Lectures on Logarithmic Algebraic Geometry

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1 Introduction

Logarithmic geometry was developed to deal with two fundamental and related problems in algebraic geometry: compactification and degeneration. One of the key aspects of algebraic geometry is that it is essentially global in nature. In particular, varieties can be compactified: any separated scheme U of finite type over a field k admits an open immersion $j: U \to X$, with X/k proper and $j(U)$ Zariski dense in X [15]. Since proper schemes are much easier to study than general schemes, it is often convenient to use such a compactification even if it is the original scheme U that is of primary interest. It then becomes necessary to keep track of the boundary $Z := X \setminus U$ and to study how functions, differential forms, sheaves, and other geometric objects on X behave near Z, and to somehow carry along the fact that it is U rather than X in which one is interested, in a functorial way.

This compactification problem is related to the phenomenon of degeneration. A scheme U often arises as a space parameterizing smooth proper schemes of a certain type, and there may be a smooth proper morphism $V \to U$ whose fibers are the objects one wants to classify. In good cases one can find a compactification X of U such that the boundary points parameterize "degenerations" of the original objects, and there is a proper and flat (but not smooth) $f: Y \to X$ which compactifies $V \to U$. Then one is left with the problem of analyzing the behavior of f along the boundary, and of comparing U to X and V to Y. A typical example is the compactification of the moduli stack of smooth curves by the moduli stack of stable curves. In this and many other cases, the addition of a canonical compactifying log structure to the total space Y and the base space X not only keeps track of the boundary data, but also gives new structure to the map along the boundary which makes it behave very much like a smooth map.

The development of logarithmic geometry, like that of any organism, began well before its official birth, and there are many classical methods to deal with the problems of compactification and degeneration. These include most notably the theories of toroidal embeddings, of differential forms and equations with log poles and/or regular singularities, and of logarithmic minimal models and Kodaira dimension. Logarithmic geometry was influenced by all these ideas and provides a language which incorporates many of them in a functorial and systematic way which extends byeond the classical theory. In particular there is a powerful version of base change for log schemes which works in arithmetic algebraic geometry, the area in which log geometry has so far enjoyed its most spectacular applications.

Logarithmic structures fit so naturally with the usual building blocks of schemes that is possible, and in most cases easy and natural, to adapt in a relatively straightforward way many of the standard techniques and intuitions of algebraic geometry to the logarithmic context. Log geometry seems to be especially compatible with the infinitesimal properties of log schemes, including the notions of smoothness, differentials, and differential operators. For example, if X is smooth over a field k and U is the complement of a divisor with normal crossings, then the resulting log scheme turns out to satisfy Grothendieck's functorial notion of smoothness. More generally any toric variety (with the log structure corresponding to the dense open torus it contains) is log smooth, and the theory of toroidal embeddings is essentially equivalent to the study of log smooth schemes over a field.

Let us illustrate how log geometry works in the most basic case of a compactification. If $j: U \to X$ is an open immersion, let $M_{U/X} \subseteq \mathcal{O}_X$ denote the subsheaf consisting of the local sections of $\mathcal{O}_X$ whose restriction to U is invertible. If f and g are sections of $M_{U/X}$, then so is fg, but $f+g$ need not be. Thus $M_{U/X}$ is not a sheaf of rings, but it is a multiplicative *submonoid* of $\mathcal{O}_X$. Note that $M_{U/X}$ contains the sheaf of units $\mathcal{O}_X^*$, and if X is integral, the quotient $M_{U/X}/\mathcal{O}_X^*$ is just the sheaf of anti-effective Cartier divisors on X with support in the complement Z of U in X. By definition, the morphism (inclusion) of sheaves of monoids $\alpha_{U/X}: M_{U/X} \to \mathcal{O}_X$ is a *logarithmic structure*, which in good cases "remembers" the inclusion $U \to X$. In the category of log schemes, the open immersion j fits into a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\tilde{j}"] \arrow[dr, "j"'] & (X, \alpha_{U/X}) \arrow[d, "\tau_{U/X}"] \\ & X \end{tikzcd}$$

This diagram provides a relative compactification of the open immersion j: the map $\tau_{U/X}$ is proper but the map $\tilde{j}$ preserves the topological nature of j, and in particular behaves like a local homotopy equivalence.

More generally, if X is any scheme, a *log structure* on X is a morphism of sheaves of commutative monoids $\alpha: M \to \mathcal{O}_X$ inducing an isomorphism

$\alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$. We do not require α to be injective. For example, let S be the spectrum of a discrete valuation ring R, let s be its closed point, let σ be its generic point, and let $j: \{\sigma\} \to S$ be the natural open immersion. The procedure described in the previous paragraph associates to the open immersion j a log structure $\alpha: M \to \mathcal{O}_S$ whose stalk at s is the inclusion $R' \to R$, where $R' := R \setminus \{0\}$. A more exotic example (the "hollow log structure") is the map $R' \to R$ which is the inclusion on the group R^* of units of R but sends all nonunits to $0 \in R$. Either of these structures can be restricted to a log structure on s, and in fact they give the same answer, a log structure $\alpha: i^*M \to k(s)$, where i^*M is the quotient of R by the group U of units congruent to 1 modulo the maximal ideal of R. Thus there is an exact sequence

$$1 \to k(s)^* \to i^*M \to \mathbf{N} \to 0$$

and α is the inclusion on $k(s)^*$ and sends all other elements of i^*M to 0.

Perhaps the most important feature of log geometry is how well it works in appropriate relative settings. Let S be the spectrum of a discrete valuation ring as above and $f: X \to S$ a proper morphism whose generic fiber X_σ is smooth and whose special fiber is a reduced divisor with normal crossings. Then the addition of the canonical compacification log structures associated with the open embeddings $X_\sigma \to X$ and $\{\sigma\} \to S$ makes the morphism $(X, \alpha_X) \to (S, \alpha_S)$ smooth in the logarithmic sense. If in the complex analytic context we replace S by a small disc D, η by the punctured disc D^*, and write D^{log} for an analytic incarnation of $(D, \alpha_{D^*/D})$ then the restriction of f to D^* is a fibration, and the cohomology sheaves $R^q f_* \mathbf{Z}$ are locally constant on D^*. Since $\tilde{j}: D^* \to D^{log}$ is a locally homotopy equivalence, the locally constant sheaf $R^q f_* \mathbf{Z}$ extends canonically to D^{log}. This extension has a geometric interpretation, coming from the fact that $(X, \alpha_X) \to (D, \alpha_D)$ is smooth in the log world. In fact, the local system on D^{log} can be entirely computed from the logarithmic special fiber $(X_s, \alpha_{X_s}) \to (s, \alpha_s)$. Arithmetic analogies of this result are valid for étale, de Rham, and crystalline cohomologies, the last playing a crucial result in the formulation and proof the the C_{st} conjecture [18].

Chapter I

The geometry of monoids

1 Basics on monoids

1.1 Limits in the category of monoids

A *monoid* is a triple $(M, \star, e_M)$ consisting of a set M, an associative binary operation $\star$, and a two-sided identity element e_M of M. A homomorphism $\theta : M \to N$ of monoids is a function $M \to N$ such that $\theta(e_M) = e_N$ and $\theta(m \star m') = \theta(m) \star \theta(m')$ for any pair of elements m and m' of M. Note that although the element e_M is the unique two-sided identity of M, compatibility of θ with e_M is not automatic from compatibility with $\star$. We write **Mon** for the category of monoids and morphisms of monoids. All monoids we consider here will be commutative unless explicitly noted otherwise.

We will often follow the common practice of writing M or $(M, \star)$ in place of $(M, \star, e_M)$ when there seems to be no danger of confusion. Similarly, if a and b are elements of a monoid $(M, \star, e_M)$, we will often write ab (or $a + b$) for $a \star b$, and 1 (or 0) for e_M.

The most basic example of a monoid is the set $\mathbf{N}$ of natural numbers, with addition as the monoid law. If M is any monoid and $m \in M$, there is a unique monoid homomorphism $\mathbf{N} \to M$ sending 1 to m: $\mathbf{N}$ is the free monoid with generator 1. More generally, if S is any set, the set $\mathbf{N}^{(S)}$ of functions $I: S \to \mathbf{N}$ such that $I_s = 0$ for almost all s, endowed with pointwise addition of functions as a binary operation, is the free (commutative) monoid with basis $S \subseteq N^{(S)}$. The functor $S \mapsto \mathbf{N}^{(S)}$ is left adjoint to the forgetful functor from monoids to sets.

Arbitrary projective limits exist in the category of monoids, and their formation commutes with the forgetful functor to the category of sets. In particular, the intersection of a set of submonoids of M is again a submonoid, and hence if S is a subset of M, the intersection of all the submonoids of M containing S is the smallest submonoid of M containing S, the *submonoid of M generated by S.* If there exists a finite subset S of M which generates M, one says that M is *finitely generated* as a monoid.

Arbitrary inductive limits of monoids also exist. This will follow from the existence of direct sums and of coequalizers. Direct sums are easy to construct: the direct sum $\bigoplus M_i$ of a family $\{M_i : i \in I\}$ of monoids is the submonoid of the product $\prod_i M_i$ consisting of those elements m such that $m_i = 0$ for almost all i. The construction of coequalizers is more difficult, and we first investigate quotients in the category of monoids.

If $\theta\colon P \to M$ is a homomorphism of monoids, then the set E of pairs $(p_1, p_2) \in P \times P$ such that $\theta(p_1) = \theta(p_2)$ is an equivalence relation on P and also a submonoid of $P \times P$, and if θ is surjective, M can be recovered as the quotient of P by the equivalence relation E. A submonoid E of $P \times P$ which is also an equivalence relation on P is called a *congruence* (or *congruence relation*) on P. One checks easily that if E is a congruence relation on P, then the set P/E of equivalence classes has a unique monoid structure making the projection $P \to P/E$ a monoid morphism. Thus there is a dictionary between congruence relations on P and isomorphism classes of surjective maps of monoids $P \to P'$. The intersection of a family of congruence relations is a congruence relation, and hence it makes sense to speak of the congruence relation generated by any subset of $P \times P$. One says that a congruence relation E is *finitely generated* if there is a finite subset S of $P \times P$ which generates E as a congruence relation; this does not imply that S generates E as a monoid.

The following proposition, whose proof is immediate, summarizes the above considerations.

Proposition 1.1.1 *Let $P \to P'$ be a surjective mapping of monoids, and let $E := P \times_{P'} P \subseteq P \times P$, i.e., the equalizer of the two maps $P \times P \to P'$.*

1. *E is a congruence relation on P.*

2. *P' is the coequalizer of the two maps $E \to P$.*

Here is a useful description of the congruence relation generated by a subset of $P \times P$.

Proposition 1.1.2 *Let P be a (commutative) monoid.*

1. *An equivalence relation $E \subseteq P \times P$ is a congruence relation if and only if $(a+p, b+p) \in E$ whenever $(a,b) \in E$ and $p \in P$.*

2. *If S is a subset of $P \times P$, let $S_P := \{(a+p, b+p) : (a,b) \in S, p \in P\}$. Then the congruence relation E generated by S is the equivalence relation generated by S_P. Explicitly, E is the union of the diagonal with the set of pairs (x,y) for which there exists a finite sequence $(s_0, \ldots, s_n)$ with $s_0 = x$ and $s_n = y$ such that for every $i > 0$, either (s_{i-1}, s_i) or (s_i, s_{i-1}) belongs to S_P.*

Proof: Suppose that an equivalence relation E is closed under addition by elements of the diagonal of $P \times P$ and that (a,b) and $(c,d) \in E$. Then $(a+c, b+c)$ and $(c+b, d+b) \in E$, and since P is commutative and E is transitive, $(a+c, b+d) \in E$. Since E contains the diagonal, the identity element $(0,0)$ of $P \times P$ belongs to E, so E is a submonoid of $P \times P$, hence a congruence relation. Conversely, if E is a congruence relation, then for any $p \in P$, $(p,p) \in E$, and hence if $(a,b) \in E$, $(a+p, b+p) \in E$. This proves (1). For (2), let E denote the congruence relation generated by S and E' the equivalence relation generated by S_P; evidently $E' \subseteq E$. It follows from the associative law that S_P is closed under addition by elements of the diagonal of $P \times P$. Hence if $(s_0, \ldots, s_n)$ is a sequence such that (s_{i-1}, s_i) or $(s_i, s_{i-1}) \in S_P$ for all $i > 0$, then $(s_0 + p, \ldots s_n + p)$ shares the same property. Thus if $(x,y) \in E'$ and $p \in P$, then $(x+p, y+p) \in E'$. Then it follows from (1) that E' is a congruence relation, and so $E' = E$. $\square$

Remark 1.1.3 If Q is an abelian group and $E \subseteq Q \times Q$ is a congruence relation on Q, then the image of E under the homomorphism $h\colon Q \oplus Q \to Q$ sending (q_1, q_2) to $q_2 - q_1$ is a subgroup K of Q, and $E = h^{-1}(K)$. Conversely the inverse image under h of any subgroup of Q is a congruence on Q. This simply makes explicit the familiar correspondence between quotients of Q, subgroups of Q, and congruence relations on Q.

If u and v are two morphisms of monoids $Q \to P$, one can construct the coequalizer of u and v as the quotient of P by the congruence relation on P

generated by the set of pairs $(u(q), v(q))$ for $q \in Q$. In general, a diagram of monoids

$$\begin{tikzcd} Q \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & P \arrow[r, "w"] & R \end{tikzcd}$$

is called *exact* if w is the coequalizer of u and v. The existence of arbitrary inductive limits follows from the existence of direct sums and coequalizers of pairs of morphisms by a standard construction.

A *presentation* of a monoid M is an exact diagram

$$\begin{tikzcd} L_1 \arrow[r, shift left] \arrow[r, shift right] & L_0 \arrow[r] & M \end{tikzcd}$$

with L_0 and L_1 free. It is equivalent to the data of a map from a set I to M whose image generates M and a map from a set J to $\mathbf{N}^{(I)} \times \mathbf{N}^{(I)}$ whose image generates the congruence relation on $N^{(I)}$ defined by the surjective monoid map $\mathbf{N}^{(I)} \to M$ corresponding to the set map $I \to M$. The monoid M is said to be *of finite presentation* if it admits a presentation as above with L_0 and L_1 free and finitely generated. We shall see in (2.1.9) that in fact every finitely generated monoid is of finite presentation.

The *amalgamated sum* $Q_1 \xrightarrow{v_1} Q \xleftarrow{v_2} Q_2$ of a pair of monoid morphisms $u_i \colon P \to Q_i$, often denoted simply by $Q_1 \oplus_P Q_2$, is the inductive limit of the diagram $Q_1 \xleftarrow{u_1} P \xrightarrow{u_2} Q_2$. That is, the pair (v_1, v_2) universally makes the diagram

$$\begin{tikzcd} P \arrow[r, "u_1"] \arrow[d, "u_2"'] & Q_1 \arrow[d, "v_1"] \\ Q_2 \arrow[r, "v_2"] & Q \end{tikzcd}$$

commute, and can be viewed as the pushout of u_1 along u_2 or the pushout of u_2 along u_1. It can also be viewed as the coequalizer of the two maps $(u_1, 0)$ and $(0, u_2)$ from P to $Q_1 \oplus Q_2$. As the following proposition shows, the calculation of Q is considerably simplified if one of the monoids in question is a group. (See (4.3.2) for a generalization.)

Proposition 1.1.4 *Let $u_i \colon P \to Q_i$ be a pair of monoid morphisms, let Q be their amalgamated sum, and let E be the congruence relation on $Q_1 \oplus Q_2$ given by the natural map $Q_1 \oplus Q_2 \to Q$.*

1. *Let E' be the set of pairs $((q_1, q_2), (q_1', q_2'))$ of elements of $Q_1 \oplus Q_2$ such that there exist a and b in P with $q_1 + u_1(b) = q_1' + u_1(a)$ and $q_2 + u_2(a) = q_2' + u_2(b)$. Then E' is a congruence relation on $Q_1 \oplus Q_2$ containing E, and if any of P, Q_1, or Q_2 is a group, then $E = E'$.*

2. *If P is a group, then two elements of $Q_1 \oplus Q_2$ are congruent modulo E if and only if they lie in the same orbit of the action of P on $Q_1 \oplus Q_2$ defined by $p(q_1, q_2) = (q_1 + u_1(p), q_2 + u_2(-p))$.*

3. *If P and Q_i are groups, then so is $Q_1 \oplus_P Q_2$, which is in fact just the fibered coproduct (amalgamated sum) in the category of abelian groups.*

Proof: If $q_1 + u_1(b) = q_1' + u_1(a)$ and $q_2 + u_2(a) = q_2' + u_2(b)$, we shall say that "$(a, b)$ links (q_1, q_2) and (q_1', q_2')." The set E' is evidently symmetric and reflexive. To prove the transitivity one checks immediately that if (a, b) links (q_1, q_2) and (q_1', q_2') and (a', b') links (q_1', q_2') and (q_1'', q_2''), then $(a + a', b + b')$ links (q_1, q_2) and (q_1'', q_2''). Moreover, if (a, b) links (q_1, q_2) and (q_1', q_2') then for any $(\tilde{q}_1, \tilde{q}_2) \in Q_1 \oplus Q_2$, (a, b) links $(q_1 + \tilde{q}_1, q_2 + \tilde{q}_2)$ and $(q_1' + \tilde{q}_1, q_2' + \tilde{q}_2)$. Then by (1.1.2) E' is a congruence relation on $Q_1 \oplus Q_2$. Furthermore, if $p \in P$, $(p, 0)$ links $(u_1(p), 0)$ and $(0, u_2(p))$, and since E is the congruence relation generated by such pairs, $E \subseteq E'$. If P or either Q_i is a group, then $v := v_i \circ u_i$ factors through the group Q^* of invertible elements of Q. If (a, b) links (q_1, q_2) and (q_1', q_2'), we find that

$$v_1(q_1) + v_2(q_2) + v(a + b) = v_1(q_1') + v_2(q_2') + v(a + b),$$

and since $v(a + b) \in Q^*$, it follows that

$$v_1(q_1) + v_2(q_2) = v_1(q_1') + v_2(q_2').$$

Thus $E' \subseteq E$. This proves (1), and (2) and (3) are immediate consequences. □

Example 1.1.5 If we take $Q_2 = 0$ in 1.1.4 one obtains the *cokernel* of the morphism $u_1 \colon P \to Q_1$, or, equivalently, the coequalizer of u_1 and the zero mapping $P \to Q_1$. If P is a submonoid of Q_1, one writes $Q_1 \to Q_1/P$ for this cokernel, and it follows from (1.1.4) that two elements q and q' of Q_1

have the same image in Q_1/P if and only if there exist p and p' in P such that $q + p = q' + p'$. If P' is a submonoid of Q_1 containing P, then P'/P is a submonoid of Q_1/P, and the natural map $(Q_1/P)/(P'/P) \to Q_1/P'$ is an isomorphism.

If S is a set, then the set of functions from S to itself forms a (not necessarily commutative) monoid $\mathrm{End}(S)$ under composition. If Q is a monoid, an *action of* Q *on* S is a morphism of monoids θ_S from Q to $\mathrm{End}(S)$. In this context we often write the monoid law on Q multiplicatively, and if $q \in Q$ and $s \in S$, qs for $\theta_S(q)(s)$. A Q*-set* is a set endowed with an action of Q, and $\mathbf{Ens}_Q$ will denote the category of Q-sets, with the evident notion of morphism. If S is a Q-set and $s \in S$, the image of the map $Q \to S$ sending q to qs is the minimal Q-invariant subset of S containing s, called the *trajectory* of s in S.

A *basis* for a Q-set (S, ρ) is a map of sets $i\colon T \to S$ such that the induced map $Q \times T \to S\colon (q, t) \mapsto \rho(q)i(t)$ is bijective; if such a basis exists, we say that (S, ρ) is a *free* Q*-set*. A free Q-set with basis $T \to S$ satisfies the usual universal property of a free object: to give a map of Q-sets $(S, \rho) \to (S', \rho')$ is the same as to give a map of sets $T \to S'$. If T is any set and if $Q \times T$ is endowed with the action ρ defined by $\rho(q')(q, t) = (q'q, t)$, then the map $T \to Q \times T$ sending t to $(1, t)$ is a basis. Thus the functor taking a set T to the free Q-set $Q \times T$ is left adjoint to the forgetful functor from the category of Q-sets to the category of sets. If G is a group and S is a G-set, then S has a basis as a G-set if and only if the action is free in the sense that $gs = s$ implies $g = 1$, but this equivalence is not true for monoids in general.

The category $\mathbf{Ens}_Q$ of Q-sets admits arbitrary projective limits, and their formation commutes with the forgetful functor to the category of sets, since the forgetful functor $\mathbf{Ens}_Q \to \mathbf{Ens}$ has a left adjoint. In particular, if S and T are Q-sets, then Q acts on $S \times T$ by $q(s, t) := (qs, qt)$, and this action makes $S \times T$ the product of S and T in $\mathbf{Ens}_Q$.

Inductive limits in the $\mathbf{Ens}_Q$ also exist. The direct sum of a family $S_i :$ $i \in I$ is just the disjoint union, with the evident Q-action. To understand the construction of quotients in the category $\mathbf{Ens}_Q$, note that if $\pi\colon S \to T$ is a surjective map of Q-sets, then the corresponding equivalence relation $E \subseteq S \times S$ is a Q-subset of $S \times S$; such an equivalence relation is called a *congruence relation* on S. Conversely, if E is any congruence relation on S, then there is a unique Q-set structure on S/E such that the projection $S \to S/E$ is a morphism of Q-sets. When $S = Q$ acting regularly on itself, the notion of a congruence relation on Q as a monoid coincides with the notion of a congruence relation as a Q-set, thanks to (1.1.2). Furthermore, the analog of (2) of (1.1.2) holds for Q-sets, and in particular the equivalence relation generated by a subset of $S \times S$ which is stable under the diagonal action of Q is already a congruence relation. If u and v are two morphisms $S' \to S$, the coequalizer of u and v is the quotient of S by the congruence relation generated by $\{(u(s'), v(s')) : s' \in S'\}$.

Suppose that S, T, and W are Q-sets. A *Q-bimorphism* $S \times T \to W$ is by definition a function $\beta: S \times T \to W$ such that $\beta(qs, t) = \beta(s, qt) = q\beta(s, t)$ for any $(s, t) \in S \times T$ and $q \in Q$. The *tensor product of S and T* is the universal Q-bimorphism $S \times T \to S \otimes_Q T$. To construct it, begin by regarding $S \times T$ as a Q-set via its action on S: $q(s, t) := (qs, t)$, and consider the equivalence relation R on $S \times T$ generated by the set of pairs

$$((qs, t), (s, qt)) \in (S \times T) \times (S \times T) \text{ for } q \in Q, s \in S, t \in T.$$

Note that this set of pairs is stable under the action of Q, since if $q' \in Q$, and if $s' := q's$, then $((q'qs, t), (q's, qt)) = ((qs', t), (s', qt))$. It follows that the equivalence relation R is a congruence relation. Then the projection $\pi: S \times T \to (S \times T)/R$ is a Q-bimorphism and satisfies the universal mapping property of the tensor product. If Q is a (commutative) group, then $S \otimes_Q T$ can be constructed in the usual way as the orbit space of the action of Q on $S \times T$ given by $q(s, t) := (qs, q^{-1}t)$.

Suppose that $\theta: Q \to P$ is a monoid homomorphism. Then θ defines an action of Q on P by $qp := \theta(q)p$. If T is a Q-set, the tensor product $P \otimes_Q T$ has a natural action of P, with $p(p' \otimes t) = (pp' \otimes t)$, and the map $T \to P \otimes_Q T$ sending t to $1 \otimes t$ is a morphism of Q-sets over the homomorphism θ. If R is the Q-set defined by a monoid homomorphism $Q \to R$, then $(p \otimes r)(p' \otimes r') = (pp' \otimes rr')$ is the unique monoid structure on $P \otimes_Q R$ for which the natural maps $P \to P \otimes_Q R$ and $R \to P \otimes_Q R$ are homomorphisms and such that the P-set structure defined above is compatible with the P-set structure coming from the homomorphism $P \to P \otimes_Q R$. It can be checked that this monoid structure makes $P \otimes_Q R$ into the amalgamated sum of P and R along Q.

Definition 1.1.6 *Let Q be a monoid and let S be a Q-set. The transporter of S is the category $\mathcal{T}_Q S$ whose objects are the elements of S, and for which the morphisms from an object s to an object t are the elements q of Q such that $qs = t$, with composition defined from the multiplication law of Q. The* *transporter of a monoid Q is the transporter of Q regarded as a Q-set, and is denoted simply by $\mathcal{T}Q$.*

Recall from [1, I,2.7]. that a category is said to be *filtering* if it satisfies the following conditions:

1. For any diagram of the form

$$\begin{tikzcd} s \arrow[r, "u_1"] \arrow[d, "u_2"'] & t_1 \\ t_2 & \end{tikzcd}$$

there exist morphisms $v_1 \colon t_1 \to t$ and $v_2 \colon t_2 \to t$ such that $v_1 u_1 = v_2 u_2$.

2. For any diagram

$$\begin{tikzcd} s \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & t, \end{tikzcd}$$

there exists a morphism $w \colon t \to t'$ such that $w \circ u = w \circ v$.

3. The category is (nonempty and) connected, *i.e.*, any two objects can be joined by a chain of arrows (in either direction).

The transporter category of any Q-set S satisfies (1), and the transporter category of Q is filtering.

Associated with the category $\mathcal{T}_Q S$ is a partially ordered set which is worthwhile making explicit.

Definition 1.1.7 *Let Q be a monoid and S a Q-set. If s and t are elements of S, we write $s \leq t$ if there exists a $q \in Q$ such that $qs = t$, and $s \sim t$ if $s \leq t$ and $t \leq s$.*

It is clear that if $s \leq t$ and $t \leq w$, then $s \leq w$, and that for every $s \in S$, $s \leq s$. Thus the relation $\leq$ defines a preordering on S. The relation $\sim$ is a congruence relation on S, and the relation $\leq$ on $S/\sim$ is a partial ordering. We shall use this notion especially when $S = Q$ with the regular representation. Since $\sim$ is a congruence relation, it follows from 1.1.2 that $Q/\sim$ inherits a monoid structure.

1.2 Integral, fine, and saturated monoids

If M is any commutative monoid, there is a universal morphism λ_M from M to a group M^{gp}. That is, M^{gp} is a group, $\lambda_M: M \to M^{gp}$ is a homomorphism of monoids, and any morphism from M to a group factors uniquely through λ_M. Thus, the functor $M \mapsto M^{gp}$ is the left adjoint of the inclusion functor from the category of groups to the category of monoids; since it has a right adjoint, it automatically commutes with the formation of direct limits. In fact, M^{gp} can be identified with the cokernel (1.1.5) of $M \times M$ by the diagonal, and λ_M with the composite of $(\mathrm{id}_M, 0)$ and the projection $M \times M \to M \times M/\Delta_M$. One can also construct M^{gp} as the set of equivalence classes of pairs (x, y) of elements of M for which (x, y) is equivalent to (x', y') if and only if there exists $z \in M$ such that $x + y' + z = x' + y + z$. The explicit description of the equivalence relation in (1.1.5) shows that the two constructions are in fact the same. One writes $x - y$ for the equivalence class containing (x, y), and $(x - y) + (x' - y') := (x + x') - (y + y')$.

If M is a monoid, let M^* denote the set of all $m \in M$ such that there exists an $n \in M$ such that $m + n = 0$. Then M^* forms a submonoid of M. It is in fact a subgroup—the largest subgroup of M. We call it the *group of units* of M; it acts naturally on M by translation. One says that M is *quasi-integral* if this action is free, *i.e.*, if whenever $u \in M^*$ and $x \in M$, $u + x = x$ implies that $u = 0$. If G is any subgroup of M, the orbit space M/G can be identified with the quotient in the category of monoids discussed in (1.1.5). In particular,we write $\overline{M}$ for M/M^*. If M is quasi-integral, the map $M \to \overline{M}$ makes M an M^*-torsor over $\overline{M}$. A monoid M is called *sharp* if 0 is its only unit. For any monoid M, the quotient $\overline{M}$ is sharp, and the map $M \to \overline{M}$ is the universal map from M to a sharp monoid.

A monoid M is called *integral* if the cancellation law holds, *i.e.*, if $x + y = x' + y$ implies that $x = x'$. Evidently any integral monoid is quasi-integral. The universal map $\lambda_M: M \to M^{gp}$ is injective if and only if M is integral, and the induced map $M^* \to M^{gp}$ is injective if and only if M is quasi-integral. For any monoid M, the monoid $M/\sim$ (see (1.1.7)) is sharp, and if M is integral, the natural map $M/M^* \to M/\sim$ is an isomorphism.

The inverse limit of a family of integral monoids is again integral. Formation of M^{gp} commutes with direct products but not with fibered products in general. For example, let $s: \mathbf{N}^2 \to \mathbf{N}$ be the map taking (a, b) to $a + b$ and let t be the map taking (a, b) to 0. Then the equalizer of s and t is zero. However, the equalizer of the associated maps on groups $\mathbf{Z}^2 \to \mathbf{Z}$ is the anti-diagonal $\mathbf{Z} \to \mathbf{Z}^2$ (sending c to $(c, -c)$.) On the other hand, it is true that an injective map $M \to N$ of integral monoids induces an injection $M^{gp} \to N^{gp}$.

Proposition 1.2.1 *If Q is an integral monoid and P is a submonoid, the natural map $Q/P \to Q^{gp}/P^{gp}$ is injective. Thus Q/P is integral and can be identified with the image of Q in Q^{gp}/P^{gp}. A monoid Q is integral if and only if it is quasi-integral and $\overline{Q}$ is integral.*

Proof: If q and q' are two elements of Q with the same image in Q^{gp}/P^{gp}, then there exist p and p' such that $q - q' = p - p'$ in Q^{gp}. Since Q is integral, $q + p' = q' + p$ in Q. Then it follows from (1.1.5) that q and q' have the same image in Q/P. In particular, if Q is integral, so is $\overline{Q}$. Conversely, suppose that Q is quasi-integral and $\overline{Q}$ is integral, and that q, q' and p are elements of Q with $q + p = q' + p$. Since $\overline{Q}$ is integral, there exists a unit u such that $q' = q + u$. Then $q' + p = q + p + u$. Since Q is quasi-integral, $u = 0$ and $q = q'$. This shows that Q is integral. □

Let $\mathbf{Mon}^{\mathrm{int}}$ denote the full subcategory of $\mathbf{Mon}$ whose objects are the integral monoids. For any monoid M, let M^{int} denote the image of $\lambda_M \colon M \to M^{gp}$. Then $M \mapsto M^{\mathrm{int}}$ is left adjoint to the inclusion functor $\mathbf{Mon}^{\mathrm{int}} \to \mathbf{Mon}$.

Proposition 1.2.2 *Let Q be the amalgamated sum of two homomorphisms $u_i \colon P \to Q_i$ in the category* $\mathbf{Mon}$*. Then Q^{int} is the amalgamated sum of $u_i^{\mathrm{int}} \colon P^{\mathrm{int}} \to Q_i^{\mathrm{int}}$ in the category* $\mathbf{Mon}^{\mathrm{int}}$*, and can be naturally identified with the image of Q in $Q_1^{gp} \oplus_{P^{gp}} Q_2^{gp}$. If P, Q_1, and Q_2 are integral and any of these monoids is a group, then Q is integral.*

Proof: The fact that Q^{int} is the amalgamated sum of u_i^{int} in $\mathbf{Mon}^{\mathrm{int}}$ is a formal consequence of the fact that $M \mapsto M^{\mathrm{int}}$ preserves inductive limits. Moreover, since $M \mapsto M^{gp}$ also preserves inductive limits, $Q^{gp} \cong Q_1^{gp} \oplus_{P^{gp}} Q_2^{gp}$. It follows that Q^{int} is the image of Q in $Q^{gp} \cong Q_1^{gp} \oplus_{P^{gp}} Q_2^{gp}$. Now suppose that any of P and Q_i is a group and that (q_1, q_2) and (q_1', q_2') are two elements of $Q_1 \oplus Q_2$ with the same image in Q^{gp}. Then $v_1(q_1) + v_2(q_2) = v_1(q_1') + v_2(q_2')$ in Q^{gp}, and so there exist elements a and b in P such that $(q_1' - q_1, q_2' - q_2) = (u_1(a-b), u_2(b-a))$. Then $q_1' + u_1(b) = q_1 + u_2(a)$ and $q_2' + u_2(a) = q_2 + u_1(b)$. It then follows from (1.1.4) that $v_1(q_1) + v_2(q_2) = v_1(q_1') + v_2(q_2')$ in Q. Thus the map $Q \to Q_1^{gp} \oplus_{P^{gp}} Q_2^{gp}$ is injective and Q is integral. □

A monoid M is said to be *fine* if it is finitely generated and integral. A monoid M is called *saturated* if it is integral and whenever $x \in M^{gp}$ is such that $mx \in M$ for some $m \in \mathbf{Z}^+$, then $x \in M$. For example, the monoid of all integers greater than or equal to some natural number d, together with zero, is not saturated if $d > 1$.

Proposition 1.2.3 *Let M be an integral monoid.*

1. *The natural map $M^{gp}/M^* \to \overline{M}^{gp}$ is an isomorphism.*
2. *If M is saturated, $\overline{M}^{gp}$ is torsion free.*
3. *The set M^{sat} of all elements x of M^{gp} such that there exists $n \in \mathbf{Z}^+$ with $nx \in M$ is a saturated submonoid of M^{gp}, and the functor $M \mapsto M^{\mathrm{sat}}$ is left adjoint to the inclusion functor from the category $\mathbf{Mon}^{\mathrm{sat}}$ of saturated monoids to $\mathbf{Mon}^{\mathrm{int}}$.*
4. *M is saturated if and only if $\overline{M}$ is saturated.*
5. *The natural map $M^{\mathrm{sat}}/M^* \to \overline{M}^{\mathrm{sat}}$ is an isomorphism. Furthermore, every unit of $\overline{M}^{\mathrm{sat}}$ is torsion, and the natural map*

$$\overline{M^{\mathrm{sat}}} \to \overline{\overline{M}^{\mathrm{sat}}}$$

is an isomorphism.

Proof: Suppose that $x_1, x_2 \in M$ and $x_2 - x_1$ maps to zero in $\overline{M}^{gp}$. Since $\overline{M} \subseteq \overline{M}^{gp}$, $\overline{x}_1 = \overline{x}_2 \in \overline{M}$, and hence there exists a $u \in M^*$ with $x_2 = u + x_1$. Then $x_2 - x_1 = u \in M^*$. This proves (1). Suppose M is saturated and $x \in M^{gp}$ maps to a torsion element of $\overline{M}^{gp}$. Then $nx \in M^*$ for some $n \in \mathbf{Z}^+$, and since M is saturated, $x \in M$. The fact that $nx \in M^*$ now implies that $x \in M^*$. Thus $\overline{M}^{gp}$ is torsion free. If x and y are elements of M^{gp} with $mx \in M$ and $ny \in M$, then $mn(x + y) \in M$, and it follows that M^{sat} is a submonoid of M^{gp}. Hence $(M^{\mathrm{sat}})^{gp} = M^{gp}$, and if $x \in M^{\mathrm{sat}}$ and $nx \in M^{\mathrm{sat}}$, then there exists an $m \in \mathbf{Z}^+$ with $mnx \in M$. It follows that $x \in M^{\mathrm{sat}}$, so M^{sat} is saturated. The verification of the adjointness of the functor $M \mapsto M^{\mathrm{sat}}$ is immediate, as is that of (4). It is clear that $M^{\mathrm{sat}}/M^* \to \overline{M}^{\mathrm{sat}}$ is surjective, and the injectivity follows from the injectivity of the map $M^{gp}/M^* \to \overline{M}^{gp}$. If $x \in M^{\mathrm{sat}}$ and $\overline{x}$ is a unit of $\overline{M}^{\mathrm{sat}}$, then there also exists an element y of

M^{sat} with $x + y \in M^*$. Then there exist m and n in $\mathbf{Z}^+$ such that mx and ny belong to M. But then $mnx + mny \in M^*$, and hence mnx is a unit of M. This shows that $\overline{x}$ is a torsion element of $\overline{M}^{\text{sat}}$. It is clear that the map in (5) is surjective. Suppose that x and y are two elements of M^{sat} with the same image in $\overline{\overline{M}^{\text{sat}}}$. Then $x - y \in M^{gp}$ maps to a unit of $\overline{M}^{\text{sat}}$, and hence to a torsion element of $\overline{M}^{\text{sat}} \subseteq \overline{M}^{gp}$. Hence $mx - my \in M^*$ for some m. Then $my - mx \in M^*$ also, so $x - y$ is a unit of M^{sat}, and x and y have the same image in $\overline{M^{\text{sat}}}$. The proves the injectivity. $\square$

Monoids which are both fine and saturated are of central importance in logarithmic geometry, and are often called *normal* or *fs-monoids*. A monoid P is said to be *toric* if it is fine and saturated and in addition P^{gp} is torsion free; in this case P^{gp} can be viewed as the character group of an algebraic torus. The schemes arising from toric monoids form the building blocks of toric geometry.

A monoid M is said to be *valuative* if it is integral and for every $x \in M^{gp}$, either x or $-x$ lies in M. This is equivalent to saying that the preorder relation (1.1.7) on M^{gp} defined by the action of M is a total preorder. The monoid $\mathbf{N}$ is valuative, and if V is a valuation ring, the submonoid V' of nonzero elements of V is valuative. Every valuative monoid is saturated.

If R is any commutative ring, its underlying multiplicative monoid $(R, \cdot, 1)$ is not quasi-integral unless $R^* = \{1\}$, since $u \cdot 0 = 0$ for any $u \in R^*$, and it is not integral unless $R = \{0\}$, since $0 \cdot 0 = 1 \cdot 0$. On the other hand, the set R' of nonzero divisors of R forms an integral submonoid of the multiplicative monoid of R. For example, $\overline{\mathbf{Z}}' = \mathbf{Z}'/(\pm)$ is a free (commutative) monoid, generated by the prime numbers. If R is a discrete valuation ring, $\overline{R}' = R'/R^*$ is freely generated by the image of a uniformizer of R'. Although there is a unique isomorphism of monoids $R'/R^* \cong \mathbf{N}$, it is not functorial: if $R \to S$ is a finite extension of valuation rings with ramification index e, the induced map $\overline{R}' \to \overline{S}'$ sends the unique generator of $\overline{R}'$ to e times that of $\overline{S}'$.

1.3 Ideals, faces, and localization

Definition 1.3.1 *An ideal of a monoid M is a subset I such that $x \in I$ and $y \in M$ implies $x + y \in I$. An ideal I is called prime if $I \neq M$ and $x + y \in I$ implies $x \in I$ or $y \in I$. A face of a monoid M is a submonoid F such that $x + y \in F$ implies that both x and y belong to F.*

Observe that a face is just a submonoid whose complement is an ideal, and a prime ideal is an ideal whose complement is a submonoid (hence a face). Thus $\mathfrak{p} \mapsto F_{\mathfrak{p}} := M \setminus \mathfrak{p}$ gives an order reversing bijection between the set of prime ideals of M and the set of faces of M. The empty set is an ideal of M—the unique minimal prime ideal. The set of units M^* is a face of M, and in fact is contained in every face. Its complement, the set M^+ of all nonunits of M, is a prime ideal of M, and in fact contains every proper ideal of M. Thus M^+ is the unique maximal ideal of M; in many respects a monoid is analogous to a local ring. In particular, a monoid homomorphism $\theta\colon M \to N$ is said to be *local* if $\theta^{-1}(N^+) = M^+$. The notion of a face of a monoid corresponds to the notion of a saturated multiplicative subset of a ring; we do not use this terminology here because of its conflict with the notion of a saturated monoid defined above.

The union of a family of ideals is an ideal, the union of a family of prime ideals is a prime ideal, and the intersection of a family of faces is a face. The intersection $\langle T\rangle$ of all the faces containing some subset T of M is a face, called the *face generated by* T. it is analogous to the multiplicatively saturated set generated by a subset of a ring. The *interior* I_M of a monoid M is the set of all elements which do not lie in a proper face of M, *i.e.*, the intersection of all the nonempty prime ideals of M.

We denote by $\mathrm{Spec}(M)$ the set of prime ideals of a monoid. If I is an ideal of M and $Z(I)$ denotes the set of primes of M containing I, one finds in the usual way that the set of subsets $Z(I)$ of $\mathrm{Spec}(M)$ defines a topology on $S := \mathrm{Spec}(M)$ (the Zariski topology), in which the irreducible closed sets correspond to the prime ideals. Since M has a unique minimal prime ideal, $\mathrm{Spec}(M)$ has a unique generic point, and in particular is irreducible. If $f \in M$ and F is the face it generates, then

$$S_f := \{\mathfrak{p} : f \notin \mathfrak{p}\} = \{\mathfrak{p} : \mathfrak{p} \cap F = \emptyset\}$$

is open in S, and the set of all such sets forms a basis for the topology on S.

If $\theta\colon M \to N$ is a morphism of monoids, then the inverse image of an ideal is an ideal, the inverse image of a prime ideal is a prime ideal, and the inverse image of a face is a face. Thus θ induces a continuous map

$$\mathrm{Spec}(N) \to \mathrm{Spec}(M) : \quad \mathfrak{p} \mapsto \theta^{-1}(\mathfrak{p}).$$

The preorder relation (1.1.7) is useful when describing ideals and faces of a monoid.

Proposition 1.3.2 *Let S be a subset of a monoid Q and let P be the submonoid of Q generated by S.*

1. *The ideal (S) of Q generated by S is the set of all $q \in Q$ such that $q \geq s$ for some $s \in S$.*

2. *The face $\langle S \rangle$ of Q generated by S is the set of elements q of Q for which there exists a $p \in P$ such that $q \leq p$. In particular, the face generated by an element p of Q is the set of all elements $q \in Q$ such that $q \leq np$ for some $n \in \mathbf{N}$.*

3. *If Q is integral, then Q/P is sharp if and only if $P^{gp} \cap Q$ is a face of Q. In particular, if F is a face of Q, then Q/F is sharp.*

Proof: The first statement follows immediately from the definitions. For the second, note that a submonoid F of Q is a face if and only if $q \leq f$ with $f \in F$ implies that $q \in F$. Hence $\langle S \rangle$ contains the set P' of all $q \in Q$ such that there exists a $p \in P$ with $q \leq p$. Since in fact P' is necessarily a submonoid of Q, it is also a face, so $P' = \langle S \rangle$. If Q is integral, Q/P can be identified with the image of Q in Q^{gp}/P^{gp}, by 1.2.1. Thus an element $q \in Q$ maps to a unit in Q/P if and only if there exists an element $q' \in Q$ such that $q + q' \in P^{gp}$, *i.e.*, if and only if $q \leq q''$ for some $q'' \in Q \cap P^{gp}$. This shows that Q/P is sharp if and only if $Q \cap P^{gp}$ is a face of Q. Finally, note that if F is a face of Q, and $q \in Q \cap F^{gp}$, then $q + f \in F$ for some $f \in F$, hence $q \in F$. $\square$

Proposition 1.3.3 *Let M be a monoid, S a subset of M, and E an M-set. Then there exists an M-set $S^{-1}E$ on which the elements of S act bijectively and a map of M-sets $\lambda_S \colon E \to S^{-1}E$ which is universal: for any morphism of M-sets $E \to E'$ such that each $s \in S$ acts bijectively on E', there is a unique M-map $S^{-1}E \to E'$ such that*

$$\begin{tikzcd} E \arrow[r, "\lambda_S"] \arrow[dr] & S^{-1}E \arrow[d] \\ & E' \end{tikzcd}$$

commutes. The morphism λ_S is called the localization of E by S.

Proof: Let us write the monoid law on M multiplicatively and θ_E for the action of M on E. Let T be the submonoid of M generated by S. The set $S^{-1}E$ can be constructed in the familiar way as the set of equivalence classes of pairs $(e,t) \in E \times T$, where $(e,t) \equiv (e',t')$ if and only if $\theta(t't'')e = \theta(tt'')e'$ for some t'' in T. Then $\lambda_S(e)$ is the class of $(e,1)$, and the action of an element m of M sends the class of (e,t) to the class of $(\theta(m)e,t)$. □

Notice that in fact every element of the face F generated by S acts bijectively on $S^{-1}E$, so that in fact $S^{-1}E \cong F^{-1}E$. Indeed, let E' be any M-set such that $\theta_{E'}(s)$ is bijective for every $s \in S$. If $f \in F$, then $f \leq t$ for some t in the submonoid T of M generated by S. Thus $t = fm$ for some $m \in M$. Then $\theta_{E'}(t) = \theta_{E'}(f)\theta_{E'}(m) = \theta_{E'}(m)\theta_{E'}(f)$, and since $\theta_{E'}(t)$ is bijective, the same is true of $\theta_{E'}(f)$. If $\mathfrak{p} := M \setminus F$ is the prime ideal of M corresponding to F, one often writes $E_{\mathfrak{p}}$ instead of $S^{-1}E$. An M-set E is called *M-integral* if the elements of M act as injections on E. If this is the case, the localization map $\lambda_S \colon E \to S^{-1}E$ is injective, for every subset S of M.

The most important case of (1.3.3) is the case where E is M itself with the action of M on itself by translations. Then $S^{-1}M$ has a unique monoid structure for which λ_S is a morphism of monoids compatible with the M-set structure defined above. The morphism $\lambda_S \colon M \to S^{-1}M$ is also characterized by a universal property: any homomorphism $\lambda \colon M \to N$ with the property that $\lambda(s) \in N^*$ for each $s \in S$ factors uniquely through $S^{-1}M$. If M is integral the natural map $S^{-1}M \to M^{gp}$ is injective, and $S^{-1}M$ can be identified with the set of elements of M^{gp} of the form $m - t$ with $m \in M$ and t belonging to the submonoid of M generated by S. If $\theta \colon M \to N$ is a morphism of monoids and S is a subset of M we write $S^{-1}N$ to mean the localization of N by the image of S, when no confusion can arise.

Let M be a monoid and $S := \operatorname{Spec} M$. If f and g are elements of M, then $S_g \subseteq S_f$ if and only if $f \in \langle g \rangle$. If this is the case, then there is a unique homomorphism $M_f \to M_g$ making the diagram

$$\begin{tikzcd} M \arrow[r] \arrow[dr] & M_f \arrow[d] \\ & M_g \end{tikzcd}$$

commute. Thus $S_f \mapsto M_f$ defines a presheaf on the base $\{S_f : f \in M\}$ for the Zariski topology on S, and we let M_S denote the corresponding sheaf.

For each $f \in M$. the prime $\mathfrak{p} := M \setminus \langle f \rangle$ is the unique closed point of S_f, and it follows that

$$\Gamma(S_f, M_S) = M_{S,\mathfrak{p}} = M_F = M_f.$$

Definition 1.3.4 *A locally monoidal space is a topological space S together with a sheaf of monoids M_S. A morphism of locally monoidal spaces*

$$f\colon (S, M_S) \to (T, M_T)$$

is a pair $(f, f^\flat)$, where $f\colon S \to T$ is a continuous map and $f^\flat\colon M_T \to f_* M_S$ is a morphism of sheaves of monoids, such that for each $t \in T$, the map $f^\flat_t\colon M_{T,t} \to M_{S,s}$ is a local homomorphism.

A morphism of monoids $\theta\colon M \to N$ induces a morphism of locally monoidal spaces $\operatorname{Spec} N \to \operatorname{Spec} M$. Locally monoidal spaces which are locally of the form $\operatorname{Spec} M$ are sometimes called "schemes over $\mathbf{F}_1$" (see [3]).

Remark 1.3.5 The localization of an integral (resp. saturated) monoid is integral (resp. saturated), but the analog for quasi-integral monoids fails, as the following example shows.

Let Q and P be monoids and let K be an ideal of Q. Let E be the subset of $(P \oplus Q)^2$ consisting of those pairs $(p \oplus q, p' \oplus q)$ such that either $p = p'$ or $q \in K$. In fact E is a congruence relation on $P \oplus Q$, and we denote the quotient $(P \oplus Q)/E$ by $P \star_K Q$ (the *join of P and Q along K*). If K is a prime ideal with complement F, then $P \star_K Q$ can be identified with the disjoint union of $P \times F$ with K, and $(p, f) + k = f + k$. Then $\mathbf{N} \star_{\mathbf{N}^+} \mathbf{N}$ is quasi-integral, but its localization by the element 1 of the "first" $\mathbf{N}$ is $\mathbf{Z} \star_{\mathbf{N}^+} \mathbf{N}$, which is not quasi-integral.

Definition 1.3.6 *Let M be a monoid.*

1. *The dimension of M is the Krull dimension of the topological space $\operatorname{Spec}(M)$, i.e., the maximum length d of a chain of prime ideals*

$$\emptyset = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_d = M^+.$$

2. *If* $\mathfrak{p} \in \mathrm{Spec}(M)$, $\mathrm{ht}(\mathfrak{p})$ *is the maximum length of a chain of prime ideals*

$$\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_h.$$

If $\mathfrak{p}$ is a prime ideal of M, the map $\mathrm{Spec}(M_\mathfrak{p}) \to \mathrm{Spec}(M)$ induced by the localization map $\lambda\colon M \to M_\mathfrak{p}$ is injective and identifies $\mathrm{Spec}(M_\mathfrak{p})$ with the subset of $\mathrm{Spec}(M)$ consisting of those primes contained in $\mathfrak{p}$. Equivalently, $F \mapsto \lambda^{-1}(F)$ is a bijection from the set of faces of $M_\mathfrak{p}$ to the set of faces of M containing $M \setminus \mathfrak{p}$. These bijections are order preserving. In particular, we have $\mathrm{ht}(\mathfrak{p}) = \dim(M_\mathfrak{p})$. If M is fine, $\mathrm{Spec}(M)$ is a finite topological space, and is catenary, of [8, 14.3.2, 14.3.3]), as the following proposition implies. We defer its proof until section (2.3), after (2.3.6).

Proposition 1.3.7 *Let M be an integral monoid.*

1. $\mathrm{Spec}\, M$ *is a finite set if M is finitely generated.*
2. $\dim(M) \le \mathrm{rank}\overline{M}^{gp}$, *where* $\overline{M}^{gp} \cong M^{gp}/M^*$, *with equality if M is fine.*
3. *If M is fine, every maximal chain* $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_d$ *of prime ideals has length* $\dim(M)$, *and for any* $\mathfrak{p} \in \mathrm{Spec}\, M$,

$$\mathrm{ht}(\mathfrak{p}) = \mathrm{rank}\overline{M}_\mathfrak{p} = \dim(M) - \dim(F_\mathfrak{p}).$$

Examples 1.3.8 The monoid $\mathbf{N}$ has just two faces, $\{0\}$ and $\mathbf{N}$. More generally, let S be a finite set and let $M = \mathbf{N}^{(S)}$, the free monoid generated by S. If T is any subset of S, $\mathbf{N}^{(T)}$ can be identified with the set of all $I \in \mathbf{N}^{(S)}$ such that $I_s = 0$ for $s \notin T$. This is a face of M, and every face of M is of this form. A more complicated example is provided by the monoid P which is given by generators x, y, z, w subject to the relation $x + y = z + w$. This is the amalgamated sum $\mathbf{N}^2 \oplus_\mathbf{N} \mathbf{N}^2$, where both maps $\mathbf{N} \to \mathbf{N}^2$ send 1 to $(1, 1)$. This monoid is isomorphic to the submonoid of $\mathbf{N}^4$ generated by $\{(1, 1, 0, 0), (0, 0, 1, 1), (1, 0, 1, 0), (0, 1, 0, 1)\}$ and to the submonoid of $\mathbf{Z}^3$ generated by $\{(1, 1, 1), (-1, -1, 1), (1, -1, 1), (-1, 1, 1)\}$. In addition to the faces $\{0\}$ and P, it has four faces of dimension one, corresponding to each of the generators, and four faces of dimension two: $\langle x, z\rangle, \langle x, w\rangle, \langle y, z\rangle, \langle y, w\rangle$. For yet another example, consider the monoid Q given by generators x, y, z, u, v subject to the relations $x + y + z = u + v$. This four-dimensional monoid has five faces of dimension one and nine of dimensions two and three.

2 Convexity, finiteness, and duality

2.1 Finiteness

Proposition 2.1.1 *A quasi-integral monoid is finitely generated as a monoid if and only if M^* is finitely generated (as a group) and $\overline{M}$ is finitely generated (as a monoid).*

Proof: If M is finitely generated as a monoid, then M^{gp} is finitely generated as a group. Since M is quasi-integral, $M^* \subseteq M^{gp}$, and it follows that M^* is finitely generated as a group. Since $M \to \overline{M}$ is surjective, $\overline{M}$ is finitely generated as a monoid. For the converse, suppose $\{s_i\}$ is a finite set of generators for the group M^* and $\{t_j\}$ is a finite subset of M whose images in $\overline{M}$ generate $\overline{M}$ as a monoid. Then $\{s_i, -s_i, t_j\}$ generates M as a monoid. □

Recall that if x and y are two elements of a monoid M, we write $x \leq y$ if there exists a $z \in M$ such that $y = x + z$. If S is a subset of a monoid M and $s \in S$, we say that s is a *minimal element of* S (or M*-minimal* if we need to specify the monoid) if whenever $s' \in S$ and $s' \leq s$, then also $s \leq s'$ (so that $s \sim s'$ in the equivalence relation corresponding to $\leq$).

An M-minimal element of the maximal ideal M^+ of an integral monoid M is called an *irreducible element* of M. An element c of M is irreducible if and only if it is not a unit and whenever $c = a + b$ in M, a or b is a unit.

Proposition 2.1.2 *Let M be a sharp integral monoid. Then every set of generators of M contains every irreducible element of M. If in addition M is finitely generated, then the set of irreducible elements of M is finite and generates M.*

Proof: The first statement is obvious. Suppose now that M is finitely generated. It is clear that every finite set of generators contains a minimal set of generators. Let S be such a minimal set; we claim that every element x of S is irreducible. If $x = y + z$ with y and z in M, we can write $y = \sum_s a_s s$ and $z = \sum_s b_s s$, where a_s and $b_s \in \mathbf{N}$ for all $s \in S$. Then $x = \sum_s c_s s$, where $c_s = a_s + b_s$. Let $S' := S \setminus \{x\}$, so that $(1 - c_x)x = \sum\{c_s s : s \in S'\}$ in M^{gp}. If $c_x > 1$ we see that x is a unit, and since M is sharp, $x = 0$ and S' generates M, a contradiction. If $c_x = 0$, $x = \sum\{c_s s : s \in S'\}$, again contradicting the minimality of S. It follows that $c_x = 1$, and hence $\sum\{a_s s + b_s s : s \in S'\} = 0$.

