $F_{i(j)}$ for some $i(j) \in I$. Denote $V_j = \mathrm{Im}(U_j \to U)$. This is an open of U by Morphisms, Lemma 25.8, and $\{U_j \to V_j\}$ is an fppf covering. Hence it follows that $V_j \to U$ factors through $F_{i(j)}$ since $F_{i(j)}$ is a subsheaf. It follows from $F_i \cap F_{i'} =$, $i \neq i'$ that $V_j \cap V_{j'} = \emptyset$ unless $i(j) = i(j')$. Hence we can take $U_i = \bigcup_{j,\ i(j)=i} V_j$ and everything is clear. □

Lemma 7.2. *Let $S \in Ob(Sch_{fppf})$. Let F be an algebraic space over S. Given a set I and sheaves F_i on $Ob(Sch/S)_{fppf}$, if $F \cong \coprod_{i \in I} F_i$ as sheaves, then each F_i is an algebraic space over S.*

Proof. It follows directly from the representability of $F \to F \times F$ that each diagonal morphism $F_i \to F_i \times F_i$ is representable. Choose a scheme U in $(Sch/S)_{fppf}$ and a surjective etale morphism $U \to \coprod F_i$ (this exist by hypothesis). By considering the inverse image of F_i we get a decomposition of U (as a sheaf) into a coproduct of sheaves. By Lemma 7.1 we get correspondingly $U \cong \coprod U_i$. Then it follows easily that $U_i \to F_i$ is surjective and etale (from the corresponding property of $U \to F$). □

The condition on the size of I in the following lemma may be ignored by those not worried about set theoretic questions.

Lemma 7.3. *Let $S \in Ob(Sch_{fppf})$. Suppose given a set I and algebraic spaces F_i, $i \in I$. Then $F = \coprod_{i \in I} F_i$ is an algebraic space provided I is not too large: for example given surjective etale morphisms $U_i \to F_i$ such that $\coprod U_i$ is isomorphic to an object of $(Sch/S)_{fppf}$, then F is an algebraic space.*

Proof. By construction F is a sheaf. We omit the verification that the diagonal morphism of F is representable. Finally, if U is an object of $(Sch/S)_{fppf}$ isomorphic to $\coprod_{i \in I} U_i$ then it is straightforward to verify that the resulting map $U \to \coprod F_i$ is surjective and etale. □

Here is the analogue of Schemes, Lemma 15.4.

Lemma 7.4. *Let $S \in Ob(Sch_{fppf})$. Let F be a presheaf of sets on $(Sch/S)_{fppf}$. Assume*

1. *F is a sheaf,*
2. *there exists an index set I and subfunctors $F_i \subset F$ such that*
   - (a) *$\coprod F_i$ is an algebraic space[1],*
   - (b) *each $F_i \to F$ is a representable,*
   - (c) *each $F_i \to F$ is an open immersion (see Definition 5.1 and Remark 4.3), and*
   - (d) *the map of sheaves $\coprod F_i \to F$ is surjective.*

[1] This basically just means each F_i is an algebraic space, see Lemmas 7.2 and 7.3.

Then F is an algebraic space.

Proof. Let T, T' be objects of $(Sch/S)_{fppf}$. Let $T \to F$, $T' \to F$ morphisms. The assumptions imply that there exists an open covering $T = \bigcup V_i$ such that $V_i = T \times_F F_i$. Note that this in particular implies that $\coprod F_i \to F$ is surjective in the Zariski topology! Also write similarly $T' = \bigcup V_i'$ with $V_i' = T' \times_F F_i$.

To show that the diagonal $F \to F \times F$ is representable we have to show that $G = T \times_F T'$ is representable. Consider the subfunctors $G_i = G \times_F F_i$. Note that $G_i = V_i \times_{F_i} V_i'$, and hence is representable as F_i is an algebraic space. By the above the G_i form a Zariski covering of F. Hence by Schemes, Lemma 15.4 we see G is representable.

Choose a scheme $U \in \mathrm{Ob}(Sch/S)_{fppf}$ and a surjective etale morphism $U \to \coprod F_i$ (this exist by hypothesis). We may write $U = \coprod U_i$ with U_i the inverse image of F_i, see Lemma 7.1. We claim that $U \to F$ is surjective and etale. Surjectivity follows as $\coprod F_i \to F$ is surjective. Consider the fibre product $U \times_F T$ where $T \to F$ is as above. We have to show that $U \times_F T \to T$ is etale. Since $U \times_F T = \coprod U_i \times_F T$ it suffices to show each $U_i \times_F T \to T$ is etale. Since $U_i \times_F T = U_i \times_{F_i} V_i$ this follows from the fact that $U_i \to F_i$ is etale and $V_i \to T$ is an open immersion (and Morphisms, Lemmas 30.8 and 30.6). □

8. Presentations of algebraic spaces

Given an algebraic space we can find a "presentation" of it.

Lemma 8.1. *Let F be an algebraic space over S. Let $f : U \to F$ be a surjective etale morphism from a scheme to F. Set $R = U \times_F U$. Then*

1. *$j : R \to U \times_S U$ defines an equivalence relation on U over S (see Groupoids, Definition 3.1).*
2. *the morphisms $s, t : R \to U$ are etale, and*
3. *the diagram*
$$R \rightrightarrows U \longrightarrow F$$
is a coequalizer diagram in $Sh((Sch/S)_{fppf})$.

Proof. Let T/S be an object of $(Sch/S)_{fppf}$. Then $R(T) = \{(a, b) \in U(T) \times U(T) \mid f \circ a = f \circ b\}$ which is clearly defines an equivalence relation on $U(T)$. The morphisms $s, t : R \to U$ are etale because the morphism $U \to F$ is etale.

To prove (3) we first show that $U \to F$ is a surjection of sheaves, see Sites, Definition 11.1. Let $\xi \in F(T)$ with T as above. Let $V = T \times_{\xi, F, f} U$. By assumption V is a scheme and $V \to T$ is surjective etale. Hence $\{V \to T\}$ is a covering for the fppf topology. Since $\xi|_V$ factors through U by construction we conclude $U \to F$ is surjective. To conclude we have to show that given any two morphisms $a, b : T \to U$ such that $f \circ a = f \circ b$ there is a morphism $c : T \to R$ such that $a = \mathrm{pr}_0 \circ c$ and $b = \mathrm{pr}_1 \circ b$. This is clear from the definition of R. □

This lemma suggests the following definitions.

Definition 8.2. Let S be a scheme. Let U be a scheme over S. An *etale equivalence relation* on U over S is an equivalence relation $j : R \to U \times_S U$ such that $s, t : R \to U$ are etale morphisms of schemes.

Definition 8.3. Let F be an algebraic space over S. A *presentation* of F is given by a scheme U over S and an etale equivalence relation R on U over S, and a surjective etale morphism $U \to F$ such that $R = U \times_F U$.

Equivalently we could ask for the existence of an isomorphism

$$U/R \cong F$$

where the quotient U/R is as defined in Groupoids, Section 8. To construct algebraic spaces we will study the converse question, namely, for which equivalence relations the quotient sheaf U/R is an algebraic space. It will finally turn out this is always the case if R is an etale equivalence relation on U over S, see Theorem 9.5.

9. Algebraic spaces and equivalence relations

Suppose given a scheme U over S and an etale equivalence relation R on U over S. We would like to show this defines an algebraic space. We will produce a series of lemmas that prove the quotient sheaf U/R (see Groupoids, Definition 8.1) has all the properties required of it in Definition 6.1.

Lemma 9.1. *Let S be a scheme. Let U be a scheme over S. Let $j = (s, t) : R \to U \times_S U$ be an etale equivalence relation on U over S. Let $U' \to U$ be an etale morphism. Let R' be the restriction of R to U', see Groupoids, Definition 3.3. Then $j' : R' \to U' \times_S U'$ is an etale equivalence relation also.*

Proof. It is clear from the description of s', t' in Groupoids, Lemma 7.1 that $s', t' : R' \to U'$ are etale as compositions of base changes of etale morphisms (see Morphisms, Lemma 30.7 and 30.6). $\square$

Lemma 9.2. *Let S be a scheme. Let U be a scheme over S. Let $j = (s, t) : R \to U \times_S U$ be a pre-relation. Let $g : U' \to U$ be a morphism. Assume*

1. *j is an equivalence relation,*
2. *$s, t : R \to U$ are surjective, flat and locally of finite presentation,*
3. *g is flat and locally of finite presentation.*

Let $R' = R|_{U'}$ be the restriction of R to U. Then $R'/U' \to R/U$ is representable, and is an open immersion.

Proof. By Groupoids, Lemma 3.2 the morphism $j' = (t', s') : R' \to U' \times_S U'$ defines an equivalence relation. Since g is flat and locally of finite presentation we see that g is universally open as well (Morphisms, Lemma 25.8). For the same reason s, t are universally open as well. Let $W^1 = g(U') \subset U$, and let $W = t(s^{-1}(W^1))$. Then W^1 and W are open in U. Moreover, as j is an equivalence relation we have $t(s^{-1}(W)) = W$.

By Groupoids, Lemma 8.4 the map of sheaves $F' = U'/R' \to F = U/R$ is injective. Let $a : T \to F$ be a morphism from a scheme into U/R. We have to show that $T \times_F F'$ is representable by an open subscheme of T.

The morphism a is given by the following data: an fppf covering $\{\varphi_j : T_j \to T\}_{j \in J}$ of T and morphsms $a_j : T_j \to U$ such that the maps

$$a_j \times a_{j'} : T_j \times_T T_{j'} \longrightarrow U \times_S U$$

factor through $j : R \to U \times_S U$ via some (unique) maps $r_{jj'} : T_j \times_T T_{j'} \to R$. The system (a_j) corresponds to a in the sense that the diagrams

$$\begin{tikzcd} T_j \arrow[r, "a_j"'] \arrow[d] & U \arrow[d] \\ T \arrow[r, "a"] & F \end{tikzcd}$$

commute.

Consider the open subsets $W_j = a_j^{-1}(W) \subset T_j$. Since $t(s^{-1}(W)) = W$ we see that

$$W_j \times_T T_{j'} = r_{jj'}^{-1}(t^{-1}(W)) = r_{jj'}^{-1}(s^{-1}(W)) = T_j \times_T W_{j'}.$$

By Morphisms, Lemma 25.9 and Descent, Lemma 5.1 applied to $\coprod T_j \to T$ this means there exists an open $W_T \subset T$ such that $\varphi_j^{-1}(W_T) = W_j$ for all $j \in J$. We claim that $W_T \to T$ represents $T \times_F F' \to T$.

First, let us show that $W_T \to T \to F$ is an element of $F'(W_T)$. Since $\{W_j \to W_T\}_{j \in J}$ is an fppf covering of W_T, it is enough to show that each $W_j \to U \to F$ is an element of $F'(W_j)$ (as F' is a sheaf for the fppf topology). Consider the commutative diagram

$$\begin{tikzcd} W'_j \arrow[rr] \arrow[dd] \arrow[dr] & & U' \arrow[d, "g"] \\ & s^{-1}(W^1) \arrow[r, "s"'] \arrow[d, "t"] & W^1 \arrow[d] \\ W_j \arrow[r, "a_j|_{W_j}"] & W \arrow[r] & F \end{tikzcd}$$

where $W'_j = W_j \times_W s^{-1}(W^1) \times_{W^1} U'$. Since t and g are surjective, flat and locally of finite presentation, so is $W'_j \to W_j$. Hence the restriction of the element $W_j \to U \to F$ to W'_j is an element of F' as desired.

Suppose that $f : T' \to T$ is a morphism of schemes such that $a|_{T'} \in F'(T')$. We have to show that f factors through the open W_T. Since $\{T' \times_T T_j \to T\}$ is an fppf covering of T' it is enough to show each $T' \times_T T_j \to T$ factors through W_T. Hence we may assume f factors as $\varphi_j \circ f_j : T' \to T_j \to T$ for some j. In this case the condition $a|_{T'} \in F'(T')$ means that there exists some fppf covering $\{\psi_i : T'_i \to T'\}_{i \in I}$ and some morphisms $b_i : T'_i \to U'$ such that

$$\begin{tikzcd} T'_i \arrow[r, "b_i"'] \arrow[d, "f_j \circ \psi_i"'] & U' \arrow[r, "g"'] & U \arrow[d] \\ T_j \arrow[r, "a_j"] & U \arrow[r] & F \end{tikzcd}$$

is commutative. This commutativity means that there exists a morphism $r'_i : T'_i \to R$ such that $t \circ r'_i = a_j \circ f_j \circ \psi_i$, and $s \circ r'_i = g \circ b_i$. This implies that $\mathrm{Im}(f_j \circ \psi_i) \subset W_j$ and we win. $\square$

The following lemma is not completely trivial although it looks like it should be trivial.