Since M is sharp, this implies that $a_s s = b_s s = 0$ for all $s \in S'$. Then $y = a_x x$ and $z = b_x x$, where $a_x + b_x = 1$. Thus exactly one of y and z is zero, so x is irreducible, as claimed. Since S contains all the irreducible elements of M, there can be only finitely many such elements. $\square$

Corollary 2.1.3 *The automorphism group of a fine sharp monoid is finite, contained in the permutation group of the set of its irreducible elements.*

Remark 2.1.4 Proposition (2.1.2) shows that every element in a fine sharp monoid can be written as a sum of irreducible elements. In fact a standard argument applies somewhat more generally. Let M be a sharp integral monoid in which every nonempty subset contains a minimal element. Then every element of M can be written as a sum of irreducible elements. (Note that 0 is by definition the sum over the empty set of irreducible elements.) Let us recall the argument. We claim that the set S of elements of M^+ which cannot be written as a sum of irreducible elements is empty. If not, by assumption it contains a minimal element s. Since s is not irreducible, $s = a + b$ where a and b are not zero. If both a and b can be written as sums of irreducible elements, then the same is true of s, a contradiction. But if for example a cannot be written as a sum of irreducible elements, $a \in S$ and $a \leq s$ with s not less than or equal to a, a contradiction of the minimality of s.

Proposition 2.1.5 *Let M be a finitely generated monoid.*

1. *Any sequence $(s(1), s(2), \ldots)$ of elements of M contains an increasing subsequence $(s(i_1) \leq s(i_2) \leq s(i_3) \leq \cdots)$.*

2. *Any decreasing sequence $s(1) \geq s(2) \geq s(3), \ldots$ in M lies eventually in a single equivalence class for the relation $\sim$.*

3. *Any nonempty subset S of M contains a minimal element, and there are only finitely many equivalence classes (for the relation $\sim$) of such elements.*

4. *If M is integral and sharp, any decreasing sequence in M is eventually constant, and any nonempty subset of M has a finite nonzero number of minimal elements.*

Proof: We begin by proving (2.1.5.1), which was pointed out to us by H. Lenstra, when $M = \mathbf{N}^r$. Let $s_1 := pr_1 \circ s$ be the sequence of first coordinates of s. Let n_1 denote the minimum of the set of all $s_1(i)$ for $i \in \mathbf{Z}^+$, and choose i_1 with $s_1(i_1) = n_1$. Let n_2 be the minimum of the set of all $s_1(i)$ with $i > i_1$, and choose $i_2 > i_1$ with $s_1(i_2) = n_2$. Continuing in this way, we find a sequence $1 \leq i_1 < i_2 < \cdots$ such that $s_1(i_1) \leq s_1(i_2) \leq \cdots$. Replacing s by its subsequence $s(i_1), s(i_2), \ldots$, we may assume that s has the property that s_1 is increasing. Now repeat this process with the sequence of second coordinates, and we find that both s_1 and s_2 are increasing. After doing this with each i in succession, we find that s_i is increasing for every i, and hence that s is increasing. If M is any finitely generated monoid, there is a surjective morphism $\theta \colon \mathbf{N}^r \to M$, and any sequence s in M can be lifted to a sequence t in $\mathbf{N}^r$. We have just seen that t has an increasing subsequence t', and the image of t' in M is an increasing subsequence of s.

The remaining statements are formal consequences of the first. To prove (2), we may replace M by its quotient $M/\sim$, so that the preorder relation $\leq$ is in fact an order relation. Let $s.$ be a decreasing sequence in M. By (1), $s.$ has an increasing subsequence $s_{i.}$, which must in fact be constant. Since the original sequence is increasing, it follows that $s(i_1) = s(i)$ for all $i \geq i_1$, so $s.$ is eventually constant.

If S is a nonempty subset of M, choose any element $s(1)$ of S. If $s(1)$ is M-minimal, we are done; if not there exists an element $s(2)$ of S such that $s(2) \leq s(1)$ and $s(2) \not\geq s(1)$. If $s(2)$ is M-minimal, we are done, and if not there exists $s(3)$ with $s(3) \leq s(2)$ and $s(3) \not\geq s(2)$. Continuing in this way, we find a decreasing sequence $s(1), \ldots, s(n)$ of elements of S with $s(i) \not\geq s(i-1)$ for $i = 1, \ldots, n$. By (2), such a sequence must terminate, and then $s(n)$ is an M-minimal element of S. If there were an infinite number of equivalence classes of such minimal elements, we could find an infinite sequence s of elements all belonging to distinct equivalence classes, and by (1) such a sequence would contain an increasing subsequence s. But then $s(1) \leq s(2)$ and $s(1) \not\sim s(2)$, contradicting the minimality of $s(2)$. This proves (3), and (4) follows. $\square$

Remark 2.1.6 An action of a monoid Q on a set S defines a preorder $\leq$ on S: $s \leq t$ if there exists $q \in Q$ such that $q + s = t$. If we let Q act on itself via the regular representation, this definition is the same as the preorder relation used for monoids. If $h \colon S \to T$ is a morphism of Q-sets, then $s \leq s'$

implies $h(s) \leq h(s')$, and conversely if h is injective. Furthermore, if Q is finitely generated, statements (1), (2), and (3) make sense and are valid for any finitely generated Q-set S. To see this, use the fact that if S is finitely generated as a Q-set, then there exists $r \in \mathbf{N}$ and a surjective map of Q-sets $f\colon \cup_r Q \to S$, where $\cup_r Q$ is the disjoint union of r copies of Q acting regularly on itself. A sequence of elements of S admits a subsequence which lies in the image of one of the copies of Q. Thus (1) for S follows from (1) for Q, and (2) and (3) are formal consequences.

Remark 2.1.7 Let S be a nonempty subset of a monoid M, and suppose that M is a submonoid of a fine sharp monoid N. Since N is fine, Proposition 2.1.5 shows that S contains an N-minimal element s, and such an element is also necessarily M-minimal. (If $s = m + s'$ with $m \in M$ and $s' \in S$, then there exist $n \in N$ such that $s' = n + s$, hence $m + n = 0$ and $m = n = 0$.) In particular, Remark 2.1.4 implies that M is generated by its irreducible elements. On the other hand, M-minimal elements of S need not be N-minimal, and it could happen that S has an infinite number of minimal elements and that M has an infinite number of irreducible elements. For example, in $N := \mathbf{N} \times \mathbf{N}$, consider the submonoid M of $\mathbf{N} \times \mathbf{N}$ consisting of $(0, 0)$ together with all pairs (m, n) such that m and n are both positive. (This submonoid is even a congruence relation on $\mathbf{N}$; the quotient $\mathbf{N}/M$ is the unique (up to isomorphism) monoid with two elements which is not a group.) Then for every $m > 0$, the element $(1, m)$ is irreducible in M, and in particular M is not finitely generated as a monoid. This situation is illuminated by the notion of *exactness*, which will turn out to be of fundamental importance in logarithmic geometry.

Definition 2.1.8 *A morphism of monoids $f\colon M \to N$ is exact if the diagram*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & N \arrow[d] \\ M^{gp} \arrow[r, "f^{gp}"] & N^{gp} \end{tikzcd}$$

is Cartesian.

Note that the diagonal morphism $\Delta_M: M \to M \times M$ is exact if and only if the map $M \to M^{gp}$ is injective, *i.e.*, if and only if M is integral. If M and N are integral, then f is exact if and only if whenever x and y are elements of M, $f(x) \leq f(y)$ implies that $x \leq y$. If M is a submonoid of an integral monoid N, then $M \to N$ is exact if and only if $M = M^{gp} \cap N$. It follows immediately that if $N' \to N$ is any morphism of integral monoids, the inverse image in N' of an exact submonoid of N is an exact submonoid of N'. Note also that if M is integral, the canonical morphism $M \to \overline{M}$ is exact.

Theorem 2.1.9

1. *Every ideal in a finitely generated monoid is finitely generated (as an ideal).*

2. *Every exact submonoid of a fine (resp. saturated) monoid is fine (resp. saturated).*

3. *A face of an integral monoid is an exact submonoid. Every face of a fine monoid is finitely generated (as a monoid), and monogenic (as a face).*

4. *Every localization (1.3.3) of a fine monoid (resp. saturated) is fine (resp. saturated).*

5. *The equalizer of two maps of integral monoids $P \to M$ is an exact submonoid of $P \times P$. The equalizer of two maps from a fine (resp. saturated) monoid to an integral monoid is fine (resp. saturated).*

6. *The fiber product of two fine (resp. saturated) monoids over an integral monoid is fine (resp. saturated).*

7. *Any congruence relation on a finitely generated monoid P is finitely generated (as a congruence relation). In particular, any finitely generated monoid is finitely presented.*

8. *Let P and Q be monoids. If Q is fine and P is finitely generated,* $\mathrm{Hom}(P, Q)$ *is also fine. If Q is saturated,* $\mathrm{Hom}(P, Q)$ *is also saturated.*

Proof: First observe that any ideal I of a finitely generated monoid M is generated by the set S of its minimal elements. Indeed, if I' is the ideal of

M generated by S, then $I' \subseteq I$, and if $I \setminus I'$ is not empty, (2.1.5.3) implies that it contains a minimal element t. Since t does not belong to S, it is not minimal as an element of I, so there exists some $q \in I$ such that $q \leq t$ and $t \not\leq q$. The minimality of t in $I \setminus I'$ implies that $q \notin I \setminus I'$. But then $q \in I'$ and consequently also $t \in I'$, which is a contradiction. Notice that two elements s and s' of S with $s \sim s'$ generate the same ideal. Thus a subset T of S containing one element from each equivalence class will still generate I and will be finite by (2.1.5.3).

Next we observe that if S is a subset of an exact submonoid M of a fine sharp monoid N, the set of M-minimal elements of S is finite. In fact, if x and y are two elements of M and $x \leq y$ in N then also $x \leq y$ in M. Thus any M-minimal element of S is also N-minimal, and by (2.1.5) the set of these is finite. In particular, the set of irreducible elements of M is finite, and by (2.1.4) it follows that M is finitely generated. This proves that every exact submonoid of a fine sharp monoid is finitely generated. Slightly more generally, if M is an exact submonoid of any fine monoid N, we can choose a surjection $\mathbf{N}^r \to N$, and the inverse image M' of M in $\mathbf{N}^r$ is an exact submonoid of $\mathbf{N}^r$. It follows that M' is finitely generated, and hence so is M. Suppose now that M is an exact submonoid of a saturated monoid N and $x \in M^{gp}$ with $nx \in M$ for some $n \in \mathbf{Z}^+$. Then $x \in N \cap M^{gp} = M$, so M is also saturated. This proves (2).

Let F be a face of an integral monoid M, let x and y be elements of F, and suppose $z := x - y \in M$. Then $x = y + z \in F$, and since F is a face, it follows that $z \in F$. Thus F is an exact submonoid of M, and hence is finitely generated as a monoid. If $f_1, \ldots, f_n$ are generators, then $f := f_1 + \cdots + f_n$ generates F as a face of M. If $S \subseteq M$ is a finite set of generators of M, then $F^{-1}M$ is generated by the set of elements $\lambda(s), s \in S$ together with $-\lambda(f)$, where f is any generator of F as a face. This proves the third and fourth statements, since localization preservations saturation.

Let $E \to P$ be the equalizer of two maps θ_1 and θ_2 from P to M, with P and M integral. Then $E \to P$ is just the pullback of the diagonal Δ_M via the map $(\theta_1, \theta_2) \colon P \to M \times M$, and since Δ_M is exact, so is $E \to P$. This proves the fifth statement, since an exact submonoid of a fine (resp. saturated) monoid is fine (resp. saturated). The sixth follows because the product of two fine (resp. saturated) monoids is fine (resp. saturated).

The following short proof of (7) is due to Pierre Grillet [6]. We may assume without loss of generality that P is finitely generated and free, hence isomorphic to $\mathbf{N}^r$. If p and q are elements of P, write $p \preceq q$ if p precedes

q in the lexicographical order of $\mathbf{N}^r$, and write $p \prec q$ if in addition $p \neq q$. If $p \preceq q$ and $p' \preceq q'$, then $p + p' \preceq q + q'$, and if $p \leq q$ in the partial order defined by the monoid structure, then $p \preceq q$. The order relation $\preceq$ is a well-orders $\mathbf{N}^r$: every nonempty subset has a unique $\preceq$-minimal element. If E is a congruence relation on P and $p \in P$, let $E(p)$ denote the E-congruence class of p, and let $\mu(p)$ denote the $\preceq$-minimal element in $E(p)$. The complement K of the image of $\mu\colon P \to P$ is the set of all elements k of P such that $\mu(k) \prec k$. Note that if $p \in P$ and $\mu(k) \prec k$, then $\mu(k) + p \prec k + p$, and since $(\mu(k) + p) \equiv_E (k + p)$, $k + p$ is not $\preceq$-minimal in $E(k + p)$. Thus $\mu(k + p) \prec k + p$ and so K is an ideal of P. The congruence relation E' on P generated by the set of pairs $(s, \mu(s))$ with s taken from a finite set S of generators for K is finitely generated and contained in E, so it will suffice to prove that $E \subseteq E'$, *i.e.*, that E' contains $(x, \mu(x))$ for every $x \in P$. If this fails, there exists an x such that $\mu(x) \notin E'(x)$ and which is $\preceq$-minimal among all such elements. Evidently x does not belong the image of μ, so $x \in K$, and hence $x = p + s$ for some $s \in S$ and $p \in P^+$. Since $\mu(s) \prec s$, $x' := p + \mu(s) \prec p + s = x$, and hence by the minimality of x, $E'(x')$ contains $\mu(x')$. But $\mu(s) \equiv_{E'} s$, so $x' \equiv_{E'} x$, and it follows that $\mu(x') = \mu(x)$ and that $\mu(x) \in E'(x)$, a contradiction.

It is clear that $\mathrm{Hom}(P, Q)$ is integral (resp. saturated) if Q is integral (resp. saturated). If P is finitely generated, choose a surjective map $\mathbf{N}^r \to P$ for some $r \in \mathbf{Z}^+$. Then $\mathrm{Hom}(P, Q)$ can be identified with the equalizer of the two maps $\mathrm{Hom}(\mathbf{N}^r, Q) \to \mathrm{Hom}(\mathbf{N}^r \times_P \mathbf{N}^r, Q)$. Since $\mathrm{Hom}(\mathbf{N}^r, Q) \cong Q^r$ is finitely generated if Q is, the same is true of $\mathrm{Hom}(P, Q)$, by (5). $\square$

Remark 2.1.10 If Q is a finitely generated monoid and S is a finitely generated Q-set, then any invariant Q-subset of S is finitely generated as a Q-set. This can be proved in the same way as (2.1.9.1), using (2.1.6).

Remark 2.1.11 Let Q be an integral monoid. A subset K of Q^{gp} which is invariant under the action of Q is called a *fractional ideal*, although sometimes this terminology is reserved for the case in which there exists an element q of Q such that $q + K \subseteq Q$. This is automatically the case if K is finitely generated as a Q-set, and the converse holds if Q is finitely generated as a monoid, by (2.1.10). Note that a fractional ideal $K \subset Q^{gp}$ need not be a submonoid of Q^{gp}. The natural map $\pi\colon Q \to \overline{Q}$ induces a bijection between the set of fractional ideals of Q and of $\overline{Q}$, and this bijection takes finitely generated fractional ideals to finitely generated fractional ideals.

Proposition 2.1.12 *Let $\theta\colon Q \to P$ be an exact homomorphism of fine monoids and let J be a finitely generated fractional ideal of P. Then $K := \theta^{g^{-1}}(J)$ is a finitely generated fractional ideal of Q.*

Proof: Replacing θ by $\overline{\theta}\colon \overline{Q} \to \overline{P}$, we may and shall assume that Q and P are sharp. If K is empty there is nothing to prove. Otherwise let $S := \theta^g(K)$, a nonempty subset of J. Since J is finitely generated as a P-set, it follows from (2.1.5) that the subset S' of minimal elements of S is finite. Let T denote the inverse image of S' in K. Since θ is exact and sharp, it is injective, so T is also finite. If k is any element of K, then there exists an element t of T such that $\theta^{gp}(k) \geq \theta^{gp}(t)$. This means that for some $p \in P$, $\theta^{gp}(k) = p + \theta^{gp}(t)$, *i.e.*, that $\theta^{gp}(k - t) \in P$. Since θ is exact, this implies that $q := k - t \in Q$, and hence that $k \in Q + T$. Thus T generates K as a Q-set. $\square$

To see that the exactness hypothess is not superfluous, note that the inverse image of the principal fractional ideal generated by 0 in $\mathbf{N}$ by the summation map $\mathbf{N} \oplus \mathbf{N} \to \mathbf{N}$ is not finitely generated as a $\mathbf{N} \oplus \mathbf{N}$-set.

Remark 2.1.13 If P is an integral monoid and E is a congruence relation on P, then P/E is integral if and only if $E \to P \times P$ is exact. Indeed the congruence relation E determined by a surjective map $\theta\colon P \to Q$ of integral monoids is just the equalizer of the two maps $P \times P \to Q$, and we saw in (2.1.9.5) that it is then an exact submonoid of $P \times P$. For the converse, suppose that $E \to P \times P$ is exact and $\theta\colon P \to Q$ is the coequalizer of the two maps $E \to P$. If $\theta(p_1) + \theta(p) = \theta(p_2) + \theta(p)$ in Q, then $e := (p_1, p_2) + (p, p) \in E$. Since $(p, p) \in E$, it follows that $(p_1, p_2) \in E^{gp} \cap P \times P$, and hence that $(p_1, p_2) \in E$. Then $\theta(p_1) = \theta(p_2)$, so Q is integral. In particular, congruence relations on P yielding integral quotients Q correspond to congruence relations on P^{gp}, and hence by (1.1.3) to subgroups of P^{gp}. Of course, the subgroup of P^{gp} corresponding to a surjective map of integral monoids $P \to Q$ is just the kernel of $P^{gp} \to Q^{gp}$.

Corollary 2.1.14 *Let P be a fine monoid and let E be a congruence relation on P such that P/E is integral. Then E is finitely generated as a monoid (not just as a congruence relation).*

Corollary 2.1.15 *Let $P \to M$ be a morphism of integral monoids. If P and $\overline{M}$ are finitely generated, then so is $P^{gp} \times_{M^{gp}} M$.*

Proof: It suffices to observe that the map $P^{gp} \times_{M^{gp}} M \to P^{gp} \times_{\overline{M}^{gp}} \overline{M}$ is an isomorphism and to apply (2.1.9.4) and (2.1.9.6). □

Proposition 2.1.16 *Let Q be a sharp valuative monoid. Then the following conditions are equivalent.*

- *Q is isomorphic to $\mathbf{N}$.*
- *Q^{gp} is isomorphic to $\mathbf{Z}$.*
- *Q is finitely generated.*

Proof: It is evident that (1) implies (2). If (2) holds, let $\nu \colon \mathbf{Q}^{gp} \to \mathbf{Z}$ be an isomorphism and choose $q \in Q^{gp}$ with $\nu(q) = 1$. Either q or $-q$ lies in Q, so by changing the signs of q and/or ν we may arrange things so that $q \in Q$ and $\nu(q) = 1$. Then the sharpness of Q implies that $\nu(q') \geq 0$ for all $q' \in Q$. Thus ν induces a homomorphism $Q \to \mathbf{N}$ which is necessarily bijective. This proves the equivalence of (1) and (2). Suppose that (3) holds. Since Q is valuative, the order relation on Q is a total order, and Proposition (2.1.5.3) implies that it is even a well-ordering. Thus Q^+ has a unique minimal element which then (freely) generates Q. This proves the equivalence of (1) and (3). □

Example 2.1.17 Let X be a normal locally noetherian scheme and Y a proper closed subset. Then it follows from Theorem (2.1.9.2) that the stalks of the sheaf $\underline{\Gamma}_Y Div_X^+$ of effective Cartier divisors on X with support in Y are fine monoids. To see this, let $\mathcal{O}'_X$ be the subsheaf of $\mathcal{O}_X$ which to each open set U of X assigns the set of sections f such that $f_x \neq 0 \in \mathcal{O}_{X,x}$ for all $x \in U$. This a sheaf of submonoids of $\mathcal{O}_X$, and Div_X^+ can be identified with the quotient $\mathcal{O}'_X/\mathcal{O}^*_X$. Let $\mathcal{W}_X^+$ be the sheaf of effective Weil divisors, *i.e.*, the sheaf associated to the presheaf which to every open U assigns the free monoid on the set of points $\eta \in U$ such that $\mathcal{O}_{U,\eta}$ has dimension one. Since X is regular in codimension one, each $\mathcal{O}_{U,\eta}$ is a discrete valuation ring, and the valuation maps induce a morphism of monoids $\nu\colon \mathcal{O}'_X \to \mathcal{W}^+_X$ [10, II §6]. The normality of X implies that for any $x \in X$, $\mathcal{O}_{X,x}$ is the intersection, in the fraction field $K_{X,x}$ of $\mathcal{O}_{X,x}$, of its localizations at height one primes. It follows that $\mathcal{O}'_{X,x}$ is the set of sections f of $K_{X,x}$ such that $\nu^{gp}(v) \in \mathcal{W}^+_X$, and that $\mathcal{O}^*_X$ is the kernel of ν. Hence the morphism $\nu_x\colon \mathcal{O}'_{X,x} \to \mathcal{W}^+_{X,x}$ is exact, and $Div^+ := \mathcal{O}'_{X,x}/\mathcal{O}^*_{X,x}$ is an exact submonoid of $\mathcal{W}^+_{X,x}$, and hence the stalk at x of $\underline{\Gamma}_Y(Div^+_X)$ is an exact submonoid of the stalk at x of $\underline{\Gamma}_Y(\mathcal{W}^+_X)$. The latter is just the free monoid on the set of prime ideals of height one in the local ring $\mathcal{O}_{X,x}$ which are contained in Y. Since Y is a proper closed subset of X, each of these is a minimal prime of the noetherian local ring $\mathcal{O}_{Y,x}$, and hence there only finitely many such primes. Thus $\underline{\Gamma}_Y(\mathcal{W}^+_X)_x$ is a fine monoid, and by (2.1.9.2), the same is true of $\underline{\Gamma}_Y(Div^+_X)_x$.

To see that the normality hypothesis is not superfluous, let X be the spectrum of the subring R of $\mathbf{C}[t]$ consisting of those polynomials whose first derivative vanishes at $t = 0$. This is a curve with a cusp at the origin x. Let $Y := \{x\}$ and for any complex number a, let D_a be the class of $t^2 - at^3$ in $Div^+_{X,x} = \mathcal{O}'_{X,x}/\mathcal{O}^*_{X,x}$. Note that in $K_{X,x}$,

$$(t^2 - at^3)/(t^2 - bt^3) = (1 - at)/(1 - bt) = 1 + (b - a)t + \cdots,$$

which does not belong to $\mathcal{O}^*_{X,x}$ if $a \neq b$. Thus $D_a \neq D_b \in \underline{\Gamma}_Y(Div^+_X)_x$. It follows that $\underline{\Gamma}_Y(Div^+_X)_x$ is uncountable and hence is not finitely generated. Similar examples can be made with local nodal curves.

2.2 Duality

Duality, and in particular the existence of “enough” homomorphisms from a fine monoid to $\mathbf{N}$, is a crucial tool in the theory of toric varieties.

Theorem 2.2.1 . *Let Q be a fine monoid, and let $H(Q) := \mathrm{Hom}(Q, \mathbf{N})$.*

1. *The monoid $H(Q)$ is fine, saturated, and sharp.*

2. *The natural map $H(Q)^{gp} \to \mathrm{Hom}(\overline{Q}^{gp}, \mathbf{Z})$ is an isomorphism.*

3. *The evaluation mapping $ev\colon Q \to H(H(Q))$ factors through an isomorphism*

$$\overline{ev}\colon \overline{Q^{\mathrm{sat}}} \to H(H(Q)).$$

The key geometric tool is the following. Let P be a submonoid of an abelian group G, and let ϕ be a homomorphism $G \to \mathbf{Z}$ which maps P to $\mathbf{N}$. Suppose that t is an element of G and $\phi(t) < 0$, and let Q be the submonoid of G generated by P and t. Then the homomorphism

$$\psi: G \to \operatorname{Ker}(\phi) : g \mapsto t\phi(g) - g\phi(t)$$

induces multiplication by $|\phi(t)|$ on $\operatorname{Ker}(\phi)$ and maps Q into P.

The following result is a corollary of the theorem, but in fact it is one of the main ingredients in the proof.

Lemma 2.2.2 *If Q is a fine monoid, there exists a local homomorphism $h: Q \to \mathbf{N}$; i.e., an element of $H(Q)$ such that $h^{-1}(0) = Q^*$.*

Proof: We may assume without loss of generality that Q is sharp, and we shall argue by induction on the number of generators of Q. If Q is zero the result is trivial. Suppose that T is a set of nonzero generators for Q, $t \in T$, and $S := T \setminus \{t\}$. Let P be the submonoid of Q generated by S. Then P is still sharp and the induction hypothesis implies that there exists a local homomorphism $h: P \to \mathbf{N}$. Then h induces a homomorphism $P^{gp} \to \mathbf{Z}$ which we denote again by h. Replacing h by nh for a suitable $n \in \mathbf{Z}^+$, we may assume that h extends to a homomorphism $Q^{gp} \to \mathbf{Z}$ we which still denote by h. If $h(t) > 0$ there is nothing more to prove. If $h(t) = 0$, choose any $h': Q^{gp} \to \mathbf{Z}$ such that $h'(t) > 0$. Then if n is a sufficiently large natural number, $nh(s) + h'(s) > 0$ for all $s \in S$ and $h'(t) > 0$, so $nh + h' \in H(Q)$ and is local. Suppose on the other hand that $h(t) < 0$. For each $s \in S$, let $s' := h(s)t - h(t)s$. Then each $s' \in Q$, and the submonoid Q' of Q generated by the set S' of all s' is sharp. Note that $h(s') = 0$ for all $s' \in S'$ and hence for all $q' \in Q'$. Thus $Q'^{gp} \subseteq \operatorname{Ker}(h) \subseteq Q^{gp}$. Since $|S'| \leq |S|$, the induction hypothesis implies that there exists a local homomorphism $g \in H(Q')$. Replacing g by ng for a suitable n, we may assume that g extends to a homomorphism $\operatorname{Ker}(h^{gp}) \to \mathbf{Z}$, which we continue to denote by g. Since $t \notin \operatorname{Ker}(h^{gp})$, the subgroup of Q^{gp} generated by t and $\operatorname{Ker} h$ is isomorphic to $\mathbf{Z} \oplus \operatorname{Ker}(h)$, and we may extend g to this subgroup by letting $g(t) = 0$. Replacing g by yet another multiple, we may assume that it extends to all of Q^{gp}. For any $s \in S$, $-h(t)g(s) = g(s') - h(s)g(t) = g(s') > 0$; since $h(t) < 0$ this implies that $g(s) > 0$. Then $ng - h \in H(Q)$ is local for n sufficiently large. $\square$

Corollary 2.2.3 *Let Q be a fine monoid and let x be an element of Q^{gp}. Then $x \in Q^{\mathrm{sat}}$ if and only if $h(x) \geq 0$ for every $h \in H(Q)$.*

Proof: If $x \in Q^{\mathrm{sat}}$ then $nx \in Q$ for some $n \in \mathbf{Z}^+$ and hence $h(x) \geq 0$ for any $h \in H(Q)$. Suppose conversely that $h(x) \geq 0$ for every $h \in H(Q)$. Let Q' be the submonoid of Q^{gp} generated by Q and $-x$, and choose a local homomorphism $h\colon Q' \to \mathbf{N}$. Then $h(x) \geq 0$ and $h(-x) \geq 0$, so that in fact $h(x) = 0$ and $-x \in Q'^*$. Then there exists an element q' of Q' such that $q' - x = 0$. Writing $q' = -mx + q$ with $m \in \mathbf{N}$ and $q \in Q$, we see that $(m+1)x = q$, so $x \in Q^{\mathrm{sat}}$. $\square$

Proof of (2.2.1) First observe that $H(Q)$ is fine, sharp, and saturated by (2.1.9.8). Since $H(Q) \to \mathrm{Hom}(Q, \mathbf{Z})$ is injective, so is the map $H(Q)^{gp} \to \mathrm{Hom}(Q, \mathbf{Z})$. Any element h of $H(Q)$ necessarily annihilates Q^*, so the image of this map is contained in $\mathrm{Hom}(\overline{Q}, \mathbf{Z})$. Suppose on the other hand that $g \in \mathrm{Hom}(\overline{Q}, \mathbf{Z})$, and let h be a local homomorphism $\overline{Q} \to \mathbf{N}$. There exists $n \in \mathbf{Z}^+$ such that $nh(\overline{q}) \geq g(\overline{q})$ for each of a finite set of nonzero generators $\overline{q}$ of $\overline{Q}$, and then $nh(\overline{q}) \geq g(\overline{q})$ for every $\overline{q} \in \overline{Q}$. This means that $h' := nh - g \in H(\overline{Q})$, so $g = nh - h' \in H(\overline{Q})^{gp} \cong H(Q)^{gp}$. It follows that the map $H(Q)^{gp} \to \mathrm{Hom}(\overline{Q}^{gp}, \mathbf{Z})$ is an isomorphism.

Since $H(H(Q))$ is fine saturated and sharp, ev factors through a map $\overline{ev}$ as claimed in the statement of the theorem. Let x_1 and x_2 be two elements of Q^{sat} with $ev(x_1) = ev(x_2)$, and let $x := x_1 - x_2 \in Q^{gp}$. Then $h(x) = 0$ for every $h \in H(Q)$. It follows that from (2.2.3) that x and $-x$ belong to Q^{sat}, so $x \in (Q^{\mathrm{sat}})^*$. Thus $\overline{x}_1 = \overline{x}_2 \in \overline{Q^{\mathrm{sat}}}$, and this proves the injectivity of $\overline{ev}$. For the surjectivity, suppose that $g \in H(H(Q))$. Since Q^{gp} is a finitely generated group, the map from Q^{gp} to its double dual is surjective. Thus there exists an element q of Q^{gp} such that $ev(q) = g$, *i.e.*, such that $h(q) = g(h)$ for all $h \in H(Q)$. Then $h(q) \geq 0$ for all h, so $q \in Q^{\mathrm{sat}}$, as required. $\square$

Corollary 2.2.4 *Let Q be a fine monoid. A subset S of Q is a face if and only if there exists an element h of $H(Q)$ such that $S = h^{-1}(0)$. For each $S \subseteq Q$, let $S^\perp$ be the set of $h \in H(Q)$ such that $h(s) = 0$ for all $s \in S$, and for $T \subseteq H(Q)$, let $T^\perp$ be the set of $q \in Q$ such that $t(q) = 0$ for all $t \in T$. Then $F \mapsto F^\perp$ induces an order reversing bijection between the set of faces of Q and the set of faces of $H(Q)$, and $F = (F^\perp)^\perp$ for any face of either.*

Proof: It is clear that $h^{-1}(0)$ is a face of Q if $h \in H(Q)$. If F is any face, Q/F is a fine sharp monoid, so by (2.2.2) there exists a local homomorphism $h: Q/F \to \mathbf{N}$. Then h can be regarded as an element of $F^\perp \subseteq H(Q)$. Since h is local, $h^{-1}(0) = F$. This proves the first statement. It is clear that $S^\perp$ is a face of $H(Q)$ if S is any subset of Q and that $T^\perp$ is a face of Q if T is any subset of $H(Q)$. Furthermore, $S_2^\perp \subseteq S_1^\perp$ if $S_1 \subseteq S_2$, and $S \subseteq (S^\perp)^\perp$. The only nontrivial thing to prove is that $F = (F^\perp)^\perp$ if F is a face of Q. But this follows immediately from the existence of an h with $F = h^{-1}(0)$.

$\square$

Corollary 2.2.5 *If Q is fine, then Q^{sat} is again fine. In fact, the action of Q on Q^{sat} defined by the homomorphism $Q \to Q^{\text{sat}}$ makes Q^{sat} a finitely generated Q-set.*

Proof: Since $(Q^{\text{sat}})^* \subseteq Q^{gp}$, it is a finitely generated abelian group. Theorem (2.2.1) implies that $\overline{Q^{\text{sat}}}$ is fine, and since Q^{sat} is integral, it follows from (2.1.1) that Q^{sat} is finitely generated, hence fine. Choose a finite set of generators T for Q^{sat} as a monoid, and for each $t \in T$, choose $n_t \in \mathbf{N}^+$ such that $n_t t \in Q$. Then $\{\sum j_t t : j_t \leq n_t, t \in T\}$ generates Q^{sat} as a Q-set. $\square$

Corollary 2.2.6 *Let P be a fine sharp monoid such that P^{gp} is torsion free (resp. which is saturated). Then P is isomorphic to a submonoid (resp. an exact submonoid) of $\mathbf{N}^r$ for some r.*

Proof: Note first that if $\pi: M \to Q$ is a surjective map of fine monoids, the dual morphism $H(Q) \to H(M)$ is injective and exact. Indeed, we can by (2.2.1) view an element h of $H(Q)^{gp}$ as a homomorphism $\overline{Q} \to \mathbf{Z}$, and we see that $h \in H(Q)$ if and only if $h \circ \pi \in H(M)$. Now let P be a fine sharp monoid such that P^{gp} is torsion free. By (2.1.9.8), $Q := H(P)$ is fine and sharp and $Q^{gp} \cong \operatorname{Hom}(P^{gp}, \mathbf{Z})$, so $P^{gp} \cong \operatorname{Hom}(Q^{gp}, \mathbf{Z}) \cong H(Q)^{gp}$. Choose a surjection $\mathbf{N}^r \to Q$. As we observed above, $H(Q)$ is then an exact submonoid of $H(\mathbf{N}^r) \cong \mathbf{N}^r$. Furthermore, the isomorphism $P^{gp} \to H(Q)^{gp}$ carries P into $H(Q)$, and in fact identifies P^{sat} with $H(Q)$ by (2.2.1). $\square$

Remark 2.2.7 If Q is a fine monoid, then an element h of $H(Q)$ lies in the interior of $H(Q)$ if and only if $h: Q \to \mathbf{N}$ is a local homomorphism. Indeed, by definition, an element h of $H(Q)$ belongs to its interior if and only if it is not contained in any proper face of Q. By (2.2.4), this is the case if and only if $h^\perp$ does not contain any nontrivial face of Q, *i.e.*, if and only if $h^\perp = Q^*$. This is exactly the condition that $h: Q \to \mathbf{N}$ be a local homomorphism.

We shall find the following crude finiteness result useful. More precise variants are available, most of which rely on the theory of Hilbert polynomials in algebraic geometry.

Corollary 2.2.8 *Let Q be a fine sharp monoid of dimension d and let $h: Q \to \mathbf{N}$ be a local homomorphism. For each real number r, let*

$$B_h(r) := \{q \in Q : h(q) < r\}.$$

Then there is a constant $c \in \mathbf{R}$ such that for all $r \in \mathbf{R}$,

$$\#B_h(r) < cr^d.$$

Proof: By (2.2.1), $H(Q)$ is finitely generated and sharp, and hence it has a unique set of minimal generators $\{h_1, \dots h_m\}$. Since h is local, (2.2.7) shows that each h_i belongs to the face generated by h. Then (1.3.2) implies that for each i there exists an integer n_i such that $n_i h \geq h_i$ in $H(Q)$. Choose $n \geq n_i$ for all i. Then for every $r \in \mathbf{R}^+$, $B_h(r) \subseteq \cap_i B_{h_i}(nr)$. Since Q is sharp, (2.2.1) implies that $H(Q)^{gp} \cong \mathrm{Hom}(Q^{gp}, \mathbf{Z})$, and consequently $\{h_i\}$ spans $\mathrm{Hom}(Q^{gp}, \mathbf{Z})$. Proposition 1.3.7 says that this group has rank d. Let $(x_1, \cdots x_d)$ be a basis for $\mathrm{Hom}(Q^{gp}, \mathbf{Z})$, find integers $a_{i,j}$ such that $x_i = \sum_j a_{i,j} h_j$, and let $a := \sum_{i,j} |a_{i,j}|$. Then if $q \in B_h(r)$,

$$|x_i(q)| \leq \sum_j |a_{i,j}| h_j(q) \leq anr.$$

Thus $B_h(r) \subseteq \cap_i B_{|x_i|}(anr)$. The cardinality of this set is bounded by $t(2anr)^d$, where t is the order of the torsion subgroup of Q^{gp}. $\square$

2.3 Monoids and cones

Let K be an Archimidean ordered field and let $K^{\geq 0}$ denote the set of nonnegative elements of K, regarded as a multiplicative monoid. Since $0 \in K^{\geq 0}$, this monoid is not quasi-integral, but $K^{\geq 0} \setminus \{0\}$ is a group. In practice here, K will be either $\mathbf{R}$ or $\mathbf{Q}$.

Definition 2.3.1 *A K-cone is an integral monoid $(C, +, 0)$ endowed with an action of $(K^{\geq 0}, \cdot, 1)$, such that*

$$\begin{aligned} (a+b)x &= ax + bx \quad \text{for } a, b \in K^{\geq 0} \text{ and } x \in C, \text{ and} \\ a(x+y) &= ax + ay \quad \text{for } a \in K^{\geq 0} \text{ and } x, y \in C. \end{aligned}$$

A morphism of K-cones is a morphism of monoids compatible with the actions of $K^{\geq 0}$.

Any K-vector space V forms a K-cone, and any nonempty subset of C of V which is stable under addition and by multiplication $K^{\geq 0}$ is a subcone. If C is any K-cone, then C^{gp} inherits a unique structure of a K-vector space such that $C \to C^{gp}$ is a morphism of K-cones, so we can regard every K-cone as sitting inside a K-vector space. If S is any subset of a K-vector space V we can define its *conical hull* $C_K(S)$ to be the set of all linear combinations of elements of S with coefficients in $K^{\geq 0}$. Then $C_K(S)$ is the smallest K-cone in V containing S. A K-cone C is called *finitely generated* if it admits a finite subset S such that $C = C_K(S)$. In the sequel we shall say "cone" instead of "K-cone," and write $C(S)$ instead of $C_K(S)$, when there seems to be no danger of confusion.

If C is a K-cone, C^* is not just a subgroup but also a vector subspace, the largest linear subspace of C. A cone is sharp if and only if $C^* = 0$; some authors call such a C a *strongly convex cone*. If C is a K-cone, then $\overline{C} := C/C^*$ is a sharp K-cone. By the *dimension* of C we mean the dimension of C^{gp} (as a K-vector space), and we call the dimension of $\overline{C}$ the *sharp dimension* of C.

Let C be a K-cone and let F be a face of C. Then F is automatically a subcone of C. Indeed, if $x \in F$ and $a \in K^{\geq 0}$, then there exists $n \in \mathbf{N}$ with $a \leq n$, since K is Archimidean. Then $ax \leq nx$ and $nx \in F$, and since F is a face, $ax \in F$ also. If F is a face of a cone C, then C/F is a sharp cone, and we call its dimension the *codimension of F*. If this codimension is one, we say that F is a *facet* of C. A one-dimensional face of C is sometimes called an *extremal ray* of C.

Let us say that an element x of a sharp cone C is *K-indecomposable in C* if it is not a unit and whenever $x = y + z$ with y and z in C, then y and z are K-multiples of x. Thus x is K-indecomposable if and only $\langle x \rangle^{gp}$ is a one-dimensional K-vector space. Notice that in the monoid P given by generators $\{x, y, z\}$ and relations $x + y = 2z$, x, y, and z are irreducible, and in the corresponding cone x and y are indecomposable, but z is not indecomposable.

Proposition 2.3.2 *Suppose that C is a finitely generated sharp cone. Then each element of every minimal set of generators for C is K-indecomposable. In particular, C is spanned by a finite number of indecomposable elements.*

Proof: The proof is essentially the same as the proof of the analogous result (2.1.2) for monoids, but we write it in detail anyway. Suppose that S is a minimal set of generators and $x \in S$. Write $x = y + z$, with $y = \sum a_s s$, $z = \sum b_s s$, and $a_s, b_s \in K^{\geq 0}$. Then $x = \sum c_s s$, with $c_s = a_s + b_s$. Let $S' := S \setminus \{x\}$, so $(1 - c_x)x = \sum_{s \in S'} c_s s$. If $c_x < 1$ we see that S' generates C, a contradiction, and if $c_x > 1$, then x is a unit, contradicting the sharpness of C. Then necessarily $c_x = 1$, so $0 = \sum_{s \in S'} a_s s + b_s s$. Since S is sharp, this implies that $a_s s = b_s s = 0$ for all $s \in S'$. Then $y = a_x x$ and $z = b_x x$, as required. □

Proposition 2.3.3 *Let C be a K-cone and S a set of generators for C.*

1. *Every face of C is generated as a cone by $F \cap S$.*

2. *If C is finitely generated, C contains only a finite number of faces.*

3. *The length d of every maximal increasing chain of faces $C^* = F_0 \subset F_1 \subset F_2 \cdots F_d = C$ is less than or equal to the K-dimension of the vector space $\overline{C}^{gp}$, with equality if C is finitely generated.*

4. *Every proper face of C is contained in a facet.*

Proof: Let F be a face of C and $x \in F$, $x \neq 0$. Then we can write $x = \sum a_s s$ with $a_s \in K^{\geq 0}$ and $s \in S$. Since F is a face, each $s \in F$ if $a_s \neq 0$. This shows that in fact F is generated as a cone by $F \cap S$. If S is finite, it has only finitely many subsets, so C can have only finitely many faces. Since there is a natural bijection between the faces of C and the faces of $\overline{C}$ we may as well assume in the proof of (3) that $C^* = 0$. Let $\mathcal{C} := F_0 \subset \cdots \subset F_d = C$ be a maximal chain of faces of C. Since each F_i is an exact submonoid of C, the inclusions $F_0^{gp} \subseteq F_1^{gp} \subset \cdots \subset F_d^{gp}$ of linear subspaces of C^{gp} are all strict. Since C^{gp} has dimension $\overline{d}$, $d \leq \overline{d}$. We prove the opposite inequality by induction on the dimension $\overline{d}$ of $\overline{C}^{gp}$. If $\overline{d} = 0$, $C = 0$ and the result is trivial. Suppose that $\overline{d} > 0$; we may assume by (2.3.2) that S is the set of indecomposable elements of C. Our assumptions imply that $d \geq 1$, and in particular $F_1 \neq 0$. Then by (1) it must contain a K-indecomposable element c. Then $\langle c \rangle \subset F_1$, and since $\mathcal{C}$ is a maximal chain, $\langle c \rangle = F_1$. Since c is K-indecomposable, $\langle c \rangle^{gp}$ is a one-dimensional K-vector space, and the dimension of $(C/F_1)^{gp} \cong C^{gp}/F_1^{gp}$ is $\overline{d} - 1$. For each i, the canonical map $(F_i/F_1)^{gp} \to F_i^{gp}/F_1^{gp}$ is an isomorphism, and it follows that the inclusions $F_1/F_1 \subset F_2/F_1 \subset \cdots \subset C/F_1$ of faces of C/F_1 are also strict. The maximality of the original chain $\mathcal{C}$ implies that this chain is also maximal, and thus the induction hypothesis implies that its length $d-1$ is less than the dimension $\overline{d} - 1$ of $(C/F_1)^{gp}$. This proves (3), and (4) is an immediate consequence. □

Proposition 2.3.4 *The interior (i.e., the complement of the union of the proper faces) of a finitely generated cone C is dense in C (in the standard topology).*

Proof: We may and shall assume without loss of generality that C is sharp. Let S be a minimal generating set of indecomposable elements of C. Then any element c of C can be written (not uniquely) as $c = \sum a_s s$ with $a_s \geq 0$, and c lies in the interior if no $a_s = 0$. Then $c_i := \sum (a_s + i^{-1})s$ lies in the interior of C and converges to c. □

Let P be an integral monoid and consider the map $P \to K \otimes P^{gp}$ sending an element p to $1 \otimes p$. Let $C_K(P)$ denote the subcone of $K \otimes P^{gp}$ generated by the image of $P \to K \otimes P^{gp}$, and

$$c \colon P \to C_K(P)$$

be the map sending $p \in P$ to $1 \otimes p \in C_K(P)$. Note that two elements p_1 and p_2 of P have the same image in $K \otimes P^{gp}$ if and only if their difference lies in the torsion subgroup of P^{gp}, *i.e.*, iff there exists an integer n, such that $np_1 = np_2$.

Proposition 2.3.5 *Let P be an integral monoid and let $c\colon P \to C_K(P)$ be the natural map described above.*

1. *If F is any face of P, the natural map $C_{\mathbf{Q}}(F) \to C_{\mathbf{Q}}(P)$ identifies $C_{\mathbf{Q}}(F)$ with a face of $C_{\mathbf{Q}}(P)$. Furthermore, $c^{-1}(C_{\mathbf{Q}}(F)) = F$, and c defines a bijection between the faces of $C_{\mathbf{Q}}(P)$ and the faces of P.*

2. *If I is an ideal of P, let $C_{\mathbf{Q}}(I) \subseteq C_{\mathbf{Q}}(P)$ denote the smallest $\mathbf{Q}^{\geq 0}$-invariant ideal of $C_{\mathbf{Q}}(P)$ containing the image of $I \to C_{\mathbf{Q}}(P)$. Then $C_{\mathbf{Q}}(I) \cap P = \sqrt{I}$.*

Proof: The proof relies on the following lemma, which is not true for a general K. However, see Proposition (2.3.17) for a partial generalization of Proposition (2.3.5).

Lemma 2.3.6 *Let P be a monoid and let $C_{\mathbf{Q}}(P) \subseteq \mathbf{Q} \otimes P^{gp}$ the corresponding cone. Then*

$$C_{\mathbf{Q}}(P) = \{x \in \mathbf{Q} \otimes P^{gp} : \text{ there exist } m \in \mathbf{Z}^+, p \in P \text{ with } mx = c(p).\}$$

If I is an ideal of P,

$$C_{\mathbf{Q}}(I) = \{x \in \mathbf{Q} \otimes P^{gp} : \text{ there exist } m \in \mathbf{Z}^+, p \in I \text{ with } mx = c(p).\}$$

Proof: If $m_1 x_1 = c(p_1)$ and $m_2 x_2 = c(p_2)$, then

$$m_1 m_2 (x_1 + x_2) = c(m_2 p_1 + m_1 p_2),$$

so the set X on the right side of the above equation is a submonoid of $\mathbf{Q} \otimes P^{gp}$. It is also stable under the action of $\mathbf{Q}^{\geq 0}$ and contains the image of P, hence contains $C_{\mathbf{Q}}(P)$. On the other hand, it is also clear that X is contained in any $\mathbf{Q}$-cone containing the image of P, hence is the smallest such cone. □

Now let F be a face of P and let x_1 and x_2 be elements of $C_{\mathbf{Q}}(P)$ whose sum y belongs to $C_{\mathbf{Q}}(F)$. Then there exist $m > 0$, $f \in F$ and $p_i \in P$ such that $my = 1 \otimes f$ and $mp_i = 1 \otimes x_i$. Hence $f - p_1 - p_2$ is a torsion element of P^{gp}, and by replacing m by a multiple, m we may assume that $f = p_1 + p_2$. Then $p_i \in F$ and hence $x_i \in C_{\mathbf{Q}}(F)$. This shows that $C_{\mathbf{Q}}(F)$ is a face of $C_{\mathbf{Q}}(P)$.

Evidently $F \subseteq c^{-1}(C_{\mathbf{Q}}(F))$. Conversely, if $p \in P$ and $c(p) \in C_{\mathbf{Q}}(F)$, then there exist an $m \in \mathbf{Z}^+$ and $f \in F$ with $c(f) = mc(p)$, hence there exist m' such that $m'f = mm'p \in P$, and hence $p \in F$. On the other hand, if G is any face of $C_{\mathbf{Q}}(P)$ and g is a generator for G as a face, then mg lies in the image of c for some m, and mg still generates G. Thus $G = C_{\mathbf{Q}}(F)$, where $F := c^{-1}(G)$. This proves (1), and the proof of (2) is similar. $\square$

Proof of (1.3.7): Because of the bijection between the prime ideals and the faces of M and the bijection (2.3.5) between the faces of M and of the cone C it spans, (1.3.7) follows from (2.3.3). Thus, M has finitely many prime ideals because C has finitely many faces, and the maximal length of a chain of prime ideals in M is the maximal length of a chain of faces of C. By (2.3.5) this is the dimension of the vector space $\overline{C}^{gp} \cong \mathbf{Q} \otimes \overline{M}^{gp}$. If $\mathfrak{p} \in \operatorname{Spec} M$, and $F_{\mathfrak{p}} = M \setminus \mathfrak{p}$ is the corresponding face of C, then by (2.3.3.3), $F_{\mathfrak{p}}$ is contained in a chain of length $\dim(\overline{C}) = \dim(M)$. Furthermore $\operatorname{ht}(\mathfrak{p})$ is by definition the maximum length h of a chain of faces $F_{\mathfrak{p}} = F_0 \subset F_1 \cdots \subset F_h = C$, *i.e.*, of a chain of faces in $C/F_{\mathfrak{p}}$. By (2.3.3.3), $h = \dim(C/F_{\mathfrak{p}})^{gp} = \dim(\overline{C}^{gp}) - \dim(F_{\mathfrak{p}}^{gp})$, so $h + \dim(F_{\mathfrak{p}}) = \dim(M)$. $\square$

Corollary 2.3.7 *Let C be a finitely generated $\mathbf{Q}$-cone and let*

$$C^{\vee} := \{\phi\colon C^{gp} \to \mathbf{Q} : \phi(c) \geq 0 \quad \text{for all } c \in C\}.$$

Then $C^{\vee}$ is also a finitely generated cone, and an element c of C^{gp} belongs to C if and only if $\phi(c) \geq 0$ (resp. $= 0$ for all $\phi \in C^{\vee}$.

Proof: Let S be a finite set of generators for C and let P the submonoid of C generated by S. Then $H(P) \subseteq C^{\vee}$ and is is finitely generated by Theorem 2.2.1. Thus it will suffice to show that $H(P)$ generates $C^{\vee}$. If $\phi \in C^{\vee}$ and $s \in S$, $\phi(s)$ is a nonnegative rational number, and hence there exists a positive integer such that $n\phi(s) \in \mathbf{N}$ for all $s \in S$. Then $n\phi \in H(P)$, and so ϕ lies in the cone generated by $H(P)$. The last statement follows from the fact that some multiple of c lies in P^{gp} and Corollary 2.2.3. $\square$

Corollary 2.3.8 *Every face of a fine monoid is the intersection of the facets containing it.*

Proof: Let G be a face of fine monoid Q. The natural map $Q \to Q/G$ induces a bijection between the facets of Q containing F and the facets of Q/F. Thus, replacing Q by Q/F, we reduce to the case in which Q is sharp and $F = 0$. We must show in this case that if $q \in Q$ belongs to every facet of Q, then $q = 0$. The complement of a facet F is a prime ideal $\mathfrak{p}$ of height one, and $q \in F$ if and only if $\nu_{\mathfrak{p}}(q) = 0$. Since the set of all such $\nu_{\mathfrak{p}}$ generates the cone $C_{\mathbf{Q}}(H(Q))$, it follows that $h(q) = 0$ for all $h \in H(Q)$. Then Lemma (2.2.2) implies that $q = 0$, since Q is sharp. $\square$

Corollary 2.3.9 *If Q is a fine monoid, the map $Q \to Q^{\mathrm{sat}}$ induces a homeomorphism* $\mathrm{Spec}(Q^{\mathrm{sat}}) \to \mathrm{Spec}(Q)$.

Corollary 2.3.10 *Let $\mathfrak{p}$ be a height one prime ideal in a fine monoid M. Then $M^{\mathrm{sat}}_{\mathfrak{p}}$ is valuative, and there is a unique isomorphism*

$$\overline{M^{\mathrm{sat}}_{\mathfrak{p}}} \cong \mathbf{N},$$

and a unique epimorphism

$$\nu_{\mathfrak{p}} \colon M^{gp} \to \mathbf{Z}$$

such that $\nu_{\mathfrak{p}}^{-1}(\mathbf{N}^+) \cap M = \mathfrak{p}$. Furthermore, $M^{\mathrm{sat}}_{\mathfrak{p}} = \{x \in M^{gp} : \nu_{\mathfrak{p}}(x) \geq 0\}$

Proof: We know that M^{sat} is fine, $M^{gp} \cong (M^{\mathrm{sat}})^{gp}$, and that $\mathrm{Spec}(M^{\mathrm{sat}}) \to \mathrm{Spec}(M)$ is a homeomorphism. Thus we may as well assume replace M by M^{sat}, and so we assume that M is saturated. Since $M_{\mathfrak{p}}$ is saturated, $\overline{M_{\mathfrak{p}}}^{gp}$ is torsion free, and since $\mathfrak{p}$ has height one, $\overline{M_{\mathfrak{p}}}^{gp}$ is isomorphic to $\mathbf{Z}$. Choose any nonzero element x of $\overline{M_{\mathfrak{p}}}$. Then there is an $n \in \mathbf{N}^+$ such that $x = ny$, where y is one of the two generators of $\overline{M_{\mathfrak{p}}}^{gp}$. Since $\overline{M_{\mathfrak{p}}}$ is saturated, $y \in \overline{M_{\mathfrak{p}}}$, and y freely generates $\overline{M_{\mathfrak{p}}}$. This shows that $M_{\mathfrak{p}}$ is saturated. Furthermore, $-y \notin \overline{M_{\mathfrak{p}}}$, so the induced isomorphism $\overline{M_{\mathfrak{p}}} \to \mathbf{N}$ is unique. Let μ be the composition $M \to \overline{M_{\mathfrak{p}}}^{gp} \to \mathbf{N}$, then $\mu^{-1}(\mathbf{N}^+) = \mathfrak{p}$, and $\nu_{\mathfrak{p}} := \mu^{gp}$ is an epimorphism such that $\nu_{\mathfrak{p}}^{-1}(\mathbf{N}^+) \cap M = \mathfrak{p}$. Suppose that $\nu \colon M^{gp} \to \mathbf{Z}$ is an epimorphism such that $\nu^{-1}(\mathbf{N}^+) \cap M = \mathfrak{p}$. Then $\nu^{-1}(0) \cap M$ is the face $F := M \setminus \mathfrak{p}$, and ν factors through $\overline{M_{\mathfrak{p}}}^{gp} \cong \mathbf{Z}$. Since ν is an epimorphism, this last map is an isomorphism, and $\nu = \pm\nu_{\mathfrak{p}}$. In fact the sign must be $+$ since $\nu^{-1}(\mathbf{N}^+) = \mathfrak{p}$. If q and p are elements of M, $\nu_{\mathfrak{p}}(p - q) = \nu_{\mathfrak{p}}(p) - \nu_{\mathfrak{p}}(q)$. Thus if $q \in M \setminus \mathfrak{p}$, $\nu_{\mathfrak{p}}(q) = 0$ and $\nu_{\mathfrak{p}}(p - q) \geq 0$. Conversely, if $x \in M^{gp}$ and $\nu_{\mathfrak{p}}(x) \geq 0$, there exists a $q \in M_{\mathfrak{p}}$ such that $\nu_{\mathfrak{p}}(q) = \nu_{\mathfrak{p}}(x)$. Then there exists $u \in M^*_{\mathfrak{p}}$ such that $x = q + u$, and $x \in M_{\mathfrak{p}}$. $\square$

Corollary 2.3.11 *Let Q be a fine saturated monoid. Then $Q = \{x \in Q^{gp} : \nu_{\mathfrak{p}}(x) \geq 0\}$, where $\mathfrak{p}$ ranges over the set of height one primes of Q (2.3.10). In other words, Q is the intersection in Q^{gp} of the set of all its localizations at height one primes.*

Proof: We know from (2.1.9.8) that $H(Q)$ is a fine sharp monoid, and from (2.3.2) that the $\mathbf{Q}$-cone C it generates is generated by a finite set $(h_1, \cdots, h_n)$ of indecomposable elements. Each h_i generates a one dimensional face of C; consequently each $h_i^{\perp}$ is a facet of Q, and $\mathfrak{p}_i := h_i^{-1}(\mathbf{N}^+)$ is a height one prime of Q. If $x \in Q_{\mathfrak{p}}$ for every height one prime $\mathfrak{p}$, then $h_i(x) \geq 0$ for every i and hence $h(x) \geq 0$ for every $h \in C$, and hence for every $h \in H(Q)$. Then $x \in Q$ by (2.2.1) $\square$

Proposition 2.3.12 *If Q is a fine monoid, let $\mathcal{W}_Q^+$ denote the free monoid on the set of height one primes of Q, and if $q \in Q$, let*

$$\nu(q) := \sum \{\nu_{\mathfrak{p}}(q)\mathfrak{p} : \mathrm{ht}(\mathfrak{p}) = 1\} \in \mathcal{W}_Q^+.$$

Then $\nu \colon Q \to \mathcal{W}_Q^+$ is a local homomorphism. Furthermore, $\nu(q_1) = \nu(q_2)$ if and only if there is some $n \in \mathbf{Z}^+$ such that $n\overline{q}_1 = n\overline{q}_2$ in $\overline{Q}$, and ν is exact if and only if Q is saturated.

Proof: It is apparent that $\nu \colon Q \to \mathcal{W}_Q^+$ is a homomorphism of monoids. Furthermore, $\{\nu_{\mathfrak{p}} : \mathrm{ht}\, \mathfrak{p} = 1\}$ generates $C_{\mathbf{Q}}(H(\mathbf{Q}))$ as a cone, so if $\nu(q) = 0$, $h(q) = 0$ for all q, and hence $q \in Q^*$ by (2.2.2). If $\nu(q_1) = \nu(q_2)$, then $h(q_1 - q_2) = 0$ for all h, hence $q_1 - q_2$ is a unit in Q^{sat} and there exists some $n \in \mathbf{Z}^+$ such that $nq_1 - nq_2 \in Q^*$. This implies that $n\overline{q}_1 = n\overline{q}_2$. The last statement follows from (2.3.11) and the fact that an exact submonoid of a saturated monoid is saturated. $\square$

Let P be a fine monoid, let $S := \operatorname{Spec} P$ with its Zariski topology, and let $\mathfrak{p}$ be a point of S. The complement F of $\mathfrak{p}$ is a face of P, and since P is finitely generated, (2.1.9) says that there exists an $f \in P$ such that $\langle f \rangle = F$. Then

$$\{\mathfrak{p}' : \mathfrak{p} \in \{\mathfrak{p}'\}^-\} = S_F := \{\mathfrak{p}' : F \cap \mathfrak{p}' = \emptyset\} = S_f := \{\mathfrak{p}' : f \notin \mathfrak{p}'\}$$

is open in S. Thus the set of generizations of each point is open, and hence a subset of S is open if and only if it is stable under generization. This shows that the topology of S is entirely determined by the order relation among the primes of P.

Let d be the Krull dimension of S and for $0 \leq i \leq d$ let

$$K_i := \cap\{\mathfrak{p} : \operatorname{ht}\mathfrak{p} = i\},$$

an ideal of P. We saw in (1.3.6) that every prime of height $i+1$ contains a prime of height i, hence $K_i \subseteq K_{i+1}$. We have

$$\emptyset = K_0 \subset I_P = K_1 \subset \cdots K_d = P^+,$$

where I_P is the interior ideal of P.

Since $\{\mathfrak{p} : \operatorname{ht}\mathfrak{p} = i\}$ is finite,

$$Z(K_i) = \cup\{Z(\mathfrak{p}) : \operatorname{ht}\mathfrak{p} = i\} = \{\mathfrak{p} : \operatorname{ht}\mathfrak{p} \geq i\}.$$

Thus we have a chain of closed sets

$$\{P^+\} = Z_d \subset Z_{d-1} \subset \cdots Z_1 \subset Z_0 = \operatorname{Spec} P.$$

If $\mathfrak{p} \in \operatorname{Spec} P$ and $F := P \setminus \mathfrak{p}$, then $\mathfrak{p}$ belongs to the open subset et S_F of S defined by P, and F is the largest face with this property. Let $\mathcal{F}_i$ denote the set of faces F of P such that $P \setminus F$ has height i, *i.e.*, such that the rank of P/F is i. A prime $\mathfrak{p}$ belongs to some S_F with $F \in \mathcal{F}_i$ if and only $\operatorname{ht}\mathfrak{p} \leq i$. This shows that

$$\cup\{S_F : F \in \mathcal{F}_i\} = \{\mathfrak{p} : \operatorname{ht}\mathfrak{p} \leq i\} = S \setminus Z_{i+1}$$

The following corollary summarizes this discussion.

Corollary 2.3.13 *Let S be the spectrum of a fine monoid P of dimension d. For each $i = 0, \ldots d$, let*

$$Z_i := \{\mathfrak{p} \in S : \operatorname{ht}\mathfrak{p} \geq i\}.$$

Then $S \setminus Z_i$ is open in S, and is the union of the set of all special affine open sets S_F as F ranges over the faces of P such that the rank of P/F is $i-1$. In particular, $S \setminus Z_2$ is the union of the sets S_F as F ranges over the facets of P. If P is toric, each P_F is a valuative monoid.

A cone is called *simplicial* if it is finitely generated and free, that is, if there exists a finite set S such that each element of C can be written uniquely as a linear combination of elements $\sum_{s\in S} a_s s$ with $a_s \in K^{\geq 0}$; such a set S necessarily forms a basis for C^{gp}. It is not hard to see that any sharp cone in K^1 or K^2 is simplicial. This is false for K^3; for example, the cone generated by the monoid P of (1.3.8) is not simplicial. For a useful criterion, see (2.3.18) below.

In fact, every finitely generated cone is a finite union of simplicial cones, as the following result of Carathéodory shows.

Theorem 2.3.14 (Carathéodory) *Let C be a K-cone and let S be a set of generators for C. Then every element of C lies in a cone generated by a linearly independent subset of S.*

Proof: If $x \in C$, we can write $x = \sum a_i s_i$ with $s_i \in S$ and $a_i > 0$. We may suppose that this has been done so that the number e of terms in the sum is minimal, and we claim that then $(s_1, s_2, \dots s_e)$ is independent in C^{gp}. In fact suppose that $\sum c_i s_i = 0$. We may choose the indexing so that c_i is positive if $1 \leq i \leq m$, negative if $m < i \leq n$, and zero if $i > n$. Furthermore, we may suppose that $a_1/c_1 \leq a_2/c_2 \cdots \leq a_m/c_m$ and that $a_n/c_n \geq a_{n-1}/c_{n-1} \cdots \geq a_{m+1}/c_{m+1}$. Suppose $m > 0$. Then for all i, $a'_i := a_i - (a_1/c_1)c_i \geq 0$, and then $x - (a_1/c_1)\sum c_i s_i = \sum\{a'_i s_i : i > 1\}$, contradicting the minimality of e. Thus $m = 0$. If $n > 0$, then for all i $a'_i = a_i - (a_n/c_n)c_i \geq 0$, and $x = \sum\{a'_i : i \neq n\}$, again a contradiction. Thus $n = 0$, all $c_i = 0$, and $(s_1, \dots s_e)$ is linearly independent. $\square$

Corollary 2.3.15 *Let C be a finitely generated sharp K-cone of dimension d. Then C is a finite union of simplicial cones of dimension d.*

Proof: Let S be a finite set of generators of C. Since the K-span of S is C^{gp}, whose dimension is d, any linearly independent subset T of S is contained in a linearly independent subset T' of cardinality d. Carathéodory's theorem implies that every element of C belongs to some $C(T)$ and hence to some $C(T')$. $\square$

Corollary 2.3.16 *Let C be be a finitely generated cone in a finite dimensional K-vector space V. Then C is closed with respect to the topology of V induced from the ordering on K. In particular, any face of a finitely generated cone C is closed in C, and the interior I_C (1.3) of C is open.*

Proof: The group C^* of units of C is a K-subspace of V, hence is closed, and hence it suffices to prove that the image of C in V/C^* is closed. Thus we may and shall assume that C is sharp. Suppose that V has dimension n and that C is simplicial of dimension d. Then there exists a basis $(v_1, \dots v_n)$ for V such that $(v_1, \dots v_d)$ spans C. Thus V can be identified with K^n and C with the subset of $v \in K^n$ such that $v_i \geq 0$ for $i \leq d$ and $v_i = 0$ for $i > d$. Since the topology on V is independent of the choice of basis, C is closed. The general case follows, since Corollary (2.3.15) shows that any C can be written as a finite union of simplicial cones. Finally we recall from (2.3.3) that a face of a finitely generated cone is finitely generated, hence closed. Since C has only a finite number of faces, I_C is open. □

Proposition 2.3.17 *Let C be a finitely generated $\mathbf{Q}$-cone and let $C_K \subseteq K \otimes C^{gp}$ be the K-cone it spans.*

1. *For every $x \in C_K$ there exists an increasing sequence $(x_i : i \in \mathbf{N})$ in C converging to x. In particular, C is dense in C_K.*
2. *If C' is any finitely generated subcone of C, $C'_K \cap C^{gp} = C'$.*
3. *The map $F \mapsto F_K$ induces a bijection between the faces of C and the faces of C_K, with inverse $G \mapsto G \cap C$.*

Proof: Let S be a finite set of generators for C; then S also generates C_K as an $\mathbf{R}$-cone. Any element x of C_K can be written $x = \sum a_s s$ with $a_s \in \mathbf{R}^{\geq 0}$. For each s there exists an increasing sequence a_{is} in $\mathbf{Q}^{\geq 0}$ converging to a_s; then $x_i := \sum a_{is} s$ is an increasing sequence in C converging to x. This proves (1). To prove (2), suppose that T is a finite set of generators for C' and $x' \in C'_K$. By (2.3.14) there exist a linearly independent subset T' of T and elements $a_t \in \mathbf{R}^{\geq 0}$ such that $x' = \sum\{a_t t : t \in T'\}$. Since C spans C^{gp}, there is a basis S' for C^{gp} which contains T' and is contained in C. If $x' \in C'_K \cap C^{gp}$, all its coordinates with respect to S' lie in $\mathbf{Q}$. In particular each $a_t \in \mathbf{Q}^{\geq 0}$, so $x' \in C'$. This proves (2).

Now suppose F is a face of C. It is clear from the definition that F_K is a submonoid of C_K; to prove that it is a face we must check that if $x \le y$ with $x \in C_K$ and $y \in F_K$, then $x \in F_K$. Recall that F is generated as a cone by $F \cap K$, so y can be written $y = \sum_s a_s s$ with $a_s \in K^{\ge 0}$ and $s \in F \cap S$. Replacing by $\sum_s a'_s s$ with $a'_s \in \mathbf{Q}$ and $a'_s \ge a_s$, we may assume that $y \in F$. By (1) we can find an increasing sequence (x_i) in C converging to x. For each fixed j, $(x_i - x_j : i \in \mathbf{N})$ is a sequence in C^{gp} which converges to $x - x_j$ and for $i > j$ lies in C; since C_K is closed it follows that $x - x_j \in C_K$ also. Then $y - x_j = y - x + x - x_j \in C_K \cap C^{gp} = C$, and since F is a face of C, $x_j \in F$ for all j. By (2.3.2) F is a finitely generated as a cone and so F_K is closed in C_K. Hence $x \in F_K$ as required. The fact that $F_K \cap C^{gp} = F$ follows from (2). Finally, if G is any face of C_K, we know from (2.3.2) that G is generated by a subset of S, hence by $G \cap C$, which is a face of C. □

Proposition 2.3.18 *Let C be a finitely generated sharp K-cone and S a finite subset. Suppose that every finite subset of S is contained in a proper face of C and that S spans C^{gp} as a vector space Then S is linearly independent and spans C as a K-cone. In particular, C is simplicial.*

Proof: Suppose that $\sum a_s s = 0$ with $a_s \in K$ and $s \in S$. Let $S' := \{s \in S : a_s > 0\}$, $S'' := \{s \in S : a_s < 0\}$, and $T := S \setminus S' \cup S''$. Then let t be the sum of all the elements of T, and let

$$f := \sum_{s \in S'} a_s s + t = \sum_{s \in S''} -a_s s + t;$$

note that $f \in C$. If S'' is not empty, then $S' \cup T$ is a proper subset of S and hence by assumption is contained in a proper face F of C. Since $f = \sum\{-a_s s : s \in S''\} \in F$ and F is a face, all the elements of S'' also belong to F. But then all of S is contained in F. Then $F^{gp} = C^{gp}$ and since F is exact in C, $F = C$, a contradiction. Thus we must have $S'' = \emptyset$. Similarly $S' = \emptyset$, and it follows that S is linearly independent.

Let c be an element of the interior of C. Then there exist disjoint subsets S' and S'' of S and elements $a_s \in K^{\ge 0}$ such that

$$c = \sum_{s \in S'} a_s s - \sum_{s \in S''} a_s s.$$

Then $c + \sum\{a_s s : s \in S''\}$ also belongs to the interior of C. If S' were a proper subset of S, it would be contained in a proper face of C, which contradicts the fact that $\sum\{a_s s : s \in S'\} = c + \sum\{a_s s : s \in S''\}$ is in the interior of C. Hence $S' = S$ and $S'' = \emptyset$. We have thus shown that every element of the interior of C lies in the the $K^{\geq 0}$-span of S. Since this span is closed, and since the interior of C is dense in C (2.3.4), S spans C, as claimed. $\Box$

Theorem 2.3.19 (Gordon's lemma) *Let L be a finitely generated abelian group, let $V := \mathbf{Q} \otimes L$, and let $C \subseteq \mathbf{Q} \otimes L$ be a finitely generated $\mathbf{Q}$-cone. Then $C_L := L \times_V C \cong L \times_{V_{\mathbf{R}}} C_{\mathbf{R}}$ is a finitely generated monoid.*

Proof: The natural map $L \times_V C \to L \times_{V_{\mathbf{R}}} C_{\mathbf{R}}$ is injective because $C \subseteq C_{\mathbf{R}}$, and it is surjective because of (2.3.17.2). Let us first treat the case in which L is free, so that it may be identified with its image in V. Let S be a finite set of generators for C, which we may as well assume contained in L. Let $S' \subseteq V_{\mathbf{R}}$ be the set of all linear combinations of elements of S with coefficients in the interval $[0,1]$. The map $[0,1]^S \to V_{\mathbf{R}}$ sending $\{a_s : s \in S\}$ to $\sum a_s s$ is continuous and maps surjectively to S'; hence S' is compact. Then $S'' := L \cap S'$ is compact and discrete, hence finite. Any element x of $C_{\mathbf{R}}$ can be written as a sum $\sum a_s s$ with $s \in S$ and $a_s \in \mathbf{R}^{\geq 0}$, and a_s can be written $a_s = m_s + a'_s$ with $m_s \in \mathbf{N}$ and $a'_s \in [0,1]$. Then $x = \sum m_s s + s'$ with $s' \in S'$; if also $x \in L$, in fact $s' \in S''$, and so x is a sum of elements of S''. Thus the monoid $C_L = L \cap C_{\mathbf{R}}$ is generated by the finite set S''. For the general case, let L_t be the torsion subgroup of L and let $L_f := L/L_t$. Notice that $L_t \subseteq C_L^*$, and the natural map $C_L \to L$ identifies C_L/L_t with $C_{L_f} = L_f \cap C$ and C_L^*/L_t with $C_{L_f}^*$. Since C_{L_f} is a fine monoid, it follows from (2.1.1) that $C_{L_f}^*$, is a finitely generated group, and since L_t is finitely generated, so is C_L^*. Now (2.1.1) implies that C_L is a finitely generated monoid. $\Box$

The finiteness of the saturation of a fine monoid also follows from Gordon's lemma.

Corollary 2.3.20 *Let M be a fine monoid and let $C \subseteq K \otimes M^{gp}$ be the K-cone it spans. Then $M^{\mathrm{sat}} = M^{gp} \times_{C^{gp}} C$ and is finitely generated as a monoid.*

Proof: The previous result implies that $M^{gp} \times_{C^{gp}} C$ is finitely generated as a monoid and is independent of the choice of K, so we may as well take $K = \mathbf{Q}$. If $x \in M^{\mathrm{sat}}$, then by definition $x \in M^{gp}$ and there exists $n > 0$

such that $nx \in M$. It follows that $1 \otimes x = (1/n)(1 \otimes nx)$ lies in C, so $x \in M^{gp} \times_{C^{gp}} C$. Conversely, if $x \in M^{gp}$ and $1 \otimes x \in C$, then there exist $x_i \in M$ and $a_i \in \mathbf{Q}^{\geq 0}$ such that $1 \otimes x = \sum a_i(1 \otimes x_i)$. Choose $n \in \mathbf{N}^+$ such that $na_i \in \mathbf{N}$ for all i. Then $1 \otimes nx = 1 \otimes y$ where $y := \sum na_i x_i \in M$. Thus $nx = y + z$ with $y \in M$ and $z \in M^{gp}_t$. If $m \in \mathbf{N}^+$ is such that $mz = 0$, then $mnx = my$, so $x \in M^{\mathrm{sat}}$. We conclude that M^{sat} is finitely generated as a monoid. $\square$

2.4 Faces and direct summands

In this section we investigate some necessary and sufficient conditions for a submonoid F of a monoid P to be a direct summand and for P to be free. Let us remark first that if F is a direct summand of P and if F contains P^*, then F is a necessarily a face of P. Indeed, suppose that $P = F \oplus Q$ and that $p_i \in P$ with $p_1 + p_2 \in F$. Write $p_i = f_i + q_i$, with $f_i \in F$ and $q_i \in Q$. Then $q_1 + q_2 = 0$, and hence $q_i \in P^* \subseteq F$; since $F \cap Q = 0$, $q_i = 0$ and so $p_i \in F$.

We begin with some results on complements of faces in cones.

Proposition 2.4.1 *Let C be a finitely generated sharp Q-cone and let F be a face of C, of codimension r.*

1. *The projection map from the union of the set of r-dimensional faces of C to C/F is surjective.*

2. *There is at least one r-dimensional face G of C such that $G^{gp} \cap F^g = 0$.*

Proof: The proof is by induction on the dimension of C, and is trivial if this is zero or one or if $F = 0$. Suppose that the result is proved for all cones of smaller dimension. Suppose further that F is an extremal ray of C and let f be a nonzero element of F. Let S be a finite set of indecomposable generators of the dual cone $C^\vee$. Since $f \notin C^*$, by (2.3.7) the set S_f of elements of S which are positive on f is not empty. If x is any element of C, choose ϕ_0 from S_f so that $\phi_0(x)/\phi_0(f)$ is minimal, and let

$$a := \phi_0(x)/\phi_0(f) \quad \text{and} \quad y := x - af \in C^{gp}$$

Then for any $\phi \in S$,

$$\phi(y) = \phi(x) - \phi(f)\phi_0(x)/\phi_0(f) \geq 0,$$

and so $y \in C$. Since $\phi_0(y) = 0$, y lies in a facet of C, and $y \equiv x \pmod F$ since $x - y \in F$. Since x was arbitrary, (1) is proved when F is one-dimensional. If the dimension of F is at least two, it contains an extremal ray R. The induction assumption applied to the face F/R of C/R implies that every element c of C is congruent modulo F to an element c' whose image in C/R is contained in an r-dimnensional face of C/R. The inverse image G in C of this face has dimension $r+1$ and contains c'. Our argument, applied to the extremal ray R of G, shows that there is an element c'' of G which is congruent to c' modulo R and which is contained in a facet G' of G. Then c'' is congruent to c modulo F and is contained in a face G' of C of dimension r. This completes the proof of (1).

For each r-dimension face G of C, the image of $G^{gp} \to (C/F)^{gp}$ is a vector subspace, and (1) implies that the latter is the union of the set of all these images. Since this set is finite, one of these spaces must be all of $(C/F)^{gp}$, so that the map $G^{gp} \to (C/F)^{gp}$ is surjective. Since the spaces have the same dimension, it is also an isomorphism, and thus $G^{gp} \cap F^{gp} = 0$. □

Proposition 2.4.2 *Let F be a face of a toric monoid P. Then the following conditions are equivalent.*

1. *F is a direct summand of P.*
2. *For every face G of P, $F + G$ is a face of P.*
3. *For every face G of P, $F + G^{gp}$ is face of P_G.*

Proof: Suppose that $P = F \oplus Q$ and G is a face of P. Any element g of G can be written uniquely as $f_0 + q_0$, where $f_0 \in F$ and $q_0 \in Q$. Since G is a face, f_0 and g_0 still belong to G. Now if p_1 and p_2 are elements of P whose sum belongs to $F + G$ it follows that we can write $p_1 + p_2 = f + g_0$, where $f \in F$ and $g_0 \in G \cap Q$. If we write $p_i = f_i + q_i$ with $f_i \in F$ and $q_i \in Q$, we see that $f = f_1 + f_2$ and $g_0 = q_1 + q_2$. It follows that each q_i belongs to G and hence that each p_i belongs to $F + G$. Thus $F + G$ is a face of P, and the implication of (2) by (1) is proved.

Suppose that F, G, and $F+G$ are faces of P. Suppose that $p, q \in P_G$ and $f \in F + G^{gp}$ with $p + q = f$. Then there exist $g_1, g_2 \in G$, such that $p + g_1$, $q + g_2$, and $f + g_1 + g_2$ belong to P. Since $(p + g_1) + (q + g_2) = f + g_1 + g_2$ in P and $F + G$ is a face of P, $p + g_1 \in F + G$. Hence $p \in F + G^g$, and this shows that $F + G^{gp}$ is a face of P. It follows that (2) implies (3).

[1] Suppose that F has codimension r. By proposition 2.4.1 above, there exists a face Q of dimension r such that $F^{gp} \cap Q^{gp} = 0$. Then $F^{gp} \oplus Q^{gp}$ maps injectively to P^{gp}. The assumption on F applied to the face Q of P implies that $F + Q^{gp}$ is a face of $P + Q^{gp}$, and since it has dimension n, $F + Q^{gp} = P + Q^{gp}$. In particular, the map $F^{gp} \oplus Q^{gp} \to P^{gp}$ is bijective. Let $P' := F \oplus Q \subseteq P$, and consider the corresponding rational cones $C(P') \subseteq C(P)$ and their duals:

$$C(P)^\vee \subseteq C(P')^\vee \subseteq \mathrm{Hom}(P^{gp}, \mathbf{Q}).$$

Let ϕ be an indecomposable element of $C(P')^\vee$. Since $P' = F \oplus Q$, $C(P')^\vee \cong C(F)^\vee \oplus C(Q)^\vee$, and since ϕ is indecompsable, ϕ either belongs to $C(F)^\vee$ or to $C(Q)^\vee$. In the first case, $\phi^\perp$ contains all of Q, and so factors through $F + Q^{gp} = P' + Q^{gp}$. As we have seen, $F + Q^{gp}$ is all of $P + Q^{gp}$, and it follows that ϕ is nonnegative on all of $P + Q^{gp}$, *i.e.*, ϕ belongs to $C(P)$. In the second case, $G := \phi^\perp \cap Q$ is a facet of Q, and hence is an $r-1$-dimensional face of P, furthermore ϕ factors through $P' + F^{gp} + G^{gp}$. Our assumption implies that $F + G^{gp}$ is a facet of $P + G^{gp}$, and hence $(P + G^{gp})/(F + G^{gp})$ is a one-dimensional sharp monoid. Since $P'/(F+G)$ is also one-dimensional, the map $P'/(F+G) \to (P+G^{gp})/(F+G^{gp})$ is almost surjective. This means that for every $p \in P$, there is a positive m such that mp belongs to $P' + F^{gp} + G^{gp}$. But this implies that $\phi(p) \geq 0$ for every p, and hence that $\phi \in C(P)^\vee$. We conclude that $C(P')^\vee = C(P)^\vee$ and hence that $C(P') = C(P)$. Hence for every $p \in P$, there exists a positive integer m such that $mp \in F \oplus Q$. Since $F \oplus Q^{gp} = P + Q^{gp}$, we can write $p = f + x$ with $f \in F$ and $x \in Q^{gp}$. But then $mp = mf + mx \in F \oplus Q$, so $mx \in Q$. Since Q is a face of P and P is saturated, Q is also saturated, so $x \in Q$ also. This proves that $P = F + Q \cong F \oplus Q$, so F is a direct summand of P. $\square$

Example 2.4.3 The saturation hypothesis is not superfluous. To see this, consider the submonid P of $\mathbf{N} \oplus \mathbf{N}$ generated by $\{(2,0), (3,0), (1,1), (0,1)\}$, and the face F generated by $(0,1)$.

[1]This proof is due to Bernd Sturmfels

Proposition 2.4.4 *Let P be a fine sharp saturated monoid. Then P is free if and only if every face of P is a direct summand.*

Proof: Suppose that every face of P is a direct summand. We prove that P is free by induction on its dimension. If the dimension of P is one, the result follows from (**??**). Assume the result is true for all monoids of smaller dimension and choose a face F of P of dimension one. Then we can write $P = F \oplus Q$, and Q is necessarily a face of P. Every face G of Q is also a face of P and hence is a direct summand of of P: $P = G \oplus Q'$. In particular, any $q \in Q$ can be written as $g + q'$ with $g \in G$ and $q' \in Q'$; since Q is a face of P, $q' \in Q$. Thus in fact we have $Q = G \oplus Q' \cap Q$, so G is a direct summand of Q. Thus Q enjoys the same property as P, and hence is free by the induction hypothesis. Since F is free and $P = F \oplus Q$, P is also free. Conversely, suppose P is the free monoid generated by a finite set S. Then because P is free, the mapping taking a subset of S to the face it generates establishes a bijection between the set of subsets of S and the set of faces of P. Furthermore, if T is a subset of S, then $P = \langle T \rangle \oplus \langle S \setminus T \rangle$. $\square$

2.5 Idealized monoids

A surjective map of commutative rings $A \to B$ induces a closed immersion $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$, but the analog for monoids is not true: if $Q \to P$ is any morphism of monoids, the generic point of $\operatorname{Spec} Q$ lies in the image of $\operatorname{Spec} P$, so the map $\operatorname{Spec} P \to \operatorname{Spec} Q$ cannot be a closed immersion unless it is bijective. To remedy this we introduce the category **Imon** of *idealized monoids*. This is the category of pairs (Q, J), where Q is a monoid and J is an ideal of Q; morphisms $(Q, J) \to (P, I)$ are morphisms $Q \to P$ sending J to I. The functor **Imon** $\to$ **Mon** taking (Q, J) to Q has a left adjoint, taking a monoid P to $(P, \emptyset)$, and we can view **Mon** as a full subcategory of **Imon**. Furthermore we have a functor from the category of commutative rings to the category **Imon**, taking a ring A to its multiplicative monoid together with the zero ideal.

If I is an ideal of a monoid Q, then the ideal of $R[Q]$ generated by $e(I)$ is free with basis $e_{|I}$, and we denote it by $R[I]$. Thus the quotient $R[Q]/R[I]$ is a free R-module with basis $Q \setminus I$. For any R-algebra A, $\operatorname{Hom}_{\mathbf{Imon}}((Q, I), (A, 0)) = \operatorname{Hom}_R(R[Q]/R[I], A)$, so that the functor $(Q, I) \mapsto R[Q]/R[I]$ is left adjoint to the functor $A \mapsto (A, 0)$.

Inductive and projective limits exist in the category of idealized monoids, and are compatible with the forgetful functor **Imon** $\to$ **Mon**. For example, if $u_i\colon (P,I) \to (Q_i, J_i)$ is a pair of morphisms and $v_i\colon Q_i \to Q$ is the pushout of the underlying monoid morphisms, then $v_i\colon (Q_i, J_i) \to (Q, J)$ is the pushout, where J is the ideal of Q generated by the images of J_i.

A morphism of idealized monoids

$$\theta\colon (Q, J) \to (P, I)$$

is said to be *ideally exact* if $J = \theta^{-1}(I)$, and to be *exact* if in addition its underlying morphism is exact.

Proposition 2.5.1 *Let $\theta\colon Q \to P$ be an exact morphism of integral monoids, let J be an ideal of Q, and let I be the ideal of P generated by the image of J. Then $\theta\colon (Q, I) \to (P, J)$ is (ideally) exact.*

Proof: Suppose that $p \in P$ and $\theta(p)$ belongs to J. Then there exists an element q of Q and an element p' of I such that $\theta(p) = q + \theta(p')$. Thus $\theta^{gp}(p - p') \in Q$ and hence $p - p' \in P$. Since $p' \in I$, this implies that $p \in I$. $\square$

3 Affine toric varieties

3.1 Monoid algebras and monoid schemes

Let R be a fixed commutative ring, usually the ring $\mathbf{Z}$ of integers or a field, and let $\mathbf{Alg}_R$ denote the category of R-algebras. If Q is any monoid and R is any commutative ring, the *R-monoid algebra of Q* is the R-algebra whose underlying R-module is free with basis Q, endowed with the unique ring structure making the inclusion map $e\colon Q \to R[Q]$ a morphism from the monoid Q into the multiplicative monoid of $R[Q]$. Thus, if p and q are elements of Q and if we use additive notation for Q, $e(p+q) = e(p)e(q)$; for this reason we sometimes write e^p for $e(p)$. For example, $R[\mathbf{N}]$ is the polynomial algebra $R[T]$ where $T = e^1$. More generally, if $\mathbf{N}^{(X)}$ is the free monoid with basis X, then $R[\mathbf{N}^{(X)}]$ is the polynomial algebra $R[X]$: if $I \in \mathbf{N}^{(X)}$, e^I corresponds to the monomial $X^I := \prod\{x^{I_x} : x \in X\}$.

The functor $Q \mapsto R[Q]$ is left adjoint to the functor taking an R-algebra to its underlying multiplicative monoid. Consequently it commutes with inductive limits. For example, if $u_i \colon P \rightrightarrows Q_i$ are maps of monoids for $i = 1, 2$ and Q is their amalgamated sum, then $R[Q] \cong R[Q_1] \otimes_{R[P]} R[Q_2]$. Similarly, if S is a Q-set, we denote by $R[S]$ the free R-module with basis S, endowed with the unique structure of $R[Q]$-module which is compatible with the action of Q on S. Then if $T \to S$ is a basis for S as a Q-set, the induced map $T \to R[S]$ is a basis for $R[S]$ as a Q-module, and if S and S' are Q-sets, there is a natural isomorphism $R[S \otimes_Q S'] \cong R[S] \otimes_{R[Q]} R[S']$.

If A is an R-algebra, a morphism from a monoid Q to the monoid $(A, \cdot, 1)$ underlying A is sometimes called an *A-valued character of Q*. The set $\underline{\mathsf{A}}_{\mathsf{Q}}(A)$ of A-valued characters of Q becomes a monoid with the multiplication law defined by the pointwise product and the identity element given by the constant function whose value is 1. Thus we can view $\underline{\mathsf{A}}_{\mathsf{Q}}$ as a functor

$$\underline{\mathsf{A}}_{\mathsf{Q}} \colon \mathbf{Alg}_R \to \mathbf{Mon}$$

from the category of R-algebras to the category of monoids. The functor $\underline{\mathsf{A}}_{\mathsf{Q}}$ taking A to the set of all the A-valued characters of Q is representable by the pair $(R[Q], e)$, where $R[Q]$ is the monoid R-algebra of Q and $e \colon Q \to R[Q]$ is the map taking an element of Q to the corresponding basis element of $R[Q]$. The resulting monoid structure on $\underline{\mathsf{A}}_{\mathsf{Q}}$ defines a structure of a monoid-scheme on $\underline{\mathsf{A}}_{\mathsf{Q}}$, whose identity section 1_Q and multiplication law μ_Q are given by the homomorphisms

$$1_Q \colon R[Q] \to R : \quad \sum_q a_q\, e^q \mapsto \sum_q a_q$$

$$\mu_Q \colon R[Q] \to R[Q] \otimes R[Q] : \quad e^q \mapsto e^q \otimes e^q.$$

In particular, we let $\underline{\mathsf{A}}_{\mathsf{m}}$ denote the functor $\underline{\mathsf{A}}_{\mathbf{N}}$, which takes an R-algebra A to the multiplicative monoid underlying A.

The following proposition shows that Q can be recovered from the functor $\underline{\mathsf{A}}_{\mathsf{Q}}$ (with its monoid-scheme structure).

Proposition 3.1.1 *Suppose that* $\operatorname{Spec} R$ *is connected. Then the functor* $P \mapsto \underline{\mathsf{A}}_{\mathsf{P}}$ *from the category of monoids to the category of monoid schemes is fully faithful:*

$$\operatorname{Hom}(Q, P) \cong \operatorname{Hom}((\underline{\mathsf{A}}_{\mathsf{P}}, 1, \cdot), (\underline{\mathsf{A}}_{\mathsf{Q}}, 1, \cdot)).$$

In particular,

1. *the monoid of characters of $\underline{\mathsf{A}}_{\mathsf{P}}$, i.e., of morphisms $\underline{\mathsf{A}}_{\mathsf{P}} \to \underline{\mathsf{A}}_{\mathsf{m}}$, is canonically isomorphic to P, and*

2. *the monoid of cocharacters of $\underline{\mathsf{A}}_{\mathsf{P}}$,* i.e. *of morphisms $\underline{\mathsf{A}}_{\mathsf{m}} \to \underline{\mathsf{A}}_{\mathsf{P}}$, is canonically isomorphic to $P^\vee$.*

Proof: A morphism of schemes $\underline{\mathsf{A}}_{\mathsf{P}} \to \underline{\mathsf{A}}_{\mathsf{Q}}$ corresponds to a morphism of rings $\theta\colon R[Q] \to R[P]$; if $q \in Q$ let us write $\theta(e^q) = \sum_{p \in P} a_p(q) e^p$ with $a_p(q) \in R$. The statement that θ corresponds to a monoid morphism is the statement that the following diagrams commute:

$$\begin{tikzcd} R[Q] \arrow[r, "\theta"] \arrow[d, "1_Q"'] & R[P] \arrow[d, "1_P"] \\ R \arrow[r, "\mathrm{id}"] & R \end{tikzcd} \qquad \begin{tikzcd} R[Q] \arrow[r, "\theta"] \arrow[d, "\mu_Q"'] & R[P] \arrow[d, "\mu_P"] \\ R[Q] \otimes R[Q] \arrow[r, "\theta \otimes \theta"] & R[P] \otimes R[P] \end{tikzcd}$$

The second diagram says that for any $q \in Q$,

$$\sum_{p,p'} a_p(q) a_{p'}(q) e^p \otimes e^{p'} = \sum_p a_p(q) e^p \otimes e^p;$$

i.e., that $\sum_{p,p'} a_p(q) a_{p'}(q)$ is zero if $p \neq p'$ and is $a_p(q)$ if $p = p'$. In other words, the $a_p(q)$'s are orthogonal idempotents. The first diagram says that for any q, $\sum_{p \in P} a_p(q) = 1$. Since $\operatorname{Spec} R$ is connected, every idempotent is either 0 or 1. Thus, there is a unique element $\beta(q) \in P$ such that $a_p(q) = 0$ if $p \neq \beta(q)$ and $a_p(q) = 1$ if $p = \beta(q)$. Thus $\theta \circ e = e \circ \beta$, where β is a function $Q \to P$. Since θ is a ring homomorphism, β is a monoid homomorphism, as required. $\square$

The proposition shows that elements of $P^\vee$ correspond precisely to morphisms of monoid schemes $\underline{\mathsf{A}}_{\mathsf{m}} \to \underline{\mathsf{A}}_{\mathsf{P}}$, *i.e.*, to "one parameter submonoids," which we call *monoidal* or *logarithmic flows* .

In order to work with modules over the monoid algebra $R[Q]$, it is helpful to recall that a monoid Q is a category with a single object. A functor from Q to the category of R-modules amounts to an R module E and a morphism of monoids $Q \to \operatorname{End}_R(E)$, *i.e.*, an $R[Q]$-module. Thus giving a quasi-coherent sheaf on $\underline{\mathsf{A}}_{\mathsf{Q}}$ is the same as giving a functor from Q to the category of R-modules. To incorporate the monoid scheme structure of $\underline{\mathsf{A}}_{\mathsf{Q}}$, we introduce the concept of a *Q-graded module.*

Definition 3.1.2 *A Q-graded $R[Q]$-module is a functor from the category $\mathcal{T}Q$ (1.1.6) to the category of R-modules.*

If Q is an integral monoid and $q_1, q_2 \in Q$, then $Mor_{\mathcal{T}Q}(q_1, q_2)$ contains a single element if $q_1 \leq q_2$ and is empty otherwise. To give a Q-graded $R[Q]$-module E is to give an R-module E_q for every $q \in Q$, and for every $q, p \in Q$ an R-linear map $h_p \colon E_q \to E_{q+p}$, compatible with composition and such that $h_0 = \mathrm{id}$. Thus $\oplus_q E_q$ becomes an $R[Q]$-module in the usual sense.

If E is any $R[Q]$-module, let $\mathbf{V}E$ denote the spectrum of the symmetric algebra of E, regarded as a $\underline{\mathsf{A}}_{\mathsf{Q}}$-scheme. For any $R[Q]$-algebra A, $\mathbf{V}E(A)$ is the set of $R[Q]$-linear maps $E \to A$, and has a natural structure of an A-module. Let $\alpha \colon Q \to A$ be the A-valued character of Q corresponding to the $R[Q]$-algebra structure of A. If E is Q-graded, then an element σ of $\mathbf{V}E(A)$ can viewed as a collection of R-linear maps

$$\sigma_q \colon E_q \to A : q \in Q$$

such that for each $p, q \in Q$, the diagram

$$\begin{tikzcd} E_q \arrow[r, "\sigma_q"] \arrow[d, "h_p"'] & A \arrow[d, "\alpha(p)"] \\ E_{p+q} \arrow[r, "\sigma_{q+p}"] & A \end{tikzcd}$$

commutes. The Q-grading of E endows the R-scheme underlying $\mathbf{V}E$ with an action μ_E of the monoid scheme $\underline{\mathsf{A}}_{\mathsf{Q}}(A)$. For any R-algebra A, the set of A-valued points of the R-scheme $\mathbf{V}E$ is the set of pairs (σ, α), where α is an A-valued character of Q and σ is a family of R-linear maps as above. Now if if $\beta \in \underline{\mathsf{A}}_{\mathsf{Q}}(A)$ and $(\sigma, \alpha) \in \mathbf{V}E(A)$, we define $\mu_E \colon \underline{\mathsf{A}}_{\mathsf{Q}} \times \mathbf{V}E \to \mathbf{V}E$ by

$$\mu_E(\beta, (\sigma, \alpha)) := (\beta\sigma, \beta\alpha) \in \mathbf{V}E(A).$$

These maps define an action of the monoid $\underline{\mathsf{A}}_{\mathsf{Q}}(A)$ on the set $\mathbf{V}E(A)$, are compatible with the A-module structure of $\mathbf{V}E(A)$, and are natural with respect to the maps induced by homomorphisms $A \to A'$. Furthermore, it follows from the definition that μ_E fits in a commutative diagram:

$$\begin{tikzcd}
\underline{\mathsf{A}}_{\mathsf{Q}} \times \mathbf{V}E \arrow[r, "\mu_E"] \arrow[d, "\mathrm{id} \times \pi"'] & \mathbf{V}E \arrow[d, "\pi"] \\
\underline{\mathsf{A}}_{\mathsf{Q}} \times \underline{\mathsf{A}}_{\mathsf{Q}} \arrow[r, "\mu_Q"] & \underline{\mathsf{A}}_{\mathsf{Q}}
\end{tikzcd}$$

Proposition 3.1.3 *The construction of the previous paragraph defines an equivalence between the category of Q-graded $R[Q]$-modules and the category of quasi-coherent sheaves E on $\underline{\mathsf{A}}_{\mathsf{Q}}$ endowed with an action of $\underline{\mathsf{A}}_{\mathsf{Q}}$ compatible with the multiplication μ_Q of the monoid scheme as in the diagram above.*

Proof: In general, if E is any $R[Q]$-module, to give an action of $\underline{\mathsf{A}}_{\mathsf{S}}$ on $\mathbf{V}E$ amounts to giving a map $\mu_E\colon E \to R[S] \otimes_R E$, linear over the comultiplication $R[Q] \to R[Q] \otimes R[Q]$, such that the diagrams below commute:

$$\begin{tikzcd}
E \arrow[r, "\mu_E"] \arrow[dr, "\mathrm{id}_E"] & R[Q] \otimes E \arrow[d, "1_Q \otimes \mathrm{id}_E"] \\
& E
\end{tikzcd}$$

$$\begin{tikzcd}
E \arrow[r, "\mu_E"] \arrow[d, "\mu_E"'] & R[Q] \otimes E \arrow[d, "\mu_Q \otimes \mathrm{id}_E"] \\
R[Q] \otimes E \arrow[r, "\mathrm{id}_{R[Q]} \otimes \mu_E"'] & R[Q] \otimes R[Q] \otimes E
\end{tikzcd}$$

If $e \in E$, write $\mu_E(e) = \sum e^q \otimes \pi_q(e)$. Then each $\pi_q\colon E \to E$ is an R-linear map, and the diagrams above say that $\sum_q \pi_q = \mathrm{id}_E$ and $\pi_q \circ \pi_p = \delta_{p,q}\pi_q$. In others words, $\{\pi_q : q \in Q\}$ is the family of projections corresponding to a direct sum decomposition $E = \oplus E_q$. If $e \in E_q$, $\mu_E(e) = e^q \otimes e$, and since μ_E is linear over μ_Q,

$$\mu_E(e^p e) = (e^p \otimes e^p)\mu_E(e) = (e^p \otimes e^p)(e^q \otimes e) = e^{p+q} \otimes e^p e,$$

so that $e^p e \in E_{q+p}$. Thus E defines a functor from $\mathcal{T}Q$ to the category of R-modules. We leave to the reader the verification that this construction is quasi-inverse to the construction described before the proposition. $\square$

For example, if V is a R-module, a *Q-filtration on V* is a family of submodules $F_q \subseteq M$ such that $F_q \subseteq F_{q'}$ whenever $q \leq q'$. Then $\bigoplus F_q \subseteq V \otimes R[Q]$ is an $R[Q]$-submodule, invariant under the action of the monoid scheme $\underline{\mathsf{A}}_{\mathsf{Q}}$. A Q-filtration on R defines an ideal of $R[Q]$, and the corresponding closed subscheme of $\underline{\mathsf{A}}_{\mathsf{Q}}$ is stable under the action of Q on itself. If K is an ideal in Q, the free R-module $R[K]$ with basis K can be viewed as an ideal of $R[Q]$ defined by the Q-filtration which is 0 for $q \notin K$ and is R if $q \in K$. When R is a field, every Q-filtration of R has this form.

If Q is a monoid and A is an R-algebra, $\underline{\mathsf{A}}_{\mathsf{Q}^{\mathsf{gp}}}(A)$ is precisely the set of invertible elements of $\underline{\mathsf{A}}_{\mathsf{Q}}(A)$, *i.e.*, $\underline{\mathsf{A}}_{\mathsf{Q}^{\mathsf{gp}}} = \underline{\mathsf{A}}^*_{\mathsf{Q}}$. If Q is fine, the localization $R[Q] \to R[Q^{gp}]$ is injective and of finite type, and hence $\underline{\mathsf{A}}^*_{\mathsf{Q}} \to \underline{\mathsf{A}}_{\mathsf{Q}}$ is a dominant and affine open immersion.

Corollary 3.1.4 *Let V be a R-module and let E be a sub-$R[Q]$-module of $V \otimes R[Q]$. Then E is invariant under the action of $\underline{\mathsf{A}}_{\mathsf{Q}}$ on $V \otimes R[Q]$ if and only if E is given by a Q-filtration on V. If Q is integral, this is the case if and only if E is invariant under the action of the subgroup $\underline{\mathsf{A}}_{\mathsf{Q}^{\mathsf{gp}}} \cong \underline{\mathsf{A}}^*_{\mathsf{Q}}$. In particular, if R is a field, the ideals of $R[Q]$ which are invariant under the action of $\underline{\mathsf{A}}_{\mathsf{Q}}$ correspond bijectively with the ideals of Q.*

Example 3.1.5 Let Q be an integral monoid such that Q^{gp} is torsion free of rank n. For each $q \in Q$, let $\langle q \rangle$ be the face of Q generated by q (1.3.2). If $q' \in Q$, $\langle q \rangle \subseteq \langle q + q' \rangle$. Hence $q \mapsto \langle q \rangle^{gp} \subseteq Q^{gp}$ defines a Q-filtration of Q^{gp} and hence a $\underline{\mathsf{A}}_{\mathsf{Q}}$-invariant submodule of $\mathbf{Z}[Q] \otimes Q^{gp}$. More generally, for any integer i,

$$q \mapsto \Lambda^i \langle q \rangle^{gp} \subseteq \Lambda^i Q^{gp}$$

defines a Q-filtration of $\Lambda^i Q^{gp}$. When $i = n$, this filtration is the filtration given by interior ideal I_Q of Q (1.3).

Remark 3.1.6 Sometimes it is natural to consider $R[Q]$-modules which are graded by a Q-set S. Such a module E has a direct sum decomposition as R-modules $E = \oplus E_s : s \in S$, and if $q \in Q$, multiplication by e^q maps E_s to E_{q+s}. In other words, E is a functor from the transporter category

$\mathcal{T}S$ (1.1.6) of S into the category of R-modules. To interpret such modules geometrically, one can associate to the Q-set S the scheme $\mathbf{V}S$ which to any A associates the set of pairs (σ, α), where α is an A-valued character of Q and $\sigma\colon Q \to A$ is a morphism of Q-sets (over α). Then $\mathbf{V}S(A)$ has a natural structure of an A-module, and also of a monoid, and the map $\mathbf{V}S \to \underline{\mathsf{A}}_{\mathsf{Q}}$ is a morphism of monoid schemes. Associated to an S-graded $R[Q]$-module is a quasi-coherent sheaf on $\mathbf{V}S$ which is invariant under the monoid structure of $\mathbf{V}S$ and compatible with the A-module structure. Since we shall not use this construction, we omit the details.

3.2 Faces, orbits, and trajectories

If K is an ideal in a monoid Q, let $\underline{\mathsf{A}}_{\mathsf{Q,K}}$ denote the functor which takes an R-algebra A to the set of maps $(Q, K) \to (A, 0)$; as we have seen, this functor is representable by $R[Q]/R[K]$. Thus $\underline{\mathsf{A}}_{\mathsf{Q,K}}$ is a closed subscheme of the monoid-scheme $\underline{\mathsf{A}}_{\mathsf{Q}}$, invariant under the action of $\underline{\mathsf{A}}_{\mathsf{Q}}$ on itself by (3.1.4). In other words, $\underline{\mathsf{A}}_{\mathsf{Q,K}}$ is an ideal-scheme of the monoid-scheme of $\underline{\mathsf{A}}_{\mathsf{Q}}$: for every A, the image of the map

$$i_K(A)\colon \underline{\mathsf{A}}_{\mathsf{Q,K}}(A) \to \underline{\mathsf{A}}_{\mathsf{Q}}(A)$$

is an ideal in the monoid $\underline{\mathsf{A}}_{\mathsf{Q}}(A)$. If Q is sharp, then $\underline{\mathsf{A}}_{\mathsf{Q,Q^+}} \cong S := \operatorname{Spec} R$. The corresponding R-valued point of Q is the homomorphism $v\colon Q \to R$ such that $v_Q(0) = 1$ and $v_Q(q) = 0$ if $q \in Q^+$. It is called the *vertex* of $\underline{\mathsf{A}}_{\mathsf{Q}}$.

In particular, let $\mathfrak{p}$ be a prime ideal of Q and let $F := Q \setminus \mathfrak{p}$ be the corresponding face. The inclusion $F \to Q$ defines a morphism of monoid algebras $R[F] \to R[Q]$ and hence a morphism of monoid schemes

$$r_F\colon \underline{\mathsf{A}}_{\mathsf{Q}} \to \underline{\mathsf{A}}_{\mathsf{F}}\,.$$

The composition of the map $R[F] \to R[Q]$ with the homomorphism $i^\sharp_{\mathfrak{p}}\colon R[Q] \to R[Q, \mathfrak{p}]$ is an algebra homomorphism and induces a bijection on the canonical basis elements, and hence induces an isomorphism of schemes $\underline{\mathsf{A}}_{\mathsf{Q},\mathfrak{p}} \to \underline{\mathsf{A}}_{\mathsf{F}}$. Let

$$i_F\colon \underline{\mathsf{A}}_{\mathsf{F}} \to \underline{\mathsf{A}}_{\mathsf{Q}}$$

be the composition of the inverse of this isomorphism with the closed immersion $i_{\mathfrak{p}}$. Thus,

$$i^\sharp_F(e^q) = \begin{cases} e^q & \text{if } q \in F \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.2.1 *Let F be a face of an integral monoid Q, let i_F and r_F be the morphisms defined above, and let $i_{Q/F}$ be the closed immersion induced by the surjection $Q \to Q/F$. These morphisms fit into a commutative diagram with Cartesian squares:*

$$\begin{tikzcd}
S \arrow[r, "v_{Q/F}"] \arrow[d, "1_F"'] & \underline{\mathsf{A}}_{\mathsf{Q/F}} \arrow[r, "\pi"] \arrow[d, "i_{Q/F}"'] & S \arrow[d, "1_F"'] \\
\underline{\mathsf{A}}_{\mathsf{F}} \arrow[r, "i_F"] & \underline{\mathsf{A}}_{\mathsf{Q}} \arrow[r, "r_F"] & \underline{\mathsf{A}}_{\mathsf{F}}
\end{tikzcd}$$

In this diagram, 1_F is the map corresponding to the identity of the monoid scheme $\underline{\mathsf{A}}_{\mathsf{F}}$ and $v_{Q/F}$ is the vertex of the (sharp) monoid scheme $\underline{\mathsf{A}}_{\mathsf{Q/F}}$. The map r_F is a morphism of monoid schemes, and the morphism i_F is compatible with the actions of the monoid scheme $\underline{\mathsf{A}}_{\mathsf{Q}}$ on itself and on $\underline{\mathsf{A}}_{\mathsf{F}} \cong \underline{\mathsf{A}}_{\mathsf{Q},\mathfrak{p}}$. If Q is fine, then i_F is a strong deformation retract.

Proof: The closed immersion i_F preserves the composition law for the monoid schemes $\underline{\mathsf{A}}_{\mathsf{F}}$ and $\underline{\mathsf{A}}_{\mathsf{Q}}$ but not the identity section of the monoid scheme structures, so that $\underline{\mathsf{A}}_{\mathsf{F}}$ cannot be regarded as a submonoid of $\underline{\mathsf{A}}_{\mathsf{Q}}$—in fact it is an ideal subscheme of the monoid scheme $\underline{\mathsf{A}}_{\mathsf{Q}}$. On the other hand, the inclusion $F \to Q$ defines a map $R[F] \to R[Q]$ and hence a map $r_F : \underline{\mathsf{A}}_{\mathsf{Q}} \to \underline{\mathsf{A}}_{\mathsf{F}}$. Since r_F is induced by a monoid homomorphism, it is a morphism of monoid schemes. It follows from the definitions that $r_F \circ i_F = \mathrm{id}_{\underline{\mathsf{A}}_{\mathsf{F}}}$. Thus r_F and i_F are morphisms of $\underline{\mathsf{A}}_{\mathsf{Q}}$-sets, and $r_F(a) = r_F(a \cdot 1) = a r_F(1_A)$ for every $a \in \underline{\mathsf{A}}_{\mathsf{Q}}(A)$. Since r_F is surjective, it follows that $\underline{\mathsf{A}}_{\mathsf{Q},\mathfrak{p}}(A)$ is a principal ideal of $\underline{\mathsf{A}}_{\mathsf{Q}}(A)$, generated by $r_F(1_A)$.

One checks immediately that the two squares in the above diagram commute. The outer rectangle is just the identity rectangle, and hence the square on the left will automatically be Cartesian if the square on the right is Cartesian. The latter is the assertion that the ideal of the closed immersion $i_{Q/F}$ is the ideal I generated by the set of all $e^f - 1$ such that $f \in F$. Indeed, it is evident that $i^\sharp_{Q/F}$ annihilates all these elements and hence factors through a map $R[Q]/I \to R[Q/F]$. On the other hand, the map $Q \to R[Q]/I$ sends F to 1, and hence factors through Q/F, by its universal mapping property. This gives the inverse map map $R[Q/F] \to R[Q]/I$.