Lemma 9.3. *Let S be a scheme. Let U be a scheme over S. Let $j = (s,t) : R \to U \times_S U$ be an etale equivalence relation on U over S. If the quotient U/R is an algebraic space, then $U \to U/R$ is etale and surjective. Hence $(U, R, U \to U/R)$ is a presentation of the algebraic space U/R.*

Proof. Denote $c : U \to U/R$ the morphism in question. Let T be a scheme and let $a : T \to U/R$ be a morphism. We have to show that the morphism (of schemes) $\pi : T \times_{a,R/U,c} U \to T$ is etale and surjective. The morphism a corresponds to an fppf covering $\{\varphi_i : T_i \to T\}$ and morphisms $a_i : T_i \to U$ such that $a_i \times a_{i'} : T_i \times_T T_{i'} \to U \times_S U$ factors through R, and such that $c \circ a_i = \varphi_i \circ a$. Hence

$$T_i \times_{\varphi_i,T} T \times_{a,R/U,c} U = T_i \times_{c\circ a_i,R/U,c} U = T_i \times_{a_i,U} U \times_{c,R/U,c} U = T_i \times_{a_i,U,t} R.$$

Since t is etale and surjective we conclude that the base change of π to T_i is surjective and etale. Since the property of being surjective and etale is local on the base in the fpqc topology (see Remark 4.3) we win. $\square$

Lemma 9.4. *Let S be a scheme. Let U be a scheme over S. Let $j = (s,t) : R \to U \times_S U$ be an etale equivalence relation on U over S. Assume that U is affine. Then the quotient $F = U/R$ is an algebraic space, and $U \to F$ is etale and surjective.*

Proof. Since $j : R \to U \times_S U$ is a monomorphism we see that j is separated (see Schemes, Lemma 23.3). Since U is affine we see that $U \times_S U$ (which comes equipped with a monomorphism into the affine scheme $U \times U$) is separated. Hence we see that R is separated. In particular the morphisms s, t are separated as well as etale.

Since the compostition $R \to U \times_S U \to U$ is locally of finite type we conclude that j is locally of finite type (see Morphisms, Lemma 15.8). As j is also a monomorphism it has finite fibres and we see that j is locally quasi-finite by Morphisms, Lemma 19.6. Alltogether we see that j is separated and locally quasi-finite.

Our first step is to show that the quotient map $c : U \to F$ is representable. Consider a scheme T and a morphism $a : T \to F$. We have to show that the sheaf $G = T \times_{a,F,c} U$ is representable. As seen in the proofs of Lemmas 9.2 and 9.3 there exists an fppf covering $\{\varphi_i : T_i \to T\}_{i\in I}$ and morphisms $a_i : T_i \to U$ such that $a_i \times a_{i'} : T_i \times_T T_{i'} \to U \times_S U$ factors through R, and such that $c \circ a_i = \varphi_i \circ a$. As in the proof of Lemma 9.3 we see that

$$\begin{aligned} T_i \times_{\varphi_i,T} G &= T_i \times_{\varphi_i,T} T \times_{a,R/U,c} U \\ &= T_i \times_{c\circ a_i,R/U,c} U \\ &= T_i \times_{a_i,U} U \times_{c,R/U,c} U \\ &= T_i \times_{a_i,U,t} R \end{aligned}$$

Since t is separated and etale, and in particular separated and locally quasi-finite (by Morphisms, Lemmas 29.9 and 30.14) we see that the restriction of G to each T_i is representable by a morphism of schemes $X_i \to T_i$ which is separated and locally quasi-finite. By Descent, Lemma 26.1 we obtain a descent datum $(X_i, \varphi_{ii'})$ relative to the fppf-covering $\{T_i \to T\}$. Since each $X_i \to T_i$ is separated and locally quasi-finite we see by More on Morphisms, Lemma 11.1 that this descent datum is effective. Hence by Descent, Lemma 26.1 (2) we conclude that G is representable as desired.

The second step of the proof is to show that $U \to F$ is surjective and etale. This is clear from the above since in the first step above we saw that $G = T \times_{a,F,c} U$ is a scheme over T which base changes to schemes $X_i \to T_i$ which are surjective and etale. Thus $G \to T$ is surjective and etale (see Remark 4.3). Alternatively one can reread the proof of Lemma 9.3 in the current situation.

The third and final step is to show that the diagonal map $F \to F \times F$ is representable. We first observe that the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, "j"'] & F \arrow[d, "\Delta"] \\ U \times_S U \arrow[r] & F \times F \end{tikzcd}$$

is a fibre product square. By Lemma 3.4 the morphism $U \times_S U \to F \times F$ is representable (note that $h_U \times h_U = h_{U \times_S U}$). Moreover, by Lemma 5.6 the morphism $U \times_S U \to F \times F$ is surjective and etale (note also that etale and surjective occur in the lists of Remarks 4.3 and 4.2). It follows either from Lemma 3.3 and the diagram above, or by writing $R \to F$ as $R \to U \to F$ and Lemmas 3.1 and 3.2 that $R \to F$ is representable as well. Let T be a scheme and let $a : T \to F \times F$ be a morphism. We have to show that $G = T \times_{a,F \times F,\Delta} F$ is representable. By what was said above the morphism (of schemes)

$$T' = (U \times_S U) \times_{F \times F,a} T \longrightarrow T$$

is surjective and etale. Hence $\{T' \to T\}$ is an etale covering of T. Note also that

$$T' \times_T G = T' \times_{U \times_S U,j} R$$

as can be seen contemplating the following cube

$$\begin{tikzcd} & R \arrow[rr] \arrow[dd] & & F \arrow[dd] \\ T' \times_T G \arrow[ur] \arrow[rr, crossing over] \arrow[dd] & & G \arrow[ur] & \\ & U \times_S U \arrow[rr] & & F \times F \\ T' \arrow[ur] \arrow[rr] & & T \arrow[ur] \arrow[from=uu, crossing over] & \end{tikzcd}$$