If Q is fine, then by (2.2.2) there exists a morphism $h: Q \to \mathbf{N}$ such that $h^{-1}(0) = F$. Then h defines a morphism $t: \underline{\mathsf{A}}_{\mathsf{m}} \to \underline{\mathsf{A}}_{\mathsf{Q}}$; on A-valued points $t(a) = a^h$; where a^h is the homomorphism $Q \to A$ sending q to $a^{h(q)}$. Let

$$f: \underline{\mathsf{A}}_{\mathsf{Q}} \times \underline{\mathsf{A}}_{\mathsf{m}} \to \underline{\mathsf{A}}_{\mathsf{Q}}$$

be the composition of $\mathrm{id}_{\underline{\mathsf{A}}_{\mathsf{Q}}} \times t$ with the multiplication map μ of the monoid structure on $\underline{\mathsf{A}}_{\mathsf{Q}}$. On A-valued points, f sends (x, a) to xa^h. Let i_0 and i_1 be the sections of $\underline{\mathsf{A}}_{\mathsf{m}}$ corresponding to 0 and 1 and let j_0 and j_1 be the corresponding maps $\underline{\mathsf{A}}_{\mathsf{Q}} \to \underline{\mathsf{A}}_{\mathsf{Q}} \times \underline{\mathsf{A}}_{\mathsf{m}}$. We check that $f \circ j_0 = i_F \circ r_F$ and that $f \circ j_1 = \mathrm{id}$ on A-valued points. The second of these calculations is obvious, and for the first, we just have to observe that $f(x, 0) = x0^h$ and remember that 0^n is 0 if $n > 0$ and is 1 if $n = 0$. Finally, if x belongs to the image of i_F, then for any a, $f(x, a)(q) = x(q)a^{h(q)} = x(q)$, since $x(q) = 0$ whenever $h(q) \neq 0$. This proves that i_F is a strong deformation retract. □

Corollary 3.2.2 *If Q is a fine sharp monoid, then $\underline{\mathsf{A}}_{\mathsf{Q}}(\mathbf{C})$, with the complex analytic topology, is contractible.*

When k is a field and Q is integral, the monoid $\underline{\mathsf{A}}_{\mathsf{Q}}(k)$ admits an explicit description in terms of the faces of Q. If $x \in \underline{\mathsf{A}}_{\mathsf{Q}}(k)$, let $F(x) := x^{-1}(k^*)$, a face of Q. If x and z are points of $\underline{\mathsf{A}}_{\mathsf{Q}}(k)$, then $F(xz) = F(x) \cap F(z)$. The map $x \mapsto F(x)$ from $\underline{\mathsf{A}}_{\mathsf{Q}}(k)$ to the set of faces of Q defines a partition of $\underline{\mathsf{A}}_{\mathsf{Q}}(k)$. Note that x is zero outside of $F(x)$ and induces a map $F^{gp} \to k^*$ which in fact determines x. Thus we can view a point of $\underline{\mathsf{A}}_{\mathsf{Q}}(k)$ as a pair (F, x'), where F is a face of Q and $x': F^{gp} \to k^*$.

Proposition 3.2.3 *Let Q be a fine monoid, let k be a field, and let F be a face of Q. Then the set of all $y \in \underline{\mathsf{A}}_{\mathsf{Q}}(k)$ such that $F(y) = F$ is a Zariski dense and open subset of $\underline{\mathsf{A}}_{\mathsf{F}}(k) \subset \underline{\mathsf{A}}_{\mathsf{Q}}(k)$. If x and y are two points of $\underline{\mathsf{A}}_{\mathsf{Q}}(k)$, then the following are equivalent:*

1. *$F(y) \subseteq F(x)$*
2. *$y \in \underline{\mathsf{A}}_{\mathsf{F(x)}}(k)$*
3. *There exists a $z \in \underline{\mathsf{A}}_{\mathsf{Q}}(k)$ such that $y = zx$.*

Furthermore, if either k is algebraically closed or $Q^{gp}/F(x)^{gp}$ is torsion free, then $F(y) = F(x)$ if and only if there exists a $z \in \underline{\mathsf{A}}^*_{\mathsf{Q}}(k)$ with $y = zx$. In particular, if k is algebraically closed or if Q^{gp}/F^{gp} is torsion free for every face F of Q, then the partition of $\underline{\mathsf{A}}_{\mathsf{Q}}(k)$ defined by the faces of Q corresponds to its orbit decomposition under the action of $\underline{\mathsf{A}}^*_{\mathsf{Q}}$, and the stratification by the closed sets $\underline{\mathsf{A}}_{\mathsf{F}}(k)$ corresponds to the trajectories under the action of $\underline{\mathsf{A}}_{\mathsf{Q}}(k)$ on itself.

Proof: We identify a point y of $\underline{\mathsf{A}}_{\mathsf{Q(k)}}$ with the corresponding character $Q \to k$. Then $F(y) \subseteq F$ if and only if $y(Q \setminus F) = 0$, *i.e.*, if and only if y factors through i_F; hence the equivalence of (1) and (2). Since F is fine, $\underline{\mathsf{A}}^*_{\mathsf{F}}$ is Zariski dense in $\underline{\mathsf{A}}_{\mathsf{F}}$, and the inclusions $\underline{\mathsf{A}}^*_{\mathsf{F}}(k) \subseteq \underline{\mathsf{A}}_{\mathsf{F}}(k) \subseteq \underline{\mathsf{A}}_{\mathsf{Q}}(k)$ identify $\underline{\mathsf{A}}^*_{\mathsf{F}}(k)$ with the set of all y such that $F(y) = F$. If $F(y) \subseteq F(x)$, define $z: Q \to k$ by $z(q) := 0$ if $q \in Q \setminus F(x)$ and $z(q) := y(q)/x(q)$ if $q \in F(x)$. Then in fact $z \in \underline{\mathsf{A}}_{\mathsf{Q}}(k)$, and $y = zx$. Thus (2) implies (3), and the converse is obvious. If $F := F(x) = F(y)$, then y/x defines a homomorphism $F^{gp} \to k^*$. If k is algebraically closed, k^* is divisible, and if (Q/F^{gp}) is torsion free, the sequence $F^{gp} \to Q^{gp} \to Q/F^{gp}$ splits. In either case, there exists an extension z of y/x to Q^{gp}, and then z defines a point of $\underline{\mathsf{A}}^*_{\mathsf{Q}}$ such that $zx = y$. $\square$

3.3 Properties of monoid algebras

Proposition 3.3.1 *Let P be an integral monoid and let R be an integral domain.*

1. *If P^{gp} is torsion free, then $R[P]$ is an integral domain.*
2. *If in addition P is finitely generated and R is normal, then $R[P^{\mathrm{sat}}]$ is the normalization of $R[P]$. In particular $R[P]$ is normal if and only if P is saturated.*

Proof: First suppose that P is finitely generated. Then if P^{gp} is torsion free, it is free of finite rank, so

$$R[P^{gp}] \cong R[T_1, T_1^{-1}, \ldots T_n, T_n^{-1}]$$

for some n. (Geometrically, $\underline{\mathsf{A}}^*_{\mathsf{P}} = \underline{\mathsf{A}}_{\mathsf{P^{gp}}}$ is a torus over $\operatorname{Spec} R$.) In particular $R[P^{gp}]$ is an integral domain, and since $R[P] \subseteq R[P^{gp}]$, $R[P]$ is also an integral domain. In general, P is the union of its finitely generated submonoids

P_λ, and each P_λ^{gp} is torsion free if P^{gp} is. Then $R[P]$ is the direct limit of the set of all $R[P_\lambda]$, each of which is an integral domain, and hence it too is an integral domain. Since $R[P^{\mathrm{sat}}]$ is generated as an $R[P]$-algebra by P^{sat} and since e^q is integral over $R[P]$ for every $p \in P^{\mathrm{sat}}$, $R[P^{\mathrm{sat}}]$ is integral over $R[P]$. Since $R[P^{\mathrm{sat}}] \subseteq R[P^{gp}]$, which is contained in the fraction field of $R[P]$, $R[P^{\mathrm{sat}}]$ is contained in the normalization of $R[P]$. It remains only to prove that $R[P^{\mathrm{sat}}]$ is normal. Since P^{sat} is fine, we may and shall assume without loss of generality that P is saturated. By (2.3.11), P is the intersection in P^{gp} of all its localizations at height one primes $\mathfrak{p}$, and hence $R[P]$ is the intersection in $R[P^{gp}]$ of the corresponding monoid algebras $R[P_\mathfrak{p}]$. Since the intersection of a family of normal subrings of a ring is normal, it will suffice to prove that each $R[P_\mathfrak{p}]$ is normal. Replacing P by $P_\mathfrak{p}$, we may assume that P is saturated and of dimension one. Then $\overline{P}$ is a one-dimensional toric monoid, hence by (2.3.10) it is isomorphic to $\mathbf{N}$. Choose any element p of P whose image in $\overline{P}$ is the generator. The corresponding map $P^* \oplus \mathbf{N} \to P$ is then an isomorphism. Since $P^* \subseteq P^{gp}$ is a finitely generated free group, $P \cong \mathbf{Z}^n \oplus \mathbf{N}$ for some n. Hence $R[P] \cong R[T_1, T_1^{-1}, \dots T_n, T_n^{-1}, T]$, which is normal since R is. (One can check easily that $R[P]$ satisfies Serre's conditions S_2 and R_1.) $\square$

To see that the hypothesis on P^{gp} is not superfluous, consider the submonoid P of $\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ generated by $p := (1,0)$ and $q := (1,1)$. This is the free monoid generated by p, q subject to the relation $2p = 2q$. It is sharp and fine, but $R[P] \cong R[x,y]/(x^2 - y^2)$, which is not an integral domain if $R \neq 0$.

A deeper theorem of Hochster whose proof [12] we cannot give here, asserts:

Theorem 3.3.2 *The monoid algebra of a fine saturated monoid over a field is Cohen-Macaulay.*

The following result is an immediate consequence of its analog (2.3.13) for monoids.

Proposition 3.3.3 *Let R be a ring and P a fine monoid of Krull dimension d. For each $i = 0, \dots d$, let $K_i := \cap\{\mathfrak{p} \in \operatorname{Spec} R : \operatorname{ht} \mathfrak{p} \leq i\}$. Then $\underline{\mathsf{A}}_{\mathsf{P}} \setminus \underline{\mathsf{A}}_{\mathsf{P},\mathsf{K}_{i+1}}$ is covered by the special affine open subsets $\underline{\mathsf{A}}_{\mathsf{P}_\mathsf{F}}$, where F ranges over the set of faces F such that $\operatorname{rk} P/F = i$. In particular, $\underline{\mathsf{A}}_{\mathsf{P}} \setminus \underline{\mathsf{A}}_{\mathsf{P},\mathsf{K}_2}$ is covered by the set of $\underline{\mathsf{A}}_{\mathsf{P}_\mathsf{F}}$ for the facets of P, and if P is toric, each of these is a product of a torus with an affine line over* $\operatorname{Spec} R$.

□

Let P be an integral monoid and R any ring. We shall find it useful to investigate further the relationship between ideals in P and ideals in $R[P]$.

Definition 3.3.4 *Let $f := \sum_P a_p(f)e^p$ be an element of $R[P]$.*

1. $\sigma(f) = \{p \in P : a_p(f) \neq 0\}$.
2. *$K(f)$ is the ideal of P generated by $\sigma(f)$.*
3. *If I is any ideal of $R[P]$, $K(I)$ is the set of all $p \in P$ for which there exists some $f \in I$ such that $a_p(f) \neq 0$.*

Note that $K(I)$ is in fact an ideal of $R[P]$. Indeed, if $p \in K(I)$ and $q \in P$, then $e^q f \in I$, and $a_{q+p}(e^q f) = a_p(f) \neq 0$. In fact, $K(I)$ is the smallest ideal K of P such that $I \subseteq R[K]$. Geometrically, $\underline{\mathsf{A}}_{\mathsf{P,K}}$ is the largest closed subscheme of $Z(I)$ which is invariant under the action of $\underline{\mathsf{A}}_{\mathsf{P}}$ on itself.

Proposition 3.3.5 *Suppose that f and g are elements of $R[P]$.*

1. *$\sigma(f+g) \subseteq \sigma(f) \cup \sigma(g)$, hence $K(f+g) \subseteq K(F) \cup K(g)$.*
2. *$\sigma(fg) \subseteq \sigma(f) + \sigma(g)$, hence $K(fg) \subseteq K(f) + K(g) \subseteq K(f) \cap K(g)$.*
3. *$K(f) = K((f))$, where (f) is the ideal of $R[P]$ generated by f.*
4. *If I and J are ideals of $R[P]$, $K(IJ) \subseteq K(I) + K(J)$.*

Proof: The first two statements follow from the fact that for every $p \in P$,

$$\begin{aligned} a_p(f+g) &= a_p(f) + a_p(g) \\ a_p(fg) &= \sum_{p_1+p_2=p} a_{p_1}(f)a_{p_2}(g) \end{aligned}$$

It is apparent from the definition that $\sigma(f) \subseteq K((f))$, and hence that $K(f) \subseteq K((f))$. On the other hand, for any $h \in (f)$, it follows from (2) that $\sigma(h) \subseteq \sigma(f)$ and hence that $K(h) \subseteq K(f)$. □

We shall be especially interested in determining when $K(f)$ is principal.

Proposition 3.3.6 *Let P an integral monoid and let R be a ring.*

1. *If $f \in R[P]$, $K(f)$ is the unit ideal of P if and only if f does not belong to the ideal $R[P^+]$ of $R[P]$.*

2. *More generally, $K(f)$ is principally generated by an element p of P if and only if $f = e^p\tilde{f}$, where $\tilde{f} \in R[P]$ and $K(\tilde{f}) = P$.*

3. *Suppose R is an integral domain and P^* is torsion free. Then if f and g are elements of $R[P]$ such that $K(f)$ and $K(g)$ are principal, the same is true of fg, and $K(fg) = K(f) + K(g)$.*

Proof: If $K := K(f)$ is generated by p, then $k - p \in P$ for every element k of $K(f)$. Hence $f = \sum_{k\in K} a_k e^k = e^p \sum_k a_k e^{k-p}$, so $f = e^p\tilde{f}$ where $\tilde{f} := \sum_k a_k e^{k-p}$. Then

$$(p) = K(f) \subseteq K(e^p) + K(\tilde{f}) = (p) + K(\tilde{f}),$$

and it follows that $K(\tilde{f}) = P$. Conversely, if $f = e^p\tilde{f}$ with $K(\tilde{f}) = P$, then certainly $K(f) \subseteq (p)$. But if $\tilde{f} = \sum \tilde{a}_q e^q$, there exists a $q \in P^*$ such that $\tilde{a}_q \neq 0$, and then $p+q \in K(f)$, so $p \in K(f)$. If $K(f)$ is principally generated by p and $K(g)$ is principally generated by q, then $f = e^p\tilde{f}$ and $g = e^q\tilde{g}$, where $\tilde{f}$ and $\tilde{g}$ belong to $R[P] \setminus R[P^+]$. The quotient of $R[P]$ by $R[P^+]$ is isomorphic to $R[P^*]$. If P^* is torsion free and R is an integral domain, then $R[P^*]$ is also an integral domain by (3.3.1). Hence $R[P^+]$ is a prime ideal, and so $K(\tilde{f}\tilde{g}) = P$. Since $fg = e^{p+q}\tilde{f}\tilde{g}$, it follows that $K(fg)$ is principally generated by $p + q$. $\square$

Consider the submonoid P of $\mathbf{N}$ generated by 2 and 3, and let $f = e(2) + e(3)$ and $g = e(2) - e(3)$. Then $K(f) = K(g)$ is the ideal $(2, 3)$ of P, which is not principal, but $(fg) = e(4) - e(6) = e(4)(1 - e(2))$, so $K(fg)$ is principally generated by 4. Thus, the converse of (3.3.6.3) is not true in general. We shall see that it does hold if P is toric.

Recall from (2.3.10) that associated to each height one prime $\mathfrak{p}$ of a fine monoid P there is a homomorphism $\nu_\mathfrak{p}\colon P \to \mathbf{N}$. If K is a nonempty ideal of P, then the ideal of $\overline{P}_\mathfrak{p}^{\mathrm{sat}} \cong \mathbf{N}$ generated by K is principal, generated by an element k such that $\nu_\mathfrak{p}(k) = \nu_\mathfrak{p}(K)$, where

$$\nu_\mathfrak{p}(K) := \inf\{\nu_\mathfrak{p}(k) : k \in K\}.$$

If $f \in R[P]$, let $\nu_\mathfrak{p}(f) := \nu_\mathfrak{p}(K(f))$. That is, $\nu_\mathfrak{p}(f)$ is the minimum of the set of all $\nu_\mathfrak{p}(p)$ such that $p \in \sigma(f)$.

Proposition 3.3.7 *Let P be a toric monoid and let R be an integral domain.*

1. *If K is an ideal of $R[P]$ and $p \in K$ is an element such that $\nu_{\mathfrak{p}}(p) = \nu_{\mathfrak{p}}(K)$, for every height one prime $\mathfrak{p}$, then K is principally generated by p.*

2. *If f and g are elements of $R[P]$, then for every height one prime $\mathfrak{p}$, $\nu_{\mathfrak{p}}(fg) = \nu_{\mathfrak{p}}(f) + \nu_{\mathfrak{p}}(g)$. Moreover, $K(fg)$ is principal if and only if $K(f)$ and $K(g)$ are.*

Proof: Suppose the hypotheses of (1) hold and $k \in K$. Then $\nu_{\mathfrak{p}}(k-p) \geq 0$ for every height one prime $\mathfrak{p}$ of P. By (2.3.11), $k - p \in P$, and it follows that K is principally generated by p. The homomorphism $\lambda_{\mathfrak{p}} \colon R[P] \to R[P_{\mathfrak{p}}]$ is injective, so $K(\lambda_{\mathfrak{p}}(f))$ is the ideal $K_{\mathfrak{p}}$ of $P_{\mathfrak{p}}$ generated by K. Since $P_{\mathfrak{p}}$ is saturated, this ideal is principal and since $P^*_{\mathfrak{p}}$ is torsion free, (3.3.6.3) implies that $K_{\mathfrak{p}}(fg) = K_{\mathfrak{p}}(f) + K_{\mathfrak{p}}(g)$ for any f and g, hence $\nu_{\mathfrak{p}}(fg) = \nu_{\mathfrak{p}}(f) + \nu_{\mathfrak{p}}(g)$. We already know that $K(fg)$ is principal if $K(f)$ and $K(g)$ are. Conversely, if $K(fg)$ is principally generated by r, (3.3.5.1) shows that r can be written as a sum $p + q$, with $p \in K(f)$ and $q \in K(g)$. Then for any $\mathfrak{p}$ of height one, $\nu_{\mathfrak{p}}(p) \geq \nu_{\mathfrak{p}}(f)$ and $\nu_{\mathfrak{p}}(q) \geq \nu_{\mathfrak{p}}(g)$. On the other hand, $\nu_{\mathfrak{p}}(p) + \nu_{\mathfrak{p}}(q) = \nu_{\mathfrak{p}}(r) = \nu_{\mathfrak{p}}(fg) = \nu_{\mathfrak{p}}(f) + \nu_{\mathfrak{p}}(g)$. Hence $\nu_{\mathfrak{p}}(p) = \nu_{\mathfrak{p}}(f)$ and $\nu_{\mathfrak{p}}(q) = \nu_{\mathfrak{p}}(g)$ for every $\mathfrak{p}$. By (1), this implies that $K(f)$ and $K(g)$ are principal. $\square$

Corollary 3.3.8 *Let R be an integral domain, P a toric monoid, and F a face of P. Then the set $\mathcal{F}$ of elements α of $R[P]$ such that $K(\alpha)$ is principally generated by an element of F is a face of the monoid $R[P]$.*

Proof: If α and β belong to $\mathcal{F}$, then $K(\alpha) = (p)$ and $K(\beta) = (q)$ with p and q in F, so by (3.3.6.3) $K(\alpha\beta) = (p+q)$ and $p + q \in F$. Thus $\mathcal{F}$ is a submonoid of $R[P]$. Conversely if $\alpha\beta \in \mathcal{F}$, then by (3.3.7.2) $K(\alpha)$ and $K(\beta)$ are principal, say generated by p and q respectively. Then $p + q$ generates $K(\alpha\beta)$ and lies in F, Since F is a face, each of p and q belongs to F and each of α and β belongs to $\mathcal{F}$. Thus $\mathcal{F}$ is a face of $R[P]$. $\square$

Our next result is a generalization of a theorem of Kato [13, 11.6]. Its goal is to compute the "compactification log structures" associated to certain open embeddings of monoid schemes.

Let P be an integral monoid, let R be a ring, and write X for $\underline{\mathsf{A}}_{\mathsf{P}}$. For each open subset U of X, we have a natural map of monoids

$$P \to R[P] = \Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X).$$

Then the set G_U of elements of P which map to a unit of $\Gamma(U, \mathcal{O}_X)$ is a face of P, and if $V \subseteq U$, $G_U \subseteq G_V$. Thus there is a natural map from the localization P_U of P by G_U to the localization P_V of P by G_V, and $U \mapsto P_U$ defines a presheaf of monoids on X. For each U, $\overline{P}_U \cong P/G_U$, and $U \mapsto \overline{P}_U$ also defines a presheaf of monoids on X. Let $\overline{M}_X$ denote the corresponding sheaf.

Now suppose that F is a face of P. For each open subset U of X, let $F(U)$ denote the face of P_U generated by F, and let $\overline{F}_X$ denote the sheaf associated to the presheaf sending U to $\overline{F}(U)$.

Theorem 3.3.9 *In the above situation, assume that R in an integral domain and that P is toric. Let F be a face of P, view $\underline{\mathsf{A}}_{\mathsf{P}_\mathsf{F}}$ as an open subset of X, and let $Y := X \setminus \underline{\mathsf{A}}_{\mathsf{P}_\mathsf{F}}$. Then the natural map $P \to \mathcal{O}_X$ induces an isomorphism of sheaves of monoids*

$$\overline{F}_X \cong \underline{\Gamma}_Y(Div_X^+).$$

Proof: The existence of the map is easy to see. If U is any open subset of X and if f belongs to the face of $P(U)$ generated by the image of $F \to P(U)$, then e^f defines an element of $\Gamma(U, \mathcal{O}_X)$ and hence a Cartier divisor D_f on U. Since e^f is invertible on $U \cap \underline{\mathsf{A}}_{\mathsf{P}_\mathsf{F}}$, D_f has support in Y, as desired. Finally, note that if f is a unit in $P(U)$, then D_f is the zero divisor on U, so our map factors through $\overline{F}(U)$. Since $\Gamma_Y(Div_X^+)$ is a sheaf, it also factors through a map $\overline{F}_X \to \Gamma_Y(Div_X^+)$, as desired.

To finish the proof of the theorem it will suffice to prove that our mapping is an isomorphism on stalks. Let x be a point of X and let G_x be the set of elements of P which map to units of $\mathcal{O}_{X,x}$. Then G_x is a face of P and the map $P \to \mathcal{O}_{X,x}$ factors through the localization P_x of P by G_x. Replacing P by P_x and F by the face it generates in P_x, we may assume without loss of generality that $G_x = P^*$, or equivalently that the ideal of $R[P]$ corresponding to x contains the ideal $R[P^+]$. An element of $\underline{\Gamma}_Y(Div_X^+)_x$ can be regarded as a principal ideal I in the local ring $\mathcal{O}_{X,x}$ which becomes the unit ideal in the localization of $\mathcal{O}_{X,x}$ by F. Suppose that p and q are elements of F and e^p

and e^q define the same ideal of $\mathcal{O}_{X,x}$. Then there exist u and $v \in R[P]$ not vanishing at x such that $ue^p = ve^q$. But then u and v belong to $R[P] \setminus R[P^+]$, and hence by (3.3.6) $K(ue^p) = (p)$ and $K(ve^q) = (q)$. Then $(p) = (q)$ and so p and q have the same image in $\overline{F}$. This proves that $\overline{e}$ is injective. To prove that it is surjective, suppose that I is a principal ideal of $\mathcal{O}_{X,x}$ which becomes a unit in its localization by F. If f generates I, there exist a $g \in \mathcal{O}_{X,x}$ and $r \in F$ such that $fg = e^r$, and there exist u and $v \in R[P]$ not vanishing at x such that $\alpha = fu$ and $\beta = gv$ belong to $R[P]$. Then $uv \notin R[P^+]$ and $\alpha\beta = fugv = e^r uv$. Thus by (3.3.6), $K(\alpha\beta)$ is generated by r, an element of F. It follows from (3.3.8), that $K(\alpha)$ and $K(\beta)$ are respectively generated by elements p and q of F, Write $\alpha = e^p\tilde{\alpha}$ and $\beta = e^q\tilde{\beta}$ with $\tilde{\alpha}$ and $\tilde{\beta}$ in $R[P] \setminus R[P^+]$ and $p + q = r$. Then $\tilde{\alpha}\tilde{\beta} = uv$, so $\tilde{\alpha}$ and $\tilde{\beta}$ do not vanish at x. Since $f = \tilde{\alpha}u^{-1}e^p$ in $\mathcal{O}_{X,x}$, e^p generates I. This proves the surjectivity. □

It will be important in the applications to know that the previous result is also true if one takes the stalks in the étale topology of X. This is not trivial because if $\eta\colon X' \to X$ is étale, the natural map $\eta^{-1}Div_X^+ \to Div_{X'}^+$ is not an isomorphism in general. However in our case the difficulty is overcome by the following observation.

Lemma 3.3.10 *Let $\eta\colon X' \to X$ be an étale morphism of normal schemes and let $Y \subseteq X$ be a closed subscheme each of whose irreducible components is purely of codimension one and unibranch. Then if $Y' := \eta^{-1}(Y)$, the natural map*

$$\eta^{-1}\underline{\Gamma}_Y(Div_X^+) \to \underline{\Gamma}_{Y'}(Div_{X'}^+)$$

is an isomorphism.

Proof: We first prove this result with Weil divisors in place of Cartier divisors. Let x' be a point in X' and let $x := \eta(x) \in X$. Since $X' \to X$ is étale, Y' is purely of codimension one in X'. The stalk of $\underline{\Gamma}_{Y'}(\mathcal{W}_{X'}^+)$ at x' is the free monoid generated by the irreducible components of Y' containing x'. If Z' is such a component, its image Z in Y is an irreducible component of Y containing x. Since η is étale and Z is unibranch, $\eta^{-1}(Z)$ has a unique irreducible component passing through x', which must therefore be Z'. This shows that in fact

$$\eta^{-1}\underline{\Gamma}_Y(\mathcal{W}_X^+) \to \underline{\Gamma}_{Y'}(\mathcal{W}_{X'}^+)$$

is an isomorphism. Since X and X' are normal, the Cartier divisors are contained in the Weil divisors, and since η is faithfully flat, a divisor on X

is Cartier if and only if its inverse on X' is. Therefore the result is also true for Cartier divisors. □

For example, in the situation of (3.3.9), the irreducible components of Y are defined by the height one primes $\mathfrak{p}$ of Q such that $\mathfrak{p} \cap F$ is not empty. If R is normal, then so is each quotient $R[Q]/R[\mathfrak{p}] \cong R[F_{\mathfrak{p}}]$, and in particular it is unibranch.

Corollary 3.3.11 *Suppose in the situation of (3.3.9) that R is normal and $\eta\colon X' \to X$ is étale. Then if $Y' := \eta^{-1}(Y)$ and x' is a point of X' mapping to X, the map*

$$\overline{F}_x \to \underline{\Gamma}_{Y'}(Div^+_{X'})_{x'}$$

is also an isomorphism.

4 Morphisms of monoids

Just as the geometry of monoids describes the skeletal structure of local models for smooth log schemes, morphisms of monoids are the basic local models for smooth morphisms of log schemes. Exact, local, and strict morphisms are basic to the vocabulary and are studied in the first section. The remaining sections are devoted to more subtle notions, including small, integral, and saturated morphisms.

4.1 Exact, sharp, and strict morphisms

Definition 4.1.1 *A homomorphism of monoids $\theta\colon Q \to P$ is sharp if the induced map $Q^* \to P^*$ is an isomorphism, and is strict if the induced map $\overline{Q} \to \overline{P}$ is an isomorphism.*

For example, the unique map from the zero monoid to M is sharp if and only if M is a sharp monoid.

Proposition 4.1.2 *Let $\theta\colon Q \to P$ be a sharp and strict monoid homomorphism. Then θ is surjective, and if P is quasi-integral θ is bijective.*

Proof: If $p \in P$ then since $\overline{\theta}$ is surjective there exist $q \in Q$ and $u \in P^*$ with $\theta(q) = p + u$. Since θ^* is surjective there exists a $v \in Q^*$ with $\theta(v) = -u$, and then $\theta(v + q) = p$. If $\theta(q_1) = \theta(q_2)$, then because $\overline{\theta}$ is injective it follows that there exists a $v \in Q^*$ such that $q_2 = q_1 + v$. Then $\theta(q_2) = \theta(q_1) + \theta(v) = \theta(q_2) + \theta(v)$. Since $\theta(v) \in P^*$ and P is quasi-integral, it follows that $\theta(v) = 0$. Since θ^* is injective, $v = 0$ and $q_2 = q_1$. $\square$

To see that the hypothesis that P be quasi-integral is not superfluous, let $\mathbf{Z} \star_{\mathbf{N}^+} \mathbf{N} \cong \mathbf{Z} \amalg \mathbf{N}^+$ be the join (1.3.5) of $\mathbf{Z}$ and $\mathbf{N}$ along $\mathbf{N}^+$. Then the morphism from $\mathbf{Z} \oplus \mathbf{N}$ to $\mathbf{Z} \star_{\mathbf{N}^+} \mathbf{N} \cong \mathbf{Z} \amalg \mathbf{N}^+$ sending (m, n) to n in $\mathbf{N}^+$ if $n > 0$ and to $m \in \mathbf{Z}$ if $n = 0$ is surjective, sharp, and strict but not bijective.

Recall from (2.1.8) that a morphism of integral monoids $\theta\colon Q \to P$ is *exact* if Q is the inverse image of P in Q^{gp}.

Proposition 4.1.3 *In the category of integral monoids:*

1. *The natural map $\pi\colon Q \to \overline{Q}$ is exact.*

2. *If $\theta: Q \to P$ and $\phi: P \to R$ are exact, then so is $\phi \circ \theta$. If $\phi \circ \theta$ and ϕ are exact, then θ is exact. If $\phi \circ \theta$ is exact and $\overline{\theta}^{gp}$ is surjective, then ϕ is exact.*

3. *A morphism θ is exact if and only if $\overline{\theta}$ is exact.*

4. *A morphism $Q \to P$ is local if it is exact, and the converse holds if Q is valuative.*

5. *An exact sharp morphism is injective. In particular, if θ is exact, then $\overline{\theta}$ is injective.*

6. *If $\theta: Q \to P$ is exact and $\beta: Q \to Q'$ is a morphism, then the pushout $\theta': Q' \to P'$ of θ (in the category of integral monoids) is again exact. If $\alpha: P' \to P$ is any morphism, then the pullback $\theta': Q' \to P'$ is exact.*

Proof: Recall that $(Q/Q^*)^{gp} \cong Q^{gp}/Q^*$. Hence if $x \in Q^{gp}$ and $\pi^{gp}(x) \in \overline{Q}$, $x = q + u$ where $q \in Q$ and $u \in Q^*$, hence in fact $x \in Q$ and π is exact. The first two parts of statement (2) follow immediately from the definitions. To prove the last part, suppose $y \in P^{gp}$ and $\phi^{gp}(y) \in R$ and write $y = \theta^{gp}(x) + v$ with $v \in P^*$ and $x \in P^{gp}$. Then since $(\phi \circ \theta)^{gp}(x) = \phi^{gp}(y) - \phi(v) \in R$ and $\phi \circ \theta$ is exact, $x \in Q$ and hence $y \in P$. For (3), note that if $\theta: Q \to P$ is any morphism of integral monoids there is a commutative diagram

$$\begin{tikzcd} Q \arrow[r, "\theta"] \arrow[d, "\pi"'] & P \arrow[d, "\pi"] \\ \overline{Q} \arrow[r, "\overline{\theta}"] & \overline{P} \end{tikzcd}$$

in which the vertical arrows are exact and surjective. Thus (2) implies that θ is exact if and only if $\overline{\theta}$ is. If θ is exact and $\theta(q) \in P^*$, then $-q \in Q^{gp}$ and $\theta(-q) \in P$, so $-q \in Q$ and $q \in Q^*$. Thus θ is local. Suppose Q is valuative, θ is local, and $x \in Q^{gp}$ with $\theta^{gp}(x) \in P$. If $x \notin Q$, then $-x \in Q$, hence $\theta(-x) \in P$, and hence $\theta(x) \in P^*$. But then $x \in Q^* \subseteq Q$. If θ is exact and sharp, and if $\theta(q) = \theta(q')$, then $q - q' \in Q^{gp}$ with $\theta(q - q') \in P$, so $q - q' \in Q$. Similarly $q' - q \in Q$, so $q - q' \in Q^*$. Since $\theta(q - q') = 0$ and θ is sharp, $q = q'$, so θ is injective.

Recall from (1.2.2) that the integral pushout P' in (6) can be identified with the image of $Q' \oplus P$ in $Q'^{gp} \oplus_{Q^{gp}} P^{gp}$. Hence if $y' \in Q'^{gp}$ and $\theta^{gp}(y') \in P'$, there exist $q' \in Q$, $p \in P$, and $y \in Q^{gp}$ such that $y' = q' + \beta(y)$ and $p = \theta(y)$. Since θ is exact, $y \in Q$ and so $y' = q' + \beta(y) \in Q'$. For the pullback statement, note that although formation of the associated group does not commute with fibered products, the natural map $Q'^{g} \to Q^{gp} \times_{P^{gp}} P'^{gp}$ is injective. Now say $x' \in Q'^{gp}$ and $p' := \theta'^{gp}(x') \in P'$. Then $\theta^{gp}\beta^{gp}(x') = \alpha^{gp}\theta'^{gp}(x) = \alpha(p') \in P$. It follows from the exactness of θ that $q := \beta'^{gp}(x') \in Q$, and there is a unique $q' \in Q'$ such that $\beta(q') = q$ and $\theta'(q') = p'$. Then q' and x' have the same image in $Q'^{gp} \times_{P^{gp}} P'^{gp}$, and hence are equal. □

In particular, the family of exact morphisms is stable under composition, pullbacks, and pushouts (in the category of integral monoids).

Examples 4.1.4 The morphism $\mathbf{N} \oplus \mathbf{N} \to \mathbf{N}$ taking (m, n) to $m+n$ is local and sharp but not exact. A localization morphism $Q \to Q_F$ is, in general, not local or exact. If K is an ideal of an integral monoid Q and a is an element of K, then $B_{K,a}(Q) := \{y \in Q^{gp} : a + y \in K\}$ is a submonoid of Q^{gp} containing Q, which corresponds to a part of the blow-up (**??**) of Q along K, and the morphism $Q \to B_{K,a}(Q)$ is in general not exact. On the other hand, the diagonal embedding $\mathbf{N} \to \mathbf{N} \oplus \mathbf{N}$ is exact.

Proposition 4.1.5 *Let $\theta\colon Q \to P$ be a morphism of integral monoids. If θ is exact, then* $\operatorname{Spec}\theta$ *is surjective. The converse holds if Q is fine and saturated.*

Proof: Suppose that θ is exact and $\mathfrak{q}$ is a prime of Q. Let $\theta_{\mathfrak{q}}\colon Q_{\mathfrak{q}} \to P_{\mathfrak{q}}$ be the localization of θ by $\mathfrak{q}$. Since $P_{\mathfrak{q}}$ can be identified with $Q_{\mathfrak{q}} \oplus_Q P$ it follows from (4.1.3) that $\theta_{\mathfrak{q}}$ is exact and hence local. Thus if $\mathfrak{p}$ is the prime of P corresponding to the maximal ideal of $P_{\mathfrak{q}}$, $\theta^{-1}(\mathfrak{p}) = \mathfrak{q}$. This proves that $\operatorname{Spec}\theta$ is surjective.

Conversely, suppose that Q is fine and saturated and that $\operatorname{Spec}\theta$ is surjective. Let x be an element of Q^{gp} such that $\theta(x) \in P$. Let $\mathfrak{q} \in \operatorname{Spec} Q$ be a prime of height one. Since $\operatorname{Spec}\theta$ is surjective, there is a prime $\mathfrak{p}$ of P lying over $\mathfrak{q}$. Then the map $Q_{\mathfrak{q}} \to P_{\mathfrak{p}}$ is local. Since $Q_{\mathfrak{q}}$ is saturated and $\mathfrak{q}$ has height one, it follows from (2.3.10) that $Q_{\mathfrak{q}}$ is valuative. Then by (4) of (4.1.3), the map $Q_{\mathfrak{q}} \to P_{\mathfrak{p}}$ is exact. Since the image of $\theta(x)$ in P^{gp} lies in $P_{\mathfrak{p}}$, it follows that the $x \in Q_{\mathfrak{q}}$. Thus $x \in Q_{\mathfrak{q}}$ for every prime of height one, and since Q is saturated, it follows from (2.3.11) that $x \in Q$. □

Definition 4.1.6 *A morphism $\theta\colon Q \to P$ of integral monoids is locally exact if for every prime $\mathfrak{p}$ of P, the localized map*

$$\theta_{\mathfrak{p}}\colon Q_{\theta^{-1}(\mathfrak{p})} \to P_{\mathfrak{p}}$$

is exact.

For example, the inclusion morphism $\mathbf{N} \to \mathbf{Z}$ is locally exact but not exact. Conversely, the morphism $\theta\colon \mathbf{N} \oplus \mathbf{N} \to \mathbf{N} \oplus \mathbf{N} \oplus \mathbf{N}$ sending (a, b) to $(a, a+b, b)$ is exact but not locally exact. (Consider the face F of $\mathbf{N} \oplus \mathbf{N} \oplus \mathbf{N}$ consisting of those elements whose first two coordinates are zero. Then $\theta^{-1}(F) = (0, 0)$, and the corresponding localized map is $\mathbf{N}\oplus\mathbf{N} \to \mathbf{N}\oplus\mathbf{N}\oplus\mathbf{Z}$, which is not exact.)

Definition 4.1.7 *A morphism $f\colon X \to Y$ of topological spaces is locally surjective if for every $x \in X$ and every generization y' of $f(x)$, there is a generization x' of x such that $f(x') = y'$.*

Proposition 4.1.8 *Let $\theta\colon Q \to P$ be a morphism of integral monoids. If θ is locally exact, then $\operatorname{Spec}\theta$ is locally surjective, and the converse is true if Q is fine and saturated.*

Proof: Suppose θ is locally exact and $\mathfrak{p} \in \operatorname{Spec} P$. Let $\mathfrak{q} := \theta^{-1}(\mathfrak{p})$ and let $\mathfrak{q}'$ be a prime of Q containing $\mathfrak{q}$. Since θ is locally exact, the localization map $\theta'\colon Q_{\mathfrak{q}} \to P_{\mathfrak{p}}$ induced by θ is exact. Hence by (4.1.5), $\operatorname{Spec}(\theta')$ is surjective, so there exists a prime $\mathfrak{p}'$ of $P_{\mathfrak{p}}$ lying over the prime $\mathfrak{q}'Q_{\mathfrak{q}}$. Then $\mathfrak{p}' \cap P$ is a prime of P which contains $\mathfrak{p}$ and lies over $\mathfrak{q}'$.

For the converse, suppose that Q is fine and saturated and that $\operatorname{Spec}(\theta)$ is locally surjective. Let $\mathfrak{p}$ be a prime of P, let $\mathfrak{q} := \theta^{-1}(\mathfrak{p})$, and let $\theta'\colon Q_{\mathfrak{q}} \to P_{\mathfrak{p}}$ be the map induced by θ. Since θ is locally surjective, θ' is surjective. Since $Q_{\mathfrak{q}}$ is fine and saturated, θ' is exact by (4.1.5) □

Corollary 4.1.9 *Let $\theta\colon Q \to P$ be a locally exact morphism of integral monoids. Let $\mathfrak{p}$ be be a prime ideal of P and let $\mathfrak{q} := \theta^{-1}(\mathfrak{p})$. Then $\operatorname{ht}\mathfrak{q} \leq \operatorname{ht}\mathfrak{p}$.*

Proof: Since θ is locally exact, $\operatorname{Spec}(\theta)$ is locally surjective. It follows that any chain of prime ideals in Q $\mathfrak{q} = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \mathfrak{q}_d$ lifts to a chain $\mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \mathfrak{p}_d$ in P. □

Definition 4.1.10 *If $\theta\colon Q \to P$ is a morphism of integral monoids, $Q^e := \{x \in Q^{gp} : \theta^{gp}(x) \in P\}$, so that θ factors*

$$Q \xrightarrow{\theta'} Q^e \xrightarrow{\theta^e} P,$$

where θ'^{gp} is an isomorphism and θ^e is exact.

For example, if $: Q \to P$ is an inclusion of integral monoids, then Q^e is the inverse image of 0 in P/Q, as we saw in our discussion of cokernels in (1.1), after (1.1.4).

Proposition 4.1.11 *Let $\theta\colon Q \to P$ be a morphism of integral monoids. Then the following conditions are equivalent:*

1. *The map of topological spaces $\operatorname{Spec}(\theta)\colon \operatorname{Spec}(P) \to \operatorname{Spec}(Q)$ is injective.*
2. *Every face F of P is generated by $\theta(\theta^{-1}(F))$.*
3. *The topology of $\operatorname{Spec}(P)$ is equal to the topology induced from the topology of $\operatorname{Spec}(Q)$ by the map $\operatorname{Spec}(\theta)$.*
4. *If $\mathfrak{p}$ and $\mathfrak{p}'$ are primes of P and $\theta^{-1}(\mathfrak{p}) \subseteq \theta^{-1}(\mathfrak{p}')$, then in fact $\mathfrak{p} \subseteq \mathfrak{p}'$.*

Proof: We may and shall assume without loss of generality that Q and P are sharp. It is obvious that (4) implies (1). Conversely, if (1) holds, and if $\mathfrak{p}$ and $\mathfrak{p}'$ are primes of P with $\theta^{-1}(\mathfrak{p}) \subseteq \theta^{-1}(\mathfrak{p}')$, then

$$\theta^{-1}(\mathfrak{p}') = \theta^{-1}(\mathfrak{p}) \cup \theta^{-1}(\mathfrak{p}') = \theta^{-1}(\mathfrak{p} \cup \mathfrak{p}').$$

Since $\mathfrak{p}'$ and $\mathfrak{p} \cup \mathfrak{p}'$ are two elements of $\operatorname{Spec}(P)$ with the same image in $\operatorname{Spec}(Q)$, it follows that $\mathfrak{p}' = \mathfrak{p} \cup \mathfrak{p}'$ and so $\mathfrak{p} \subseteq \mathfrak{p}'$. Thus (1) is equivalent to (4). Let F be a face of P, let $G := \theta^{-1}(F)$, and F' be the face of P generated by $\theta(G)$. Then $F' \subseteq F$ and $\theta^{-1}(F') = G = \theta^{-1}(F)$. If $\operatorname{Spec}(\theta)$ is injective, it follows that $F' = F$, and so (1) implies (2). To prove that (2) implies (3), suppose that f is an element of P, and let F be the face of P generated by f. Then by (2), $\theta(\theta^{-1}(F))$ is a submonoid of P which generates F as a face, so there exists $g \in \theta^{-1}(F)$ such that $\theta(g) \geq f$. Thus f belongs to the face generated by $\theta(g)$ and since also $\theta(g) \in F$, $\theta(g)$ generates F. Thus $D(f) = D(\theta(g)) = (\operatorname{Spec}\theta(g))^{-1}D(g)$. Thus every basic open set of $\operatorname{Spec}(P)$ is pulled back from $\operatorname{Spec}(Q)$, and (3) follows. To prove that (3) implies (4), suppose that $\mathfrak{p}$ and $\mathfrak{p}'$ are primes of P with $\theta^{-1}(\mathfrak{p}) \subseteq \theta^{-1}(\mathfrak{p}')$. Then $\theta^{-1}(\mathfrak{p}')$ is a point of $\mathrm{Spec}(Q)$ belonging to the closure of $\theta^{-1}(\mathfrak{p})$, and (3) implies that $\mathfrak{p}'$ belongs to the closure of $\mathfrak{p}$, *i.e.*, that $\mathfrak{p} \subseteq \mathfrak{p}'$. This concludes the proof of the equivalence of (1)–(4). □

Corollary 4.1.12 *Let $\theta \colon Q \to P$ be a morphism of integral monoids.*

1. *If* $\mathrm{Spec}\,\theta$ *is injective and θ is exact, then θ is locally exact.*
2. *If Q is fine and saturated and* $\mathrm{Spec}\,\theta$ *is bijective, then θ is exact and locally exact.*

Proof: Suppose θ is exact and $\mathrm{Spec}\,\theta$ is injective. If G is a face of Q, then its localization $G^{-1}Q \to G^{-1}P$ is exact, by (6) of (4.1.3). If F is any face of P and $G = \theta^{-1}(F)$, condition (2) of (4.1.11) implies that the map $G^{-1}P \to F^{-1}P$ is an isomorphism, and consequently $G^{-1}Q \to F^{-1}P$ is exact. Thus θ is locally exact. If Q is fine and saturated and $\mathrm{Spec}\,\theta$ is bijective then θ is also exact by (4.1.5). □

4.2 Small and almost surjective morphisms

Definition 4.2.1 *A morphism of integral monoids $\theta \colon Q \to P$ is almost surjective if it satisfies the following equivalent conditions:*

1. *For every $p \in P$, there exists $n \in \mathbf{N}^+$, $u \in P^*$, and $q \in Q$ such that $\theta(q) = u + np$.*
2. *The corresponding map of sharp cones $C_{\mathbf{Q}}(\overline{\theta}) \colon C_{\mathbf{Q}}(\overline{Q}) \to C_{\mathbf{Q}}(\overline{P})$ is surjective.*

If $\theta \colon Q \to P$ and $\phi \colon P \to R$ are morphisms of integral monoids, then $\phi \circ \theta$ is almost surjective if ϕ and θ are almost surjective. Conversely, if $\phi \circ \theta$ is almost surjective, then ϕ is almost surjective, and if addition $\overline{\phi}$ is injective then θ is also almost surjective.

Proposition 4.2.2 *Let $\theta \colon Q \to P$ be a morphism of integral monoids. If θ is small,* $\mathrm{Spec}\,\theta$ *is injective, and the converse holds if P is finitely generated.*

Proof: If θ is almost surjective and $\mathfrak{p}$ and $\mathfrak{p}'$ are primes of P and $\theta^{-1}(\mathfrak{p}) \subseteq \theta^{-1}(\mathfrak{p}')$, then for any $p \in \mathfrak{p}$, there exist $n \in \mathbf{N}^+$ and q in Q with $np = \theta(q)$. Then $q \in \theta^{-1}(\mathfrak{p}) \subseteq \theta^{-1}(\mathfrak{p}')$, and hence $np = \theta(q) \in \mathfrak{p}'$. Since $\mathfrak{p}'$ is prime, it follows that $p \in \mathfrak{p}'$, so (4) holds. Finally, observe that (2) implies that if $p \in C(P)$ is any $\mathbf{Q}$-indecomposable element of $C(P)$, and $F := \mathbf{Q}^{\geq 0}p$ is the face it generates, then $\theta^{-1}(F)$ contains a nonzero element q. But then $\theta(q) = rp$ for some positive rational number r, and $p = \theta(r^{-1}q)$. Since a finitely generated cone is generated by its $\mathbf{Q}$-indecomposable elements, we see that θ is almost surjective if P is finitely generated. $\square$

Corollary 4.2.3 *Let $\theta\colon Q \to P$ be a morphism of fine saturated monoids. Then $\mathrm{Spec}(\theta)\colon \mathrm{Spec}(P) \to \mathrm{Spec}(Q)$ is a homeomorphism if and only if θ is exact and almost surjective.*

Proposition 4.2.4 *Let $\theta\colon Q \to P$ be a morphism of fine monoids. Then the following are equivalent.*

1. $\mathrm{Spec}(\theta)\colon \mathrm{Spec}(P) \to \mathrm{Spec}(Q)$ *is injective.*
2. *The action of Q on $\overline{P}$ induced by θ makes $\overline{P}$ into a finitely generated Q-set.*
3. $\overline{\theta}\colon \overline{Q} \to \overline{P}$ *is almost surjective.*

Proof: The equivalence of (1) and (2) has already been proved in (4.1.11) above. To prove the equivalence of (2) and (3), we may replace Q and P by $\overline{Q}$ and $\overline{P}$, respectively, so that we may assume that Q and P are sharp. Assume that (2) holds, let S be a finite set of generators for P as a Q-set, and let p be an element of P. Since S generates P as a Q-set, there exist $q_1 \in Q$ and $p_1 \in S$ such that $p = q_1 + p_1$. Similarly, there exist $q_2 \in Q$ and $p_2 \in S$ such that $2p_1 = q_2 + p_2$. Continuing in this way, we construct a sequence $(p_1, p_2, \ldots)$ in S and a sequence $(q_1, q_2, \ldots)$ in Q such that $2p_i = q_i + p_{i+1}$ for all i. Note that $4p_1 = 2q_1 + 2p_2 = 2q_1 + q_2 + p_3$, and in fact for each i and k, there exist $q_{i,k} \in Q$ such that $2^k p_i = q_{i,k} + p_{i+k}$. Since S is finite, there exists $i \in \mathbf{N}$ and $k \in \mathbf{Z}^+$ such that $p_i = p_{i+k}$. Then $2^k p_i = q_{i,k} + p_i$ and so $(2^k - 1)p_i \in Q$. On the other hand, $2^{i-1}p = 2^{i-1}p_1 + 2^{i-1}q_1 = p_i + q_{i-1,1} + 2^{i-1}q_1$ and thus $2^{i-1}(2^k - 1)p \in Q$. Conversely, if $Q \to P$ is almost surjective and

S is a finite set of generators for P as a monoid, then there exists $n \in \mathbf{Z}^+$ such that $ns \in Q$ for every $s \in S$, and the finite set of all is with $0 \leq i < n$ generates P as a Q-set. $\square$

Definition 4.2.5 *A morphism of integral monoids $\theta\colon Q \to P$ is small if* $\mathrm{Cok}(\theta^{gp})$ *is a torsion group.*

Lemma 4.2.6 *Let $\theta\colon Q \to P$ be a morphism of integral monoids.*

1. *If θ is almost surjective, $\overline{\theta}$ is small.*
2. *If θ is small, $\overline{\theta}$ is small.*
3. *If $\overline{\theta}$ is exact and small, then θ is almost surjective.*
4. *If $\overline{\theta}$ is small, then the induced map $\theta^e\colon Q^e \to P$ is almost surjective.*

Proof: Only (3) and (4) require proof. For (3), observe that if $\overline{\theta}$ is small and $p \in P$, there exists $n > 0$ and $q_1, q_2 \in Q$ such that $np = \overline{\theta}(q_1) - \overline{\theta}(q_2)$. If $\overline{\theta}$ is exact, it follows that $q_1 - q_2 \in Q$. Thus θ is almost surjective. Now if $\overline{\theta}$ is small so is $\overline{\theta}^e$, and since $\overline{\theta}^e$ is exact, (4) follows from (3). $\square$

Proposition 4.2.7 *Suppose that $\theta\colon Q \to P$ is a morphism of integral monoids such that either*

1. $\mathrm{Spec}(\theta)$ *is injective, or*
2. *$\overline{\theta}$ is small.*

Then the corresponding map $\theta^e\colon Q^e \to P$ is locally exact (as well as exact).

Proof: In the first case $\mathrm{Spec}(\theta) = \mathrm{Spec}(\theta') \circ \mathrm{Spec}(\theta^e)$, and since $\mathrm{Spec}(\theta)$ is injective, so is $\mathrm{Spec}(\theta^e)$. Then θ^e is locally exact by (4.1.12). In the second case, we apply (4.2.6) to see that $\overline{\theta}^e$ is almost surjective, hence by (**??**) that $\mathrm{Spec}\,\theta^e$ is injective, and again it follows that θ^e is locally exact. $\square$

4.3 Integral actions and morphisms

In this section we study conditions which guarantee that the amalgamated sum of integral monoids again be integral. The conditions which emerge turn out to be related to flatness, and just as flatness is best understood in the context of R-modules, we have found that integrality is best studied in the context of Q-sets.

Let Q be a monoid and let S be a Q-set. We shall say that an element q of Q is *S-regular* if the endomorphism of S induced by the action of q is injective. We say that S is *Q-integral* if every q in Q is S-regular. Of course, this is automatic if Q is a group.

Proposition 4.3.1 *Let Q be a monoid.*

1. *The inclusion functor from the category of Q-integral Q-sets to the category of all Q-sets has a left adjoint $S \mapsto S^{\text{int}} = S/E$, where E is the congruence relation on S consisting of the set E of pairs (s_1, s_2) of elements of S such that there exists some $q \in Q$ with $qs_1 = qs_2$. Furthermore, S^{int} can be identified with the image of the localization map $S \to Q^{-1}S$.*

2. *If S and T are Q-sets, $(S \otimes_Q T)^{\text{int}}$ can be identified with the image of the natural map*
$$S \times T \to Q^{-1}S \otimes_{Q^{gp}} Q^{-1}T.$$

3. *Two elements (s_1, t_1) and (s_2, t_2) of $S \times T$, have the same image in $(S \otimes_Q T)^{\text{int}}$ if and only there exist $q_1, q_2 \in Q$ such that $q_1 s_1 = q_2 s_2$ and $q_2 t_1 = q_1 t_2$.*

Proof: We must first verify that the set E described in (1) really is a congruence relation on S. It is clear that E is symmetric and reflexive. If $qs_1 = qs_2$ and $q's_2 = q's_3$, then it follows from the commutativity of Q that

$$qq's_1 = q'qs_1 = q'qs_2 = qq's_2 = qq's_3,$$

so $(s_1, s_3) \in E$ and E is transitive. Furthermore for any $p \in Q$,

$$qps_1 = pqs_1 = pqs_2 = qps_2$$

so $(ps_1, ps_s) \in E$, and so by the analog of (1.1.2) for Q-sets, E is a congruence relation. Evidently any morphism from Q to a Q-integral S-set factors uniquely through S/E. If s_1 and s_2 are elements of S and if there exists a $q' \in Q$ such that $q's_1 \equiv q's_2 \pmod{E}$, then there exists $q \in Q$ such that $qq's_1 = qq's_2$, and hence $s_1 \equiv s_2 \pmod{E}$. Thus S/E is integral, and $S/E = S^{\text{int}}$. The identification of S/E with the image of S in $Q^{-1}S$ then follows from the explicit construction (1.2) of $Q^{-1}S$.

The action of Q on $Q^{-1}S \otimes_{Q^{gp}} Q^{-1}T$ is integral, so the natural map

$$\alpha\colon S \otimes_Q T \to Q^{-1}S \otimes_{Q^{gp}} Q^{-1}T$$

factors through $(S \otimes_Q T)^{\text{int}}$. In fact there is a commutative diagram:

$$\begin{tikzcd}
S \otimes_Q T \arrow[r] \arrow[d, "\alpha"] & (S \otimes_Q T)^{\text{int}} \arrow[d, "\beta"'] \\
Q^{-1}S \otimes_{Q^{gp}} Q^{-1}T & Q^{-1}(S \otimes_Q T)^{\text{int}}. \arrow[l, "\gamma"']
\end{tikzcd}$$

On the other hand, if $t \in T$, then the map $S \to Q^{-1}(S \otimes_Q T)^{\text{int}}$ sending s to the class of $s \otimes t$ factors through $Q^{-1}S$, and the induced map

$$Q^{-1}S \times T \to Q^{-1}(S \otimes_Q T)^{\text{int}}$$

factors through $Q^{-1}S \otimes_{Q^{gp}} Q^{-1}T$ and is inverse to γ. Thus γ is an isomorphism, and since β is injective, $\gamma \circ \beta$ is injective. Since $S \times T \to (S \otimes_Q T)^{\text{int}}$ is surjective, (2) follows.

For the third statement, recall from (1.1) that $Q^{-1}S \otimes_{Q^{gp}} Q^{-1}T$ is isomorphic to the orbit space of $S \times T$ by the antidiagonal action of Q^{gp}. Thus (s_1, t_1) and (s_2, t_2) have the same image in $Q^{-1}S \otimes_{Q^{gp}} Q^{-1}T$ if and only if there exist q_1 and q_2 in Q such that $(q_1 q_2^{-1})s_1 = s_2$ in $Q^{-1}S$ and $(q_2 q_1^{-1})t_1 = t_2$ in $Q^{-1}T$, *i.e.* if and only if there exist q_1 and q_2 such that $q_1 s_1 = q_2 s_2$ and $q_2 t_1 = q_1 t_2$. □

Corollary 4.3.2 *If $u_1\colon Q \to P_1$ and $u_1\colon Q \to P_2$ are morphisms of integral monoids, then the Q-set underlying the monoid $(P_1 \oplus_Q P_2)^{\text{int}}$ is $(P_1 \otimes_Q P_2)^{\text{int}}$. In particular, $(P_1 \oplus_Q P_2)^{\text{int}}$ is an integral monoid if and only if $(P_1 \otimes_Q P_2)^{\text{int}}$ is a Q-integral Q-set.*

Proof: As we have already observed in (1.1), the Q-set $P_1 \otimes_Q P_2$ has a monoid structure, and in fact $P_1 \oplus_Q P_2 \cong P_1 \otimes_Q P_2$. Dividing by the congruence relation E of (4.3.1), we find a monoid structure on $(P_1 \otimes_Q P_2)^{\text{int}}$, which by (4.3.1.2) is the image of $P_1 \times P_2$ in $Q^{-1}P_1 \otimes_{Q^{gp}} Q^{-1}P_2 \subseteq P_1^{gp} \oplus_{Q^{gp}} P_2^{gp}$. By (1.2.2), the pushout of P_1 and P_2 in the category of integral monoids can be identified with the image of $P_1 \times P_2$ in $P_1^{gp} \oplus_{Q^{gp}} P_2^{gp}$, so

$$(P_1 \oplus_Q P_2)^{\text{int}} \cong (P_1 \otimes_Q P_2)^{\text{int}}.$$

$\square$

Definition 4.3.3 *Let Q be an integral monoid. A Q-set S is said to be universally integral if for every homomorphism of integral monoids $Q \to Q'$, the Q'-set $Q' \otimes_Q S$ is again integral. A homomorphism $\theta: Q \to P$ of integral monoids is said to be universally integral (or just integral) if the corresponding action of Q on P makes P a universally integral Q-set.*

The following corollary, which explains the equivalence of the above definition with the original one due to Kato, is an immediate consequence of (4.3.2).

Corollary 4.3.4 *If $\theta: Q \to P$ is a homomorphism of integral monoids, then the following are equivalent:*

1. *θ is (universally) integral.*
2. *For every homomorphism $Q \to Q'$ of integral monoids, the pushout $Q' \oplus_Q P$ is an integral monoid.*

Proof: If the action of Q on P induced by θ is universally integral, then the action of Q on $Q' \otimes_Q P$ is Q-integral, and hence by (4.3.2), $Q' \oplus_Q P$ is integral as a monoid. The converse follows immediately from the implication (3) implies (1) of (4.3.11). $\square$

We shall see later that an integral and local homomorphism of integral nonoids is exact (4.3.14). On the other hand, an exact morphism of fine monoids need not be integral. For example, in the monoid P with generators $\{x, y, z, w\}$ and relations $x + y = z + w$, the submonoid F generated by x

and z is a face. Hence by (2.1.9) the inclusion $F \to P$ is exact, but it is not integral, since y and w are irreducible.

Our next goal is to make the conditions in (4.3.5) more explicit and to relate them to flatness.

Definition 4.3.5 *We say that a Q-set S satisfies:*

1. *condition I1 if whenever q_1 and q_2 are elements in Q and s_1 and s_2 are elements of S with $q_1 s_1 = q_2 s_2$, then there exist $s \in S$ and $q_1', q_2' \in Q$ such that $s_i = q_i' s$ and $q_1 q_1' = q_2 q_2'$.*

2. *condition I2 if whenever $q_1 s = q_2 s$, there exist $s' \in S$ and $q' \in Q$ with $s = q' s'$ and $q' q_1 = q' q_2$.*

When Q is an integral monoid, the condition I2 is equivalent to saying that whenever q_1 and q_2 are elements of Q and s is an element of S, $q_1 s = q_2 s$ implies $q_1 = q_2$. (We are reluctant to call such an action "free" because it does not imply that S is free as a Q-set, in general.) If Q is integral and S satisfies I1, then it is Q-integral: if $q s_1 = q s_2$, then there exist $s \in S$ and $q_i' \in Q$ such that $s_i = q_i' s$ and $q q_2' = q q_1'$, hence $q_1' = q_2'$ and $s_1 = s_2$.

Remark 4.3.6 Let $\mathcal{T} S$ be the transporter of S (1.1.6). Then S satisfies condition I1 if and only if every pair morphisms with the same target fits into a commutative square:

$$
\begin{tikzcd}
 & s_2 \arrow[d, "q_2"] \\
s_1 \arrow[r, "q_1"] & s'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
s \arrow[r, "q_2'"] \arrow[d, "q_1'"'] & s_2 \arrow[d, "q_2"] \\
s_1 \arrow[r, "q_1"] & s'
\end{tikzcd}
$$

The action satisfies I2 if and only if given any two morphisms $q_1, q_2 \colon s \to s''$, there exists a $q' \colon s' \to s$ with $q_1 \circ q' = q_2 \circ q'$. Note that conditions I1 and I2 are the opposites (duals) of the axioms PS1 and PS2 defining a filtering category [1, I,2.7].

The following proposition can be thought of as an analog for monoids of Lazard's theorem in commutative algebra. Notice first that if S is a Q-set and R is any nonzero ring, that an element q of Q is S-regular if and only if $e(q) \in R[Q]$ is $R[S]$-regular.

Proposition 4.3.7 *Let Q be a monoid and let S be a Q-set. Then the following conditions are equivalent.*

1. *S satisfies I1 and I2.*

2. *S is a direct limit of free Q-sets.*

If Q is integral, (1) and (2) are also equivalent to:

3. *$\mathbf{Z}[S]$ is flat over $\mathbf{Z}[Q]$.*

4. *For every field k, $k[S]$ is flat over $k[Q]$.*

Proof: We begin with a generality.

Lemma 4.3.8 *Let Q be a monoid and let S be a Q-set. For each s in S, let*

$$i_s \colon F(s) := Q \to S$$

denote the unique morphism of Q-sets sending 1 to s, and for each $p \in P$, consider the commutative diagram

$$\begin{tikzcd} F(ps) \arrow[r, "i_{ps}"] \arrow[d, "F(p)"'] & S \\ F(p), \arrow[ur, "i_s"'] & \end{tikzcd}$$

where $F(p)$ is multiplication by p. Then the corresponding map of Q-sets

$$f \colon \varinjlim F \to S$$

is an isomorphism.

Proof: Note that F is a functor from the category $\mathcal{T}S^{op}$ to the category of free Q-sets. It is clear that f is surjective. To see that it is an isomorphism, let $\eta_s \colon F(s) \to \varinjlim F$ be the natural map to the direct limit, and let $g \colon S \to \varinjlim F$ be the map sending s to $\eta_s(1)$. Then if $p \in Q$, $g(ps) = \eta_{ps}(1) = \eta_s(F(p)(1)) = \eta_s(p) = p\eta_s(1) = pg(s)$, so g is a morphism of Q-sets. Since $g \circ i_s = \eta_s$ for all s, $g \circ f = \mathrm{id}$, and it follows that f is injective, hence an isomorphism. $\square$

Let $E \subseteq S \times S$ be the set of pairs (s, s') such that there exists a sequence $(s_0, \ldots s_n)$ in S and a sequence $(q_1, \ldots q_n)$ in Q such that $s_{i-1} = q_i + s_i$ and $s_{i+1} = q_{i+1} + s_i$ for all i. Then E is a congruence relation on S, and the action of Q on the quotient is trivial, so that the equivalence classes are Q-subsets. Let us call these subsets the "connected components of S." If S satisfies I1 (resp I2), then so does each of its connected components. Since S is the disjoint union of its connected components, S will be a direct limit of free Q-sets if each of its connected components is, and so it will suffice to prove that (1) implies (2) if S is connected. Now if S satisfies I1 and I2 and is connected, $\mathcal{T}S^{op}$ is filtering, so the inductive limit $\varinjlim F$ in Lemma 4.3.8is a direct limit, and hence and S is a direct limit of free Q-sets.

Conversely, any free Q-set satisfies I1 and I2, and the direct limit of any family of Q-sets satisfying I1 (resp. I2) again satisfies I1 (resp. I2). If S is a free Q-set, $k[S]$ is a free $k[Q]$-module, and since a direct limit of free modules is flat, (2) implies (3).

Since it is trivial that (3) implies (4), it remains only to prove that if Q is an integral monoid and S is a Q-set such that $k[S]$ is $k[Q]$-flat for every field k, then S satisfies I1 and I2. Let us begin by showing that, when G is a group, condition (3) implies that S satisfies I2, *i.e.* that G acts freely on S. Suppose that $g \in G$, $s \in S$, and $gs = s$. Then $(e^g - 1)e^s = 0$ in the $k[G]$-module $k[S]$, and since $k[S]$ is flat over $k[G]$, we can write $e^s = \sum_i \alpha_i \sigma_i$ where $\alpha_i \in k[G]$ is killed by $e^g - 1$ and $\sigma_i \in k[S]$. But if $\alpha := \sum c_h e^h \in k[G]$, α is annihilated by $e^g - 1$ if and only if $\sum c_h e^{gh} = \sum c_h e^h$, *i.e.* if and only if $c_{g^{-1}h} = c_h$ for all h. This means that α is a linear combination of g-orbits for the regular representation of G on itself; since only finite sums are allowed, either α is zero or g has finite order. Thus if g has infinite order all α_i are zero, so $e^s = 0$, a contradiction. If g has order n, then each α_i is a multiple of $\alpha := \sum_{i=0}^{n-1} e^{g^i}$, and hence we can write $e^s = \alpha\sigma$ for some $\sigma := \sum c_t e^t \in k[S]$. Then $e^s = \sum_{i,t} c_t e^{g^i t} = \sum c'_t e^t$ where $c'_t := \sum_i c_{g^i t}$. Comparing the coefficients of e^s, we find that $1 = c'_s := \sum_i c_{g^i s}$; since $gs = 1$, we find that $1 = nc_s$. Thus n is invertible in k, and since this is true for every field k, $n = 1$ and g is the identity, as required.

Now suppose that Q is any integral monoid and that $k[S]$ is flat for every field k. Let $S \to S'$ be the localization of S by Q, so that the action of Q on S extends to an action of Q^{gp}. Then $k[S'] \cong k[Q^{gp}] \otimes_{k[Q]} k[S]$, and by flatness of $k[S]$, $k[S]$ injects in $k[S']$, and $k[S']$ is flat over $k[Q^{gp}]$. Then as we saw in the previous paragraph, the action of Q^{gp} on S' is free. It follows that the action of Q on S satisfies I2.

It remains to prove that the flatness of $k[S]$ implies that S satisfies I1. First let us check that S is Q-integral. If $q \in Q$ and s_i in S with $qs_1 = qs_2$, then $e^q(e^{s_1} - e^{s_2}) = 0$ in $k[S]$, and since $k[S]$, is flat, $e^{s_1} - e^{s_2} = \sum \alpha_i \sigma_i$ where $\alpha_i \in k[Q]$ is killed by e^q and $\sigma_i \in k[S]$. But if $\alpha = \sum c_p e^p \in k[Q]$ is killed by e^q, then $\sum c_q e^{qp} = 0$, and since Q is integral, each $c_q = 0$, hence $s_1 = s_2$ as required.

Suppose now that s_1 and s_2 are elements of S and q_1 and q_2 are elements of Q with $q_1 s_1 = q_2 s_2$. Let K be the $k[Q]$-module defined by the exact sequence

$$0 \longrightarrow K \longrightarrow k[Q] \oplus k[Q] \xrightarrow{q_1 - q_2} k[Q].$$

Tensoring by $k[S]$ we get by flatness of $k[S]$ an exact sequence

$$0 \longrightarrow K \otimes_{k[Q]} k[S] \longrightarrow k[S] \oplus k[S] \xrightarrow{q_1 - q_2} k[S].$$

Hence we have $(e^{s_1}, e^{s_2}) \in K \otimes_{k[Q]} k[S]$, and we can find elements (α_i, β_i) of K and σ_i of $k[S]$ with $e^{s_1} = \sum \alpha_i \sigma_i$ and $e^{s_2} = \sum \beta_i \sigma_i$. Examining the homogeneous pieces of the first of these equations, we see that for some i there exists q'_1 appearing in α_i and s' appearing in σ_i such that $s_1 = q'_1 s'$. Since $q_1 \alpha_i = q_2 \beta_i$, there exist q'_2 appearing in β_i such that $q_1 q'_1 = q_2 q'_2$. But $q_2 s_2 = q_1 s_1 = q_1 q'_1 s' = q_2 q'_2 s'$, and since S is Q-integral, $s_2 = q'_2 s'$. □

Proposition 4.3.9 *Suppose that Q is an integral monoid and S is a Q-set. Then $k[S]$ is faithfully flat over $k[Q]$ if and only if S satisfies I1 and I2 and in addition $Q^+ S$ is properly contained in S.*

Proof: We begin with the following lemma, which may be of independent interest.

Lemma 4.3.10 *Let Q be an integral monoid, let S be a Q-set satisfying I1, and let $\mathfrak{p}$ be a prime ideal of Q with complementary face F. Then $T := S \setminus \mathfrak{p}S$ is stable under the action of F, and the action of F on T satisfies I1. Let $k[Q] \to k[F]$ be the homomorphism induced by the isomorphism $k[Q]/\mathfrak{p}k[Q] \cong k[F]$ of (3.2.1). Then there is a natural isomorphism of $k[F]$-modules*

$$k[S] \otimes_{k[Q]} k[F] \cong k[T].$$

Proof: Suppose that $s, t \in S$, $f \in F$, and $p \in \mathfrak{p}$ with $ft = ps$. Then by I1 there exist $s' \in S$ and $q_i \in Q$ such that $t = q_1 s'$, $s = q_2 s'$, and $fq_1 = pq_2$. Since $p \in \mathfrak{p}$ and $f \in F$, we conclude that $q_1 \in \mathfrak{p}$. Thus $t \in \mathfrak{p}S$. This shows that in fact T is stable under the action of F. If $t_i \in T$ and $f_i \in F$ with $t_1 f_1 = t_2 f_2$, then there exist $t \in S$ and $q_i \in Q$ with $t_i = q_i t$ and $f_1 q_1 = f_2 q_2$. Since $t_i \in T$, $q_i \in F$ and $t \in T$, so that the F-set T again satisfies I1. For the last statement, observe that $\mathfrak{p}S$ is a k-basis for the $k[Q]$-submodule $k[\mathfrak{p}]k[S]$ of $k[S]$, and hence that T is a basis for the quotient, with the induced action of F. $\square$

If S satisfies I1 and I2, we know that $k[S]$ is flat over $k[Q]$, and for the faithfulness it will suffice to prove that for every field extension k' of k and every k-homomorphism $k[Q] \to k'$, the tensor product $k[S] \otimes_{k[Q]} k'$ is not zero. Such a homomorphism amounts to the choice of a face F of Q and a morphism $F^{gp} \to k'^*$; it then sends the complement $\mathfrak{p}$ of F to zero (3.2.3). If we let $T := S \setminus \mathfrak{p}S$ as in the above lemma, $k[S] \otimes_{k[Q]} k'$ becomes identified with $k[T] \otimes_{k[F]} k'$. By assumption, T is not empty, and consequently $T' := F^{-1}T$ is not empty. Property I2 for S implies property I2 for F acting on T and for F^{gp} acting on T', and hence the action of the group F^{gp} on T' is free. Thus $k[T']$ is a nonzero free $k[F^{gp}]$-module, hence is faithfully flat. It follows that

$$k[T] \otimes_{k[F]} k' \cong k[T'] \otimes_{k[F^{gp}]} k'$$

is nonzero. Conversely, if $k[S]$ is faithfully flat, then $k[S] \otimes_{k[Q]} k[Q^*] \cong k[S \setminus Q^+ S]$ is not zero. $\square$

Proposition 4.3.11 *Let Q be an integral monoid acting on a set S. Then the following conditions are equivalent:*

1. *S satisfies I1.*
2. *For every homomorphism of integral monoids $Q \to Q'$ the action of Q' on $Q' \otimes_Q S$ satisfies I1.*
3. *For every homomorphism of integral monoids $Q \to Q'$, the action of Q' on $Q' \otimes_Q S$ is Q'-integral.*

We begin with a lemma that takes place entirely in the realm of Q-sets.

Lemma 4.3.12 *Let Q be an integral monoid, let T be an integral Q-set, and let S be a Q-set satisfying I1. Then $S \otimes_Q T$ is Q-integral. In particular, if $s_1, s_2 \in S$ and t_1, t_2 in T, then the following are equivalent.*

1. *$s_1 \otimes t_1 = s_2 \otimes t_2$ in $S \otimes_Q T$.*
2. *$(s_1 \otimes t_2)^{\text{int}} = (s_2 \otimes t_2)^{\text{int}}$ in $(S \otimes_Q T)^{\text{int}}$.*
3. *There exist $q_1, q_2 \in Q$ such that $q_1 s_1 = q_2 s_2$ and $q_2 t_1 = q_1 t_2$.*
4. *There exist $s \in S$ and $q'_1, q'_2 \in Q$ such that $s_i = q'_i s$ and $q'_1 t_1 = q'_2 t_2$.*

Proof: It is obvious that (1) implies (2). The equivalence of (2) and (3) has already been proved in (4.3.1). Since S satisfies I1, (3) implies that there exist $s \in S$ and $q'_i \in Q$ such that $s_i = q'_i s$ and $q'_1 q_1 = q'_2 q_2$. Then $q'_2 q_2 t_2 = q'_1 q_1 t_2 = q'_1 q_2 t_1$, and since T is Q-integral, $q'_2 t_2 = q'_1 t_1$. Thus (3) implies (4). Finally, if (4) holds, we have in $S \otimes_Q T$:

$$s_1 \otimes t_1 = (q'_1 s) \otimes t_1 = s \otimes (q'_1 t_1) = s \otimes (q'_2 t_2) = (q'_2 s) \otimes t_2 = s_2 \otimes t_2.$$

This completes the proof that (1) through (4) are equivalent. Now the equivalence of (1) and (2) implies that the natural map $S \otimes_Q T \to (S \otimes_Q T)^{\text{int}}$ is an isomorphism and hence that $S \otimes_Q T$ is Q-integral. $\square$

Proof of (4.3.11) Suppose that $(t_i, s_i) \in Q' \times S$ and $p_i \in Q'$ with $p_1(t_1 \otimes s_1) = p_2(t_2 \otimes s_2)$ in $Q' \otimes_Q S$. Let $t'_i := p_i t_i$, so that $(t'_1 \otimes s_1) = (t'_2 \otimes s_2)$ in $Q' \otimes_Q S$. Then because (1) implies (4) in (4.3.12), there exist $s \in S$ and $q'_i \in Q$ such that $s_i = q'_i s$ in S and $q'_2 t'_2 = q'_1 t'_1$ in Q'. Set $p'_i := q'_i t_i \in Q'$. Then for $i = 1, 2$, $t_i \otimes s_i = p'_i (1 \otimes s)$, and

$$p_1 p'_1 = p_1 q'_1 t_1 = q'_1 t'_1 = q'_2 t'_2 = p_2 p'_2,$$

so $Q' \otimes_Q S$ satisfies I1.

As we have already noted, condition I1 implies Q'-integrality, and consequently (2) implies (3). To prove that (3) implies (1), suppose that (3) holds and that x and y are elements of S and a and b are elements of Q such that $ax = by$. To show that (1) holds we construct a morphism of monoids $Q \to Q'$, a Q-set S', and a Q-morphism $Q' \otimes_Q S \to S'$ as follows. Let E be the subset of $(Q \oplus \mathbf{N}^2) \times (Q \oplus \mathbf{N}^2)$ consisting of those pairs $((c, m, n), (c', m', n'))$ such that $m + n = m' + n'$ and $c a^m b^n = c' a^{m'} b^{n'}$. In fact E is a congruence relation and an exact submonoid of $(Q \oplus \mathbf{N}^2) \times (Q \oplus \mathbf{N}^2)$, so by (2.1.13) the quotient $Q' := (Q \oplus \mathbf{N}^2)/E$ is an integral monoid. Let $[c, m, n]$ denote the class in Q' of an element (c, m, n) of $Q \oplus \mathbf{N}^2$. Let Q act on $S \times \mathbf{N}^2$ via its action on S, and let R be the subset of $(S \times \mathbf{N}^2) \times (S \times \mathbf{N}^2)$ consisting of those pairs $((s, m, n), (s', m', n'))$ such that $m + n = m' + n'$ and such that there exist c,c' in Q and t in S such that $s = ct$, $s' = c't$ and $ca^m b^n = c'a'^{m'} b'^{n'}$. This subset is symmetric, contains the diagonal, and is invariant under the action of Q. It follows from the anlog of (1.1.2 for Q-sets that the congruence relation E' it generates is just the set of pairs (e, f) such that there exists a sequence $(r_0, \cdots r_k)$ with $(r_{i-1}, r_i) \in R$ for $i > 0$ and $r_0 = e$, $r_k = f$. Write $[s, m, n]$ for the class in $S' := (S \times \mathbf{N}^2)/E'$ of (s, m, n). Then the map $Q \oplus \mathbf{N}^2 \times S \to S'$ sending (c, m, n, s) to $[cs, m, n]$ factors through $Q' \times S$, and furthermore the corresponding map $Q' \times S \to S'$ is a Q-bimorphism. Thus there is a map $g \colon Q' \otimes_Q S \to S'$ sending each $[c, m, n] \otimes s$ to $[cs, m, n]$.

It follows from the definition of E and the fact that $ba = ab$ that $p := [b, 1, 0] = [a, 0, 1]$ in Q'. Since $ax = by$ in S, we find that in $Q' \otimes_Q S$,

$$\begin{aligned} p([1,1,0] \otimes x) &= [a,1,1] \otimes x \\ &= [1,1,1] \otimes (ax) \\ &= [1,1,1] \otimes (by)] \\ &= [b,1,1] \otimes y \\ &= p([1,0,1] \otimes y). \end{aligned}$$

Since the action of Q' on $Q' \otimes_Q S$ is Q'-integral, it follows that

$$[1,1,0] \otimes x = [1,0,1] \otimes y \quad \text{in } Q' \otimes_Q S,$$

and hence that $[x, 1, 0] = [y, 0, 1]$ in S'. Then there exists a sequence $r := (r_0, \ldots r_k)$ as above, with $r_i = (s_i, m_i, n_i)$ and $r_0 = (x, 1, 0)$ and $r_k = (y, 0, 1)$. Then for all i, $m_i + n_i = 1$, so that $(m_i, n_i) = (1, 0)$ or $(0, 1)$. Suppose that for some i, $(m_{i-1}, n_{i-1}) = (m_i, n_i)$. Then there exist c, c', t with $s_{i-1} = ct$, $s_i = c't$ and $ca = c'a$ or $cb = c'b$. But then $c = c'$ and hence $s_{i-1} = s_i$, $r_{i-1} = r_i$, and in fact r_i can be omitted from the sequence r. Consequently we may assume that for all i, $m_{i-1} \neq m_i$. Since $m_0 = 1$, it follows that $m_i = 1$ if i is even and $n_i = 1$ if i is odd. If $k \geq 2$ and $i > 0$ is odd, we find that $r_{i-1} = (ct, 1, 0)$, $r_i = (c't, 0, 1) = (dt', 0, 1)$ and $r_{i+1} = (d't', 1, 0)$, with $ca = c'b$ and $db = d'a$. But then $act = c'bt = bdt' = ad't'$, and hence $ct = d't'$ and $r_{i-1} = r_{i+1}$. In this case r_i and r_{i+1} can be omitted from r. Thus we may assume without loss of generality that $k = 1$. Then there exist c, c', t such that $x = ct$, $y = c't$, and $ac = bc'$, as claimed. $\square$

Remark 4.3.13 If $\theta: Q \to P$ is homomorphism of integral monoids, then the corresponding action of Q on P satisfies I2 if and only if θ is injective. Thus, we see that θ is injective and integral if and only if $\mathbf{Z}[P]$ is flat over $\mathbf{Z}[Q]$.

Proposition 4.3.14 *Let $\theta: Q \to P$ be a morphism of fine monoids. Then the following are equivalent.*

1. *θ is integral and local.*
2. *θ is exact, and for every $p \in P$, there exists a $p' \in P$ such that*

$$S_p := (Q^{gp} + p) \cap P = \theta(Q) + p'.$$

In particular, an integral morphism of integral monoids is exact if and only if it is local.

Proof: Suppose that θ is local and integral and $q_1 - q_2 \in Q^{gp}$ is such that $\theta(q_1) - \theta(q_2)$ is an element p of P. Then in P we have $\theta(q_1) + 0 = \theta(q_2) + p$, and since θ is integral there exist $p' \in P$, $q'_i \in Q$ with $0 = \theta(q'_1) + p'$, $p = \theta(q'_2) + p'$, and $q'_1 + q_1 = q'_2 + q_2$. But then p' is a unit of P, and since θ is local q'_1 is a unit of Q. Then $q_1 - q_2 = q'_2 - q'_1 \in Q$, so θ is exact. Then the rest of the implication of (2) by (1) follows from the following lemma.

Lemma 4.3.15 *Let $\theta: Q \to P$ be an exact and injective homomorphims of fine sharp monoids. For each $p \in P$, let $S_p := (Q^{gp} + p) \cap P$. Then the set of all such S_p forms a partion of P, and each S_p is a finitely generated Q-subset of P. If θ is integral, each S_p is free and monogenic as a Q-set.*

Proof: It is clear that S_p is stable under the action of Q on P and that the set of all such sets S_p forms a partition of P. Let $J_p := P - p \subseteq P^{gp}$ be the principal fractional ideal of P^g generated by $-p$ and let K_p be its inverse image in Q^{gp}. Then θ^{gp} followed by translation by p induces an isomorphism of Q-sets $K_p \to S_p$. Since θ is exact, it follows from (2.1.12) that K_p is finitely generated as a Q-set, and hence so is S_p. Now suppose that θ is integral and that s_1 and s_2 are two elements of the set S'_p of minimal generators for S_p. Since Q^{gp} acts transitively on S_p, there exist q_1 and q_2 in

Q such that $q_1 + s_1 = q_2 + s_2$. Since θ is integral, there exist $p' \in P$ and $q'_1, q'_2 \in Q$ such that $s_i = q'_i + p'$. But then $p' \in S$ and $p' \leq s_i$, so by the minimality of s_i we must have $p' = s_i$. Thus S' has just one element, and S is the free Q-set generated by this element □

Now suppose that (2) holds. We already know that any exact morphism of integral monoids is local (4.1.3). Suppose that $p_1, p_2 \in P$ and $q_1, q_2 \in Q$ with $\theta(q_1) + p_1 = \theta(q_2) + p_2$. Then $S_{p_1} = S_{p_2}$, so there exist $p' \in P$ and $q'_1, q'_2 \in Q$ such that $p_i = \theta(q'_i) + p'$. Then $\theta(q'_1 + q_1) + p' = \theta(q'_2 + q_2)$, and hence $\theta^{gp}(q'_1 + q_1 - q'_2 - q_2) = 0$. Since θ is exact, this implies that $u := q'_1 + q_1 - q'_2 - q_2 \in Q^*$. Replacing q'_2 by $q'_2 + u$, we find that $q_1 + q'_1 = q_2 + q'_2$. This shows that θ is integral. □

Corollary 4.3.16 *Let $\theta\colon Q \to P$ be a local homomorphism of fine sharp monoids. Then the following are equivalent.*

1. *θ is integral.*
2. *θ makes P into a free Q-set.*
3. *The homomorphism $\mathbf{Z}[\theta]\colon \mathbf{Z}[P] \to \mathbf{Z}[Q]$ makes $\mathbf{Z}[P]$ a free $\mathbf{Z}[Q]$-module.*
4. *The map $\mathbf{Z}[\theta]\colon \mathbf{Z}[P] \to \mathbf{Z}[Q]$ is flat.*

Proof: If (1) holds, then by (4.3.14) and (4.1.3), θ is exact and injective. Then it is clear from Lemma 4.3.15 that (1) implies (2). The implications of (4) by (3) and (3) by (2) are trivial, and the implication of (1) by (4) was explained in (4.3.7). □

One verifies immediately that the regular representation of an integral monoid Q on Q is universally integral, and that if S is any universally integral Q-set and F is a face of S, then S_F is universally integral as a Q-set.

Proposition 4.3.17 *Let $u\colon Q \to P$ and $v\colon P \to R$ be a morphisms of integral monoids.*

1. *If u and v are integral then $v \circ u$ is integral. If $v \circ u$ is integral and u is surjective, then v is integral, and if v is exact then u is integral.*

2. *The natural maps* $\pi\colon Q \to \overline{Q}$ *and* $P \to \overline{P}$ *are integral, and* u *is integral if and only if* $\overline{u}$ *is integral.*

3. *If* Q *is valuative (1.2), for example if* $Q \cong \mathbf{N}$*, then* u *is integral.*

4. *If* u *is local, sharp, and integral, then it is injective. In particular, if* u *is local and integral,* $\overline{u}$ *is injective.*

5. *If either* Q *or* P *is a group, then* u *is integral.*

Proof: The proof of the first part of (1) follows either from direct calculation or (more quickly) from the fact that the composition of two cocartesian squares is cocartesian and (4.3.4). Suppose that $v \circ u$ is integral. It is obvious that if u is surjective, then v is integral. Suppose that v is exact and that $p_1, p_2 \in P$ and $q_1, q_2 \in Q$ with $p_1 + u(q_1) = p_2 + u(q_2)$. Then $v(p_1) + v(u(q_1)) = v(p_2) + v(u(q_2))$, and since $u \circ v$ is integral there exist $r \in R$ and $q'_1, q'_2 \in Q$ with $q_1 + q'_1 = q_2 + q'_2$ and $v(p_i) = r + v(u(q'_i))$. Then $v(p_i - u(q'_i)) = r \in R$, and since v is exact, we see that $p_i - u(q'_i) \in P$. In fact,

$$p := p_1 - u(q'_1) = p_1 + u(q_1) - u(q_2) - u(q'_2) = p_2 - u(q'_2).$$

It follows that $p_i = p + u(q'_i)$ in P, and since $q_1 + q'_1 = q_2 + q'_2$, that u is integral.

The first part of (2) is an immediate verification. For the second part, observe that in the diagram

$$\begin{tikzcd} Q \arrow[r, "u"] \arrow[d] & P \arrow[d] \\ \overline{Q} \arrow[r, "\overline{u}"] & \overline{P} \end{tikzcd}$$

the vertical arrows are integral and exactd and apply (1).

For (3), suppose that $q_1, q_2 \in Q$ and $p_1, p_2 \in P$ with $u(q_1)+p_1 = u(q_2)+p_2$. Since Q is valuative, $q_1 - q_2 \in Q$ or $q_2 - q_1 \in Q$, say without loss of generality that $q_2 = q + q_1$. Then $u(q_1) + p_1 = u(q) + u(q_1) + p_2$, and since P is integral $p_1 = u(q) + p_2$. Set $p = p_2$, $q'_1 = q$ and $q'_2 = 0$, so that $p_i = u(q'_i) + p$ and $q'_1 + q_1 = q'_2 + q_2$. This shows that u is integral.

If u is local and integral, it is exact by (4.3.14), and if it is sharp it is then injective by (4.1.3). If u is local and integral, then $\overline{u}$ is integral, local, and sharp, hence injective. This completes the proof of (4), and statement (5) follows from (2) and the trivial case in which either P or Q is 0. $\square$

Corollary 4.3.18 *If P is an integral monoid and Q is a submonoid of P, then the localization map $P \to Q^{-1}P$ and the quotient morphism $P \to P/Q$ are integral.*

Proof: In fact, $Q \to Q^{gp}$ and $Q \to 0$ are integral by (4.3.17), and hence so are the corresponding pushouts by $Q \to P$. $\square$

Proposition 4.3.19 *Let $\theta\colon Q \to P$ be a morphism of integral monoids, let Q^{loc} be the localization of Q by $\theta^{-1}(P^*)$, and let $\theta^{loc}\colon Q^{loc} \to P$ be the map induced by θ. Then θ is integral if and only if θ^{loc} is.*

Proof: Corollary (4.3.18) says that the localization map $Q \to Q^{loc}$ is integral. Since the composition of integral morphisms is integral, it follows that if $Q^{loc} \to P$ is integral, then so is $Q \to P$. Conversely, suppose $Q \to P$ is integral and let $Q^{loc} \to Q'$ be any morphism of integral monoids. It follows from the universal mapping properties of pushouts and localizations that the natural map $Q' \oplus_Q P \to Q' \oplus_{Q^{loc}} P$ is an isomorphism. Since $Q \to P$ is integral, $Q' \oplus_Q P$ is integral, and hence so is $Q' \oplus_{Q^{loc}} P$. $\square$

Corollary 4.3.20 *Let $\theta\colon Q \to P$ be a morphism of integral monoids. Then θ is integral if and only if for every face F of P, the localization $Q_{\theta^{-1}}(F) \to P_F$ is integral.*

Proof: Suppose θ is integral and F is a face of P. By the previous result, $P \to P_F$ is integral, and hence so is $Q \to P_F$. Then it follows from (4.3.19) that $Q_{\theta^{-1}(F)} \to P_F$ is integral. Suppose conversely that each such localization is integral. Then in particular $Q_{\theta^{-1}(P^*)} \to P$ is integral, and hence by (4.3.19) θ is integral. $\square$

Theorem 4.3.21 *Let $\theta\colon Q \to P$ be a morphism of fine saturated monoids. Then the following conditions are equivalent.*

1. $\operatorname{Spec}\theta$ *is locally surjective.*

2. θ *is locally exact.*

3. $C_{\mathbf{Q}}(\theta)\colon C_{\mathbf{Q}}(Q) \to C_{\mathbf{Q}}(P)$ *is integral.*

Proof: Suppose $C_{\mathbf{Q}}(\theta)$ is integral, and let F be any face of P. Then by (4.3.20) the map $Q_{\theta^{-1}(F)} \to P_F$ is again integral. It is local by construction, and so it follows from (4.3.14) that it is also exact. Thus $C_{\mathbf{Q}}(\theta)$ is locally exact, and hence locally surjective. Since the maps $Q \to C_{\mathbf{Q}}(Q)$ and $P \to C_{\mathbf{Q}}(P)$ induce homeomorphisms on the associated topological spaces, compatible with the maps induced by θ, it follows that $\operatorname{Spec}(\theta)$ is locally surjective. This proves that (3) implies (1). The implication of (2) by (1) was proved in (4.1.8).

It remains to prove that (2) implies (3). We may assume that Q and P are sharp, by (2) of (4.3.17). Then θ is injective, by (4.1.3), and to simplify the notation we shall identify Q with its image in P. Suppose q_1 and q_2 are elements of Q and p_1 and p_2 are elements of P such that $\theta(q_1)+p_1 = \theta(q_2)+p_2$. We shall show that there exist $p \in C(P)$ and $q'_i \in C(Q)$ with $p_i = q'_i + p$ and $q_1 + q'_1 = q_2 + q'_2$. Let L be the subgroup of P^{gp} generated by the image of Q and p_1 and let $P' := L \cap P$. Evidently $p_i \in P'$, and P' is an exact submonoid of P. Hence P' is again finitely generated by (2.1.9). Furthermore, since $P' \to P$ is exact, the map $\operatorname{Spec} P \to \operatorname{Spec} P'$ is surjective, and since the map $\operatorname{Spec} P \to \operatorname{Spec} Q$ is locally surjective, it follows that the map $\operatorname{Spec} P' \to \operatorname{Spec} Q$ is also locally surjective. Since it will suffice to find the desired p in $C(P')$, we may replace P by P'. Thus we may assume that the group P^{gp}/Q^{gp} is generated by p_1. Note that if $p_1 \in C(Q)^{gp}$, then in fact $C(Q)^{gp} = C(P)^{gp}$ and since $C(P) \to C(Q)$ is exact, $C(P) = C(Q)$ and there is nothing to prove. Thus we may assume that $C(P)^{gp}/C(Q)^{gp}$ has dimension one.

Claim 4.3.22 *For each indecomposable element a of $C(P)$ which does not lie in $C(Q)$, there exist unique $r(a) \in \mathbf{Q}$ and $q_1(a), q_2(a) \in C(Q)$ such that $p_i = q_i(a) + r(a)a$ and $q_1(a) + q_1 = q_2(a) + q_2$.*

To prove this claim, let a be an indecomposable eleement of $C(P)$ which does not belong to $C(Q)$. Note that $a \notin C(Q)^{gp}$, because otherwise it would belong to $C(Q)$, since $C(Q)$ is exact in $C(P)$. Since a is indecomposable, $\langle a\rangle$ is one-dimensional, and hence $\langle a\rangle \cap C(Q)^{gp} = \{0\}$ and it follows that the natural map $\langle a\rangle^{gp} \oplus C(Q)^{gp} \cong C(P)^{gp}$ is an isomorphism. Moreover, since $Q \cap \langle a\rangle = \{0\}$, the map $Q \to P_{\langle a\rangle}$ is still local and hence exact, since $Q \to P$ is locally exact. Then the map $C(Q) \to C(P/\langle a\rangle)$ is an isomorphism, since it is exact and injective and the induced map on groups is bijective. In particular, there exist $q_1(a), q_2(a) \in C(Q)$ such that $q_i(a)$ and p_i have the same image in $C(P/\langle a\rangle)$. Since $\langle a\rangle$ is one-dimensional, this means that $p_i = q_i(a) + r_i a$ for some $r_i \in \mathbf{Q}$. Then

$$q_1 + q_1(a) + r_1 a = q_1 + p_1 = q_2 + p_2 = q_2 + q_2(a) + r_2 a,$$

so that $(r_1 - r_2)a \in C(Q)^{gp}$. Since $a \notin C(Q)^{gp}$, it follows that $r_1 = r_2$ and $q_1(a) + q_1 = q_2(a) + q_2$. This completes the proof of the claim.

Every element of $C(P)$ can be written as a sum of indecomposable elements, by (2.3.2). In particular, write $p_1 = \sum_i a_i + \sum_i b_i$, where a_i and b_i are indecomposable and $a_i \notin C(Q), b_i \in C(Q)$. For each i, write $p_1 = q_1(a_i) + r(a_i)a_i$ as above. Since $p_1 \notin C(Q)$, $r(a_i) \neq 0$, and we can also write $a_i = r(a_i)^{-1}(p_1 - q_1(a_i))$. Hence

$$p_1 = \sum a_i + \sum b_i = \sum_i r(a_i)^{-1} p_1 - \sum_i r(a_i)^{-1} q_1(a_i) + \sum_i b_i.$$

Since $p_1 \notin C(Q)^{gp}$, it follows that $\sum_i r(a_i)^{-1} = 1$ and hence that for some i, $r(a_i) > 0$. Then $p_i = q_i(a_i) + r(a_i)a_i$, so we can set $p := ra$ and $q'_i := q(a)$, and the proof is complete. □

Corollary 4.3.23 *Let $\theta \colon Q \to P$ be a morphism of fine monoids, where Q is free and P is saturated. Then θ is integral if and only if $C(\theta) \colon C_{\mathbf{Q}}(Q) \to C_{\mathbf{Q}}(P)$ is integral. In particular, this is the case if and only if θ is locally exact.*

Proof: [2] Suppowse that $C(\theta)$ is integral and that $q_1, q_2 \in Q$ and $p_1, p_2 \in P$ with $\theta(q_1) + p_1 = \theta(q_2) + p_2$. Since $C(\theta)$ is integral, so there exist $a \in C(P)$

[2]This proof is due to Aaron Gray.

and $b_i \in C(Q)$ with $b_1 + q_1 = b_2 + q_2$ and $p_i = \theta(b_i) + a$. Choose a positive integer n such that $q'_i := nb_i \in Q$ and $p := na \in P$. Then $q'_1 + nq_1 = q'_2 + nq_2$. It follows that for every $\phi\colon Q \to \mathbf{N}$, $\phi(q'_1) \equiv \phi(q'_1) \pmod{n}$. Let $(e_1, \dots e_r)$ be a basis for Q and let $(\phi_1, \dots \phi_r)$ be the dual basis for $H(Q)$. For each i, write $\phi_i(q'_1) = nm_i + r_i$ with $m_i, r_i \in \mathbf{N}$ and $r_i < n$, and let $r := \sum r_i e_i$ and $q''_1 = \sum m_i e_i$ in Q. Then $q'_1 = nq''_1 + r$ Since $\phi_i(q'_1) \equiv \phi_i(q''_i) \pmod{n}$, we can also write $q'_2 = nq''_2 + r$ with $q''_2 \in Q$. Then $nq''_1 + r + nq_1 = nq''_2 + r + nq_2$, and hence $q''_1 + q_1 = q''_2 + q_2$. Now let $x_i := p_i - \theta(q''_i) \in P^{gp}$. Note that

$$x_1 + \theta(q_1) + \theta(q''_1) = p_1 + \theta(q_1) = p_2 + \theta(q_2) = x_2 + \theta(q_1) + \theta(q''_2),$$

and hence $x_1 = x_2$. Furthermore,

$$nx_1 = np_1 - n\theta(q''_1) = np_1 - \theta(q'_1 - r) = p + \theta(r) \in P$$

Since p is saturated, $p := x_1 = x_2 \in P$. Since $p_i = q''_i + p$ and $q_1 + q''_1 = q_2 + p''_2$, θ is integral. □

4.4 Saturated morphisms

This section has not yet been written, or even understood.

Chapter II

Log structures and charts

1 Log structures and log schemes

Although the concepts of logarithmic geometry apply potentially to a wide range of situations, we shall not attempt to develop a language to carry this out in great generality here. We restrict ourselves to the case of algebraic geometry using the language of schemes, leaving to the future the task of building a foundation for logarithmic algebraic spaces, algebraic stacks, analytic varieties, *etc.* It is often very convenient to work with with logarithmic structures in the étale topology, and we shall do allow ourselves to do so here. In particular, if X is a scheme and x is a scheme-theoretic point, we shall write $\overline{x} \to X$ for a geometric point lying over x, *i.e.*, a separably closed field extension of the residue field $k(x)$. The stalk of $\mathcal{O}_X$ at such a point $\overline{x}$ is a Henselization of $\mathcal{O}_{X,x}$, with residue field the separable closure of $k(x)$ in $k(\overline{x})$. We refer the reader to Chapter I of [5] for an introduction to the étale topology.

1.1 Logarithmic structures

Let $(X, \mathcal{O}_X)$ be a scheme, and let $\mathbf{Mon}_X$ denote the category of sheaves of (commutative) monoids on $X_{\acute{e}t}$.

Definition 1.1.1 *A prelogarithmic structure on* $X := (X, \mathcal{O}_X)$ *is a homomorphism of sheaves of monoids* $\alpha\colon P \to (\mathcal{O}_X, \cdot, 1)$ *on* $X_{\acute{e}t}$. *A logarithmic structure is a prelogarithmic structure such that the induced map* $\alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$ *is an isomorphism.*

A *morphism* of prelogarithmic or logarithmic structures is a commutative diagram

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d] & \mathcal{O}_X \arrow[d, "\mathrm{id}"] \\ Q \arrow[r, "\beta"] & \mathcal{O}_X. \end{tikzcd}$$

To save space and time, one often writes "log" instead of "logarithmic."

Note that the addition law in the sheaf of rings $\mathcal{O}_X$ is not used in the definition of (pre)log structures. Thus it will make sense to speak of a *logarithmic morphism of sheaves of monoids*, as follows.

Definition 1.1.2 *A homomorphism of sheaves of monoids $\theta\colon Q \to P$ is:*

1. *local if the induced map $Q^* \to \theta^{-1}(P^*)$ is an isomorphism,*

2. *sharp if the induced map $Q^* \to P^*$ is an isomorphism,*

3. *logarithmic if the induced map $\theta^{-1}(P^*) \to P^*$ is an isomorphism.*

Note that each of the above conditions can be checked on the stalks.

Proposition 1.1.3 *Let $\theta\colon Q \to P$ be a homomorphism of sheaves of monoids. Then the following conditions are equivalent:*

1. *θ is sharp and local.*

2. *θ is logarithmic.*

3. *$\theta^*\colon Q^* \to P^*$ is surjective and $\theta^{-1}(0) = 0$.*

4. *$\theta^{-1}(0) = 0$ and $P^* \subseteq Q$, i.e., the inclusion $P^* \to P$ factors through θ.*

A homomorphism $\theta\colon Q \to P$ satisfying these conditions is called a logarithmic structure over *P.*

Proof: It is enough to check the equivalence on the stalks, so we may assume that X is a point. If θ is local, $\theta^{-1}(P^*) = Q^*$, and if θ is also sharp, it induces an isomorphism $Q^* \to P^*$, so (2) holds. If (2) holds, then $\theta^{-1}(P^*)$ is a subgroup of Q containing Q^*, hence equal to Q^*, and it follows that (3) holds. If (3) holds then θ^* is a surjective group homomorphism whose kernel is zero, and hence it induces an isomorphism $Q^* \to P^*$, so (4) holds. Finally, if (4) is true, let q be an element of Q with $\theta(q) \in P^*$. Then there exists a $p' \in P^*$ with $p' + \theta(q) = 0$ and by assumption a $q' \in Q$ with $\theta(q') = p'$. Then $\theta(q + q') = 0$, hence $q + q' = 0$, so $q \in Q^*$ and θ is local. Since $\mathrm{Ker}(\theta^*)$ is zero, θ^* is injective. The assumption also implies that θ^* is surjective, hence an isomorphism, *i.e.*, θ is also sharp. $\square$

The category of log structures on X has an initial element, called the *trivial log structure*: the inclusion $\mathcal{O}_X^* \to \mathcal{O}_X$. It also has a final element: the identity map $\mathcal{O}_X \to \mathcal{O}_X$ (which is rarely used).

A *log scheme* is a scheme X endowed with a log structure $\alpha_X \colon M_X \to \mathcal{O}_X$ on its small étale topos $X_{ét}$. Sometimes it is convenient to work with the Zariski, fppf, fpqc, or other topologies in place of the étale topology. A *morphism of log schemes* is a morphism $f \colon X \to Y$ of the underlying schemes together with a morphism $f^\flat \colon M_Y \to f_*(M_X)$ such that the diagram

$$\begin{tikzcd} M_Y \arrow[r, "f^\flat"] \arrow[d, "\alpha_Y"'] & f_*(M_X) \arrow[d, "f_*(\alpha_X)"] \\ \mathcal{O}_Y \arrow[r, "f^\sharp"] & f_*(\mathcal{O}_X) \end{tikzcd}$$

If X is a log scheme, α_X induces an isomorphism $M_X^* \to \mathcal{O}_X^*$, and it is common practice to identify $\mathcal{O}_X^*$ and M_X^*. Doing so requires requires the use of multiplicative notation for the monoid law on M_X. When using additive notation for M_X, we shall write λ_X for the mapping $\mathcal{O}_X^* \to M_X$ induced by the inverse of α_X. Then $\lambda_X(uv) = \lambda_X(u) + \lambda(v)$, and $\lambda(u)$ can be thought of as the logarithm of the invertible function u. For any section f of $\mathcal{O}_X$, $\alpha_X^{-1}(f)$ is then the (possibly empty) set of logarithms of the function f.

Corollary 1.1.4 *If $(M, \alpha) \to (N, \beta)$ is a morphism of log structures, then the underlying homomorphism $\theta \colon M \to N$ is sharp and local. If $f \colon (X, M_X) \to$*

(Y, M_Y) is a morphism of log schemes, then the induced homomorphism $f^{-1}M_Y \to M_X$ is local.

Proof: A morphism $(M, \alpha) \to (N, \beta)$ is a commutative diagram

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\theta"'] & \mathcal{O}_X \arrow[d, "\mathrm{id}"] \\ N \arrow[r, "\beta"] & \mathcal{O}_X \end{tikzcd}$$

In this diagram α and β are sharp and local, and it follows that the same is true of θ. If f is a morphism of log schemes and x is a point (or geometric point) of X and $y = f(x)$, then the induced homomorphism $(f^{-1}M_Y)_x \to M_{X,x}$ can be identified with the map $M_{Y,y} \to M_{X,x}$, which fits into the commutative diagram:

$$\begin{tikzcd} M_{Y,y} \arrow[r] \arrow[d] & \mathcal{O}_{Y,y} \arrow[d] \\ M_{X,x} \arrow[r] & \mathcal{O}_{X,x} \end{tikzcd}$$

Since f is a morphism of locally ringed spaces, the map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is local, and since $M_Y \to \mathcal{O}_Y$ is a log structure, the map $M_{Y,y} \to \mathcal{O}_{X,x}$ is also local. It follows that the map $M_{Y,y} \to M_{X,x}$ is local. $\square$

Proposition 1.1.5 *Let X be a scheme. The inclusion functor from the category of log structures to the category of prelog structures on X admits a left adjoint $(Q, \beta) \mapsto (Q^\beta, \beta^a)$, where Q^β is the amalgamated sum of Q and $\mathcal{O}_X^*$ along $\beta^{-1}(\mathcal{O}_X^*)$ and $\beta^a \colon Q^\beta \to \mathcal{O}_X$ is the morphism defined by β and the inclusion of $\mathcal{O}_X^*$ in $\mathcal{O}_X$.*

Proof: The construction makes no use of the addition law on $\mathcal{O}_X$, so we consider an arbitrary morphism $\beta \colon Q \to P$ of sheaves of monoids. Form the pushout

$$\begin{tikzcd}
\beta^{-1}(P^*) \arrow[r] \arrow[d] & P^* \arrow[d, "i"] \arrow[dr] & \\
Q \arrow[r, "\gamma"] & Q^\beta \arrow[r, "\beta^a"] & P
\end{tikzcd} \tag{II.1}$$

Let us verify that the map $\beta^a\colon Q^\beta \to P$ is a log structure over P. Since $P^* \subseteq Q$, (1.1.3) shows that it will suffice to check that ${\beta^a}^{-1}(0) = \{0\}$. If $\tilde{q} \in Q^\beta$ then locally there exist $q \in Q$, $u \in P^*$ such that $\tilde{q} = \gamma(q)+i(u)$, and if $\beta^a(\tilde{q}) = 0$, then $\beta(q)+u = 0$. In this case $q \in \beta^{-1}(P^*)$ and $i\beta(q) = \gamma(q) \in Q^\beta$. Then $\tilde{q} = i\beta(q) + i(u) = i(\beta(q) + u) = 0$. Furthermore, note that the factorization $\beta = \beta^a \circ \gamma$ is universal: give any other factorization $\beta = \beta' \circ \gamma'$ with β' a log structure, there is a unique morphism $v\colon Q^\beta \to Q'$ such that the diagram

$$\begin{tikzcd}
Q \arrow[r, "\gamma"] \arrow[dr, "\gamma'"'] & Q^\beta \arrow[r, "\beta^a"] \arrow[d, "v"] & P \\
& Q' \arrow[ur, "\beta'"'] &
\end{tikzcd}$$

commutes. □

One calls β^a the *log structure associated to* β. If there is no danger of confusion we write Q^a instead of Q^β.

Remark 1.1.6 Formation of the log structure $P^\beta \to \mathcal{O}_X$ associated to a prelog structure $\beta\colon P \to \mathcal{O}_X$ involves a pushout in the category of sheaves of monoids: this is the sheaf associated to the presheaf which sends each open set to the pushout in the category of monoids. We shall see later that, if Q is integral, then this sheafification yields the same result when carried out in the Zariski or the étale topology. More precisely, let $Q^\beta_{\acute{e}t}$ denote the log structure on $X_{\acute{e}t}$ associated to β and for each étale $f\colon X' \to X$, let $Q^\beta_{X'}$ denote the log structure on X_{zar} associated to $Q \to f^{-1}(\mathcal{O}_X) \to \mathcal{O}_{X'}$. Then in fact $Q^\beta_{X'} = Q^\beta_{\acute{e}t}$ as sheaves on Z'_{zar}. This follows from the fact that $X' \mapsto Q^\beta_{X'}(X')$ defines a sheaf on $X_{\acute{e}t}$, as we shall see in (1.2.11).

Remark 1.1.7 Since one of the corners of the pushout square in (II.1) is a group, the computation of Q^β is relatively easy: Proposition (I,1.1.4) shows that it is the quotient of $P^* \oplus Q$ in the category of sheaves of monoids by the equivalence relation which identifies (u, q) with (u', q') if and only if locally there exist sections v and v' of $\beta^{-1}(P^*)$ such that $u + \beta(v') = u' + \beta(v)$ and $v + q = v' + q'$. This construction is especially simple if θ is local.