Hence we see that the restriction of G to T' is representable by a scheme X, and moreover that the morphism $X \to T'$ is a base change of the morphism j. Hence $X \to T'$ is separated and locally quasi-finite (see second paragraph of the proof). By Descent, Lemma 26.1 we obtain a descent datum (X, φ) relative to the fppf-covering $\{T' \to T\}$. Since $X \to T$ is separated and locally quasi-finite we see by More on Morphisms, Lemma 11.1 that this descent datum is effective. Hence by Descent, Lemma 26.1 (2) we conclude that G is representable as desired. $\square$

Theorem 9.5. *Let S be a scheme. Let U be a scheme over S. Let $j = (s,t) : R \to U \times_S U$ be an etale equivalence relation on U over S. Then the quotient U/R is an algebraic space, and $U \to U/R$ is etale and surjective, in other words $(U, R, U \to U/R)$ is a presentation of U/R.*

Proof. By Lemma 9.3 it suffice to just prove that U/R is an algebraic space. Let $U' \to U$ be a surjective, etale morphism. Then $\{U' \to U\}$ is in particular an fppf covering. Let R' be the restriction of R to U', see Groupoids, Definition 3.3. According to Groupoids, Lemma 8.4 we see that $U/R \cong U'/R'$. By Lemma 9.1 R' is an etale equivalence relation on U'. Thus we may replace U by U'.

We apply the previous remark to $U' = \coprod U_i$, where $U = \bigcup U_i$ is an affine open covering of S. Hence we may and do assume that $U = \coprod U_i$ where each U_i is an affine scheme.

Consider the restriction R_i of R to U_i. By Lemma 9.1 this is an etale equivalence relation. Set $F_i = U_i/R_i$ and $F = U/R$. It is clear that $\coprod F_i \to F$ is surjective. By Lemma 9.2 each $F_i \to F$ is representable, and an open immersion. By Lemma 9.4 applied to (U_i, R_i) we see that F_i is an algebraic space. Then by Lemma 9.3 we see that $U_i \to F_i$ is etale and surjective. From Lemma 7.3 it follows that $\coprod F_i$ is an algebraic space. Finally, we have verified all hypotheses of Lemma 7.4 and it follows that $F = U/R$ is an algebraic space. $\square$

10. Algebraic spaces, retrofitted

We start building our arsenal of lemmas dealing with algebraic spaces.

Lemma 10.1. *Let S be a scheme contained in Sch_{fppf}. Let F be an algebraic space over S. Let $G \to F$ be a representable transformation of functors. Then G is an algebraic space.*

Proof. By Lemma 3.5 we see that G is a sheaf. The diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\Delta_G"'] & F \arrow[d, "\Delta_F"] \\ G \times G \arrow[r] & F \times F \end{tikzcd}$$

is cartesian. Hence we see that Δ_G is representable by Lemma 3.3. Finally, let U be an object of $(Sch/S)_{fppf}$ and let $U \to F$ be surjective and etale. By assumption $U \times_F G$ is representable by a scheme U'. By Lemma 5.5 the morphism $U' \to G$ is surjective and etale. This verifies the final condition of Definition 6.1 and we win. $\square$

Lemma 10.2. *Let S be a scheme contained in Sch_{fppf}. Let F, G be algebraic spaces over S. Let $G \to F$ be a representable morphism. Let $U \in Ob((Sch/S)_{fppf})$, and $q : U \to F$ surjective and etale. Set $V = G \times_F U$. Finally, let $\mathcal{P}$ be a property of morphisms of schemes as in Definition 5.1. Then $G \to F$ has property $\mathcal{P}$ if and only if $V \to U$ has property $\mathcal{P}$.*

Proof. It is clear from the definitions that if $G \to F$ has property $\mathcal{P}$, then $V \to U$ has property $\mathcal{P}$. Conversely, assume $V \to U$ has property $\mathcal{P}$. Let $T \to F$ be a morphism from a scheme to F. Let $T' = T \times_F G$ which is a scheme since $G \to F$ is representable. We have to show that $T' \to T$ has property T. Consider the commutative diagram of schemes

$$\begin{tikzcd} V \arrow[d] & T \times_F V \arrow[l] \arrow[r] \arrow[d] & T \times_F G \arrow[r, equal] \arrow[d] & T' \\ U & T \times_F U \arrow[l] \arrow[r] & T & \end{tikzcd}$$

where both squares are fibre product squares. Hence we conclude the middle arrow has property $\mathcal{P}$ as a base change of $V \to U$. Finally, $\{T \times_F U \to T\}$ is a fppf covering as it is surjective etale, and hence we conclude that $T' \to T$ has property $\mathcal{P}$ as it is local on the base in the fppf topology. □

Lemma 10.3. *Let S be a scheme contained in Sch_{fppf}. Let F, G be algebraic spaces over S. Then $F \times G$ is an algebraic space, and is a product in the category of algebraic spaces over S.*

Proof. It is clear that $H = F \times G$ is a sheaf. The diagonal of H is simply the product of the diagonals of F and G. Hence it is representable by Lemma 3.4. Finally, if $U \to F$ and $V \to G$ are surjective etale morphisms, with $U, V \in \mathrm{Ob}((Sch/S)_{fppf})$, then $U \times V \to F \times G$ is surjective etale by Lemma 5.6. □

Lemma 10.4. *Let S be a scheme contained in Sch_{fppf}. Let F, G be algebraic spaces over S. Let $a : F \to G$ be a morphism. Given any $V \in Ob((Sch/S)_{fppf})$ and a surjective etale morphism $q : V \to G$ there exists a $U \in Ob((Sch/S)_{fppf})$ and a commutative diagram*

$$\begin{tikzcd} U \arrow[r, "\alpha"] \arrow[d, "p"'] & V \arrow[d, "q"] \\ F \arrow[r, "a"] & G \end{tikzcd}$$

with p surjective and etale.

Proof. First choose $W \in \mathrm{Ob}((Sch/S)_{fppf})$ with surjective etale morphism $W \to F$. Next, put $U = W \times_G V$. Since G is an algebraic space we see that U is isomorphic to an object of $(Sch/S)_{fppf}$. As q is surjective etale, we see that $U \to W$ is surjective etale (see Lemma 5.5). Thus $U \to F$ is surjective etale as a composition of surjective etale morphisms (see Lemma 5.4). □

Lemma 10.5. *Let S be a scheme contained in Sch_{fppf}. Let F, G, H be algebraic spaces over S. Let $a : F \to H$, $b : G \to H$ be morphisms of algebraic spaces. Then $F \times_H G$ is an algebraic space, and is a fibre product in the category of algebraic spaces over S.*

Proof. It is clear that $E = F \times_H G$ is a sheaf. The diagonal of E is the left vertical arrow in

$$\begin{tikzcd} F \times_H G \arrow[r] \arrow[d, "\Delta"'] & F \times G \arrow[d, "\Delta"] \\ (F \times F) \times_{(H \times H)} (G \times G) \arrow[r] & (F \times F) \times (G \times G) \end{tikzcd}$$

which is cartesian. Hence the diagonal Δ_E is representable as the base change of the morphism on the right which is representable (use Lemmas 3.4 and 3.3). Finally, let $W \in \mathrm{Ob}((Sch/S)_{fppf})$ and $q : W \to H$ be surjective and etale. By Lemma 10.4 there exist $U, V \in \mathrm{Ob}((Sch/S)_{fppf})$, morphisms $\alpha : U \to W$ and $\beta : V \to W$ and surjective etale morphisms $p : U \to F$ and $r : V \to G$ such that $q \circ \alpha = a \circ p$, and $q \circ \beta = b \circ r$. We claim that the morphism $U \times_{\alpha, W, \beta} V \to F \times_{a, H, b} G$ is surjective and etale. OK, and now we see that the diagrams

$$\begin{array}{ccc} U \times_H V & \longrightarrow & U \times V \\ \downarrow & & \downarrow \\ F \times_H G & \longrightarrow & F \times G \end{array} \qquad \begin{array}{ccc} U \times_W V & \longrightarrow & W \\ \downarrow & & \downarrow \\ U \times_H V & \longrightarrow & W \times_H W \end{array}$$

are cartesian. Hence it suffices (by Lemmas 5.5 and Lemmas 5.4) to show that $U \times V \to F \times G$ and $W \to W \times_H W$ are surjective and etale. For the first we use Lemma 5.6. Denote $R = W \times_H W$. Then R is an etale equivalence relation on W over S, see Lemma 8.1. Each of the morphisms $s, t : R \to W$ is etale and the composition $W \to R \to W$ is the identity. Hence $W \to R$ is etale by Morphisms, Lemma 30.15. This proves that E is an algebraic space. It is clear that E is a fibre product in the category of algebraic spaces over S since that is a full subcategory of the category of (pre)sheaves of sets on $(Sch/S)_{fppf}$. ☐

11. Morphisms representable by algebraic spaces

Here we define the notion of one presheaf being relatively representabe by algebraic spaces over another, and we prove some properties of this notion.

Definition 11.1. Let S be a scheme contained in Sch_{fppf}. Let F, G be presheaves on Sch_{fppf}/S. We say a morphism $a : F \to G$ is *representable by algebraic spaces* if for every $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $\xi \in G(U)$ the fiber product $h_U \times_{\xi, G} F$ is an algebraic space.

Here is a sanity check.

Lemma 11.2. *Let S be a scheme in Sch_{fppf}. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then f is representable by algebraic spaces.*

Proof. This is formal and relies only on the fact that the category of algebraic spaces over S has fibre products, see Lemma 10.5. ☐

Lemma 11.3. *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$ be representable by algebraic spaces. If G is a sheaf, then so is F.*

Proof. (Same as the proof of Lemma 3.5.) Let $\{\varphi_i : T_i \to T\}$ be a covering of the site $(Sch/S)_{fppf}$. Let $s_i \in F(T_i)$ which satisfy the sheaf condition. Then $\sigma_i = a(s_i) \in G(T_i)$ satisfy the sheaf condition also. Hence there exists a unique $\sigma \in G(T)$ such that $\sigma_i = \sigma|_{T_i}$. By assumption $F' = h_T \times_{\sigma, G, a} F$ is a sheaf. Note that $(\varphi_i, s_i) \in F'(T_i)$ satisfy the sheaf condition also, and hence come from some unique $(\mathrm{id}_T, s) \in F'(T)$. Clearly s is the section of F we are looking for. ☐

The following lemma is actually slightly tricky.

Lemma 11.4. *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$ be representable by algebraic spaces. If G is an algebraic space, then so is F.*

Proof. We have seen in Lemma 11.3 that F is a sheaf.

Let U be a scheme and let $U \to G$ be a surjective etale morphism. In this case $U \times_G F$ is an algebraic space. Let W be a scheme and let $W \to U \times_G F$ be a surjective etale morphism.

First we claim that $W \to G$ is representable. To see this let X be a scheme and let $X \to F$ be a morphism. Then

$$W \times_F X = W \times_{U \times_G F} U \times_G F \times_F X = W \times_{U \times_G F} (U \times_G X)$$

Since both $U \times_G F$ and G are algebraic spaces we see that this is a scheme.

Next, we claim that $W \to G$ is surjective and etale (this makes sense now that we know it is representable). This follows from the formula above since both $W \to U \times_G F$ and $U \to G$ are etale and surjective, hence $W \times_{U \times_G F} (U \times_G X) \to U \times_G X$ and $U \times_G X \to X$ are surjective and etale, and the composition of surjective etale morphisms is surjective and etale.

Set $R = W \times_F W$. By the above the projections $t, s : R \to W$ are etale. It is clear that R is an equivalence relation, and $W \to F$ is a surjection of sheaves. Hence R is an etale equivalence relation and $F = W/R$. Hence F is an algebraic space by Theorem 9.5. ☐

12. Immersions and Zariski coverings of algebraic spaces

At this point an intersting phenomenon occurs. We have already defined the notion of an open immersion of algebraic spaces (through Definition 5.1) but we have yet to define the notion of a *point*[2]. Thus the *Zariski topology* of an algebraic space has already been defined, but there is no space yet!