It is sometimes helpful to construct the log structure θ^a associated to a morphism $\theta\colon Q \to P$ in two steps: first localize, then sharpen. Thus, if $\theta\colon M \to N$ is a homomorphism of sheaves of monoids, let M^{loc} be the sheaf associated to the presheaf which assigns to each U the localization of $M(U)$ by $\theta^{-1}(N^*(U))$ (I,1.3). Then $M \to N$ factors as

$$M \xrightarrow{\lambda} M^{loc} \xrightarrow{\theta^{loc}} N,$$

and this factorization is the universal factorization of M through a local homomorphism of sheaves of monoids. We call θ^{loc} the *localization* of θ. It can also be viewed as a pushout:

$$\begin{tikzcd}
\theta^{-1}(N^*) \arrow[r] \arrow[d] & M \arrow[d] \\
\theta^{-1}(N^*)^{gp} \arrow[r] & M^{loc}
\end{tikzcd}$$

Similarly, if $\theta\colon Q \to P$ is a morphism of sheaves of monoids, consider the pushout diagram

$$\begin{tikzcd}
Q^* \arrow[r, "\theta^*"] \arrow[d] & P^* \arrow[d] \arrow[dr] & \\
Q \arrow[r, "\sigma"] & Q^{sh} \arrow[r, "\theta^{sh}"] & P
\end{tikzcd} \tag{II.2}$$

Then θ^{sh} is sharp, and the factorization $\theta = \theta^{sh} \circ \sigma$ is the universal factorization of θ through a sharp morphism. In this construction Q^{sh} is just the orbit space of the natural action of Q^* on $P^* \oplus Q$, and the natural map $\overline{Q} \to \overline{Q}^{sh}$ is an isomorphism. In particular $Q \to Q^{sh}$ is local, and $Q^{sh} \to P$ is local if and only if $Q \to P$ is local. If we start with any map $Q \to P$, then the map $(Q^{loc})^{sh} \to P$ is sharp and local, hence by (1.1.3) a log structure, and it follows from the universal mapping properties of these constructions that there is a unique isomorphism $(Q^{loc})^{sh} \to Q^a$ making the diagram

$$\begin{tikzcd}
& (Q^{loc})^{sh} \arrow[dr, "(\theta^{loc})^{sh}"] \arrow[dd] & \\
Q \arrow[ur] \arrow[rr, "\theta"] \arrow[dr] & & P \\
& Q^a \arrow[ur, "\theta^a"'] &
\end{tikzcd}$$

commute. We sometimes refer to θ^a as the *sharp localization of* θ instead of the log structure associated to θ.

Definition 1.1.8 *A log ring is a homomorphism* β *from a monoid* P *to the multiplicative monoid of a ring* A*. If* $P \to A$ *is a log ring,* $\mathrm{Spec}(P \to A)$ *is the log scheme whose underlying scheme is* $X := \mathrm{Spec}\, A$ *with the log structure associated to the prelog structure* $P \to \mathcal{O}_X$ *induced by the map* $P \to A$*. In particular,* $\mathsf{A_P} := \mathrm{Spec}(e_P \colon P \to \mathbf{Z}[P])$.

Let P be a monoid and let $\alpha_P \colon M_P \to \mathcal{O}_{\mathsf{A_P}}$ the log structure of $\mathsf{A_P}$. The construction of M_P shows that there is a natural homomorphism

$$e_P \colon P \to \Gamma(\mathsf{A_P}, M_P).$$

We omit the proof of the following proposition.

Proposition 1.1.9 *Let* T *be a log scheme and* P *a monoid. For each morphism* $f \colon T \to \mathsf{A_P}$ *of log schemes, consider the composition*

$$e_f \colon P \to \Gamma(\mathsf{A_P}, M_P) \xrightarrow{f^\flat} \Gamma(T, M_T).$$

Then $f \mapsto e_f$ *defines a bijection*

$$\mathrm{Mor}(T, \mathsf{A_P}) \xrightarrow{\sim} \mathrm{Hom}(P, \Gamma(T, M_T)).$$

Corollary 1.1.10 *Let T be a scheme with trivial log structure and let P be a monoid. Then every morphism of log schemes $T \to \mathsf{A}_\mathsf{P}$ factors uniquely through $\underline{\mathsf{A}}^*_\mathsf{P} \to \mathsf{A}_\mathsf{P}$, and in fact*

$$\mathsf{A}_\mathsf{P}(T) \cong \mathsf{A}_{\mathsf{P}^{\mathsf{gp}}}(T) \cong \underline{\mathsf{A}}^*_\mathsf{P}(T).$$

If P is fine, $\underline{\mathsf{A}}^*_\mathsf{P}$ is the largest open subscheme of A_P on which the log structure is trivial, and the corollary says that the set of T-valued points of A_P is the same as the set of T-valued points of A^*_P.

Proposition 1.1.11 *Let $\beta\colon Q \to P$ be a morphism of sheaves of monoids on X and $\beta^a\colon Q^\beta \to P$ be its sharp localization. Then:*

1. *The map $Q \to \overline{Q^\beta}$ factors through an isomorphism*

   $$Q/\beta^{-1}(P^*) \to \overline{Q^\beta}.$$

   In particular, the map $\overline{Q} \to \overline{Q^\beta}$ is surjective, and if β is local it is an isomorphism.

2. *Q^β is integral (resp. saturated) if Q is, and conversely if β is local and Q is quasi-integral.*

Proof: It suffices to check the stalks. The first statement follows from the construction on Q^β as the sharp localization of Q by β. If β is local, then $Q^\beta \cong Q^{sh}$, so $\overline{Q} \to \overline{Q^\beta}$ is an isomorphism. If Q is integral then by (I,1.2.2), so is Q^β. If Q is saturated, then so is its localization Q^{loc} with respect to β. Since $\overline{Q^\beta} \cong \overline{Q^{loc}}$ and an integral M monoid is saturated if and only if $\overline{M}$ is, it follows that Q^β is saturated. Conversely, if β is local, then $\overline{Q} \cong \overline{Q^\beta}$. Then if Q is quasi-integral and Q^β is integral, Q is integral by (I,1.2.1), and is saturated if Q^β is. $\square$

A warning: If Q is quasi-integral, it does not follows that Q^β is also quasi-integral, since localization can destroy quasi-integrality, as we saw in (I 1.3.5).

Corollary 1.1.12 *Let $\theta\colon Q \to M$ be a morphism of sheaves of quasi-integral monoids whose sharp localization $Q^\theta \to M$ is an isomorphism. Then the following are equivalent:*

1. *$\overline{\theta}\colon \overline{Q} \to \overline{M}$ is an isomorphism.*

2. *$\theta\colon Q \to M$ is exact.*

3. *$\theta\colon Q \to M$ is local.*

Proof: If $\overline{\theta}$ is an isomorphism, then θ is exact by (I,4.1.3). If θ is exact, then it is local by (I,4.1.3). If θ is local, then by (1.1.11) the map $\overline{Q} \to \overline{Q^\theta}$ is an isomorphism. By assumption the map $Q^\theta \to M$ is an isomorphism, hence so is $\overline{Q^\theta} \to \overline{M}$, and hence also $\overline{Q} \to \overline{M}$. $\square$

1.2 Direct and inverse images

If $f\colon X \to Y$ is a morphism of schemes and $\alpha_X\colon M_X \to \mathcal{O}_X$ is a log (resp. pre-log) structure on X, then the natural map β in the diagram below

$$\begin{tikzcd} f_*M_X \times_{f_*\mathcal{O}_X} \mathcal{O}_Y \arrow[r, "\beta"] \arrow[d] & \mathcal{O}_Y \arrow[d] \\ f_*M_X \arrow[r, "f_*\alpha_X"'] & f_*\mathcal{O}_X \end{tikzcd}$$

is a log (resp. prelog) structure on Y, called the *direct image structure* induced by α_X, which we denote by

$$f_*^{log}(\alpha_X)\colon f_*^{log}(M_X) \to \mathcal{O}_Y.$$

There is a morphism of prelog schemes $(X, \alpha_X) \to (Y, f_*^{log}(\alpha_X))$, and in fact $f_*^{log}(\alpha_X)$ is the final object in the category of log structures on Y for which such a morphism exists.

If $\alpha_Y\colon M_Y \to \mathcal{O}_Y$ is a log structure on Y, then the composite

$$f^{-1}(M_Y) \xrightarrow{f^{-1}(\alpha_Y)} f^{-1}(\mathcal{O}_Y) \longrightarrow \mathcal{O}_X$$

is a prelog structure on X; the associated log structure (1.1.5) will be denoted by

$$f^*(\alpha_Y)\colon f^*M_Y \to \mathcal{O}_X$$

and called the *inverse image of* α_Y or the *log structure induced by* α_Y. If X and Y are log schemes, it follows from the definitions that there are natural isomorphisms

$$\mathrm{Hom}(\alpha_Y, f_*^{log}\alpha_X) \cong \mathrm{Hom}(f^{-1}\alpha_Y, \alpha_X) \cong \mathrm{Hom}(f^*\alpha_Y, \alpha_X).$$

In particular, if $f\colon X \to Y$ is a morphism of log schemes, the corresponding homomorphism of sheaves of monoids $f^{-1}M_Y \to M_X$ factors canonically through $f^*M_Y \to M_X$.

Remark 1.2.1 If $f\colon X \to Y$ is a morphism of log schemes and $\alpha_Y\colon M_Y \to \mathcal{O}_X$ is a log structure on X, then the maps $f^{-1}M_Y \to f^{-1}\mathcal{O}_Y$ and $f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ are both local, and hence so is the composite $f^{-1}M_Y \to \mathcal{O}_X$. It follows that the construction of the associated log structure $f^*M_Y \to \mathcal{O}_X$ is accomplished just by sharpening, and in particular the map

$$\overline{f}^{-1}\overline{M}_Y \to \overline{f^*M_Y}.$$

is an isomorphism.

Definition 1.2.2 *A morphism of log schemes $f\colon X \to Y$ is strict if the induced map : $f^*M_Y \to M_X$ is an isomorphism.*

Evidently the composite of strict morphisms is strict. The following result is an immediate consequence of (1.2.1) and (I, 4.1.2).

Corollary 1.2.3 *Let $f\colon X \to Y$ be a morphism of log schemes. If f is strict, the induced map $f^{-1}\overline{M}_Y \to \overline{M}_X$ is an isomorphism, and the converse holds if M_X is quasi-integral.*

□

In general, a morphism $f\colon (X, \alpha_X) \to (Y, \alpha_Y)$ of log schemes has a *canonical factorization*

$$(X, \alpha_X) \xrightarrow{i} (X, f^*\alpha_Y) \xrightarrow{f^s} (Y, \alpha_Y).$$

This factorization is uniquely determined by the fact that i is the identity on underlying schemes and and f^s is strict. There is a similarly factorization through the direct image log structure, and in fact f fits into a commutative diagram:

$$\begin{tikzcd}
(X, \alpha_X) \arrow[r, "i"] \arrow[d] & (X, f^*\alpha_Y) \arrow[d, "f^s"] \\
(Y, f_*^{log}\alpha_X) \arrow[r, "j"] & (Y, \alpha_Y),
\end{tikzcd}$$

where i and j are the identity on the underlying schemes. In some sense, $f_*^{log}\alpha_X$ is the log structure on Y which makes it as close as possible to X, and $f^*\alpha_Y$ is the log structure on X which makes it as close as possible to Y.

Definition 1.2.4 *If X is a log scheme, $\underline{X}$ is the underlying scheme of X, (often viewed as a log scheme with the trivial log structure), and X^* denotes the set of all points x of X such that $M^*_{X,\overline{x}} = M_{X,\overline{x}}$ for every (equivalently, for some) geometric point $\overline{x}$ lying over x.*

Proposition 1.2.5 *Let $f\colon X \to Y$ be a morphism of log schemes. Then f maps the subset X^* of X to the subset Y^* of Y. In particular, if the log structure on Y is trivial, so is the log structure on X.*

Proof: Let $f\colon X \to Y$ be a morphism of log schemes and let $\overline{x}$ be a geometric point of X. Then $f^\flat_{\overline{x}}\colon M_{Y,f(\overline{x})} \to M_{X,\overline{x}}$ is by (1.1.4) a local homomorphism of monoids, so if $\overline{M}^*_{X,\overline{x}} = 0$, the same is true of $\overline{M}_{Y,f(\overline{x})}$. Thus the function f takes X^* into Y^*. $\square$

Proposition 1.2.6 *Let U be a nonempty Zariski open subset of a scheme X and let $j\colon U \to X$ be the inclusion. Let*

$$\alpha_{U/X}\colon j^{log*}(\mathcal{O}_U^*) \to \mathcal{O}_X$$

denote the direct image of the trivial log structure on U. Then for any log scheme Y, the natural map

$$\mathrm{Mor}(X, Y) \to \{g \in \mathrm{Mor}(\underline{X}, \underline{Y}) : g(U) \subseteq Y^*\}$$

is bijective.

Proof: □

Examples 1.2.7 Thus, $j^{log*}(\mathcal{O}_U^*)$ is the inverse image of $j_*\mathcal{O}_U^* \to j_*\mathcal{O}_U$ via the natural map $\mathcal{O}_X \to j_*\mathcal{O}_U$. Note that $\alpha_{U/X}: j_*^{log}(\mathcal{O}_U^*) \to \mathcal{O}_X$ is injective and that its image is a sheaf of faces in the monoid $\mathcal{O}_X$. If X is integral and U is not empty, there is a canonical isomorphism $j^{log*}(\mathcal{O}_U^*)/\mathcal{O}_X^* \cong \underline{\Gamma}_Y(Div_X^+)$, where $\underline{\Gamma}_Y Div_X^+$ is the sheaf of effective Cartier divisors on X with support on $Y := X \setminus U$. To see this, note that since X is integral, $\alpha_{U/X}(m)$ lies in the sheaf $\mathcal{O}'_X$ of nonzero divisors for every $m \in j_*^{log}(\mathcal{O}_U^*)$. Since $\alpha_{U/X}$ is injective and $j_*^{log}(\mathcal{O}_U^*)^* \cong \mathcal{O}_X^*$, $\alpha_{U/X}$ induces an injection

$$\overline{\alpha}: j_*^{log}(\mathcal{O}_U^*)/\mathcal{O}_X^* \to \mathcal{O}'_X/\mathcal{O}_X^* \cong Div_X^+,$$

and since each $\alpha_{U/X}(m)$ restricts to a unit on U, $\overline{\alpha}(\overline{m})$ has support in Y. Conversely, if D is an effective Cartier divisor, then locally D can be expressed at the class of an element f of $\mathcal{O}'_X$, and D has support in Y if and only if f $f_{|U}$ is a unit, *i.e.*, if and only if $f \in j_*^{log}(\mathcal{O}_U^*)$.

A *log point* is a log scheme whose underlying scheme is the spectrum of a field. If P is a sharp monoid and $\xi := \operatorname{Spec} k$, the map $k^* \oplus P$ sending (u, p) to u if $p = 0$ and to 0 otherwise defines a log point, denoted ξ_P. In particular, $\xi_{\mathbf{N}}$ is sometimes called the *standard log point.*

Let S be the spectrum of a discrete valuation ring A and let X be an S-scheme. One says that X has *semistable reduction* if, locally for the étale topology on X and S, X is isomorphic to an S-scheme of the form $\operatorname{Spec} A[t_1, \ldots t_n]/(t_1, \ldots t_r - \pi)$, where π is a uniformizer of A. Then if η is the generic point of S, the open immersions $X_\eta \to X$ and $\eta \to S$ define log structures $\alpha_{X_\eta/X}$ and $\alpha_{\eta/S}$ on X and S, and the morphism X/S underlies a morphism of the corresponding log schemes. For example, the morphism of schemes $\underline{\mathsf{A}}_{\mathbf{N}^r} \to \underline{\mathsf{A}}_{\mathbf{N}}$ corresponding to the morphism $\mathbf{N} \to \mathbf{N}^r$ sending 1 to $(1, 1, \ldots 1)$, when localized at the origin of the base, has semistable reduction. We shall see that the corresponding morphism of log schemes $\mathsf{A}_{\mathbf{N}^r} \to \mathsf{A}_{\mathbf{N}}$ is much better behaved than the underlying morphism of schemes.

Definition 1.2.8 *Let $f: X \to Y$ be a morphism of log schemes. Then $M_{X/Y}$ is the cokernel of $f^*M_Y \to M_X$ in the category of sheaves of monoids. The inverse image M_X^v in M_X of $M_{X/Y}^*$ is called the vertical part of the log structure of X relative to Y, and $\overline{M}_{X/Y}$ is called the horizontal part.*

Notice that $M_{X/\underline{X}} \cong \overline{M}_X$. More generally, since f^*M_Y contains M^*_X, in fact $M_{X/Y}$ is canonically isomorphic to the cokernel of the natural map $f^*\overline{M}_Y \to \overline{M}_X$. Recall from (I,1.2.1) that if M_Y and M_X are integral, so is $M_{X/Y}$; furthermore $M^{gp}_{X/Y}$ is isomorphic to the cokernel of $f^*M^{gp}_Y \to M^{gp}_X$ and $M_{X/Y}$ can be identified with the image of M_X in this sheaf of groups. By way of an example, observe that if $f\colon X \to Y$ is a morphism of log schemes associated with semi-stable reduction (1.2.7), then $M_{X/Y}$ is entirely vertical, because the quotient of the map $\mathbf{N} \to \mathbf{N}^r$ sending 1 to $(1, 1, \ldots 1)$ is $\mathbf{Z}^{r-1}$.

The following result is helpful in comparing notions of log structures on different topologies, for example, the Zariski and étale topologies. The situation is the following. Let $f\colon X' \to X$ be a morphism, let $X'' := X' \times_X X'$, let $p_i\colon X'' \to X'$, $i = 1, 2$ be the two projections, and let

$$g := f \circ p_1 = f \circ p_2 : X'' \to X.$$

If $\alpha_X\colon M_X \to \mathcal{O}_X$ is any log structure on X, let $M_{X'} := f^*M_X$ and let $M_{X''} := g^*M_X$. Then there are canonical isomorphisms $M_{X''} \cong p_i^*M_{X'}$, and hence each of the maps p_i induces a morphism of sheaves of monoids $f_*M_{X'} \to g_*M_{X''}$.

Proposition 1.2.9 *Let $f\colon X' \to X$ be a faithfully flat and quasi-compact morphism of schemes, and let $\alpha_X\colon M_X \to \mathcal{O}_X$ be a quasi-integral log structure on the Zariski topology of X. Then the natural map*

$$M_X \to \mathrm{Eq}\,(f_*M_{X'} \rightrightarrows g_*M_{X''})$$

is an isomorphism.

Proof: This proposition is a simple consequence of faithfully flat descent and the following elementary lemma about sheaves of sets.

Lemma 1.2.10 *Let S be a sheaf of sets on X. Then the natural map*

$$S \to \mathrm{Eq}\left(f_*f^{-1}S \rightrightarrows g_*g^{-1}S\right)$$

is an isomorphism.

Proof: The injectivity of $F \to f_* f^{-1} S$ is clear from the surjectivity of f. For the surjectivity, recall that since f is faithfully flat and quasi-compact, the underlying map on topological spaces is open and surjective [,]. Let s' be a section of $f_* f^{-1}(S)$ such that $p_1^*(s') = p_2^*(s')$ in $g_* g^{-1}(S)$. For any point x of X there is at least one point x' of X' such that $f(x') = x$, and for any pair (x'_1, x'_2) of such points, there is a point x'' of X'' such that $p_i(x'') = x'_i$. The natural maps $F_x \to f^{-1} S_{x'_i} \to g^{-1} S_{x''}$, are isomorphisms, and because $p_1^*(s') = p_2^*(s')$, the stalks of s' at x'_1 and x'_2 correspond to the same element of F_x, which we denote by $s(x)$. Thus $x \mapsto s(x) \in \prod_x F_x$ is a "discontinuous section" of F such that $s(x) = s_{x'}$ whenever $f(x') = x$. It remains only to prove that s is in fact continuous. If $x \in X$, there exist a neighborhood U of x in X and a section t of S over U whose stalk at x is $s(x)$. Choose a point x' of $f^{-1}(U)$ mapping to x. Then the stalk of s' at x' agrees with the stalk of $f^*(t)$ at x', and hence there is a neighborhood U' of x' in X' such that $f^*(t)_{|_{U'}} = s'_{|_{U'}}$. Then if $y' \in U'$, the stalk of t at $f(y')$ equals the stalk of s' at y', so $t_{f(y')} = s(f(y'))$. In other words, $t_y = s(y)$ for all y in the image of U'. Since f is open, this image contains a neighborhood of x, and so s is continuous, as required. $\square$

Now to prove the proposition, note that since M_X is quasi-integral, it is an $\mathcal{O}_X^*$-torsor over $\overline{M}_X$, and similarly $M_{X'}$ (resp. $M_{X''}$) is an $\mathcal{O}_{X'}^*$-torsor (resp., an $\mathcal{O}_{X''}$-torsor) over $\overline{M}_{X'}$ (resp., $\overline{M}_{X''}$). and $M_{X''}$. Consequently the rows of the diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X^* \arrow[r] \arrow[d] & M_X \arrow[r] \arrow[d] & \overline{M}_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f_*\mathcal{O}_{X'}^* \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & f_* M_{X'} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & f_*\overline{M}_{X'} \arrow[d, shift left] \arrow[d, shift right] & \\
0 \arrow[r] & g_*\mathcal{O}_{X''}^* \arrow[r] & g_* M_{X''} \arrow[r] & g_*\overline{M}_{X''} &
\end{tikzcd}
$$

are exact. The column on the left is exact by standard descent theory for $\mathcal{O}_X^*$. The argument of the previous paragraph shows that the column on the right is exact, because $\overline{M}_{X'} \cong f^{-1}\overline{M}_X$ and $\overline{M}_{X''} \cong g^{-1}\overline{M}_X$. Now the exactness of the middle column follows by chasing the diagram (locally in the Zariski topology on X). □

Corollary 1.2.11 *Let (X, M_{zar}) be a quasi-integral log scheme for the Zariski topology, and for each étale $U \to X$, let $\alpha_U : M_U \to \mathcal{O}_U$ denote the inverse image log structure. Then $U \mapsto M_U(U)$ is a sheaf in the étale topology of X and defines a log structure $M_{\acute{e}t}$ for the étale topology of X.*

Corollary 1.2.12 *If X is a log scheme, then the functor on the category of quasi-integral log schemes sending T to the set of morphisms $T \to X$ forms a sheaf in the topology whose open sets are Zariski open (resp. étale resp. fppf...).*

2 Charts and coherence

2.1 Coherent, fine, and saturated log structures

Definition 2.1.1 *Let $\alpha : M \to \mathcal{O}_X$ be a log structure on a scheme X and let P be a monoid. A chart for α subordinate to P is a morphism of prelog structures*

$$\begin{tikzcd} P \arrow[r, "\theta"] \arrow[dr, "\beta"'] & M \arrow[d, "\alpha"] \\ & \mathcal{O}_X \end{tikzcd}$$

such that $\theta^a : P^a \to M$ (1.1.5) is an isomorphism. A log structure α is called quasi-coherent (resp. coherent) if locally on X it admits a chart (resp. a chart subordinate to a finitely generated monoid).

A chart for α subordinate to P is determined by the morphism $\theta : P \to M$ (but not by the morphism $\alpha \circ \theta$, in general) and we shall sometimes identify the chart with the morphism θ.

One says that a chart $P \to M$ is *coherent* (resp. *integral, fine, saturated*) if P is of finite type, (resp. integral, fine, saturated).

Remark 2.1.2 Let $\alpha\colon M \to \mathcal{O}_X$ be a log structure on X, let $\theta\colon P \to M$ be morphism from a constant monoid P to M, and let $\beta := \alpha \circ \theta$. Because α is a log structure, it is sharp and local, and it follows that the natural map $P^\theta \to P^\beta$ in the diagram below is an isomorphism.

$$\begin{tikzcd}
P \arrow[r] \arrow[dr] & P^\theta \arrow[r, "\theta^a"] \arrow[d] & M \arrow[d] \\
 & P^\beta \arrow[r, "\beta^a"] & \mathcal{O}_X
\end{tikzcd}$$

If $\beta\colon Q \to M$ is a chart for a log structure $\alpha\colon M \to \mathcal{O}_X$, then $M \cong Q^{\beta\alpha}$, and because α is strict and local, $Q^\beta \cong Q^{\beta\alpha} \cong M$. Thus neither the sheaf $\mathcal{O}_X$ nor the map α is needed to compute P^β, and it makes sense to define a *chart for a sheaf of monoids* M on a topos X as a morphism from a constant monoid Q to M inducing an isomorphism $P^a \to M$ and to say that a sheaf of monoids is *quasi-coherent* (resp. *coherent*) if locally on X it admits a chart (resp. a chart subordinate to a finitely generated monoid). Then $\theta\colon P \to M$ is a chart for the log structure α if and only if it is a chart for the sheaf of monoids M. Note that with this definition any sheaf of abelian groups defines a coherent sheaf of monoids.

Remark 2.1.3 If X is a log scheme, then a morphism from a monoid P to $\Gamma(X, M_X)$ induces a commutative diagram

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "e_P"'] & \Gamma(X, M_X) \arrow[d, "\Gamma(\alpha_X)"] \\
\mathbf{Z}[P] \arrow[r] & \Gamma(X, \mathcal{O}_X),
\end{tikzcd}$$

and hence a morphism of log schemes $X \to \mathsf{A_P}$. It follows from the definitions that $P \to M_X$ is a chart for α_X if and only if $X \to \mathsf{A_P}$ is strict, and in this case we say that $X \to \mathsf{A_P}$ is a *chart for the log scheme* X. As a matter of fact, the map $P \to M_X$ defines a morphism of monoidal spaces $g\colon (X, M_X) \to (S, M_S)$, where $(S, M_S) := \operatorname{Spec} P$ described in section (1.3), and $P \to M_X$ is a chart for M_X in the sense of (2.1.2) if and only if $g^* M_S \to M_X$ is an isomorphism.

The notion of a chart for a log structure is due to Kato and is central to the theory. Note that (in contrast to the notion of a chart in differential geometry), a chart is not an isomorphism and only describes the log structure of X.

Proposition 2.1.4 *Let $\beta\colon P \to M$ be a morphism from a constant monoid P to a sheaf of monoids M on a scheme X, and let $s\colon (X, M) \to (S, M_S) :=$ $\operatorname{Spec} P$ be the map of locally monoidal spaces corresponding to β. If β is a chart for M, then s induces an isomorphism $s^{-1}(\overline{M}_S) \to \overline{M}$, i.e., for every geometric point $\overline{x}$ of X, the map $P \to M_{\overline{x}}$ induces an isomorphism $P/F_{\overline{x}} \to \overline{M}_{\overline{x}}$, where $F_{\overline{x}} := \beta_{\overline{x}}^{-1}(M^*_{\overline{x}})$. The converse holds if M is quasi-integral.*

Proof: If $P^a \to M$ is the sharp localization of $\beta\colon P \to M \to \mathcal{O}_X$, then β is a chart if and only if β^a is an isomorphism; by . Thus if β is a chart $\overline{\beta}^a$ is an isomorphism, and the converse holds if M is quasi-integral by $(I, 4.1.2)$. According to (1.1.7), the stalk of $\overline{P}^a$ at a point x is exactly $P/F_{\overline{x}}$. On the other hand, the point of $\operatorname{Spec} P$ corresponding to $s(x)$ is the prime ideal $\mathfrak{p} := P \setminus F_{\overline{x}}$, and the stalk of M_S at $\mathfrak{p}$ also identifies with $P/F_{\overline{x}}$. Thus $\overline{\beta}^a$ is an isomorphism if and only if the map $P/F_{\overline{x}} \to \overline{M}_{X,\overline{x}}$ is an isomorphism. $\square$

Corollary 2.1.5 *Let $\beta\colon Q \to M$ be a chart for a log structure $\alpha\colon M \to \mathcal{O}_X$ on a scheme X, and let x be a point of X. Then there is a natural isomorphism $Q/F_x \cong \overline{M}_x$, where $F_x := \beta^{-1}(M^*_x) = (\alpha \circ \beta)^{-1}(\mathcal{O}^*_X)$.*

Proposition 2.1.6 *If X is a coherent log scheme, X^* is an open subset of X, and the inclusion $j_X\colon X^* \to X$ is an affine morphism. A morphism of coherent log schemes $f\colon X \to Y$ fits into a commutative diagram*

$$\begin{tikzcd} X^* \arrow[r, "j_X"] \arrow[d, "f^*"'] & X \arrow[d, "f"] \\ Y^* \arrow[r, "j_Y"] & Y \end{tikzcd}$$

which is Cartesian if f is strict.

Proof: To prove that $X^* \to X$ is open and affine when X is coherent is a local problem on X, so we may assume that X admits a chart, *i.e.*, a strict map $f: X \to Y$, where $Y := \mathsf{A}_\mathsf{P}$ for some finitely generated monoid P. Then $Y^* := \mathsf{A}_{\mathsf{P}^{\mathsf{gp}}}$, and since P is finitely generated, Y^* is a special affine open subset of Y, and consequently X^* is an affine open subset of X. We have already seen in (1.2.5) that f maps X^* into Y^* set-theoretically. If f is strict, $f^\flat$ induces an isomorphism $\overline{M}_{Y,f(\overline{x})} \to \overline{M}_{X,\overline{x}}$, so that the diagram in the proposition is set-theoretically Cartesian. Since $Y^* \to Y$ is an open immersion, the diagram of underlying schemes is Cartesian. If $g: T \to Y^*$ an $h: TtoX$ with $f \circ h = g$, in the category of log schemes, then the log structure on T must be trivial, so h factors uniquely through X^* and the diagram is also Cartesian in the category of log schemes. $\square$

Definition 2.1.7 *Let $f: X \to Y$ be a morphism of log schemes and let $\theta: Q \to P$ be a morphism of monoids. A chart for f subordinate to θ is a commutative diagram*

$$\begin{array}{ccc} Q & \xrightarrow{\gamma} & \Gamma(Y, M_Y) \\ \theta \downarrow & & \downarrow f^\flat \\ P & \xrightarrow{\beta} & \Gamma(X, M_X), \end{array}$$

where γ and β are charts for α_Y and α_X, respectively.

Definition 2.1.8 *Let X be a scheme. One says that a log structure (M, α) on X is integral, (resp. saturated) if M_X is integral (resp. saturated), and that (M, α) is fine (resp. saturated) if it is coherent and integral (resp. saturated).*

Proposition 2.1.9 *Let U be an open subset of a locally noetherian and locally factorial scheme X. Then the direct image log structure (1.2.7) $M_X := j_*^{log}(\mathcal{O}_U^*) \to \mathcal{O}_X$ is coherent. For each $x \in X$, $\overline{M}_{X,x} \cong \mathbf{N}^r$, where r is the number of irreducible components of codimension one of $X \setminus U$ passing through x.*

Proof: Let Y be the union of the irreducible components of codimension one of $X \setminus U$, let Z be the union of the irreducible components of codimension at least two, and let $U' := X \setminus Y$. Then $U = U' \setminus Z$, and since Z has codimension at least two and U' is normal, the natural map $\mathcal{O}_{U'} \to j'_*\mathcal{O}_U$ is an isomorphism. It follows that the natural map $j_*^{log}(\mathcal{O}_{U'}) \to j_*^{log}(\mathcal{O}_U)$ is an isomorphism, and so without loss of generality we may assume that Z is empty. We may also assume that X is affine; since X is locally factorial, the ideals $\{\mathfrak{p}_i : i = 1 \ldots n\}$ defining the irreducible components of Y are invertible, and we may assume that they are principle, say $\mathfrak{p}_i = (t_i)$. Then t_i defines a global section of M_X. We shall see that the map $\beta \colon \mathbf{N}^n \to M_{U/X}$ sending the i^{th} standard basis element e_i of $\mathbf{N}^n$ to t_i is a chart for M_X. The stalk of $M_{U/X}$ at a point x consists of the set of all elements of $\mathcal{O}_{X,x}$ which become units in the localization of $\mathcal{O}_{X,x}$ by $t := t_1 t_2, \cdots t_n$. Because $\mathcal{O}_{X,x}$ is factorial, an element of this localization be written uniquely as a product $a t_1^{e_1} \cdots t_n^{e_n}$ with $e_i \in \mathbf{Z}$ and $a \in \mathcal{O}_{X,x}$. Such an element lies in $M_{U/X,x}$ if and only if $a \in \mathcal{O}^*_{X,x}$ and $e_i \geq 0$ for all i, and it lies in $M^*_{X,x}$ if and only if $e_i = 0$ whenever t_i is not a unit in $\mathcal{O}_X$. Thus $\overline{M}_{U/X} \cong \mathbf{N}^r \cong \mathbf{N}^n/\beta^{-1}(M^*_X)$, and β is chart by (2.1.5). □

Corollary 2.1.10 *The log structures associated to a semistable reduction over a DVR (1.2.7) are fine, and the associated morphism of log schemes locally admits a chart of the form* $\mathbf{N} \to \mathbf{N}^r : 1 \mapsto (1, 1, \ldots 1)$.

Corollary 2.1.11 *If* $(X_{\acute{e}t}, M_{\acute{e}t})$ *is a fine log scheme for the étale topology, there exist an étale cover* $f \colon X' \to X$ *and a log structure* $M_{X'}$ *on* X'_{zar} *such that* $f^* M_{\acute{e}t}$ *is the étale log structure associated to* $M_{X'}$.

Proof: Without loss of generality we may assume that X admits a chart. Let M_{zar} denote the corresponding Zariski log structure Then the previous result shows that for every étale $U \to X$, $\Gamma(U, M_{\acute{e}t}) \cong \Gamma(U, M_{\mathrm{zar}})$ so $M_{\acute{e}t}$ is the étale log structure associated to M_{zar}. □

2.2 Construction and comparison of charts

Proposition 2.2.1 *Let $\beta\colon Q \to M$ be a chart for a sheaf of monoids M on as scheme X. Suppose that β factors:*

$$\beta = Q \xrightarrow{\theta} Q' \xrightarrow{\beta} M,$$

where Q' is finitely generated. Then, locally on X, β' can be factored

$$\beta' = Q' \xrightarrow{\theta'} Q'' \xrightarrow{\beta''} M$$

where β'' a finitely generated chart for M. In particular, M is coherent.

Proof: Let $\{q'_i : i \in I\}$ be a finite system of generators for Q', and let $\overline{x}$ be a geometric point of X. Because β is a chart, it follows from (1.1.11) that the map $\overline{\beta}_{\overline{x}}$ is surjective. Hence for each $i \in I$ there exist an element $q_i \in Q$, a neighborhood U_i of x, and a section u_i of $M^*(U_i)$ such that $\beta'(q'_i) = \beta(q_i) + u_i$. Replacing X by $\prod_X U_i$, we may assume that the u_i are global sections of M^*. Let Q'' be the quotient of $Q' \oplus \mathbf{Z}^I$ by the relation identifying $(q'_i, 0)$ with $(\theta(q_i), e_i)$. Then there are commutative diagrams

$$\begin{tikzcd} Q \arrow[r, "\theta"] \arrow[d, "\beta"'] \arrow[dr, "\theta'"] & Q' \arrow[d, "\theta''"] \\ M & Q'' \arrow[l, "\beta''"] \end{tikzcd} \qquad \begin{tikzcd} Q^a \arrow[r, "\theta^a"] \arrow[d, "\beta^a"'] \arrow[dr, "\theta'^a"] & Q'^a \arrow[d, "\theta''^a"] \\ M & Q''^a \arrow[l, "\beta''^a"] \end{tikzcd}$$

where β'' sends the class of any $(q', 0)$ to $\beta'(q')$ and the class of $(0, e_i)$ to u_i. Then $\overline{Q}''$ is generated by the elements $\overline{q}'_i = \overline{\theta}(q_i)$, and so $\overline{\theta}'\colon \overline{Q} \to \overline{Q}''$ is surjective, and it follows from (II, 4.1.2) that $\theta'^a\colon Q^a \to Q''^a$ is also surjective. But $\beta''^a \circ \theta'^a = \beta^a$ which is an isomorphism, so θ'^a is also bijective, and so $\beta''\colon Q'' \to M$ is again a chart. $\square$

Let M be a sheaf of monoids on X. If $\overline{x}$ is a geometric point of X, a *germ of a chart at* $\overline{x}$ is a chart of the restriction of M to some open neighborhood of $\overline{x}$ in X, and a morphism of such germs $\beta \to \beta'$ is an element of the direct limit $\varinjlim \operatorname{Hom}_M(\beta_{|U}, \beta'_{|U})$, where U ranges over the étale neighborhoods of $\overline{x}$.

Corollary 2.2.2 *Let M be a coherent sheaf of monoids on X and let $\overline{x}$ be a geometric point of X. Then the category of germs of coherent charts for M at $\overline{x}$ is filtering.*

Proof: Let $\beta_i\colon Q_i \to M_{|U_i}$ be finitely generated charts for the restrictions of M to neighborhoods U_i of $\overline{x}$ in X, for $i = 1,2$ and let $U := U_1 \times_X U_2$. Then $Q' := Q_1 \oplus Q_2$ is finitely generated and β_i factors through the map $\beta'\colon Q' \to M_{|U}$ induced by β_1 and β_2. By (2.2.1), β' factors through a coherent chart $\beta''\colon Q'' \to M$ in some neighborhood of $\overline{x}$, and so there is a commutative diagram:

$$\begin{tikzcd}
 & Q_1 \arrow[dl, "\theta_1"'] \arrow[d, "\theta_1'"] \arrow[dr, "\beta_1"] & \\
Q' \arrow[r, "\theta'"] & Q'' \arrow[r, "\beta''"] & M \\
 & Q_2 \arrow[ul, "\theta_2"] \arrow[u, "\theta_2'"'] \arrow[ur, "\beta_2"'] &
\end{tikzcd}$$

where β'' is a coherent chart for α.

Similarly, if $\theta_i\colon \beta \to \beta'$ is a pair of morphisms of coherent charts, the coequalizer Q'' of θ_1 and θ_2 is finitely generated, and there is a diagram

$$\begin{tikzcd}
Q \arrow[r, shift left] \arrow[r, shift right] \arrow[dr, "\beta"] & Q' \arrow[r] \arrow[d, "\beta'"'] & Q'' \arrow[dl, "\beta''"] \\
 & M. &
\end{tikzcd}$$

Then by (2.2.1), β'' factors through a coherent chart $Q''' \to M$. Combining these two constructions, we see that any diagram of charts

$$\begin{tikzcd}
Q \arrow[r] \arrow[d] & Q_1 \\
Q_2 &
\end{tikzcd}$$

fits into a commutative square in a neighborhood of $\overline{x}$. Finally, since M is assumed to be coherent, there is a chart of M in some neighborhood of every point. Thus the category of germs of charts at $\overline{x}$ is nonempty, and hence is filtering. $\square$

Corollary 2.2.3 *Let $\theta: M \to M'$ be a morphism of coherent sheaves of monoids on a scheme X and let $\beta: Q \to M$ be a chart for M. Then locally on X there exists a commutative diagram*

$$\begin{tikzcd} Q \arrow[r, "\phi"] \arrow[d, "\beta"'] & Q' \arrow[d, "\beta'"] \\ M \arrow[r, "\theta"] & M' \end{tikzcd}$$

where β' is a coherent chart for α'. If $f: X \to Y$ is a morphism of coherent log schemes and $Q \to M_Y$ is a coherent chart for Y, then locally on X there exists a coherent chart for f subordinate to a morphism of finitely generated monoids $Q \to P$.

Proof: Since (M', α') is coherent, and the assertion is local on X, we may assume that M' admits a coherent chart $\beta'': Q'' \to M'$. Consider the commutative diagram

$$\begin{tikzcd} Q \arrow[r] \arrow[d, "\beta"'] & Q \oplus Q'' \arrow[d, "\gamma"] \\ M \arrow[r, "\theta"] & M', \end{tikzcd}$$

where γ is $\theta \circ \beta$ on Q and β'' on Q''. Since $Q \oplus Q''$ is finitely generated, (2.2.1) implies that γ factors through a chart Q' of M', and $\phi: Q \to Q \oplus Q'' \to Q'$ is the desired map of finitely generated monoids. To deduce the second statement, observe that the morphism $Q \to f^*(M_Y)$ deduced from $Q \to M_Y$ is a chart for the log structure $f^*(M_Y)$ on Y, and apply the first statement to the morphism $f^*(M_Y) \to M_X$. $\square$

The next result allows us to extend charts from a stalk to a neighborhood.

Proposition 2.2.4 *Let M be a coherent sheaf of monoids on a scheme X and let $\overline{x}$ be a geometric point of X. Then the evident functor from the category of germs of coherent charts of M at $\overline{x}$ to the category of finitely generated charts of $M_{\overline{x}}$ is an equivalence.*

The proof of this proposition will depend on some preliminary results.

Lemma 2.2.5 *Let M be a sheaf of monoids on X and let $\overline{x}$ be a geometric point of X. If P is a finitely generated monoid, the natural map*

$$\mathrm{Hom}(P, M)_{\overline{x}} \to \mathrm{Hom}(P, M_{\overline{x}})$$

is an isomorphism.

Proof: By (I,2.1.9.7) P is of finite presentation, so the functor $\mathrm{Hom}(P, \ )$ commutes with direct limits. □

Lemma 2.2.6 *Let M_1, M_2, and N be sheaves of monoids on X, let $\alpha_i \colon M_i \to N$ be logarithmic morphims, and let $\overline{x}$ be a geometric point of X.*

1. *If M_1 is coherent, the natural map*

$$\underline{\mathrm{Hom}}_N(M_1, M_2)_{\overline{x}} \to \mathrm{Hom}_{N_{\overline{x}}}(M_{1_{\overline{x}}}, M_{2_{\overline{x}}})$$

   is an isomorphism.

2. *If M_1 and M_2 are coherent, then a homomorphism $\theta \colon M_1 \to M_2$ over N is an isomorphism in a neighborhood of $\overline{x}$ if and only if its stalk $\theta_{\overline{x}}$ is an isomorphism.*

Proof: Let $\beta_1 \colon Q_1 \to M_1$ be a coherent chart for M_1. Since α_1 and α_2 are log structures over N and β_1 is a chart for M_1, any morphism from Q_1 to M_2 over N factors uniquely through M_1. That is,

$$\mathrm{Hom}_N(Q_1, M_2) \cong \mathrm{Hom}_N(M_1, M_2).$$

This remains true on any neighborhood of $\overline{x}$ in X, so passing to the limit and applying (2.2.5) with $M = M_2$ and with $M = N$, we get

$$\underline{\mathrm{Hom}}_N(M_1, M_2)_{\overline{x}} \cong \underline{\mathrm{Hom}}_N(Q_1, M_2)_{\overline{x}} \cong \mathrm{Hom}_{N_{\overline{x}}}(Q_1, M_{2,\overline{x}}).$$

But $Q_1 \to M_{1_{\overline{x}}}$ is also a chart for $M_{1_{\overline{x}}}$, and so

$$\mathrm{Hom}_{N_{\overline{x}}}(Q_1, M_{2_{\overline{x}}}) \cong \mathrm{Hom}_{N_{\overline{x}}}(M_{1_{\overline{x}}}, M_{2_{\overline{x}}}),$$

proving (1). Statement (2) is an immediate consequence. $\square$

Lemma 2.2.7 *Let $\theta: M_1 \to M_2$ be a logarithmic homomorphism of coherent sheaves of monoids. If the stalk of θ at a point $\overline{x}$ of X is an isomorphism, then θ is an isomorphism in some neighborhood of $\overline{x}$.*

Proof: This is an immediate consequence of (2.2.6.2), with $\alpha_1 = \theta$ and $\alpha_2 = \mathrm{id}_{M_2}$. $\square$

Proof of (2.2.4): Let $\beta: Q \to M_{|U}$ be a chart for $M_{|U}$. Then $\beta_{\overline{x}}: Q \to M_{\overline{x}}$ is a chart of $M_{\overline{x}}$. A morphism of germs of charts $\beta \to \beta'$ comes from a morphism of charts

$$\begin{tikzcd} Q \arrow[r, "\beta"] \arrow[d, "\theta"'] & M_{|U} \\ Q' \arrow[ur, "\beta'"'] & \end{tikzcd}$$

in some neighborhood and hence induces a morphism on stalks $\beta_{\overline{x}} \to \beta'_{\overline{x}}$. This defines our functor. On the other hand, if $\theta: Q \to Q'$ is such that $\beta'_{\overline{x}} \circ \theta = \beta_{\overline{x}}$ and Q is finitely generated, then in fact this equality holds in some neighborhood of $\overline{x}$. This shows that the functor is fully faithful. To show that it is essentially surjective, let $\beta_{\overline{x}}$ be a chart for $M_{\overline{x}}$. Then by (2.2.5), β extends to a homomorphism from Q to M in some neighborhood of $\overline{x}$. Moreover $\beta^a_{\overline{x}}$ is an isomorphism, and since β^a is logarithmic, it follows from (2.2.7) that β^a is an isomorphism in some neighborhood U of x. Thus $\beta_{|U}$ is a chart for $M_{|U}$. $\square$

It is often desirable to construct charts for a log structure that are as close as possible to its stalk at some given point. We shall now discuss some of the techniques for doing so, restricting ourselves to the context of fine log schemes.

Definition 2.2.8 *Let M be a sheaf of integral monoids on a scheme X, let $\overline{x}$ be a geometric point of X and let $\theta\colon P \to M$ be an integral chart for M.*

1. *θ is exact at $\overline{x}$ if it satisfies the following equivalent conditions:*
   (a) *$\theta_{\overline{x}}\colon P \to M_{\overline{x}}$ is exact (2.1.8).*
   (b) *$\theta_{\overline{x}}\colon P \to M_{\overline{x}}$ is local.*
   (c) *$\overline{\theta}_{\overline{x}}\colon \overline{P} \to \overline{M}_{\overline{x}}$ is an isomorphism.*
2. *θ is good at $\overline{x}$ if it satisfies the following equivalent conditions:*
   (a) *P is sharp and θ is exact at $\overline{x}$.*
   (b) *$\pi \circ \theta_{\overline{x}}\colon P \to \overline{M}_{\overline{x}}$ is an isomorphism.*
   (c) *$\pi^{gp} \circ \theta^{gp}_{\overline{x}}\colon P^{gp} \to \overline{M}^{gp}_{\overline{x}}$ is an isomorphism.*

The equivalence of the conditions in (1) follows immediately from (1.1.12). To check the equivalences in (2), note that (a) implies (b), because (1c) holds, and (b) trivially implies (c). If (c) is true, then $P \to \overline{M}_{\overline{x}}$ is injective, so P is sharp. Since θ is a chart, $\pi \circ \theta_{\overline{x}}$ is surjective, hence bijective, so $\theta_{\overline{x}}$ is exact by (I, 4.1.3). Thus (c) implies (a).

Remark 2.2.9 Let $\theta\colon P \to M$ be a fine chart for M and let $\overline{x}$ be a geometric point of X. Then $F := \theta^{-1}M^*_{\overline{x}}$ is a face of P, and hence by (I, 2.1.9) there exists a $p \in F$ such that $\langle p \rangle = F$. Since $\theta(p)_{\overline{x}} \in M^*_{\overline{x}}$, there exists a neighborhood $U \to X$ of $\overline{x}$ on which $\theta(p)$ is a unit, and then θ factors through a map $\theta'\colon P_F \to M_{|_U}$. Then $\theta'_{\overline{x}}$ is exact. In other words, any fine chart for M factors locally through a chart which is exact at $\overline{x}$.

Definition 2.2.10 *A markup of an integral monoid P is a homomorphism ϕ from a finitely generated abelian group L to P^{gp} which induces a surjection $L \to \overline{P}^{gp}$. A morphism of markups of P is a homomorphism of abelian groups $\theta\colon L_1 \to L_2$ such that $\phi_2 \circ \theta = \phi_1$.* or *layout*?

If $\phi_i\colon L_i \to P^{gp}$, $i = 1, 2$, is a pair of markups of P, then so is the map $(\phi_1, \phi_2)\colon L_1 \oplus L_2 \to P^{gp}$. If θ and θ' are morphisms of markups $\phi_1 \to \phi_2$, then the induced map from the coequalizer of θ and θ' to P^{gp} is also a markup. The category of markups of P is nonempty, and hence filtering, if and only if $\overline{P}^{gp}$ is finitely generated.

Theorem 2.2.11 *Let M be a fine sheaf of monoids on X and let $\overline{x}$ be a geometric point of X.*

1. *If $\phi\colon L \to M^{gp}_{\overline{x}}$ is a markup of $M_{\overline{x}}$, consider the induced map*

$$\theta\colon Q := L \times_{M^{gp}_{\overline{x}}} M_{\overline{x}} \to M_{\overline{x}}.$$

   Then the natural map $Q^{gp} \to L$ is an isomorphism, and θ is the germ of a fine exact chart for M at $\overline{x}$.

2. *Conversely, if $\theta\colon Q \to M$ is a fine exact chart at $\overline{x}$, then $\theta^{gp}_{\overline{x}}\colon Q^{gp} \to M^{gp}_{\overline{x}}$ is a markup of $M_{\overline{x}}$. The correspondence $\phi \mapsto \theta$ gives a equivalence between the category of germs of fine exact charts for M at $\overline{x}$ of α which are exact at $\overline{x}$ and the category of markups of $M_{\overline{x}}$.*

Proof: Let ϕ be a markup of $M_{\overline{x}}$ and let $\theta\colon Q \to M_{\overline{x}}$ be the map described in (1). Note first that since $M_{\overline{x}} \to \overline{M}_{\overline{x}}$ is exact,

$$Q := L \times_{M^{gp}_{\overline{x}}} M_{\overline{x}} = L \times_{M^{gp}_{\overline{x}}} M^{gp}_{\overline{x}} \times_{\overline{M}^{gp}_{\overline{x}}} \overline{M}_{\overline{x}} = L \times_{\overline{M}^{gp}_{\overline{x}}} \overline{M}_{\overline{x}}.$$

Thus Q is a fibered product of fine monoids and hence by (I, 2.1.9), Q is fine. The integrality of $M_{\overline{x}}$ implies that $Q \subseteq L$ and hence $Q^{gp} \subseteq L$. If $z \in L$, $\phi(z) \in M^{gp}_{\overline{x}}$ can be written as $m_1 - m_2$ with $m_i \in M^{gp}_{\overline{x}}$. Then there exist $z_i \in L$ and $u_i \in M^*_{\overline{x}}$ such that $\phi(z_i) = m_i + u_i$, hence $z_i \in Q$ and $\phi(z - z_1 + z_2) = u_1 - u_2 \in M^*_{\overline{x}}$. Thus $w := z - z_1 + z_2 \in Q^{gp}$ and $z = w + z_1 - z_2$, so $Q^{gp} \cong L$. It follows that θ is exact, and so by (I, 4.1.3) $\overline{\theta}\colon \overline{Q} \to \overline{M}_{\overline{x}}$ is injective. Since ϕ is surjective, $\overline{\theta}$ is surjective, hence an isomorphism. Since θ is exact, it is local, and so by (1.1.11) the map $\overline{Q} \to \overline{Q}^a$ is an isomorphism. Then $\overline{\theta}^a_{\overline{x}}$ is an isomorphism, and since θ^a is sharp and $M_{\overline{x}}$ is integral, it follows from (I, 4.1.2) that $\theta_{\overline{x}}$ is an isomorphism. By (2.2.4), θ defines a chart in some neighborhood of x; θ is exact at x by construction. This proves (1). Conversely, if $\theta\colon Q \to M$ is a chart which is exact at $\overline{x}$, then $\overline{Q} \cong \overline{Q}^a \cong \overline{M}_{\overline{x}}$ by (1.1.12). Thus the map $Q^{gp} \to M^{gp}$ is a markup, and this construction is quasi-inverse to the functor taking a markup to a chart. □

Corollary 2.2.12 *Suppose that X is a fine (resp. fine and saturated) log scheme and $\overline{x}$ is a geometric point of X. Then, in some neighborhood of $\overline{x}$, X admits a fine (resp. fine and saturated) chart which is exact at $\overline{x}$, and the category of germs of such charts is filtering.*

Corollary 2.2.13 *A log structure on a scheme X is fine (resp. fine and saturated) if and only if locally it admits a fine (resp. fine and saturated) chart.*

Corollary 2.2.14 *Let $f\colon X \to Y$ be a morphism of fine log schemes, let $\gamma\colon Q \to M_Y$ be a fine chart for M_Y and let $\overline{x}$ be a geometric point of X. Then in some neighborhood of $\overline{x}$ in X, γ fits into a fine chart for f which is exact at $\overline{x}$.*

Proof: Since M_X is fine, $\overline{M}_{X,\overline{x}}$ is fine, and admits a markup $L \to M^{gp}_{X,\overline{x}}$. Then

$$(f^\flat_{\overline{x}} \circ \gamma, \phi)\colon L' := Q^{gp} \oplus L \to M^{gp}_{X,\overline{x}}$$

is also a markup of $M_{X,\overline{x}}$, and so corresponds by (2.2.11) to a chart $\beta\colon P \to M_X$ in some neighborhood of $\overline{x}$. Then the map $Q^{gp} \to L'$ induces a map $\theta\colon Q \to P := L' \times_{M^{gp}_{X,\overline{x}}} M_{X,\overline{x}}$. Since $\beta_{\overline{x}} \circ \theta = f^\# \circ \gamma$ and Q is fine, it follows from (2.2.4) that, after further shrinking X, $\beta \circ \theta = f^\# \circ \gamma$. □

Proposition 2.2.15 *Let X be a fine log scheme such that $\overline{M}^{gp}_X$ is torsion free (for example, a fine and saturated log scheme) and let $\overline{x}$ be a geometric point of X. Then in a neighborhood of $\overline{x}$, there is a chart for M_X which is good at $\overline{x}$.*

Proof: Let $P =: \overline{M}_{X,\overline{x}}$. Since M_X is fine, P is fine, and hence P^{gp} is a finitely generated abelian group. Since $\overline{M}^{gp}_X$ is torsion free, $\overline{M}^{gp}_{X,\overline{x}} \cong P^{gp}$ is torsion free, hence free, and the exact sequence

$$0 \to M^*_{X,\overline{x}} \to M^{gp}_{X,\overline{x}} \to \overline{M}^{gp}_{X,\overline{x}} \to 0$$

splits. Choose a splitting $\phi\colon P^{gp} \to M^{gp}_{X,\overline{x}}$; then ϕ is a markup (2.2.10) of $M_{X,\overline{x}}$. The inverse image of $\overline{M}_{X,\overline{x}}$ in P^{gp} is just P, and so by (2.2.11), $P \to M_{X,\overline{x}}$ extends to a chart β for X in some neighborhood of x; evidently β is good at x. □

To produce good charts in a more general setting we shall use the following lemma.

Lemma 2.2.16 *Suppose that G is a finitely generated abelian group (resp. a finitely generated abelian group whose torsion part is killed by an integer n invertible in $\mathcal{O}_X$). Let G' be any abelian sheaf in the fppf (resp. étale) topology on X. Then as sheaves in the fppf (resp. étale) topology on a scheme X we have*

1. $\underline{\mathrm{Ext}}^2(G, G') = 0$.
2. $\underline{\mathrm{Ext}}^1(G, G')$ *is right exact.*
3. $\underline{\mathrm{Ext}}^1(G, G') = 0$ *if G' is any quotient of $\mathcal{O}_X^*$.*

Proof: This is certainly true if G is free, and since G is a direct sum of a free abelian group and a torsion group, we may as well assume that G is a torsion group. Since G admits a finite free resolution of length 1, $\underline{\mathrm{Ext}}^2(G, \ ) = 0$ and consequently $\underline{\mathrm{Ext}}^1(G, \ )$ is right exact. Thus we have already proved (1) and (20. If n is the order of G, multiplication by n on $\mathcal{O}_X^*$ is surjective in the fppf (resp étale) topology, and it follows from (2) that it is also surjective on $\underline{\mathrm{Ext}}^1(G, \mathcal{O}_X^*)$. Since n annihilates G, it also annihilates $\underline{\mathrm{Ext}}^1(G, \mathcal{O}_X^*)$, and consequently $\underline{\mathrm{Ext}}^1(G, \mathcal{O}_X^*) = 0$. Then the right exactness of $\underline{\mathrm{Ext}}^1(G, \ )$ implies that the same is true with $\mathcal{O}_X^*$ replaced by any quotient G'. □

Proposition 2.2.17 *Let X be a fine log scheme and let $\overline{x} \to X$ be a geometric point. Suppose that the order of the torsion subgroup of $\overline{M}^{gp}_{X,\overline{x}}$ is invertible in $k(\overline{x})$. Then locally in an étale neighborhood of x in $\underline{X}$, M_X admits a chart which is good at $\overline{x}$.*

Proof: Let $\overline{x}$ be a geometric point of X lying over x, and consider the exact sequence of abelian groups:

$$0 \longrightarrow \mathcal{O}^*_{X,\overline{x}} \xrightarrow{\lambda} M^{gp}_{X,\overline{x}} \xrightarrow{\pi} \overline{M}^{gp}_{X,\overline{x}} \longrightarrow 0$$

Let $L := \overline{M}^{gp}_{X,\overline{x}}$; then by (2.2.16.3) (applied with $G' = \mathcal{O}_X^*$), there is a map $\phi\colon G \to M^{gp}_{X,\overline{x}}$ such that $\pi \circ \phi$ is the identity. Then ϕ is a markup of $M_{X,\overline{x}}$, and, just as in the proof of (2.2.15), the corresponding chart in a neighborhood of $\overline{x}$ is good at x. □

We now turn to the considerably more complicated relative case. The charts constructed in the following theorem, due to K. Kato, are sometimes called *neat* charts. Recall from (1.2.8) that if $f: X \to Y$ is a morphism of log schemes, $M_{X/Y}$ is the cokernel of $f^*M_Y \to M_X$.

Theorem 2.2.18 *Let $f: X \to Y$ be a morphism of fine log schemes, and let $\gamma: Q \to M_Y$ be a fine chart for M_Y. Then in a flat neighborhood of any geometric point $\overline{x}$ of X, there exists a* neat chart *for f, i.e., a chart for f*

$$\begin{tikzcd} P \arrow[r, "\beta"] & M_X \\ Q \arrow[u, "\theta"] \arrow[r, "\gamma"] & M_Y \arrow[u, "f^\flat"'] \end{tikzcd}$$

with the following properties:

1. *$\theta^{gp}: Q^{gp} \to P^{gp}$ is injective,*
2. *the map $P^{gp}/Q^{gp} \to M^{gp}_{X/Y,\overline{x}}$ induced by β is bijective, and*
3. *β is exact at $\overline{x}$.*

If the order of the torsion part of $M^{gp}_{X/Y,\overline{x}}$ is a unit in $k(\overline{x})$, then such a chart exists in an étale neighborhood of $\overline{x}$.

Proof: Let $\overline{y} := f(\overline{x})$, let $N_{\overline{x}}$ denote the image of $(f^*M_Y)_{\overline{y}}$ in $M_{X,\overline{x}}$ and let Q' denote the image of Q in $M_{X,\overline{x}}$. Consider the exact sequences:

$$0 \to N^{gp}_{\overline{x}} \to M^{gp}_{X,\overline{x}} \to M^{gp}_{X/Y,\overline{x}} \to 0$$

$$0 \to \mathcal{O}^*_{X,\overline{x}} \to f^*M^{gp}_{Y,\overline{x}} \to \overline{M}^{gp}_{Y,y} \to 0.$$

Because $Q \to M_{Y,y}$ is a chart, the map $Q^{gp} \to \overline{M}^{gp}_{Y,y}$ is surjective, and consequently $N^{gp}_{\overline{x}}$ is the subgroup of $M^{gp}_{X,\overline{x}}$ generated by $\mathcal{O}^*_{X,\overline{x}}$ and Q'^{gp}. Thus the map $\mathcal{O}^*_{X,\overline{x}} \to N^{gp}_{\overline{x}}/Q'^{gp}$ is surjective, and it follows from (2.2.16.3) that $\mathrm{Ext}^1(M^{gp}_{X/Y,\overline{x}}, N^{gp}_{\overline{x}}/Q'^{gp})$ vanishes in the appropriate topology. Then the exact sequence

$$\mathrm{Ext}^1(M^{gp}_{X/Y,\overline{x}}, Q'^{gp}) \to \mathrm{Ext}^1(M^{gp}_{X/Y,\overline{x}}, N^{gp}_{\overline{x}}) \to \mathrm{Ext}^1(M^{gp}_{X/Y,\overline{x}}, N^{gp}_{\overline{x}}/Q'^{gp})$$

shows that the extension class in $\mathrm{Ext}^1(M^{gp}_{X/Y,\overline{x}}, N^{gp}_{\overline{x}})$ corresponding to the first of the exact sequences above lifts to a class in $\mathrm{Ext}^1(M^{gp}_{X/Y,\overline{x}}, Q'^{gp})$. Since $Q^{gp} \to Q'^{gp}$ is surjective, it follows from (2.2.16.2) that this class lifts to a class in $\mathrm{Ext}^1(M^{gp}_{X/Y,\overline{x}}, Q^{gp})$. In other words, there is a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & Q^{gp} \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, "\phi"'] & M^{gp}_{X/Y,\overline{x}} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & N^{gp}_{\overline{x}} \arrow[r] & M^{gp}_{X,\overline{x}} \arrow[r] & M^{gp}_{X/Y,\overline{x}} \arrow[r] & 0
\end{tikzcd}$$

Since the map $M_X \to M_{X/Y}$ factors through $\overline{M}_X$ and M_X is fine, the monoid $M^{gp}_{X/Y,\overline{x}}$ is also fine, and in particular $M^{gp}_{X/Y,\overline{x}}$ is a finitely generated group. Since Q is fine, Q^{gp} is also finitely generated, and it follows that the same is true of L. Moreover, the map $Q^{gp} \to N^{gp}_{\overline{x}}$ is surjective, and it follows from the diagram that $L \to \overline{M}^{gp}_{X,\overline{x}}$ is also surjective. Thus ϕ is a markup of $M_{X,\overline{x}}$. It follows immediately that the corresponding chart $P \to M_{X,\overline{x}}$ fits into the diagram in the statement of the theorem and satisfies conditions (1)–(3). $\square$

Remark 2.2.19 Suppose in the situation of the previous theorem that f induces an injection $\overline{M}_{Y,\overline{y}} \to \overline{M}_{X,\overline{x}}$ and that $Q \to M_Y$ is good at $\overline{y}$. Then $P \to M_X$ is also good at $\overline{x}$. Indeed, we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & Q^{gp} \arrow[r] \arrow[d, "\cong"'] & L \arrow[r] \arrow[d] & M^{gp}_{X/Y,\overline{x}} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \overline{M}^{gp}_{Y,\overline{y}} \arrow[r] & \overline{M}^{gp}_{X,\overline{x}} \arrow[r] & M^{gp}_{X/Y,\overline{x}} \arrow[r] & 0
\end{tikzcd}$$

This shows that $L \to \overline{M}^{gp}_{X,\overline{x}}$ is an isomorphism.

If $\theta\colon P \to M$ is a chart for M and $\gamma\colon P \to M^*$ is any homomorphism, then $\theta+\gamma$ is again a chart for M. In fact it is almost true that any two charts can be compared in this way. For the sake of simplicity of exposition, we begin with the following easy special case, which we shall generalize later.

Proposition 2.2.20 *Let $\theta\colon P \to M$ and $\theta'\colon P' \to M$ be fine charts for a fine sheaf of monoids M on X. Suppose that P^{gp} is torsion free and that θ' is exact at a geometric point $\overline{x}$ of X. Then in some neighborhood of $\overline{x}$ in X, there exist maps $\kappa\colon P \to P'$ and $\gamma\colon P \to M^*$ such that $\theta = \theta' \circ \kappa + \gamma$.*

Proof: The fact that θ' is exact and x implies that $\overline{\theta}'\colon \overline{P}' \to \overline{M}_{\overline{x}}$ is an isomorphism. Let $\overline{\kappa}$ denote the composition of the θ with the map $M_{\overline{x}} \to \overline{M}_{\overline{x}}$ followed by the inverse of $\overline{\theta}'$. Then $\overline{\theta}' \circ \overline{\kappa}$ is the map $P \to \overline{M}_{\overline{x}}$ induced by θ. Since P^{gp} is a finitely generated free abelian group, there exists a map $\kappa^{gp}\colon P^{gp} \to P'^{gp}$ lifting $\overline{\kappa}^{gp}$. By the exactness of θ', κ^{gp} maps $P \to P'$. Thus there is a map $\kappa\colon P \to P'$ such that $\overline{\theta}' \circ \overline{\kappa} = \overline{\theta}$. Then for every $p \in P$, $\gamma(p) := \theta(p) - \theta'\kappa(p) \in M^*$. □

More generally, the existence of torsion may necessitate a localization the étale or flat topology.

Proposition 2.2.21 *Let $f\colon X \to Y$ be a morphism of fine log schemes with two fine charts*

$$\begin{tikzcd} P \arrow[r, "\alpha"] & M_X \\ Q \arrow[u, "\theta"] \arrow[r, "\beta"] & f^{-1}(M_Y) \arrow[u, "f^\flat"'] \end{tikzcd} \qquad \begin{tikzcd} P' \arrow[r, "\alpha'"] & M_X \\ Q \arrow[u, "\theta'"] \arrow[r, "\beta"] & f^{-1}(M_Y) \arrow[u, "f^\flat"'] \end{tikzcd}$$

for f. Suppose that α' is exact at a geometric point $\overline{x}$ of X and that θ^{gp} is injective. Then after replacing P' by a mild pushout and X by a quasi-finite and flat neighborhood of $\overline{x}$, there exist maps $\kappa\colon P \to P'$ and $\gamma\colon P \to M^*$ such that

$$\kappa \circ \theta = \theta', \quad \alpha' \circ \kappa = \gamma + \alpha, \quad \text{and} \quad \gamma^{gp} \circ \theta^{gp} = 0.$$

If the order of the torsion of the cokernel of θ^{gp} is invertible on X then the neighborhood $\tilde{X} \to X$ can be taken to be étale.

We begin with the following elementary construction.

Definition 2.2.22 *A morphism $P \to \tilde{P}$ of integral monoids is said to be a mild pushout if the diagram*

$$\begin{tikzcd} P^* \arrow[r] \arrow[d] & \tilde{P}^* \arrow[d] \\ P \arrow[r] & \tilde{P} \end{tikzcd}$$

is cocartesian and the quotient $\tilde{P}^*/P^*$ is a finite group.

Lemma 2.2.23 *Let $P \to \tilde{P}$ be a mild pushout and let R be a ring. Then the map $R[P] \to R[\tilde{P}]$ is finite and flat, and it is étale if the order of $\tilde{P}^*/P^*$ is invertible in R.*

Proof: Because $\tilde{P}$ is the pushout, the map

$$R[P] \otimes_{R[P^*]} R[\tilde{P}^*] \to R[\tilde{P}]$$

is an isomorphism. Thus we are reduced showing that $R[P^*] \to R[\tilde{P}^*]$ is flat or étale. The flatness follows from (**??**). It can also be seen directly from the fact that as a P-set, $\tilde{P}$ is a union of its P-cosets, each of which is a free P-set, and so as an $R[P]$-module, $R[\tilde{P}]$ is a direct sum of free $R[P]$-modules, hence is free. For the last statement, it is enough to show that if the order of $\tilde{P}^*/P^*$ is invertible in R, then the map $R[P] \to R[\tilde{P}]$ is unramified. The easiest way to see this is to use the fact (**??**) that, for any abelian group G, there is a natural isomorphism $\Omega^1_{R[G]/R} \to R \otimes G$. Then the module of relative Kahler differentials of our map can be identified with $R \otimes \tilde{P}^*/P^*$, which vanishes if the order of $\tilde{P}^*/P^*$ is invertible in R. □

Lemma 2.2.24 *Let P be an integral monoid and let $P^* \to G$ be an injective homomorphism of abelian groups such that G/P^* is finitely generated. Then there is a mild pushout $P \to \tilde{P}$ such that the induced map $P^* \to \tilde{P}^*$ factors through G and such that the quotient $\tilde{P}^*/P^*$ is isomorphic to the torsion subgroup of G/P^*.*

Proof: Let G' be the inverse image in G of the torsion subgroup of G/P^*. Then G/G' is finitely generated and torsion free, hence free, so there is a splitting of the inclusion $G' \to G$, and the map $P^* \to G'$ factors through G. Let $\tilde{P}$ be the pushout of $P^* \to P$ by the map $P^* \to G'$. Then $\tilde{P}^* - G'$, and $P \to \tilde{P}$ is a mild pushout as required. $\square$

Proof of (2.2.21): Since α' is an exact chart, $\overline{\alpha}'$ is an isomorphism. Let $\overline{\kappa}$ be the composition of $P \to M \to \overline{M}$ with the inverse of $\overline{\alpha}'$ and let $\phi := f^\flat \circ \beta$. Then we have a diagram:

$$\begin{tikzcd}
0 \arrow[r] & P'^* \arrow[r] & P'^{gp} \arrow[r, "\pi'"] & \overline{P}'^{gp} \arrow[r] & 0 & \\
 & 0 \arrow[r] & Q^{gp} \arrow[r] \arrow[u, "\theta'"] & P^{gp} \arrow[r] \arrow[u, "\overline{\kappa}"'] & C \arrow[r] & 0
\end{tikzcd}$$

The obstruction to lifting $\overline{\kappa}$ to a map $P^{gp} \to P'^{gp}$ lies in $Ext^1(P^{gp}, P'^*)$, and is in fact the pullback of the upper row of the diagram by means of $\overline{\kappa}^{gp}$ However, because of the existence of θ', this obstruction dies in $Ext^1(Q^{gp}, P'^*)$, and hence comes from an element in $\operatorname{Ext}^1(C, P'^*)$. By lemma (2.2.23), a mild pushout along P'^* kills this element, so that we may assume that there exists $\kappa\colon P^{gp} \to P'^{gp}$ with $\pi'\kappa = \overline{\kappa}$. Since α' is exact, κ in fact maps P to P'. Now let $\delta := \kappa\theta - \theta'$. Then $\pi'\delta = 0$, so that in fact δ is a map from Q to P'^*. The obstruction to extending it to P lies in $Ext^1(C, P'^*)$, and another mild pushout $P' \to \tilde{P}'$ kills it. Since the composition of mild pushouts is another mild pushout, this is allowed. But now if δ' extends δ, we may replace κ by $\kappa - \delta$, and then $\kappa\theta' = \theta$.

The chart α' for the log structure $\alpha_X\colon M_X \to \mathcal{O}_X$ defines a strict morphism of log schemes $X \to \mathsf{A}_{\mathsf{P}'}$. Let $\tilde{X}$ be the Cartesian product

$$\begin{tikzcd}
\tilde{X} \arrow[r] \arrow[d] & \mathsf{A}_{\tilde{\mathsf{P}}'} \arrow[d] \\
X \arrow[r] & \mathsf{A}_{\mathsf{P}'},
\end{tikzcd}$$

where the map on the right is induced by the mild pushout $P' \to \tilde{P}'$. This map is finite and flat by (2.2.23). Suppose be the order m of the torsion subgroup of C is invertible in $k(\overline{x})$. Then in some neighborhood of $\overline{x}$, the map $X \to \operatorname{Spec} \mathbf{Z}$ lifts to $\operatorname{Spec} \mathbf{Z}[1/m]$, and if we work over this base, everything becomes étale. Since the map $\tilde{P}' \to M_{\tilde{X}}$ is again a chart, we may as well assume that $\tilde{P}' = P'$ and that $\tilde{X} = X$.

Finally, observe that, from the definition of κ, it follows that $\overline{\alpha}' \circ \overline{\kappa} = \overline{\alpha}$, and hence that $\alpha' \circ \kappa - \alpha$ factors through M^*. In fact, since $\kappa'\theta = \theta'$, $\alpha' \circ \kappa - \alpha$ also factors through the cokernel of θ. This shows that there is a map γ with the desired properties. □

Remark 2.2.25 If in the situation of the proposition (2.2.21) θ is neat (2.2.18) and θ'^{gp} is injective, then κ^{gp} is also injective. Indeed, if $p \in P^{gp}$ and $\kappa^{gp}(p) = 0$, then $\pi\alpha(p) = \pi'\alpha'\kappa(p) = 0$, and since θ is neat, it follows that p maps to zero in P^{gp}/Q^{gp}. Thus $p = \theta(q)$ for some $q \in Q^{gp}$, and so $0 = \kappa(p) = \kappa(\theta(q)) = \theta'(q) = 0$. Since θ' is injective, it follows that $q = 0$. We should also remark that if α' is good, no mild pushouts are necessary, and the construction of κ and γ is much simpler.

2.3 Constructibility and coherence

This section has not yet been rewritten or covered in lectures

It is possible to give a fairly explicit description of what it means for a sheaf of integral monoids to be coherent. As we saw in (), a log structure for the étale topology on X is coherent if and only if X admits an étale covering on which the associated Zariski log structure is coherent. Since coherence is a condition that can be verified étale locally, it therefore will be sufficient to work with the Zariski topology, and we shall do so in the current section.

Recall from [9, 0 (2.1.1)] that a topological space is said to be *sober* if every irreducible subset contains a unique generic point.

Definition 2.3.1 *Let X be a sober noetherian topological space and let E be a sheaf of sets on X. A trivializing stratification for E is a finite subset Σ of locally closed connected subsets S of X such that*

1. *$X = \cup_\Sigma S$ and $S \cap T = \emptyset$ if S and T are distinct elements of Σ.*

2. *If S_1 and S_2 are elements of Σ and $S_1 \cap \overline{S}_2 \neq \emptyset$, then $S_1 \subseteq \overline{S}_2$.*

3. *The restriction of E to each $S \in \Sigma$ is constant.*

We say that a sheaf E on X is quasi-constructible *if X has a trivializing stratification for E.*

For example, if X is a finite Kolmogoroff space, each point is locally closed, and the set Σ of singleton subsets of X is a stratification of X satisfying the above conditions. Thus any sheaf on X admits a trivializing stratification. Furthermore, if $X \to Y$ is a continuous map and Σ is a trivializing partition for E on Y, then the set of connected components of the elements of $f^{-1}(\Sigma)$ is a trivializing stratification for $f^{-1}(E)$ on X.

If Σ is a trivializing stratification for E and $s \in S \in \Sigma$, then since $E_{|S}$ is constant and S is connected, the natural map $E(S) \to E_s$ is an isomorphism. We write E_S for $E(S)$ to emphasize this. If x and y are points of X such that $x \in y^-$, then every neighborhood U of x contains y, and the compatible family of maps $E(U) \to E_y$ induce a *cospecialization map*

$$cosp_{x,y} \colon E_x \to E_y.$$

If S and T are elements of S with $S \subseteq T^-$, and $s \in S$ and $t \in T$, there is a commutative diagram

$$\begin{tikzcd}
E_S \arrow[r, "cosp_{S,T}"] \arrow[d, "\cong"'] & E_T \arrow[d, "\cong"] \\
E_s \arrow[r, "cosp_{s,t}"] & E_t
\end{tikzcd}$$

Theorem 2.3.2 *An integral sheaf of monoids M on a locally noetherian sober topological space X is fine if and only if it satisfies the following three conditions:*

1. *X admits an open covering on which E is quasi-constructible.*

2. *For each $x \in X$, $\overline{M}_x$ is finitely generated.*

3. *Whenever x and ξ are points of X with $x \in \overline{\xi}$, the cospecialization map $cosp_{x,\xi} \colon \overline{M}_x \to \overline{M}_\xi$ identifies $\overline{M}_\xi$ with the quotient of $\overline{M}_x$ by a face.*

Proof: Suppose that M is fine. Properties (1) through (3) are local on X, so we may assume that X is noetherian and by (2.2.13) that M admits a fine chart $P \to M$. Let $h\colon X \to S := \mathrm{Spec}(P)$ be the corresponding map of locally monoidal spaces. Then by (2.1.4), $\overline{M} \cong h^{-1}\overline{M}_S$. Since S is a finite Kolmogoroff space, $\overline{M}_S$ is quasi-constructible, and hence so is $\overline{M}$. Furthermore, properties (2) and (3) hold for $\overline{M}_S$, and hence also for $\overline{M}$.

Now suppose that M satisfies the conditions (1) through (3) and let x be a point of X. Since $\overline{M}_x^{gp}$ is finitely generated, M_x admits a markup $L \to M_x$, and since $\overline{M}_x$ is finitely generated, $P := L \times_{\overline{M}_x^{gp}} \overline{M}_x$ is a fine monoid by (2.1.15). By (2.2.5), there exist an open neighborhood U of x and a map $\beta\colon P \to M(U)$ inducing the map $P \to M_x$. If $y \in U$, let $\overline{P}_y := \overline{P_y^a} \cong P/(\beta^{-1}M_y^*)$, and let W be the set of y such that the map $\overline{P}_y \to \overline{M}_y$ is an isomorphism. It will suffice to prove that W is open in X.

If y and ξ are points of X and $y \in \xi^-$, there is a commutative diagram:

$$\begin{tikzcd}
\overline{P}_y \arrow[r, "\overline{\beta}_y^a"] \arrow[d, "cosp_P"'] & \overline{M}_y \arrow[d, "cosp_M"] \\
\overline{P}_\xi^a \arrow[r, "\overline{\beta}_\xi^a"'] & \overline{M}_\xi
\end{tikzcd}$$

If $y \in W$, then β_y^a is an isomorphism. By condition (3), $cosp_M$ is the quotient by a face, and since P^a is coherent, the same is true of $cosp_P$. It follows that β_ξ^a is also an isomorphism, so that W is stable under generization. If $\xi \in W$, let S (resp. T) denote the stratum of the trivializing partition for $\overline{P}_X^a$ (resp. for $\overline{M}$) containing ξ. Since S and T are locally closed, $S \cap T$ contains a neighborhood U of ξ in ξ^-. Then for any point $y \in U \subseteq \xi^-$, the $cosp_P$ and $cosp_M$ are isomorphisms. Since β_ξ is an isomorphism, it follows that β_x is also an isomorphism, so $y \in W$. This shows that if $\xi \in W$, W contains a nonempty open subset of ξ^-. Since W is also stable under generization, it is open, by [7, 0_{III},9.2.6], and $P_X \to M$ is a chart of M on W. $\square$

Definition 2.3.3 *A sheaf of monoids M on a locally noetherian sober topological space is locally constructible if it satisfies (1) and (2) of (2.3.2).*

Corollary 2.3.4 *If X is a fine log scheme then $\overline{M}_X$ satisfies the conditions (1)–(3) of the Theorem (2.3.2).*

Corollary 2.3.5 *If X is a fine log scheme and n is an integer, then*

$$X^{(n)} := \{x \in X : \mathrm{rk}(\overline{M}^{gp}_{X,\overline{x}}) \leq n\}$$

is an open subset of X.

Proof: We may assume without loss of generality that X is noetherian. By (3), if x in $X^{(n)}$ and $x \in \xi^-$, then $\xi \in X^{(n)}$, *i.e.,* $X^{(n)}$ is stable under generization. Also if $\xi \in X^{(n)}$ and S is the stratum containing ξ, then S contains a dense open subset of ξ^-, and for each point s of S, $\mathrm{rk}(\overline{M}^{gp}_{X,s}) = \mathrm{rk}(\overline{M}^{gp}_{X,\xi}) \leq n$. Then by [7, 0_{III}, 9.2.6], $X^{(n)}$ is open. $\square$

We shall say that a stratum S of a trivializing partition Σ for E is a *central stratum* if S is contained in the closure of every element of Σ, and we say that a point x is a *central point* of Σ if x belongs to the closure of every element of Σ. It follows from (2) in the definition of a trivializing partition that x is a central point of Σ if and only if the stratum containing it is a central stratum for Σ. Any point of X has a neighborhood U such that x is a central point for $\Sigma_{|_U}$: it suffices to take U to be the complement of the closures of all the strata whose closures don't contain x.

Proposition 2.3.6 *Let E be a quasi-constructible sheaf on a noetherian topological space X and let x be a point of X. Then for all sufficiently small neighborhoods U of x in X, the natural map $E(U) \to E_x$ is an isomorphism.*

Proof: For each $S \in \Sigma$, let F_S be the set of irreducible components of S^-. Then $\{F \in F_S : x \notin F\}$ is a finite set of closed subsets of X not containing x. Removing all these from X, we may without loss of generality assume that x belongs to the closure of every element of F_S. This remains true on every open neighborhood of x in X, so it will suffice to prove that the map $E(X) \to E_x$ is an isomorphism. Note that x is necessarily a central point of X.

Lemma 2.3.7 *If z is a central point of X, the map $E(X) \to E_z$ is injective.*

Proof: For each $y \in X$, let $S(y)$ be the stratum containing y. Then $z \in S(y)^-$, so there is a commutative diagram:

$$\begin{tikzcd} E(X) \arrow[r] \arrow[d] & E_z \arrow[d, "cosp_{z,S(y)}"] \\ E_y \arrow[r, "\cong"] & E_{S(y)} \end{tikzcd}$$

Hence if e and e' are elements of $E(X)$ with the same stalk at z, they have the same stalk at every $y \in X$, hence they are equal. □

Applying this lemma with $z = x$, we see that the map $E(X) \to E_x$ is injective. For the surjectivity, suppose $s \in E_x$, and let U be a neighborhood of x in X and $e \in E(U)$ such that $e_x = s$; e is unique by (2.3.7). Since X is quasi-compact, we may suppose that U is a maximal open subset of X to which e extends, and we claim that in fact $U = X$. If not, let z be a point in $X \setminus U$, let Y be the irreducible component of the stratum S in Σ containing z, and let η be the generic point of Y. Then x and z both belong to η^-, and $cosp_{z,\eta}$ is an isomorphism. Hence there exist an open neighborhood V of z in X and an element $f \in E(V)$ such that

$$cosp_{z,\eta}(f_z) = cosp_{x,\eta}(e_x).$$

Shrinking V, we may assume that z is a central point for $\Sigma_{|V}$, and then η is a central point for $\Sigma_{|U\cap V}$. Since $e_{|U\cap V}$ and $f_{|U\cap V}$ have the same stalk at $\sigma(z)$, it follows from (2.3.7) that they agree on $V \cap U$, hence patch to a section of $E(U \cup V)$, contradicting the maximality of U. □

Proposition 2.3.8 *If M_X is a fine log structure on a locally noetherian scheme X, then for every quasi-compact open set U of X, $\Gamma(U, \overline{M}_X)$ is fine.*

Proof: Suppose first that M_X is a fine log structure for the Zariski topology. Then U is noetherian, and by (2.3.2) U admits a trivializing partition for $\overline{M}_X$. Then by (2.3.6), every point x admits an open neighborhood U_x contained in U such that the map $\overline{M}_X(U_x) \to \overline{M}_{X,\overline{x}}$ is an isomorphism. In particular, $\overline{M}_X(U_x)$ is a fine monoid. Since U is quasi-compact, there exists a finite set $\{U_{x_1}, \ldots U_{x_n}\}$ of these neighborhoods which cover U, and we prove that $\Gamma(U_m, \overline{M}_X)$ is fine by induction on m, where $U_m := \cup\{U_{x_i} : i \leq m\}$. In fact, $\Gamma(U_m, \overline{M}_X)$ is the fiber product of $\Gamma(U_{m-1}, \overline{M}_X)$ and $\Gamma(U_{x_m}, \overline{M}_X)$ over the integral monoid $\Gamma(U_m \cap U_{x_m}, \overline{M}_X)$, so it is fine by (I, 2.1.9.6).

Now suppose that M_X is a fine log structure for the étale topology. Then U admits an étale covering $U' \to U$ over which M_X is a fine log structure for the Zariski topology (2.1.11); U' is quasi-compact since U is. Since $\Gamma(U, \overline{M}_X)$ is the equalizer of the two maps $\Gamma(U', \overline{M}_X) \to \Gamma(U' \times_U U', \overline{M}_X)$ and since $\Gamma(U', \overline{M}_X)$ is fine and $\Gamma(U' \times_U U', \overline{M}_X)$ is integral, $\Gamma(U, \overline{M}_X)$ is fine by (I,2.1.9.5) $\square$

2.4 Fibered products of log schemes

Just as in the case of ordinary schemes, the existence of products in the category of log schemes has deep consequences and many subtleties.

Proposition 2.4.1 *Let X be a scheme. Then the category of prelog (resp. log) structures on X admits inductive limits. The inductive limit of a finite family of coherent log structures is coherent.*

Proposition 2.4.2 *The category of log schemes admits fibered products, and the functor $X \to \underline{X}$ taking a log scheme to its underlying scheme commutes with fibered products. The fibered product of coherent log schemes is coherent.*

Proof: Let $\{\alpha_i : M_i \to \mathcal{O}_X : i \in I\}$ be an inductive family of prelog structures on X and let M be the inductive limit of the system M_i in the category of sheaves of monoids on X. Then the maps α_i induce a map $\beta: M \to \mathcal{O}_X$, and β is the inductive limit of $\{\alpha_i : i \in I\}$ in the category of prelog structures on X. If each α_i is in fact a log structure, then the log structure $\alpha := \beta^a$ associated to β is the limit of $\{\alpha_i : i \in I\}$ in the category of log structures on X. It remains to show that α is coherent if each α_i is coherent and I is finite. It suffices to treat the case of amalgamated sums. Given a pair of maps of coherent log structures

$$\begin{tikzcd} \alpha_0 \arrow[r, "\theta_1"] \arrow[d, "\theta_2"'] & \alpha_1 \\ \alpha_2, & \end{tikzcd}$$

let α be the amalgamated sum in the category of log structures. We may assume that α_0 admits a chart β_0 subordinate to a finitely generated monoid Q_0. By (2.2.3) we may, after shrinking X if necessary, find coherent charts $\phi_i: Q_0 \to Q_i$ for the morphisms θ_i. Let Q be the amalgamated sum $Q_1 \oplus_{Q_0} Q_2$, with its canonical map $\beta: Q \to M$. Because the functor $\beta \mapsto \beta^a$ is a left adjoint, it commutes with inductive limits, and it follows that $\beta^a \cong \alpha$, in other words, that β is a chart for α. This proves (1). For (2), it suffices to construct fibered products, and if $f: X \to Z$ and $g: Y \to Z$ are morphisms of schemes with coherent log structures, then on the fibered product X' of underlying schemes we have a pair of morphisms of log structures $pr_Z^*\alpha_Z \to pr_X^*\alpha_X$ and $pr_Z^*\alpha_X \to pr_Y^*\alpha_Y$. One checks immediately that, if $\alpha_{X'}$ is the inductive limit of this family in the category of log structures, then $(X', \alpha_{X'})$ together with the induced maps to X, Y, and Z, is the fibered product in the category of coherent log schemes. $\square$

Remark 2.4.3 It follows from the construction of fibered products that the family of strict maps is stable under base extension.

Remark 2.4.4 If X is a log scheme, let $\underline{X}$ denote the log scheme with the same underlying scheme but with trivial log structure. Then there is a natural morphism of log schemes $X \to \underline{X}$, and a morphism $f: X \to Y$ of log schemes fits into a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ \underline{X} \arrow[r, "\underline{f}"] & \underline{Y} \end{tikzcd}$$

If f is strict, this diagram is Cartesian. In particular, if $Y = \mathrm{Spec}(P \to \mathbf{Z}[P])$ and f is a chart for X corresponding to a morphism $\beta\colon P \to \Gamma(X, M_X)$, then for any log scheme T, to give a morphism $g\colon T \to X$ is the same as to give a morphism $\underline{g}\underline{T} \to \underline{X}$ and a morphism $\gamma\colon P \to \Gamma(T, M_T)$ such that the following diagram commutes:

$$\begin{tikzcd}
P \arrow[r, "\gamma"] \arrow[d] & \Gamma(T, M_T) \arrow[d, "\alpha_T"] \\
\Gamma(X, \mathcal{O}_X) \arrow[r, "\underline{g}^\sharp"] & \Gamma(T, \mathcal{O}_T).
\end{tikzcd}$$

If $f\colon X \to Y$ is any morphism of log schemes, let $i\colon X \to X'$ and $f^s\colon X_Y \to Y$ be the canonical factorization of f, with f^s strict. These maps fit into a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[dr, "f"'] & X' \arrow[r] \arrow[d, "f^s"'] & \underline{X} \arrow[d, "\underline{f}"] \\
 & Y \arrow[r] & \underline{Y},
\end{tikzcd}$$

in which the square is Cartesian.

Since the amalgamated sum of integral (resp. saturated) monoids need not be integral (resp. saturated), the construction of fibered products in the category of fine (or fs) log schemes is more delicate, and in fact involves some of the main technical difficulties of logarithmic algebraic geometry. We will make use of the following construction

Proposition 2.4.5

1. *The inclusion functor from the category of fine log schemes to the category of coherent log schemes admits a right adjoint $X \mapsto X^{\mathrm{int}}$, and the corresponding morphism of underlying schemes $\underline{X}^{\mathrm{int}} \to \underline{X}$ is a closed immersion.*

2. *The inclusion functor from the category of fs log schemes to the category of fine log schemes admits a right adjoint $X \mapsto X^{\mathrm{sat}}$, and the corresponding morphism of underlying schemes $\underline{X}^{\mathrm{sat}} \to \underline{X}$ is finite and surjective.*

Proof: Suppose that X is a coherent (resp. fine) log scheme, and let F be the functor on the category of fine (resp. fine saturated) log schemes sending T to the set of morphisms $T \to X$. We wish to prove that F is representable. Suppose first that there is a coherent (resp. fine) chart $f\colon X \to \mathsf{A_P}$ for X, where $\mathsf{A_P} := \mathrm{Spec}(P \to \mathbf{Z}[P])$. Notice that if F is representable by a some $X' \to X$, then $X' \to X$ is unique up to unique isomorphism, independent of the choice of f. Let $P' := P^{\mathrm{int}}$ (resp. P^{sat}). Since $\mathbf{Z}[P] \to \mathbf{Z}[P']$ is surjective (resp. injective and finite (2.2.5)), the natural map $\mathsf{A_{P'}} \to \mathsf{A_P}$ is a closed immersion (resp. a finite surjective morphism). Let $X' := X \times_{\mathsf{A_P}} \mathsf{A_{P'}}$. Since $X \to \mathsf{A_P}$ is strict, it follows that $X' \to \mathsf{A_{P'}}$ is strict, and hence by (2.4.4) that X' is integral (resp. saturated). If T is a fine (resp. fine and saturated) log scheme, then by (2.4.4) a morphism $f\colon T \to X$ can be viewed as a morphism $\underline{f}\colon \underline{T} \to \underline{X}$ together with a compatible map $P \to \Gamma(T, M_T)$. Since $\Gamma(T, M_T)$ is integral (resp. saturated), the map $P \to \Gamma(T, M_T)$ factors uniquely through P', and it follows that the map $T \to \mathsf{A_P}$ factors uniquely through $\mathsf{A_{P'}}$ and hence that the map $T \to X$ factors uniquely through X'. Thus X' represents the functor F. In the general case, X admits an étale covering $\tilde{X} \to X$, where $\tilde{X}$ is a union of open sets each of which admits a chart. It follows that the functor $\tilde{F}$ corresponding to $\tilde{X}$ is representable by a fine (resp. fine and saturated) log scheme $\tilde{X}' \to \tilde{X}$. Furthermore, the underlying morphism of schemes $\underline{\tilde{X}}' \to \underline{\tilde{X}}$ is a closed immersion (resp. a finite surjective morphism), and in either case is relatively affine over $\tilde{X}$. The functorial interpretation of $\tilde{X}'$ provides it with descent data for the covering $\tilde{X} \to X$. It follows from the descent of relatively affine schemes for the étale toplogy [5, I-2] that there is an affine morphism $\underline{X}' \to \underline{X}$ corresponding to $\underline{\tilde{X}'} \to \tilde{X}$, and the log structure on $\tilde{X}'$ descends to X' since sheaves in the étale topology also satisfy étale descent. $\square$

Notice that the morphisms of topological spaces underlying the maps $X^{\mathrm{int}} \to X$ and $X^{\mathrm{sat}} \to X^{\mathrm{int}}$ are not in general homeomorphisms, and in particular that we cannot identify $M_{X^{\mathrm{int}}}$ with M_X^{int} or $M_{X^{\mathrm{sat}}}$ with M_X^{sat}, in general.

Corollary 2.4.6 *The category of fine log schemes (resp. of fine and saturated log schemes) admits finite projective limits. If X and Y are fine (resp. fine and saturated) log schemes over a fine (resp. fine and saturated) log scheme Z then the natural map from the underlying scheme of the fibered product $X \times_Z Y$ to the fibered product of underlying schemes is a closed immersion (resp. a finite morphism).*

Proof: If $X \to Z$ and $Y \to Z$ is a pair of morphisms of fine log schemes, then it follows from the universal mapping properties that $(X \times_Z Y)^{\mathrm{int}}$, together with its induced maps to X, Y, and Z, is the fibered product of X and Y over Z in the category of fine log schemes. The analogous construction works for fine and saturated log schemes. □

2.5 Coherent sheaves of ideals and faces

This section has not been rewritten or lectured on.

Let $\theta\colon P \to M$ be a homomorphism from a constant monoid P to a sheaf of monoids M on a topos X and let I be an ideal of P. We denote by I_θ or $\tilde{I}$ the sheaf associated to the presheaf taking an open set U to the ideal of $M(U)$ generated by $\theta_U(I)$. In particular, if $\beta\colon P \to M$ is a chart for M and $K \cong I_\beta$, we say that (P, I) *is a chart for* (M, K).

Definition 2.5.1 *A sheaf of ideals in a sheaf of monoids is* coherent *if it is locally generated by a finite number of sections.*

Theorem 2.5.2 *Let M be a sheaf of monoids on a locally noetherian sober topological space X such that $\overline{M}$ is locally constructible (2.3.3) and let K be a sheaf of ideals in M. Then the following are equivalent:*

1. *K is coherent.*

2. *X can be covered by open sets U for each of which there exists an ideal $I \subseteq M(U)$ such that $K_{|_U} = \tilde{I}$.*

3. *For every pair of points x and ξ of X with $x \in \xi^-$, the image of*

$$cosp_{x,\xi}\colon K_x \to K_\xi$$

generates K_ξ as an ideal in M_ξ.

Proof: If K is coherent and $x \in X$, then x admits a neighborhood U on which K is generated by a finite number of sections $(s_1, \ldots s_n)$. Let I be the ideal of $M(U)$ generated by $(s_1, \ldots s_n)$; then $\tilde{I} \cong K_{|_U}$, so (2) holds. Assuming that $I \subseteq M(U)$ generates $K_{|_U}$ on a neighborhood U of x, then every generization ξ of x is contained in U, so I generates K_ξ and (3) holds. Supposing that (3) is satisfied, let x be a point of X. Since $\overline{M}_x$ is a finitely generated monoid, it follows from (I,2.1.9) that the stalk of K at x is finitely generated as an ideal. Hence there exist an open neighborhood U of x in X and a finite set of sections $(s_1, \ldots s_n)$ of $K(U)$ which generate K_x. Shrinking further, we may assume that x is a central point for some trivializing partition of $\overline{M}$. It will suffice to prove that $\tilde{I} = K_{|_U}$, where I is the ideal of $M(U)$ generated by $(s_1, \ldots s_n)$. We just have to check the stalks, *i.e.*, that for every point x' of U, the map $I \to \overline{K}_{x'}$ generates $\overline{K}_{x'}$ as an ideal of $\overline{M}_{x'}$. Let S be the stratum containing x'. Then x belongs to the closure of S, and hence we have a commutative diagram:

$$\begin{tikzcd}
I \arrow[r, "\beta_{x'}"] \arrow[d, "\beta_x"'] & \overline{K}_{x'} \arrow[r, tail] \arrow[d, "\gamma"'] & \overline{M}_{x'} \arrow[d, "\delta"] \\
\overline{K}_x \arrow[r, "\sigma"] & \overline{K}_S \arrow[r, tail] & \overline{M}_S
\end{tikzcd}$$

The assumption (3) implies that image of γ generates the ideal $\overline{K}_S$ in $\overline{M}_S$. But δ is an isomorphism because $\overline{M}_S$ is constant and $x' \in S$, and it follows that γ is bijective. Furthermore the image of β_x generates $\overline{K}_x$ by construction and the image of σ generates $\overline{K}_S$ by (3). It follows that $\overline{K}_{x'}$ is generated by the image of $\beta_{x'}$, as required. □

Corollary 2.5.3 *Let X be a locally noetherian fine log scheme and let $K \subseteq M_X$ be a coherent sheaf of ideals. Let $\overline{x} \to X$ be a geometric point of X and let $\beta\colon P \to M_X$ be a chart for M_X. Then in some neighborhood of $\overline{x} \to X$, $K \cong I_\beta$, where $I := \beta_{\overline{x}}^{-1}(K_{\overline{x}})$.*

Proof: Replacing X by some étale neighborhood, we may by (2.1.11) assume that M_X is a log structure for the Zariski topology. Arguing as in the proof of (2.5.2), we see that $K \cong I_\beta$ in some neighborhood of x. □

It is also sometimes useful to work with sheaves of faces. For example, let X be a log scheme and U be an open subset of X. Then the subsheaf $\mathcal{F}$ of M_X consisting of those sections whose restriction to U lies in $\mathcal{O}_X^*$ is a sheaf of faces of M_X. Even if $\mathcal{F}$ is not coherent as a sheaf of monoids, it often is "relatively coherent" as a sheaf of faces, as we shall explain now.

Suppose that $\beta\colon P \to M_X$ is a fine chart for a sheaf of monoids. If F is a face of P, let $\tilde{F}$ denote the sheaf associated to the presheaf which to every open set U assigns the face of $M_X(U)$ generated by the image of F in $M_X(U)$. Then $\tilde{F}$ is a sheaf of faces in M_X.

Definition 2.5.4 *Suppose that M is a sheaf of integral monoids on X and $\mathcal{F} \subseteq M$ is a sheaf of faces of M. Then a relative chart for $\mathcal{F}$ is a chart $P \to M$ for M together with a face $G \subseteq P$ such that $F = \tilde{G}$. A sheaf of faces F in a quasi-coherent (resp. coherent) sheaf of monoids M is said to be relatively quasi-coherent (resp. relatively coherent) if locally on X it admits a relative chart.*

A relatively coherent sheaf of faces in a coherent sheaf of monoids need not be coherent as a sheaf of monoids. For a simple example, consider the monoid P given by generators x, y, z and relations $x + y = 2z$. Let F be the face of P generated by $x = 2z - y$ and let $\mathfrak{p}$ be the complement of the face of P generated by y. Then the stalk of $\tilde{F}$ at $\mathfrak{p}$ is the face of P_y generated by x, which is the monoid generated by z, y, and $-y$. Thus $\tilde{F}_\mathfrak{p}/\tilde{F}_\mathfrak{p}^* \cong \mathbf{N}$, with generator the class of z. At the closed point $\mathfrak{m} := P^+$, $\tilde{F}_\mathfrak{m}$ is the free monoid generated by x. Thus the map $\tilde{F}_\mathfrak{m} \to \tilde{F}_\mathfrak{p}/\tilde{F}_\mathfrak{p}^*$ identifies with $\mathbf{N} \xrightarrow{\cdot 2} \mathbf{N}$, and so (2.3.2) shows that $\tilde{F}$ is not coherent. Other examples can be constructed from the nonsimplicial monoid given by x, y, z, w with $x + y = w + z$.

Lemma 2.5.5 *Let M be an integral sheaf of monoids on X and let $\theta\colon G \to M$ be a morphism from a constant monoid to M. Let $\tilde{G}$ denote the sheaf associated to the presheaf which to every object U of X assigns the face of $M(U)$ generated by the image of $G \to M(U)$. Then $\tilde{G}$ is a sheaf of faces of M, and for every quasi-compact object U of X, $\tilde{G}(U)$ is the face of $M(U)$ generated by the image of $G \to M(U)$.*

Proof: For each point x of X, the stalk of $\tilde{G}$ at x is the face of M_x generated by the image of $G \to M_x$. If m_1 and m_2 are elements of $M(U)$ with m_1 and $m_2 \in G(U)$, then $m_1 + m_2 \in \tilde{G}(U)$ if and only if each $m_i \in \tilde{G}(U)$ (check the stalks), so that $\tilde{G}$ is a sheaf of faces of M. If U is quasi-compact and $m \in \tilde{G}(U)$, then there exists a finite cover $(U_i : i \in I)$ such that each $m_{|U_i}$ belongs to the face of $M(U_i)$ generated by the image of $\theta_i \colon G \to M(U_i)$, and hence there exist $g_i \in G$ and $m_i \in M(U_i)$ such that $\theta_i(g_i) = m_i + m_{|U_i}$. Let $g := \sum g_j$, and for each i let $m'_i := m_i + \sum_{j \neq i} \theta_i(g_j) \in M(U_i)$. Then $\theta_i(g) = m'_i + m_{|U_i}$ for all i. Let $m' := \theta_U(g) - m \in M^{gp}(U)$. Then $m'_{|U_i} = m'_i$ for all i, so $m' \in M(U)$. Since $\theta_U(g) = m' + m$ on U, this shows that m belongs to the face of $M(U)$ generated by the image of G. □

Here is an analog of Theorem (2.5.2) for faces; the proof is so similar that we omit it.

Theorem 2.5.6 *Let M be a fine sheaf of monoids on a locally noetherian sober topological space X and let F be a sheaf of faces in M. Then the following are equivalent*

1. *F is relatively coherent.*
2. *F is locally generated (as a sheaf of faces) by a finite set of sections.*
3. *Whenever x and ξ are points of X with $x \in \xi^-$, the image of*

   $$cosp_{x,\xi} \colon F_x \to F_\xi$$

   generates F_ξ as a face in F_ξ.
4. *For every $x \in X$ and every fine chart $\beta \colon P \to M$ in neighborhood U of x, $\tilde{G} \cong F$ in some neighborhood U' of x, where $G := \beta_x^{-1}(F_x)$. In particular, a relatively coherent sheaf of faces satisfies conditions (1) and (2) of Theorem 2.3.2.*

□

Corollary 2.5.7 *Let F be a face of a fine sharp monoid P and let $X := \mathsf{A_P}$ over R, where R is a nonzero ring. Then the relatively coherent sheaf of faces $\mathcal{F}$ in M_X generated by F is coherent if and only if F is a direct summand of P.*

Proof: By Theorems 2.3.2 and 2.5.6, $\mathcal{F}$ is coherent if and only for each specialization pair x, ξ, the map $cosp_{x,\xi}$ identifies $\mathcal{F}_\xi$ with the quotient of $\mathcal{F}_x$ by a face. For each $x \in X$, let G_x denote the face of P consisting of those elements which map to a unit in M_x. Then $\overline{M}_x \cong P/G_x$ and $\overline{\mathcal{F}}_x$ is the face of $\overline{M}_x$ generated by $F \mapsto P/G_x$. Since R is nonzero, the map $X \to \operatorname{Spec} P$ is surjective, and in fact there exists a point x of X such that for every G of P, there exists a generization ξ of x with $G_\xi = G$, in particular $G_x = P^*$. If $\mathcal{F}$ is coherent, it follows that the image of F in P/G is a face of P/G face for every G. Then Proposition 2.4.2 implies that F is a direct summand of P. Conversely, if F is a direct summand, then (2.4.2) implies that $F + G^{gp}$ is a face of P_G for every G, and hence that F defines a chart for $\mathcal{F}$. $\square$

2.6 Relatively coherent log structures

If $\mathcal{F}$ is a sheaf of faces in a sheaf of monoids M, then $\mathcal{F}^* = M^*$, and if $M \to \mathcal{O}_X$ is a log structure, so is the composition $\mathcal{F} \to M \to \mathcal{O}_X$.

Definition 2.6.1 *Let X be a log scheme and let $\mathcal{F} \subseteq M_X$ be a sheaf of faces. Then $\underline{X}(\mathcal{F})$ is the log scheme whose underlying scheme is $\underline{X}$ and with log structure the composed map $\mathcal{F} \to M_X \to \mathcal{O}_X$.*

Note that the canonical map $X \to \underline{X}$ factors uniquely: $X \to \underline{X}(\mathcal{F}) \to \underline{X}$. The morphism $X \to \underline{X}(\mathcal{F})$ is an epimorphism in the category of log schemes, since the underlying map of schemes is an isomorphism and the map of sheaves of monoids $\mathcal{F} \to M_X$ is injective. If $\mathcal{G}$ is another sheaf of faces of M_X and if $\mathcal{F} \subseteq \mathcal{G}$, there is a corresponding commutative diagram:

$$\begin{tikzcd}
\underline{X}(\mathcal{G})^* \arrow[r] \arrow[d] & \underline{X}(\mathcal{G}) \arrow[d] \\
\underline{X}(\mathcal{F})^* \arrow[r] & \underline{X}(\mathcal{F}).
\end{tikzcd}$$

Remark 2.6.2 If F and G are relatively coherent and M_X is fine, the horizontal maps in the diagram above are affine open immersions. This statement may be verified étale locally on X, so by (2.5.6) we may assume that there exists a fine chart $\beta: P \to M_X$ for X with a face $F \subseteq P$ which generates $\mathcal{F}$.

Since P is fine, there exists an $f \in F$ with $\langle f \rangle = F$. Then for each geometric point $\overline{x}$ of X lying over a point x of X, $\mathcal{F}_{\overline{x}}$ is the face of $M_{\overline{x}}$ generated by the image $f_{\overline{x}}$ in $M_{\overline{x}}$. Hence x belongs to $\underline{X}(\mathcal{F})^*$ if and only if $f_{\overline{x}} \in M^*_{\overline{x}}$, *i.e.*, if and only if $\alpha_X(f_{\overline{x}}) \in \mathcal{O}^*_{X,\overline{x}}$. Thus $\underline{X}(\mathcal{F})^*$ is the special affine open subset of X defined by the invertibility of $\alpha_X(\beta(f))$.

The following result illustrates an important example in which relatively coherent log structures arise naturally.

Theorem 2.6.3 *Let P be a toric monoid and let $X := \operatorname{Spec}(e_P \colon P \to R[P])$, where R is an integral domain. Let F be a face of P, $\mathcal{F} := \tilde{G}$, $U := \underline{X}^*(\mathcal{F})$, and let $\alpha_{U/X} \colon j_*^{log}(\mathcal{O}^*_U) \to \mathcal{O}_X$ be the direct image log structure (1.2.7). Then the natural map $F \to \Gamma(X, \mathcal{O}_X)$ induces an isomorphism $\gamma \colon \mathcal{F} \to j_*^{log}(\mathcal{O}^*_U)$. In particular $j_*^{log}(\mathcal{O}^*_U) \subseteq M_X$ is relatively coherent.*

Taking the special case when $F = P$, we find the following theorem of Kato [13, 11.6].

Corollary 2.6.4 *With the notation of (2.6.3), there is a natural isomorphism: $j_*^{\log}(\mathcal{O}^*_X) \cong M_X$.*

Proof of (2.6.3) Choose a generator h for F as a face of P. Then $X^*(\mathcal{F})$ is the special affine open subset of X corresponding to h. If f is any element of F, f maps to a unit in $k[P]_h$, and consequently $e_P(f) \in \Gamma(X, j_*^{log}(\mathcal{O}^*_U))$. Since $j_*^{log}(\mathcal{O}^*_U) \subseteq \mathcal{O}_X$ is a sheaf of faces in the multiplicative monoid $\mathcal{O}_X$, e_P induces a morphism of sheaves of monoids $\gamma \colon \mathcal{F} \to j_*^{log}(\mathcal{O}^*_U)$. The morphism γ is sharp and $j_*^{log}(\mathcal{O}^*_U)$ is integral, so by (I,4.1.2), it will suffice to prove that $\overline{\gamma}$ is an isomorphism. Since P is torsion free, X is integral, and hence by (1.2.7) $j_*^{log}(\mathcal{O}^*_U)/\mathcal{O}^*_X \cong \underline{\Gamma}_Y Div^+$, where $Y := X \setminus U$. Thus the theorem follows from (I, 3.3.9). $\square$

Let P be a fine monoid and let $X := \mathsf{A_P}$ be the log scheme $\operatorname{Spec}(P \to R[P])$. If F is a face of P, let $\mathcal{F} \subseteq M_X$ denote the relatively coherent sheaf of faces by F. The coherent log scheme

$$\mathsf{A_P}(F) := \operatorname{Spec}(F \to R[P])$$

has the same underlying scheme as $\mathsf{A_P}$ and $\underline{\mathsf{A}}_\mathsf{P}(\mathcal{F})$. The sheaf of monoids defining the log structure of $\mathsf{A_P}(F)$ is coherent and is contained in $\mathcal{F}$; it generates the latter as a sheaf of faces of M_X. Thus there are morphisms of log schemes:

$$\mathsf{A}_{\mathsf{P}} \to \underline{\mathsf{A}}_{\mathsf{P}}(\mathcal{F}) \to \underline{\mathsf{A}}_{\mathsf{P}}(F) \to \underline{\mathsf{A}}_{\mathsf{P}};$$

the arrow $\underline{\mathsf{A}}_{\mathsf{P}}(F) \to \mathsf{A}_{\mathsf{P}}(F)$ is not an isomorphism in general.