Perhaps superfluously we formally introduce immersions as follows.

Definition 12.1. Let $S \in \mathrm{Ob}(Sch_{fppf})$ be a scheme. Let F be an algebraic space over S.

(1) A morphism of algebraic spaces over S is called an *open immersion* if it is an open immersion in the sense of Definition 5.1.
(2) An *open subspace* of F is a subfunctor $F' \subset F$ such that F' is an algebraic space and $F' \to F$ is an open immersion.
(3) A morphism of algebraic spaces over S is called a *closed immersion* if it is a closed immersion in the sense of Definition 5.1.
(4) A *closed subspace* of F is a subfunctor $F' \subset F$ such that F' is an algebraic space and $F' \to F$ is a closed immersion.
(5) A morphism of algebraic spaces over S is called an *immersion* if it is an immersion in the sense of Definition 5.1.
(6) A *locally closed subspace* of F is a subfunctor $F' \subset F$ such that F' is an algebraic space and $F' \to F$ is an immersion.

We note that these definitions make sense since an immersion is in particular a monomorphism (see Schemes, Lemma 23.7 and Lemma 5.7), and hence the image of an immersion $G \to F$ of algebraic spaces is a subfunctor $F' \subset F$ which is (canonically) isomorphic to G. Thus some of the discussion of Schemes, Section 10 carries over to the setting of algebraic spaces.

Lemma 12.2. *Let $S \in Ob(Sch_{fppf})$ be a scheme. A composition of (closed, resp. open) immersions of algebraic spaces over S is a (closed, resp. open) immersion of algebraic spaces over S.*

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[2]We will associate a topological space to an algebraic space in Properties of Algebraic Spaces, Section 4, and its opens will correspond exactly to the open subspaces defined below.

Proof. See Lemma 5.4 and Remarks 4.3 (see very last line of that remark) and 4.2. □

Lemma 12.3. *Let $S \in Ob(Sch_{fppf})$ be a scheme. A base change of a (closed, resp. open) immersion of algebraic spaces over S is a (closed, resp. open) immersion of algebraic spaces over S.*

Proof. See Lemma 5.5 and Remark 4.3 (see very last line of that remark). □

Lemma 12.4. *Let $S \in Ob(Sch_{fppf})$ be a scheme. Let F be an algebraic space over S. Let F_1, F_2 be locally closed subspaces of F. If $F_1 \subset F_2$ as subfunctors of F, then F_1 is a locally closed subspace of F_2. Similarly for closed and open subspaces.*

Proof. Let $T \to F_2$ be a morphism with T a scheme. Since $F_2 \to F$ is a monomorphism, we see that $T \times_{F_2} F_1 = T \times_F F_1$. The lemma follows formally from this. □

Let us formally define the notion of a Zariski open covering of algebraic spaces. Note that in Lemma 7.4 we have already encountered such open coverings as a method for constructing algebraic spaces.

Definition 12.5. Let $S \in \mathrm{Ob}(Sch_{fppf})$ be a scheme. Let F be an algebraic space over S. A *Zariski covering* $\{F_i \subset F\}_{i \in I}$ of F is given by a set I, a collection of open subspaces $F_i \subset F$ such that $\coprod F_i \to F$ is a surjective map of sheaves.

Note that if T is a schemes, and $a : T \to F$ is a morphism, then each of the fibre products $T \times_F F_i$ is identified with an open subscheme $T_i \subset T$. The final condition of the definition signifies exactly that $T = \bigcup_{i \in I} T_i$.

It is clear that the collection $\mathcal{T}$ of open subspaces of F is a set (as $(Sch/S)_{fppf}$ is a site, hence a set). Moreover, we can turn $\mathcal{T}$ into a category by letting the morphisms be inclusions of subfunctors (which are automatically open immersions by Lemma 12.4). Finally, Definition 12.5 provides the notion of a Zariski covering $\{F_i \to F'\}_{i \in I}$ in the category $\mathcal{T}$. Hence, just as in the case of a topological space (see Sites, Example 6.4) by suitably choosing a set of coverings we may obtain a Zariski site of the algebraic space F.

Definition 12.6. Let $S \in \mathrm{Ob}(Sch_{fppf})$ be a scheme. Let F be an algebraic space over S. A *small Zariski site* F_{Zar} of an algebraic space F is one of the sites $\mathcal{T}$ described above.

Hence this gives a notion of what it means for something to be true Zariski locally on an algebraic space, which is how we will use this notion. In general the Zariski topology is not fine enough for our purposes. For example we can consider the category of Zariski sheaves on an algebraic space. It will turn out that this is not the correct thing to consider, even for quasi-coherent sheaves. One only gets the desired result when using the etale or fppf site of F to define quasi-coherent sheaves.

13. Separation conditions on algebraic spaces

A separation condition on an algebraic space F is a condition on the diagonal morphism $F \to F \times F$. Let us first list the properties the diagonal has automatically. Since the diagonal is representable by definition the following lemma makes sense (through Definition 5.1).

Lemma 13.1. *Let S be a scheme contained in Sch_{fppf}. Let F be an algebraic space over S. Let $\Delta : F \to F \times F$ be the diagonal morphism. Then*

1. *Δ is locally of finite type,*
2. *Δ is a monomorphism,*
3. *Δ is separated, and*
4. *Δ is locally quasi-finite.*

Proof. Let $F = U/R$ be a presentation of F. As in the proof of Lemma 9.4 the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, "j"'] & F \arrow[d, "\Delta"] \\ U \times_S U \arrow[r] & F \times F \end{tikzcd}$$

is cartesian. Hence according to Lemma 10.2 it suffices to show that j has the properties listed in the lemma. (Note that each of the properties (1) – (4) occur in the lists of Remarks 4.1 and 4.3.) Since j is an equivalence relation it is a monomorphism. Hence it is separated by Schemes, Lemma 23.3. As R is an etale equivalence relation we see that $s, t : R \to U$ are etale. Hence s, t are locally of finite type. Then it follows from Morphisms, Lemma 15.8 that j is locally of finite type. Finally, as it is a monomorphism its fibres are finite. Thus we conclude that it is locally quasi-finite by Morphisms, Lemma 19.6. $\square$

Here are some common types of separation conditions. We will later discuss the relative version (insert future reference here).

Definition 13.2. Let S be a scheme contained in Sch_{fppf}. Let F be an algebraic space over S. Let $\Delta : F \to F \times F$ be the diagonal morphism.

1. We say F is *separated* if Δ is a closed immersion.
2. We say F is *weakly locally separated*[3] if Δ is an immersion.
3. We say F is *locally separated* if Δ is a quasi-compact immersion.
4. We say F is *quasi-separated* if Δ is quasi-compact.
5. We say F is *Zariski locally quasi-separated*[4] if there exists a Zariski covering $F = \bigcup_{i \in I} F_i$ such that each F_i is quasi-separated.

Note that if the diagonal is quasi-compact (when F is separated, locally separated or quasi-separated) then the diagonal is actually quasi-finite and separated, hence quasi-affine (by More on Morphisms, Lemma 10.1).

14. Examples of algebraic spaces

In this section we construct some examples of algebraic spaces. Some of these were suggested by B. Conrad. Since we do not yet have a lot of theory at our disposal the discussion is a bit awkward in some places.

Example 14.1. Let k be a field. Let $U = \mathbf{A}^1_k$. Set

$$j : R = \Delta \coprod \Gamma \longrightarrow U \times_k U$$

where $\Delta = \{(x, x) \mid x \in \mathbf{A}^1_k\}$ and $\Gamma = \{(x, -x) \mid x \in \mathbf{A}^1_k, x \neq 0\}$. It is clear that $s, t : R \to U$ are etale, and hence j is an etale equivalence relation. The quotient

[3] This is probably nonstandard notation.

[4] This definition was suggested by B. Conrad.

$X = U/R$ is an algebraic space by Theorem 9.5. Since R is quasi-compact we see that X is quasi-separated. On the other hand, X is not locally separated because the morphism j is not an immersion.

We will use the following lemma as a convenient way to construct algebraic spaces as quotients of schemes by free group actions.

Lemma 14.2. *Let $U \to S$ be a morphism of Sch_{fppf}. Let G be an abstract group. Let $G \to Aut_S(U)$ be a group homomorphism. Assume*

(1) *if $u \in U$ is a point, and $g(u) = u$ for some non-identity element $g \in G$, then g induces a nontrivial automorphism of $\kappa(u)$.*

Then

$$j : R = \coprod\nolimits_{g \in G} U \longrightarrow U \times_S U, \quad (g, x) \longmapsto (g(x), x)$$

is an etale equivalence relation and hence

$$F = U/R$$

is an algebraic space by Theorem 9.5.

Proof. In the statement of the lemma the symbol $\text{Aut}_S(U)$ denotes the group of automorphisms of U over S. Assume (1) holds. Let us show that

$$j : R = \coprod\nolimits_{g \in G} U \longrightarrow U \times_S U, \quad (g, x) \longmapsto (g(x), x)$$

is a monomorphism. This signifies that if T is a nonempty scheme, and $h : T \to U$ is a T-valued point such that $g \circ h = g' \circ h$ then $g = g'$. Suppose $T \neq \emptyset$, $h : T \to U$ and $g \circ h = g' \circ h$. Let $t \in T$. Consider the composition $\text{Spec}(\kappa(t)) \to \text{Spec}(\kappa(h(t))) \to U$. Then we conclude that $g' \circ g^{-1}$ fixes $u = h(t)$ and acts as the identity on its residue field. Hence $g = g'$ by (1).

Thus if (1) holds we see that j is a relation (see Groupoids, Definition 3.1). Moreover, it is an equivalence relation since on T-valued points for a connected scheme T we see that $R(T) = G \times U(T) \to U(T) \times U(T)$ (recall that we always work over S). Moreover, the morphisms $s, t : R \to U$ are etale since R is a disjoint product of copies of U. This proves that $j : R \to U \times_S U$ is an etale equivalence relation. $\square$

Given a scheme U and an action of a group G on U we say the action of G on U is *free* if condition (1) of Lemma 14.2 holds. Thus the lemma says that quotients of schemes by free actions of groups exist in the category of algebraic spaces.

Definition 14.3. Notation $U \to S$, G, R and assumptions as in Lemma 14.2. The algebraic space U/R is denoted $[U/G]$ and is called the *quotient of U by G*.

We will later make sense of the quotient $[U/G]$ as an algebraic stack without any assumptions on the action whatsoever (insert future reference here). Before we discuss the examples we prove some more lemmas to facilitate the discussion. Here is a lemma discussing the various separation conditions for this quotient when G is finite.

Lemma 14.4. *Notation and assumptions as in Lemma 14.2. Assume G is finite. Then*

(1) *if $U \to S$ is quasi-separated, then $[U/G]$ is quasi-separated, and*
(2) *if $U \to S$ is separated, then $[U/G]$ is separated.*

Proof. In the proof of Lemma 13.1 we saw that it suffices to prove the corresponding properties for the morphism $j : R \to U \times_S U$. If $U \to S$ is quasi-separated, then for every affine open $V \subset U$ the opens $g(V) \cap V$ are quasi-compact. It follows that j is quasi-compact. If $U \to S$ is separated, the the diagonal $\Delta_{U/S}$ is a closed immersion. Hence $j : R \to U \times_S U$ is a finite coproduct of closed immersions with disjoint images. Hence j is a closed immersion. □

Lemma 14.5. *Notation and assumptions as in Lemma 14.2. If Spec(k) $\to [U/G]$ is a morphism, then there exist*

1. *a finite Galois extension $k \subset k'$,*
2. *a finite subgroup $H \subset G$,*
3. *an isomorphism $H \to Gal(k'/k)$, and*
4. *an H-equivariant morphism Spec(k') $\to U$.*

Conversely, such data determine a morphism Spec(k) $\to [U/G]$.

Proof. Consider the fibre product $V = \mathrm{Spec}(k) \times_{[U/G]} U$. Here is a diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & {[U/G]} \end{tikzcd}$$

This is a nonempty scheme etale over Spec(k) and hence is a disjoint union of spectra of fields finite separable over k (Morphisms, Lemma 30.4). So write $V = \coprod_{i \in I} \mathrm{Spec}(k_i)$. The action of G on U induces an action of G on $V = \coprod \mathrm{Spec}(k_i)$. Pick an i, and let $H \subset G$ be the stabilizer of i. Since

$$V \times_{\mathrm{Spec}(k)} V = \mathrm{Spec}(k) \times_{[U/G]} U \times_{[U/G]} U = \mathrm{Spec}(k) \times_{[U/G]} U \times G = V \times G$$

we see that (a) the orbit of Spec(k_i) is V and (b) $\mathrm{Spec}(k_i \otimes_k k_i) = \mathrm{Spec}(k_i) \times H$. Thus H is finite and is the Galois group of k_i/k. We omit the converse construction. □

It follows from this lemma for example that if k'/k is a finite Galois extension, then $[\mathrm{Spec}(k')/\mathrm{Gal}(k'/k)] \cong \mathrm{Spec}(k)$. What happens if the extension is infinite? Here is an example.

Example 14.6. Let $S = \mathrm{Spec}(\mathbf{Q})$. Let $U = \mathrm{Spec}(\overline{\mathbf{Q}})$. Let $G = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ with obvious action on U. Then by construction property (1) of Lemma 14.2 holds and we obtain an algebraic space

$$X = [\mathrm{Spec}(\overline{\mathbf{Q}})/\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})] \longrightarrow S = \mathrm{Spec}(\mathbf{Q}).$$

Of course this is totally ridiculus as an approximation of S! Namely, by the Artin-Schreier theorem, see [Jac64, Theorem 17, page 316], the only finite subgroups of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ are $\{1\}$ and the conjugates of the order two group $\mathrm{Gal}(\overline{\mathbf{Q}}/\overline{\mathbf{Q}} \cap \mathbf{R})$. Hence, if Spec($k$) $\to X$ is a morphism with k algebraic over $\mathbf{Q}$, then it follows from Lemma 14.5 and the theorem just mentioned that either k is $\overline{\mathbf{Q}}$ or isomorphic to $\overline{\mathbf{Q}} \cap \mathbf{R}$.

What is wrong with the example above is that the Galois group comes equipped with a topology, and this should somehow be part of any construction of a quotient of Spec($\overline{\mathbf{Q}}$). The following example is much more reasonable in my opinion and may actually occur in "nature".

Example 14.7. Let k be a field of characteristic zero. Let $U = \mathbf{A}^1_k$ and let $G = \mathbf{Z}$. As action we take $n(x) = x+n$, i.e., the action of $\mathbf{Z}$ on the affine line by translation. The only fixed point is the generic point and it is clearly the case that $\mathbf{Z}$ injects into the automorphism group of the field $k(x)$. (This is where we use the characteristic zero assumption.) Consider the morphism

$$\gamma : \mathrm{Spec}(k(x)) \longrightarrow X = [\mathbf{A}^1_k/\mathbf{Z}]$$

of the generic point of the affine line into the quotient. We claim that this morphism does not factor through any monomorphism $\mathrm{Spec}(L) \to X$ of the spectrum of a field to X. (Contrary to what happens for schemes, see Schemes, Section 13.) In fact, since $\mathbf{Z}$ does not have any finite subgroups we see from Lemma 14.5 that for any such factorization $k(x) = L$. Finally, γ is not a monomorphism since

$$\mathrm{Spec}(k(x)) \times_{\gamma, X, \gamma} \mathrm{Spec}(k(x)) \cong \mathrm{Spec}(k(x)) \times \mathbf{Z}.$$

This example suggests that in order to define points of an algebraic space X we should consider equivalence classes of morphisms from spectra of fields into X and not the set of monomorphisms from spectra of fields.

We finish with a truly awful example.

Example 14.8. Let k be a field. Let $A = \prod_{n \in \mathbf{N}} k$ be the infinite product. Set $U = \mathrm{Spec}(A)$ seen as a scheme over $S = \mathrm{Spec}(k)$. Note that the projection maps $\mathrm{pr}_n : A \to k$ define open and closed immersions $f_n : S \to U$. Set

$$R = U \amalg \coprod\nolimits_{(n,m) \in \mathbf{N}^2,\ n \neq m} S$$

with morphism j equal to $\Delta_{U/S}$ on the component U and $j = (f_n, f_m)$ on the component S corresponding to (m, m). It is clear from the remark above that s, t are etale. It is also clear that j is an equivalence relation. Hence we obtain an algebraic space

$$X = U/R.$$

To see what this means we specialize to the case where the field k is finite with q elements. Let us first discuss the topological space $|U|$ associated to the scheme U a little bit. All elements of A satisfy $x^q = x$. Hence every residue field of A is isomorphic to k, and all points of U are closed. But the topology on U isn't the discrete topology. Let $u_n \in |U|$ be the point corresponding to f_n. As mentioned above the points u_n are the open points (and hence isolated). This implies there have to be other points since we know U is quasi-compact, see Algebra, Lemma 8.10 (hence not equal to an infinite discrete set). Another way to see this is because the (proper) ideal

$$I = \{x = (x_n) \in A \mid \text{all but a finite number of } x_n \text{ are zero}\}$$

is contained in a maximal ideal. Note also that every element x of A is of the form $x = ue$ where u is a unit and e is an idempotent. Hence a basis for the topology of A consists of open and closed subsets (see Algebra, Lemma 9.1.) So the topology on $|U|$ is totally disconnected, but nontrivial. Finally, note that $\{u_n\}$ is dense in $|U|$.

We will later define a topological space $|X|$ associated to X, see Properties of Spaces, Section 4. What can we say about $|X|$? It turns out that the map $|U| \to |X|$ is surjective and continuous. All the points u_n map to the same point x_0 of $|X|$, and none of the other points get identified. Since $\{u_n\}$ is dense in $|U|$ we conclude that the closure of x_0 in $|X|$ is $|X|$. In other words $|X|$ is irreducible and x_0 is a generic point of $|X|$. This seems bizarre since also x_0 is the image of a section $S \to X$ of the structure morphism $X \to S$ (and in the case of schemes this would imply it was a closed point, see Morphisms, Lemma 19.2).

Whatever you think is actually going on in this example, it certainly shows that some care has to be exercised when defining irreducible components, connectedness, etc of algebraic spaces.

15. Other chapters

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