Recall from (1.1.9) that for any log scheme T, the set of morphisms of log schemes $T \to P$ can be identified with the set of morphisms $P \to \Gamma(T, M_T)$, and hence has a natural monoid structure. Thus A_{P} becomes a monoid object in the category of log schemes over R; the multiplication map $\mu \colon \mathsf{A}_{\mathsf{P}} \times \mathsf{A}_{\mathsf{P}} \to \mathsf{A}_{\mathsf{P}}$ is just the map induced by the diagonal morphism $P \to P \oplus P$. It is not easy to describe the functor of log points of the log scheme $\underline{\mathsf{A}}_{\mathsf{P}}(\mathcal{F})$ in general, but let us observe that $\underline{\mathsf{A}}_{\mathsf{P}}(\mathcal{F})$ is also a monoid object.

Proposition 2.6.5 *For any face F of a fine monoid P, there is a unique monoid structure on the log scheme $\underline{\mathsf{A}}_{\mathsf{P}}(\mathcal{F})$ compatible with the monoid structure on the log scheme A_{P}.*

Proof: The proposition asserts the unique existence of the bottom arrow making the following diagram commute:

$$\begin{tikzcd}
\mathsf{A}_{\mathsf{P}} \times \mathsf{A}_{\mathsf{P}} \arrow[r, "\mu"] \arrow[d] & \mathsf{A}_{\mathsf{P}} \arrow[d] \\
\underline{\mathsf{A}}_{\mathsf{P}}(\mathcal{F}) \times \underline{\mathsf{A}}_{\mathsf{P}}(\mathcal{F}) \arrow[r] & \underline{\mathsf{A}}_{\mathsf{P}}(\mathcal{F}).
\end{tikzcd}$$

Lemma 2.6.6 *Let P_1 and P_2 be fine monoids, with respective faces F_1 and F_2. Then $F := F_1 \oplus F_2$ is a face of $P := P_1 \oplus P_2$, and the evident map*

$$\underline{\mathsf{A}}_{\mathsf{P}}(\mathcal{F}) \to \underline{\mathsf{A}}_{\mathsf{P}_1}(\mathcal{F}_1) \times \underline{\mathsf{A}}_{\mathsf{P}_2}(\mathcal{F}_2)$$

is an isomorphism.

Proof: Let G be a face of P, and let $G_i := G \cap P_i$. Then $G = G_1 \oplus G_2$. It follows easily from this that the face of P_G generated by F is the sum of the faces of P_i generated by F_i, and the lemma follows. $\square$

It follows from Lemma 2.6.6 that the map on the left is an epimorphism, and this gives the uniqueness. The lemma also implies that the morphism of sheaves of monoids on $\mathsf{A_P} \times \mathsf{A_P}$

$$\mu^\flat \colon M \to pr_1^* M \oplus pr_2^* M$$

maps $\mu^\flat \mathcal{F}$ to $pr_1^*\mathcal{F} \oplus pr_2^*\mathcal{F}$; this gives the existence of the arrow. Finally, we should observe that the identity section factors through $\underline{\mathsf{A}}_{\mathsf{P}}^*$ and in particular through $\underline{\mathsf{A}}_{\mathsf{P}}(\mathcal{F})$. $\square$

Let P be a fine monoid, let F be a face of P, and let $\mathfrak{p}$ be the complement of F. The morphism of monoids $F \to P$ induces morphisms of prelog rings

$$(F \to R[F]) \longrightarrow (F \to R[P]) \longrightarrow P \to R[P])$$

and hence also morphisms of log schemes

$$\mathsf{A_P} \to \underline{\mathsf{A}}_{\mathsf{P}}(\mathcal{F}) \to \mathsf{A_P}(F) \to \mathsf{A_F}\,.$$

In particular, we have a morphism of log schemes

$$r_F \colon \underline{\mathsf{A}}_{\mathsf{P}}(\mathcal{F}) \to \mathsf{A_F}\,.$$

Lemma 2.6.7 *The map $\underline{\mathsf{A}}_{\mathsf{P}}(\mathcal{F}) \to \mathsf{A_P}(F)$ is strict at each point of the closed log subscheme $Y \subseteq \underline{\mathsf{A}}_{\mathsf{P}}(\mathcal{F})$ define by $\mathfrak{p}$. There is a unique strict closed immersion*

$$i_F \colon \mathsf{A_F} \to \underline{\mathsf{A}}_{\mathsf{P}}(\mathcal{F})$$

such that $r_F \circ i_F = \mathrm{id}_{\mathsf{A_F}}$.

Proof: A point y of Y is a prime ideal of $R[P]$ containing $R[\mathfrak{p}]$. Thus every element of $\mathfrak{p}$ maps to zero in $k(y)$, so the set G of elements of P which map to a unit in $k(y)$ is contained in F. It follows that the face of $P_y := P_G$ generated by F is just F_G. This shows that the map is strict. Recall from the discussion preceeding (3.2.1) that the map $R[F] \to R[P]/R[\mathfrak{p}]$ is an isomorphism of R-algebras, and hence induces an isomorphism of log schemes

$$\mathrm{Spec}(F \to R[P]/R[\mathfrak{p}]) \to \mathrm{Spec}(F \to R[F]),$$

i.e., an isomorphism of log schemes $Y \to \mathsf{A_F}$. We define i_F to be the inverse of this isomorphism followed by the strict closed immersion $Y \to \underline{\mathsf{A}}_{\mathsf{P}}(\mathcal{F})$. $\square$

Proposition 2.6.8 *With the notation and hypotheses above, The composite*

$$i := i_F \circ r_F \colon \underline{\mathsf{A}}_{\mathsf{P}}(\mathcal{F}) \to \underline{\mathsf{A}}_{\mathsf{P}}(\mathcal{F})$$

is homotopic to the identity, with $\underline{\mathsf{A}}_{\mathsf{m}} := \underline{\mathsf{A}}_{\mathbf{N}}$ as a base for the homotopy.

Proof: We follow the method of proof of (3.2.1). Let $h \colon P \to \mathbf{N}$ be a homomorphism such that $h^{-1}(0) = F$, and consider the commutative diagram

$$\begin{tikzcd} P \arrow[r, "h"] \arrow[d] & \mathbf{N} \arrow[d] \\ R[P] \arrow[r] & R[\mathbf{N}] \end{tikzcd}$$

Let x be a point of $\mathsf{A}_{\mathbf{N}}$ at which the log structure is not trivial. Then the set G_x of elements of $\mathbf{N}$ which map to units of $k(x)$ must be a proper face of $\mathbf{N}$, and hence $G_x = \{0\}$. Let $y \in \mathsf{A}_{\mathsf{P}}$ be the image of x under the map A_{h} and let G_y be the set of elements of P which map to units in $k(y)$. The diagram shows that $G_y \subseteq h^{-1}(0) = F$. As we saw above, this implies that F is a chart for the stalk of $\mathcal{F}$ at y. Since $h(F) = 0$, this implies that the composite $\mathsf{A}_{\mathbf{N}} \to \mathsf{A}_{\mathsf{P}} \to \underline{\mathsf{A}}_{\mathsf{P}}(\mathcal{F})$ factors through a map $t \colon \underline{\mathsf{A}}_{\mathbf{N}} \to \underline{\mathsf{A}}_{\mathsf{P}}(\mathcal{F})$. Let

$$f \colon \underline{\mathsf{A}}_{\mathsf{P}}(F) \times \underline{\mathsf{A}}_{\mathsf{m}} \to \underline{\mathsf{A}}_{\mathsf{P}}(\mathcal{F})$$

be the composition of $\mathrm{id} \times t$ with the multiplication map of the monoid log scheme $\underline{\mathsf{A}}_{\mathsf{P}}(\mathcal{F})$. Since t takes the identity section of A_{m} to the identity section of $\underline{\mathsf{A}}_{\mathsf{P}}(\mathcal{F})$, $f \circ (\mathrm{id} \times 1_{\mathsf{A}_{\mathsf{m}}}) = \mathrm{id}$. We already saw in the proof of (3.2.1) that $f \circ (\mathrm{id} \times 0_{\mathsf{A}_{\mathsf{m}}})$ is i on the underlying schemes. Since the map $\alpha_{\underline{X}(F)} \colon \mathcal{F} \to \mathcal{O}_X$ is injective, this implies that the same equality holds in the category of log schemes. $\square$

Remark 2.6.9 Note that since $i_F \colon \mathsf{A}_{\mathsf{F}} \to \underline{\mathsf{A}}_{\mathsf{P}}(\mathcal{F})$ is strict, there is a Cartesian diagram

$$\begin{tikzcd} \underline{\mathsf{A}}^*_{\mathsf{F}} \arrow[r] \arrow[d] & \underline{\mathsf{A}}^*_{\mathsf{P}}(\mathcal{F}) \arrow[d] \\ \mathsf{A}_{\mathsf{F}} \arrow[r] & \underline{\mathsf{A}}_{\mathsf{P}}(\mathcal{F}) \end{tikzcd}$$

in which the vertical maps are open immersions. Note also that $\underline{\mathsf{A}}^*_{\mathsf{P}}(\mathcal{F}) \cong \underline{\mathsf{A}}_{\mathsf{P}_\mathsf{F}}$. We should also remark that the homotopies preserve the open subsets $\underline{\mathsf{A}}^*_{\mathsf{F}}$ and $\underline{\mathsf{A}}^*_{\mathsf{P}}(\mathcal{F})$, by functoriality.

2.7 Idealized log schemes

It is sometimes convenient to add an ideal to the data of a log structure. To avoid overburdening the exposition, we shall limit ourselves to explaining the main definitions and concepts.

Definition 2.7.1 *An idealized log scheme is a log scheme (X, α_X) endowed with a sheaf of ideals $K_X \subseteq M_X$ such that $\alpha_X(k) = 0$ for all local sections k of K_X. A morphism of idealized log schemes is a morphism which is compatible with ideals.*

The functor which endows a log scheme X with the empty sheaf of ideals defines a fully faithful functor from the category of log schemes to the category of idealized log schemes. This functor is left adjoint to the functor from idealized log schemes to log schemes which forgets the ideal.

Let K be an ideal of a monoid P and let $\mathbf{Z}[P, K]$ be the quotient of the monoid algebra $\mathbf{Z}[P]$ by the ideal generated by the image of K. The map $P \to \mathbf{Z}[P, K]$ sends the elements of K to zero. We denote by $\mathsf{A}_{\mathsf{P},\mathsf{K}}$ the idealized log scheme whose underlying scheme is $\operatorname{Spec} \mathbf{Z}[P, K]$, with log structure associated to the prelog structure coming from the map $P \to \mathbf{Z}[P, K]$, and with the sheaf of ideas $K_{\mathsf{A}_{\mathsf{P}},K}$ in M_P generated by the image of K. If T is any idealized log scheme, then we can argue as in (1.1.5) to see that the set of morphisms $T \to \mathsf{A}_{\mathsf{P},\mathsf{K}}$ can be identified with the set of morphisms of monoids $P \to \Gamma(T, M_T)$ sending K to $\Gamma(T, K_T)$.

direct and inverse images, fibered products, strict maps, exact maps.

If (X, M_X) is a log scheme, $\alpha^{-1}(0)$ defines a sheaf of ideals in M_X, and it is often convenient to specify a distinguished subsheaf of ideals.

In general, if K is an ideal in a monoid Q, then the equivalence relation on Q which collapses the elements of K to a single point defines a congruence relation on Q, and the class of K in the quotient monoid acts as a "zero element." If K is nonempty this quotient monoid is not integral so we do not find it convenient to work with directly. Instead we consider the category $Imon$ of *idealized monoids*. This is just the category of pairs (Q, J), where Q is a monoid and J is an ideal of Q; morphisms $(Q, J) \to (P, I)$ are morphisms

$Q \to P$ sending J to I. The functor **Imon** $\to$ **Mon** taking (Q, J) to Q has a left adjoint, taking a monoid P to $(P, \emptyset)$, and we can view **Mon** as a full subcategory of **Imon**. Furthermore we have a functor from the category of commutative rings to the category **Imon**, taking a ring A to its multiplicative monoid together with the zero ideal.

If I is an ideal of a monoid Q, then the ideal of $R[Q]$ generated by $e(I)$ is free with basis $e_{|I}$, and we denote it by $R[I]$. Thus the quotient $R[Q]/R[I]$ is a free R-module with basis $Q \setminus I$. For any R-algebra A, $\mathrm{Hom}_{\mathbf{Imon}}((Q, I), (A, 0)) = \mathrm{Hom}_R(R[Q]/R[I], A)$, so that the functor $(Q, I) \mapsto R[Q]/R[I]$ is left adjoint to the functor $A \mapsto (A, 0)$.

Inductive and projective limits exist in the category of idealized monoids, and are compatible with the forgetful functor **Imon** $\to$ **Mon**. For example, if $u_i \colon (P, I) \to (Q_i, J_i)$ is a pair of morphisms and $v_i \colon Q_i \to Q$ is the pushout of the underlying monoid morphisms, then $v_i \colon (Q_i, J_i) \to (Q, J)$ is the pushout, where J is the ideal of Q generated by the images of J_i.

A morphism $\theta \colon (Q, J) \to (P, I)$ is *ideally strict* if I is generated by the image of J, and is *strict* if in addition its underlying morphism is strict. Note that θ is ideally strict if and only if $\overline{\theta}$ is. We say that θ is *ideally exact* if $J = \theta^{-1} I$, and that it is *exact* if in addition its underlying morphism is exact. Note that if the underlying morphism of θ is strict, then $\overline{\theta}$ is bijective, and hence θ is ideally strict if and only if it is ideally exact.

Definition 2.7.2 *An idealized log scheme is a log scheme (X, M_X) equipped with a sheaf of ideals $K_X \subseteq M_X$ such that $K_X \subseteq \alpha_X^{-1}(0)$. A morphism of idealized log schemes $f \colon X \to Y$ is a morphism of log schemes such that $f^\flat$ maps $f^{-1} K_Y$ into K_X.*

If X is a fine log scheme, the inverse image in M_X of the zero ideal of $\mathcal{O}_X$ need not be coherent. For example, let $X := \mathrm{Spec}(\mathbf{N} \to k[X, Y]/(XY))$, where n is sent to x^n. Then the stalk of $\alpha_X^{-1}(0)$ at the origin is empty, but the stalk at a point on the y-axis is not. Hence $\alpha_X^{-1}(0)$ is not coherent, by (2.5.2). On the other hand, the following analog of (2.6.3) shows that $\alpha_X^{-1}(0)$ is sometimes coherent.

Proposition 2.7.3 *Suppose that K is an ideal in a fine monoid P, R is a ring, and $X := \mathsf{A}_{\mathsf{P},\mathsf{K}}$ Then $\tilde{K} \cong \alpha_X^{-1}(0) \subseteq M_X$, and in particular $\alpha_X^{-1}(0)$ is coherent.*

Proof: If x is a point of X, let β_x be the map $P \to \mathcal{O}_{X,x}$, and let $F_x := \beta_x^{-1}(\mathcal{O}^*_{X,x})$. Recall that F_x is a face of P and that $\overline{M}_{X,x}$ identifies with P/F_x, where P/F_x is the quotient of P by the equivalence relation $p_1 \cong p_2$ if and only if there exist $f_1, f_2 \in F_x$ such that $p_1 + f_1 = p_2 + f_2$. Evidently $\tilde{K}$ maps injectively to $\alpha_X^{-1}(0)$; to prove that the map is an isomorphism, suppose that $m \in M_{X,x}$ and $\alpha_{X,x}(m) = 0$. Since $P \to \overline{M}_{X,x}$ is surjective, there exists a $p \in P$ mapping to $\overline{m}$, and it will suffice to prove that $p \cong k \bmod F$ for some $k \in K$. Let $\mathfrak{m}_x \subseteq R[P]$ be the prime ideal corresponding to the point x. Then $\mathcal{O}_{X,x}$ is the localization of $R[P]/R[K]$ at $\mathfrak{m}_x$, and since $e(p)$ maps to zero in $\mathcal{O}_{X,x}$, there exists an $f \in A[P] \setminus \mathfrak{m}_x$ such that $fe(p) \in R[K]$. Write $f := \sum a_q e(q)$; then since $f(x) \neq 0$, there exists some $q \in F_x$ such that $a_q \neq 0$. Since $fe(p) \in R[K]$, $q + p \in K$, and it follows that $p \in \tilde{K}_x$. $\square$

3 Betti realizations of log schemes over C

3.1 $\mathbf{C}^{log}$ and $X(\mathbf{C}_{log})$

In this section we explain the Betti realization of a log scheme X of finite type over the field $\mathbf{C}$ of complex numbers. This construction, due to Kato and Nakayama [14], associates to X a topological space X_{log} which gives a good geoemtric picture of the log structure of X. In particular, the topology of X_{log} explains why the factorization $X^* \to X_{log} \to \underline{X}$ serves as a compactification of the open immersion $X^* \to \underline{X}$.

Let $\mathbf{R}^{\geq}$ denote the set of nonnegative real numbers, endowed with the monoid structure given by multiplication. (This monoid is neither finitely generated, integral, or even quasi-integral.) Let $\mathbf{S}^1$ denote the set of all complex numbers of absolute value 1, also with the monoid structure given by multiplication. Consider the prelog ring:

$$\mathbf{C}_{log} := \mu \colon \mathbf{R}^{\geq} \times \mathbf{S}^1 \to \mathbf{C} \quad (r, \zeta) \mapsto r\zeta.$$

Then $\mu^{-1}(\mathbf{C}^*) = \mathbf{R}^+ \times \mathbf{S}^1$, and the map

$$\mathbf{R}^+ \times \mathbf{S}^1 \to \mathbf{C}^* \quad (r, \zeta) \mapsto r\zeta$$

is an isomorphism, with inverse

$$\mathbf{C}^* \to \mathbf{R}^+ \times \mathbf{S}^1 \quad z \mapsto (|z|, \arg(z)),$$

where $\arg(z) := z|z|^{-1}$ if $z \in \mathbf{C}^*$. Thus the prelog structure μ is in fact a log structure. Note also that

$$\mu^{-1}(0) = \{0\} \times \mathbf{S}^1,$$

the maximal ideal of $\mathbf{R}^{\geq 0} \times \mathbf{S}^1$.

Definition 3.1.1 *If X is a log scheme over $\mathbf{C}$, $X(\mathbf{C}_{log})$ denotes the set of $\mathbf{C}$-morphisms* $\operatorname{Spec}(\mathbf{C}_{log}) \to X$, *and*

$$\tau_X \colon X(\mathbf{C}_{log}) \to \underline{X}(\mathbf{C})$$

is the map taking a morphism x to the underlying morphism of schemes $\underline{x}$.

Thus, a point x of $X(\mathbf{C}_{log})$ mapping to a point $\underline{x}$ of $X(\mathbf{C})$ is a commutative diagram

$$\begin{tikzcd} M_{X,\underline{x}} \arrow[r, "x^\flat"] \arrow[d, "\alpha_X"'] & \mathbf{R}^{\geq} \times \mathbf{S}^1 \arrow[d, "\mu"] \\ \mathcal{O}_{X,\underline{x}} \arrow[r, "x^\sharp"] & \mathbf{C} \end{tikzcd} \tag{II.3}$$

Here we have identified the $\mathbf{C}$-valued point $\underline{x}$ with the corresponding closed point of the scheme $\underline{X}$. In the future, we will allow ourselves to write $M_{X,x}$ in place of $M_{X,\underline{x}}$ if no confusion seems to result. The morphism $x^\flat$ above can be viewed as a pair:

$$x^\flat = (\rho_x, \sigma_x) \in \operatorname{Hom}(M_{X,\underline{x}}) \times \operatorname{Hom}(M_{X,\underline{x}}, \mathbf{S}^1).$$

Proposition 3.1.2 *Let X be a quasi-integral log scheme over $\mathbf{C}$, and let*

$$\lambda \colon \mathcal{O}_X^* \to M_{X,\underline{x}}$$

denote the map such that $\alpha_X \circ \lambda$ is the inclusion $\mathcal{O}_X^* \to \mathcal{O}_X$.

1. *The set $X(\mathbf{C}_{log})$ can be identified with the set of pairs $(\underline{x}, \sigma_x)$, where $\underline{x} \in \underline{X}(\mathbf{C})$ and $\sigma_x \colon M_{X,\underline{x}}^{gp} \to \mathbf{S}^1$ is a homomorphism such that for every $u \in \mathcal{O}_{X,\underline{x}}^*$,*

$$\sigma_x(\lambda(u)) = \arg(u(\underline{x})).$$

2. *The map $\tau_X\colon X(\mathbf{C}_{log}) \to \underline{X}(\mathbf{C})$ is surjective, and the fiber over a point $\underline{x}$ is naturally a torsor under the group*

$$\mathbf{S}_{X,\underline{x}} := \mathrm{Hom}(\overline{M}^{gp}_{X,\underline{x}}, \mathbf{S}^1).$$

The action of this group on the fiber is given, via the identification in (1), by the natural action of the subgroup $\mathbf{S}_{X,\underline{x}} \subseteq \mathrm{Hom}(M^{g}_{X,\underline{x}}, \mathbf{S}^1)$:

$$(h\sigma)(m) := \sigma(m)h(\overline{m}) \quad \text{for } h \in \mathrm{Hom}(\overline{M}^{gp}_{X,\underline{x}}\mathbf{S}^1),\ \sigma \in \mathrm{Hom}(M^{gp}_{X,\underline{x}}, \mathbf{S}^1).$$

Proof: Let x be an element of $X(\mathbf{C}_{log})$. The diagram (II.3) can be expanded:

$$\begin{tikzcd}
 & \mathbf{C}^* \arrow[r, "\arg"] & \mathbf{S}^1 & \\
\mathcal{O}^*_{X,x} \arrow[ur, "x^\sharp"] \arrow[r, "\lambda"] \arrow[dr] & M_{X,x} \arrow[r, "(\rho_x, \sigma_x)"] \arrow[d, "\alpha_X"] & \mathbf{R}^{\geq} \times \mathbf{S}^1 \arrow[u, "pr_2"'] \arrow[d, "\mu"] \arrow[dr, "pr_1"] & \\
 & \mathcal{O}_{X,x} \arrow[r, "x^\sharp"] & \mathbf{C} \arrow[r, "\mathrm{abs}"] & \mathbf{R}^{\geq}
\end{tikzcd}$$

This diagram shows that $\rho_x = \mathrm{abs} \circ x^\sharp \circ \alpha_X$, and hence is determined entirely by $\underline{x}$. Thus x is determined by $\underline{x}$ and σ_x. The diagram also show that if $u \in \mathcal{O}^*_{X,\underline{x}}$, $u(\underline{x}) = \sigma_x(\lambda(u))|u(\underline{x})|$. Conversely, if $\sigma\colon M^{gp}_{X,\underline{x}} \to \mathbf{S}^1$ satisfies $\sigma \circ \lambda = x^\sharp \circ \arg$ as in (2), we can let $\rho := \mathrm{abs} \circ x^\sharp \circ \alpha_X$. Then then we get a commutative square as in the diagram above, hence a morphism $x\colon \mathrm{Spec}(\mathbf{C}_{log}) \to X$. This proves (1).

Since M_X is quasi-integral, the sequence

$$1 \to \mathcal{O}^*_{X,\underline{x}} \to M^{gp}_{X,\underline{x}} \to \overline{M}_{X,\underline{x}} \to 0$$

is exact, and since $\mathbf{S}^1$ is divisible, this yields an exact sequence

$$0 \to \mathrm{Hom}(\overline{M}^{gp}_{X,x}, \mathbf{S}^1) \longrightarrow \mathrm{Hom}(M^{gp}_{X,x}, \mathbf{S}^1) \longrightarrow \mathrm{Hom}(\mathcal{O}^*_{X,x}, \mathbf{S}^1) \longrightarrow 0.$$

We have just seen that an element $\sigma \in \mathrm{Hom}(M^{gp}_{X,x}, \mathbf{S}^1)$, corresponds to a point x of X_{log} lying over $\underline{x}$ if and only if its image $\sigma \circ \lambda$ in $\mathrm{Hom}(\mathcal{O}^*_{X,\underline{x}}, \mathbf{S}^1)$ is $\underline{x}^\sharp \circ \arg$. The exact sequence shows that the set of all such σ is a torsor under the kernel, with the action as described. In particular, the surjectivity of the map τ_X follows from the right exactness of the above sequence. □

Let X be a log scheme over $\mathbf{C}$ and let m be a global section of M_X. Define

$$\begin{aligned} \rho(m)\colon X(\mathbf{C}_{log}) &\to \mathbf{R}^{\geq} : x \mapsto \rho_x(m_x) \\ \sigma(m)\colon X(\mathbf{C}_{log}) &\to \mathbf{S}^1 : x \mapsto \sigma_x(m_x) \end{aligned}$$

Note that for any $x \in X(\mathbf{C}_{log})$ and $m \in M_X(X)$,

$$(\sigma(m)\rho(m))(x) = \alpha(m)(\tau_X(x)).$$

Remark 3.1.3 If P is a monoid, the set $\mathrm{Hom}(P, \mathbf{S}^1) = \mathrm{Hom}(P^{gp}, \mathbf{S}^1)$ endowed with the product (pointwise) topology and the pointwise product law becomes a compact topological group. Since $\mathbf{S}^1$ is a divisible group, an exact sequence of abelian groups $0 \to G' \to G \to G'' \to 0$ yields an exact sequence

$$0 \to \mathrm{Hom}(G'', \mathbf{S}^1) \to \mathrm{Hom}(G, \mathbf{S}^1) \to \mathrm{Hom}(G', \mathbf{S}^1) \to 0.$$

These maps are continuous, and since the groups are compact, the topologies on the extremes are induced by the topology in the middle. For example, if G is a finitely generated abelian group, G_{tor} is its torsion subgroup and $G_f := G/G_{tor}$, there is a canonical exact sequenced

$$0 \to \mathrm{Hom}(G, \mathbf{S}^1) \to \mathrm{Hom}(G, \mathbf{S}^1) \to \mathrm{Hom}(G_{tor}, \mathbf{S}^1) \to 0$$

Here $\mathrm{Hom}(G_f, \mathbf{S}^1)$ is isomorphic to a compact torus (a product of copies of $\mathbf{S}^1$), and is the connected component of $\mathrm{Hom}(G, \mathbf{S}^1)$ containing the identity. The finite quotient $\mathrm{Hom}(G_{tor}, \mathbf{S}^1)$ of $(G, \mathbf{S}^1)$ is its group of connected components. In particular, if M_X is a fine sheaf of monoids, then $\mathbf{S}_{X,x}$ is a torus whose dimension is the dimension of the monoid $\overline{M}_{X,x}$, and it is connected if and only if $\overline{M}_{X,x}$ is torison free.

3.2 X_{an} and X_{log}

Let X be a scheme of finite type over $\mathbf{C}$. The set $X(\mathbf{C})$ of $\mathbf{C}$-valued points is classically endowed with the topology induced from the classical (strong) topology on $\mathbf{C}$. This is the weakest topology with the property that for every Zariski open subset U of X and every section f of $\mathcal{O}_X(U)$, the function $U(\mathbf{C}) \to \mathbf{C}$ given by f is continuous. We should remark that one gets the same result if one uses étale open sets $U \to X$ instead of Zariski open sets. This follows from the implicit function theorem in complex analysis, which says that if $U \to X$ is étale, then every point of $U(\mathbf{C})$ has a strong neighborhood basis of open sets V such that the restriction $V \to X(\mathbf{C})$ is an open embedding. If U is affine and $(f_1, \ldots, f_n)$ is a finite set of generators for $\mathcal{O}_X(U)$ over $\mathbf{C}$, the topology on $U(\mathbf{C})$ is also the weakest topology such that each f_i is continuous, and it is the topology induced from $\mathbf{C}^n$ via the closed immersion $U(\mathbf{C}) \to \mathbf{C}^n$ given by $(f_1, \ldots f_n)$. We denote by X_{an} or X^{an} the topological space $X(\mathbf{C})$ with this topology.

When P is a fine monoid and $X = \mathsf{A_P}$, the topology on $X(\mathbf{C})$ has a useful explicit description, which follows immediately from the previous discussion.

Proposition 3.2.1 *Let P be a fine monoid, let $X := \mathsf{A_P}$, and let $x_0 \colon P \to \mathbf{C}$ be an element of X_{an}.*

1. *Let S be a finite set of generators for P, and for each $\delta > 0$, let*

$$U_\delta := \{x \in X_{an} : |x(s) - x_0(s)| < \delta \text{ for all } s \in S.\}$$

Then the set of all such U_δ forms a neighborhood basis for x_0 in X_{an}.

2. *In particular, if P is sharp and S is the set of irreducible elements of P, then the set of all*

$$U_\delta := \{x : |x(s)| < \delta \text{ for all } x \in S\}$$

forms a neighborhood basis for the vertex of X_{an}.

□

The neighborhood bases described above allow us to give a useful local version of the deformation retracts associated to a face of P (3.2.1).

Proposition 3.2.2 *Let P be a fine monoid, let F be a face of P, and let x_0 be a point of $\underline{\mathsf{A}}_{\mathsf{F}}$, viewed as an element of $\underline{\mathsf{A}}_{\mathsf{P}}$ via the closed embedding i_F*

(3.2.1). Then x_0 has a neighborhood basis of open sets which are stable under the retraction $r_F : \underline{\mathsf{A}}_{\mathsf{P}} \to \underline{\mathsf{A}}_{\mathsf{F}}$ as well as a homotopy $[0,1] \times \underline{\mathsf{A}}_{\mathsf{P}}(\mathbf{C}) \to \underline{\mathsf{A}}_{\mathsf{P}}(\mathbf{C})$ carrying $\mathrm{id}_{\underline{\mathsf{A}}_{\mathsf{P}}}$ to $i_F \circ r_F$.

Proof: Let S be any finite set of generators for P and for each $\delta > 0$ let U_δ be the open neighborhood of x_0 defined in (3.2.1). Recall that if $x \in \mathsf{A}_{\mathsf{P}}(\mathbf{C})$, $i_F r_F(x)$ is the map $P \to \mathbf{C}$ sending p to $0 = x_0(p)$ if $p \notin F$ and to $x(p)$ if $p \in F$. Thus $i_F r_F(x) \in U_\delta$ if $x \in U_\delta$. Recall that in (3.2.1) we used the existence of a homomorphism $h : P \to \mathbf{N}$ with $h^{-1}(0) = F$ to construct a homotopy $f : \underline{\mathsf{A}}_{\mathsf{m}} \times \underline{\mathsf{A}}_{\mathsf{P}} \to \underline{\mathsf{A}}_{\mathsf{P}}$ between the identity and $i_F \circ r_F$. Let us verify that this map induces a map $[0,1] \times U_\delta \to U_\delta$. Indeed, if $t \in [0,1]$ and $x \in U_\delta$, then $y := f(t,x)$ is the map sending each $p \in P$ to $t^{h(p)}x$. If $p \notin F$, then $x_0(p) = 0$ and

$$|y(p) - x_0(p)| = |t^{h(p)}x(p)| \leq |x(p)| < \delta,$$

and if $p \in F$,

$$|y(p) - x_0(p)| = |t^0 x(p) - x_0(p)| = |x(p) - x_0(p)| < \delta.$$

□

When X is a log scheme, we can also endow $X(\mathbf{C}_{log})$ with a canonical topology.

Definition 3.2.3 *Let X be a log scheme over $\mathbf{C}$. Then X_{log} (or X^{log}) is the set $X(\mathbf{C}_{log})$ endowed with the weakest topology such that for every étale $U \to X$ and every section m of $M_X(U)$, the functions*

$$\rho(m) : U(\mathbf{C}_{log}) \to \mathbf{R}^{\geq} \quad \textit{and} \quad \sigma(m) : U(\mathbf{C}_{log}) \to \mathbf{S}^1$$

are continuous.

To make this definition more explicit, let x be a point of X_{log}, let m be a section of M_X defined in some étale neighbborhood of $\underline{x}$, and let U and V be neighborhoods of $\rho(m)(x)$ and $\sigma(m)(x)$ Then the set of all $x' \in X_{log}$ such that m is defined at x' and $(\rho(m)(x'), \sigma(m)(x')) \in U \times V$ is an open neighborhood of x in X_{log}, and the family of finite intersections of such sets forms a neighborhood basis for x in X_{log}.

Example 3.2.4 It follows from (2.4.4) that if P is a fine monoid, $\mathsf{A}_{\mathsf{P}}^{\mathsf{log}}$ can be identified with $\mathrm{Hom}(P, \mathbf{R}^{\geq}) \times \mathrm{Hom}(P, \mathbf{S}^1)$, and the map $\mathsf{A}_{\mathsf{P}}^{\mathsf{log}} \to \mathsf{A}_{\mathsf{P}}^{\mathsf{an}}$ with the map

$$\mathrm{Hom}(P, \mathbf{R}^{\geq}) \times \mathrm{Hom}(P, \mathbf{S}^1) \to \mathrm{Hom}(P, \mathbf{C})$$

induced by multiplication $\mathbf{R}^{\geq} \times \mathbf{S}^1 \to \mathbf{C}$. Here all the sets are endowed with the product (weak topology) coming from the standard topologies on $\mathbf{R}^{\geq}$, $\mathbf{S}^1$, and $\mathbf{C}$. For example, if $P = \mathbf{N}$, $X_{log} \cong \mathbf{R}^{\geq} \times \mathbf{S}^1$, which can be viewed either as a half-closed cylinder, the complex plane with an open disc removed. The map τ in this case becomes real blowup of the complex plane at the origin: the fiber over the origin is the set of real rays emanating from the origin.

It is clear from the definition that a morphims of log schemes over $\mathbf{C}$ $X \to Y$ induces a continuous map $X_{log} \to Y_{log}$. Let us note that the topology on X_{log} is necessarily Hausdorff. This is a formal consequence of the definition and the fact that the topolgies of $\mathbf{R}^{\geq}$ and $\mathbf{S}^1$ are Hausdorff.

Lemma 3.2.5 *If X is a scheme over $\mathbf{C}$ with trivial log structure, then the map $\tau_X \colon X_{log} \to X_{an}$ is a homeomorphism.*

Proof: It is clear that τ_X is bijective. The topology on X_{log} is the weak topology defined by the functions $\mathrm{abs} \circ u$ and $\arg \circ u$ for every section u of $\mathcal{O}_X^*$. Since $u = (\mathrm{abs} \circ u)(\arg \circ u)$ and since abs, arg, and μ are continuous, this is the same as the weak topology defined by the sections of $\mathcal{O}_X^*$. The topology on X_{an} is the weak topology defined by the sections of $\mathcal{O}_X$. Thus it is certainly true that the inverse map $X_{an} \to X_{log}$ is continuous. To prove that τ_X is continuous, observe that if f is any local section of $\mathcal{O}_X$, and x is any point of X, then $f + c$ is invertible in a neighborhood of x for some constant c, and hence continuity of $f + c$ implies the continuity of f. $\square$

Proposition 3.2.6 *Let X be a quasi-integral log scheme of finite type over $\mathbf{C}$.*

1. *The map $\tau_X \colon X_{log} \to X_{an}$ is continuous, and for each $\underline{x} \in X_{an}$, the action of the topological group $\mathbf{S}_{X,\underline{x}}$ on the fiber $\tau^{-1}(\underline{x})$ is continuous. If M_X is coherent, the map τ_X is proper.*

2. *Suppose M_X admits a chart $P \to M_X$. Then X_{log} has the topology induced from the (injective) mapping*

$$X(\mathbf{C}_{log}) \to X_{an} \times \mathrm{Hom}(P^{gp}, \mathbf{S}^1) : x \mapsto (\underline{x}, \sigma_x)$$

Proof: The canonical map $p\colon X \to \underline{X}$ can be viewed as a map of log schemes, where $\underline{X}$ is given the trivial log structure. By functoriality, p induces a continuous map $X_{log} \to \underline{X}_{log}$, and by the previous lemma, $\underline{X}_{log}$ can be identified with X_{an}. The proves the continuity of τ_X.

A morphism $\theta\colon P \to M_X$ induces a continuous map

$$X(\mathbf{C}_{log}) \to \mathrm{Hom}(P^{gp}, \mathbf{S}^1).$$

Since θ is a chart, the map $P \to \overline{M}_X$ is surjective, so every local section m of M_X can locally be written $m = u + \theta(p)$, where $u \in M_X^*$ and $p \in P$. Since σ_x is fixed on $M^*_{X,\underline{x}}$, σ_x is determined uniquely by $\sigma_x \circ \theta$. Thus it follows from (3.1.2) above that the resulting map in statement (2) is injective. To complete the proof of (2), we must show that if for every local section f of $\mathcal{O}_X$ and for every $p \in P$, $f \circ \tau$ and $\sigma(\theta(p))$ are continuous is some topology on $X(\mathbf{C}_{log})$, then the same is true of $\rho(m)$ and $\sigma(m)$ for every local section m of M_X. Since $\rho(m) = |\alpha_X(m)|$, $\rho(m)$ will be continuous, and if m is a unit of M_X, $\alpha(m) \in \mathcal{O}_X^*$ and $\sigma(m) = \arg \circ u$ is continuous. Since any m is locally a sum of a unit and an element in the image of θ, $\sigma(m)$ will also be continuous.

We have now proved (2) and the first part of (1). The properness of τ_X can be checked locally on $\underline{X}$ with the aid of a chart. It then suffices to observe that in the commutative diagram

$$\begin{tikzcd}
X_{log} \arrow[r] \arrow[d] & X_{an} \times \mathrm{Hom}(P^{gp}, \mathbf{S}^1) \arrow[dl, "pr_1"] \\
X_{an}, &
\end{tikzcd}$$

the top arrow is a closed immersion and the map pr_1 is proper because $\mathrm{Hom}(P^{gp}, \mathbf{S}^1)$ is compact. The continuity of the action of $\mathbf{S}_{X,x}$ on $\tau^{-1}(\underline{x})$ is clear from the definitions. $\square$

The construction of X_{log} is functorial in X, and the map τ is natural: if $X \to Y$ is a map of log schemes, we find a commutative diagram

$$\begin{tikzcd} X_{log} \arrow[r] \arrow[d, "\tau_X"'] & Y_{log} \arrow[d, "\tau_Y"] \\ X_{an} \arrow[r] & Y_{an} \end{tikzcd} \tag{II.4}$$

Furthermore, if $X \to Y$ is strict, it is easy to verify that this diagram is Cartesian. In particular, if $P \to M_X$ is a chart, the map

$$X_{log} \to X_{an} \times_{\underline{\mathsf{A}}_{\mathsf{P}an}} \mathsf{A}_{\mathsf{P}log}$$

is a homeomorphism.

I hope that a Cartesian diagram of log schemes gives a Cartersian diagram of Betti realizations in general.

Corollary 3.2.7 *Let X be a quasi-integral coherent log scheme. Then the maps*

$$X^{\mathrm{int}}_{log} \to X_{log} \quad \textit{and} \quad X^{\mathrm{sat}} \to X^{\mathrm{int}}_{log}$$

are homeomorphisms.

Remark 3.2.8 Let $f\colon X \to Y$ be a morphism of integral log schemes such that $\underline{f}$ is an isomorphism. Then $X_{an} \cong Y_{an}$ and $\tau_Y \circ f_{log}$ can be identified with τ_X. If X is coherent, τ_X is proper, and since Y_{log} is Hausdorff, it follows that f_{log} is also proper. If in addition $f^\flat\colon f^*M_Y \to M_X$ is injective, the map $f_{log}\colon X_{log} \to Y_{log}$ is surjective, and it follows that τ_Y is also proper. Finally, note that if $f^\flat$ is surjective, then f_{log} is a closed immersion.

Let $\mathbf{R}(1)$ denote the set of purely imaginary complex numbers z and let $\mathbf{Z}(1) \subseteq \mathbf{R}(1)$ the subroup generated by $2\pi i$. The exponential mapping $z \mapsto \exp z := e^z$ defines an exact sequence

$$0 \to \mathbf{Z}(1) \to \mathbf{R}(1) \to \mathbf{S}^1 \to 0,$$

and the map $\mathbf{R}(1) \to \mathbf{S}^1$ is a universal covering of $\mathbf{S}^1$. Thus the automorphism group of $\mathbf{R}(1)$ over $\mathbf{S}^1$ can be viewed as the fundamental group of $\mathbf{S}^1$ and is canonically isomorphic to $\mathbf{Z}(1)$, via the action of $\mathbf{Z}(1)$ on $\mathbf{R}(1)$ by translation. (Since the fundamental group is abelian, the choice of a base point is not relevant.)

Proposition 3.2.9 *Let X be a fine saturated log scheme over* **C** *and let x be a point of X_{log}. The isomorphism $\mathbf{S}_{X,\underline{x}} \to \tau^{-1}(\underline{x})$ defined by the point x and the action (3.1.2) induces a canonical isomorphism:*

$$\pi_1(\tau^{-1}(\underline{x}), x) \cong \mathrm{Hom}(\overline{M}^{gp}_{X,\underline{x}}, \mathbf{Z}(1)).$$

This fundamental group is called the logarithmic inertia group of X at x.

Proof: Since X is fine and saturated, $\overline{M}^{gp}_{X,\underline{x}}$ is a finitely generated free abelian group, so the sequence of abelian groups:

$$0 \to \mathrm{Hom}(\overline{M}^{gp}_{X,\underline{x}}, \mathbf{Z}(1)) \to \mathrm{Hom}(\overline{M}^{gp}_{X,\underline{x}}, \mathbf{R}(1)) \to \mathrm{Hom}(\overline{M}^{gp}_{X,\underline{x}}, \mathbf{S}^1) \to 0$$

is exact. Thus the vector space $\mathrm{Hom}(\overline{M}^{gp}_{X,\underline{x}}, \mathbf{R}(1))$ becomes a universal covering of $\mathbf{S}_{X,\underline{x}}$, with covering group $\mathrm{Hom}(\overline{M}^{gp}_{X,\underline{x}}, \mathbf{Z}(1))$. The point x induces a homeomorphism $\mathbf{S}_{X,\underline{x}} \to \tau^{-1}(\underline{x})$, and hence the isomorphim on fundamental groups. In fact this isomorphism is independent of the choice of covering space and of $x \in \tau_X^{-1}(\underline{x})$, again because the fundamental group is abelian. $\square$

3.3 Asphericity of j_{log}

Now let X be a fine log scheme over **C**, so that X^* is an open subset of X. The restriction of τ_X to X^* is an isomorphism onto X^*. Thus there is commutative diagram:

$$\begin{tikzcd} X^*_{an} \arrow[r, "j_{log}"] \arrow[dr, "j_{an}"'] & X_{log} \arrow[d, "\tau_X"] \\ & X_{an} \end{tikzcd}$$

As we have seen, τ_X is proper and surjective. We shall see later in (**??**) that if $X/\mathbf{C}$ is "log smooth," the map j_{log} preserves the topological nature of X^*_{an}. At present we content ourselves with the following special case, which will serve as a model for the log smooth case later.

Theorem 3.3.1 *Let P be a fine monoid and let $X := \mathsf{A_P}$. Then the map $j_{log}\colon X^*_{an} \to X_{log}$ is aspheric. That is, any point of X_{log} has a basis of neighborhoods U such that $j^{-1}_{log}(U)$ is (nonempty and) contractible.*

Proof: We have a commutative diagram:

$$\begin{tikzcd}
X^*_{an} \arrow[r, "j_{log}"] \arrow[d, "\cong"'] & X_{log} \arrow[d, "\cong"] \\
\mathrm{Hom}(P, \mathbf{R}^{>0}) \times \mathrm{Hom}(P, \mathbf{S}^1) \arrow[r, "(j_Y, \mathrm{id})"] & \mathrm{Hom}(P, \mathbf{R}^{\geq}) \times \mathrm{Hom}(P, \mathbf{S}^1)
\end{tikzcd}$$

A point x of X_{log} corresponds to a pair (y, s) with $y \in \mathrm{Hom}(P, \mathbf{R}^{\geq})$ and $s \in \mathrm{Hom}(P^{gp}, \mathbf{S}^1)$. Since s has a neighborhood basis of contractible sets, it is enough to prove that the map

$$j_Y\colon \mathrm{Hom}(P, \mathbf{R}^{>0}) \to \mathrm{Hom}(P, \mathbf{R}^{\geq})$$

is aspheric.

Let $Y_P := \mathrm{Hom}(P, \mathbf{R}^{\geq})$ and let $Y^*_P := \mathrm{Hom}(P, \mathbf{R}^{>0})$. An element p of P defines a function $\hat{p}\colon Y_P \to \mathbf{R}^{\geq}$, and Y_P has the weak topology defined by the set of such functions, where p ranges over any S set of generators S for P. Let us make this explicit, assuming for convenience that S is finite. Choose some $y_0 \in Y_P$, and for each s in S, choose real numbers $a(s)$ and $b(s)$ with $a(s) < y_0(s) < b(s)$. Then

$$Y(a, b) := \{y\colon P \to \mathbf{R}^{\geq} : a(s) < y(s) < b(s) \text{ for all } s \in S\}$$

is an open neighborhood of y_0 in Y_P, and the family of all such $Y(a, b)$ forms a basis for the family of neighborhoods of y_0. Thus it will suffice to show that each $Y^*(a, b) := Y(a, b) \cap Y^*_P$ is nonemepty and contractible.

The logarithm map $\log \mathbf{R}^{>0} \to \mathbf{R}$ is an order preserving topological isomorphism of groups, and it induces an isomorphism of topological groups

$$Y^*_P := \mathrm{Hom}(P^{gp}, \mathbf{R}^{>0}) \xrightarrow{\ \ell\ } \mathrm{Hom}(P^{gp}, \mathbf{R}).$$

Under this identification, Y^*_P becomes a finite dimensional real vector space V with its standard topology, which is the weak topology induced by evaluation e_s at elements of S. If $s \in S$ and $y \in Y^*_P$, then

$$e_s(\ell(y)) = \ell(y)(s) = \log(y(p))$$

Thus ℓ takes the set $Y^*(a,b)$ isomorphically to

$$V(a,b) := \{v \in V : \log a(s) < e_s(v) < \log b(s), \forall s \in S\}.$$

For any linear function $\phi\colon V \to \mathbf{R}$ and any $r \in R$, $\{v \in V : \phi(v) < r\}$ is a convex subset of V. This remark applies to each e_s and each $-e_s$, and it follows that $V(a,b)$ is an intersection of convex sets. Then $V(a,b)$ is also convex, hence contractible.

It remains to prove that $Y^*(a,b)$ is not empty, assuming again that $Y(a,b)$ is a neighborhood of $y_0 \in Y$. Let $F := \{p \in P : y(p) > 0\}$, and let $\mathfrak{p} := P \setminus F$. The sequence

I said this was "clear" in class, but the proof isn't so trivial.

$$1 \to \mathrm{Hom}(P^{gp}/F^{gp}, \mathbf{R}^+) \to \mathrm{Hom}(P^{gp}, \mathbf{R}^+) \to \mathrm{Hom}(F^{gp}, \mathbf{R}^+) \to 1$$

is exact, and so there exists an element y^* of $\mathrm{Hom}(P^{gp}, \mathbf{R}^+)$ such that $y^*(f) = y_0(f)$ for all $f \in F$. Since F is a face of P, by (2.2.4) there exists a homomorphism $h\colon P \to \mathbf{N}$ such that $h^{-1}(0) = F$. For each $r \in \mathbf{R}^+$ with $r < 1$, let $y_r := y^* r^h \in Y^*_P$. Then if $p \in P$,

$$y(p) = \begin{cases} y^*(p)r^0 = y_0(p) & \text{if } p \in F \\ y^*(p)r^{h(p)} \leq y^*(p)r & \text{if } p \in \mathfrak{p}. \end{cases}$$

In particular, we can choose r small enough so that $y_r(s) < b(s)$ for all $s \in S \cap \mathfrak{p}$. Since $y_0(s) = 0$ for such s, and since $y_r(s) = y_0(s)$ if $s \in S \setminus P$, it follows that $y_r \in Y^*(a,b)$, as required. □

Theorem **??** is also true for relatively coherent log structures.

Proposition 3.3.2 *Let P be a fine monoid, let F be a face of P, and let $\mathcal{F}$ denote the relatively coherent sheaf of faces of $\mathsf{A_P}$ generated by F.*

1. *The map*
$$i^{log}_F\colon \mathsf{A}^{\mathsf{log}}_{\mathsf{F}} \to \underline{\mathsf{A}}^{\mathsf{log}}_{\mathsf{P}}(\mathcal{F})$$
induced by i_F (3.2.2) is a strong deformation retract. Moreover the standard neighborhood basis of each point of the image of i^{log}_F is stable under the homotopy $i^{log}_F \circ r^{log}_F \sim \mathrm{id}$.

2. *The map $j^*_{log}\colon \underline{\mathsf{A}}_{\mathsf{P_F}} \to \mathsf{A}^{\mathsf{log}}_{\mathsf{P}}(\mathcal{F})$ is aspheric.*

Proof: As in the proof of (3.2.2), let $h\colon P \to \mathbf{N}$ be a homomorphism such that $h^{-1}(0) = F$. Then the monoid structures on $\mathsf{A_P}$ and $\underline{\mathsf{A}}_\mathsf{P}(\mathcal{F})$ and the morphism $\mathsf{A_h}$ induce by functoriality the following diagram:

$$\begin{tikzcd}
\mathsf{A}^{\mathsf{log}}_{\mathbf{N}} \times \mathsf{A}^{\mathsf{log}}_{\mathsf{P}} \arrow[r] \arrow[d] & \mathsf{A}^{\mathsf{log}}_{\mathsf{P}} \arrow[d] \\
\underline{\mathsf{A}}^{\mathsf{log}}_{\mathbf{N}} \times \underline{\mathsf{A}}^{\mathsf{log}}_{\mathsf{P}}(\mathcal{F}) \arrow[r] & \underline{\mathsf{A}}^{\mathsf{log}}_{\mathsf{P}}(\mathcal{F})
\end{tikzcd}$$

The top arrow is the map

$$(r, \zeta) \times (\rho, \sigma) \mapsto (r^h \rho, \zeta^h \sigma).$$

Its restriction along the path $[0,1] \to \mathsf{A}^{\mathsf{log}}_{\mathbf{N}}$ given by taking $r \in [0,1]$ and $\zeta = 1$ gives a map $[0,1] \times \mathsf{A}^{\mathsf{log}}_{\mathsf{P}} \to \mathsf{A}^{\mathsf{log}}_{\mathsf{P}}$. Suppose that y_0 is a point of $\underline{\mathsf{A}}^{\mathsf{log}}_{\mathsf{P}}(\mathcal{F})$ in the image of i^{log}_F and (ρ_0, σ_0) is a point of $\mathsf{A}^{\mathsf{log}}_{\mathsf{P}}$ which maps to y_0. Then $\rho_0(p) = 0$ for all $p \in \mathfrak{p}$. If S is a finite set of generators for P and $\epsilon > 0$, then

$$V_\epsilon := \{(\rho, \sigma) : |\rho(s) - \rho_0(s)| < \epsilon, |\sigma(s) - \sigma_0(s)| < \epsilon$$

is a typical open neighborhood of (ρ_0, σ_0) in $\mathsf{A}^{\mathsf{log}}_{\mathsf{P}}$. The homotopy takes a point (ρ, σ) in V_ϵ to $(r^h\rho, \sigma)$. Since $h(p) = 0$ for $p \in F$, this point still lies in V_ϵ if $r \in [0,1]$. The image $\overline{V}_\epsilon$ of V_ϵ is a typical open neighborhood of y_0 in $\underline{\mathsf{A}}^{\mathsf{log}}_{\mathsf{P}}(\mathcal{F})$, and is also stable under the homotopy, as is its intersection $\overline{V}^*_\epsilon$ with $\underline{\mathsf{A}}^{\mathsf{log}}_{\mathsf{P_F}}$. Thus the pair $(\overline{V}_\epsilon, \overline{V}^*_\epsilon)$ is homotopy equivalent to its intersection with $\mathsf{A}^{\mathsf{log}}_{\mathsf{F}}$ The proof of Theorem shows that $\overline{V}^*_\epsilon \cap \mathsf{A_F}$ is contractible, and hence so is $\overline{V}^*_\epsilon$ $\square$

3.4 $\mathcal{O}^{an}_X$ and $\mathcal{O}^{log}_X$

So far we have discussed only the toplogical space X_{an} associated to a scheme of finite type over $\mathbf{C}$. To truly pass into the realm of analytic geoemtry, we need to introduce the sheaf $\mathcal{O}^{an}_X$ of analytic functions on X_{an}. We refer to [10, Appendix B] for precise definitions. Let us note here the following explicit description for the ring of germs of analytic functions at the vertex of a monoid scheme over $\mathbf{C}$.

Proposition 3.4.1 *Let P be a fine sharp monoid, let $x_0 \in \underline{\mathsf{A}}_{\mathsf{P}}(\mathbf{C})$ denote the point sending P^+ to 0. If h is an element of the interior of $H(P)$, i.e., a local homomorphism $P \to \mathbf{N}$, then a formal power series $\alpha := \sum_p a_p e^p$ converges in some neighborhood of x_0 if and only if the set*

$$\left\{ \frac{\log |a_p|}{h(p)} : p \in P^+ \right\}$$

is bounded above.

Proof: We let T be the set of irreducible elements of P, and use the notation of (3.2.1). Suppose that $\alpha = \sum_p a_p e^p$, and that $b \in \mathbf{R}$ is an upper bound for the set of all $\frac{\log |a_p|}{h(p)}$ with $p \in P^+$. Choose $\epsilon > 0$, let $\lambda_t := -(b+\epsilon)h(t)$ for each $t \in T$, and choose a positive number δ such that $\delta < e^{\lambda_t}$ for all t. Then U_δ is an open neighborhood of s in $\underline{\mathsf{A}}_{\mathsf{P}}(\mathbf{C})$, and if $x \in U_\delta$, $\log |x(t)| < \lambda_t$ for all $t \in T$. Any $p \in P$ can be written $p = \sum n_t t$. Hence for any $x \in U_\delta$,

$$\begin{aligned}
\log |a_p x(p)| &= \log |x(p)| + \log |a_p| \\
&\leq \log |x(p)| + bh(p) \\
&\leq \sum_t \left(n_t \log |x(t)| + b n_t h(t) \right) \\
&\leq \sum_t n_t \left(\lambda_t + bh(t) \right) \\
&\leq \sum_t n_t (-\epsilon h(t)) \\
&\leq -\epsilon h(p)
\end{aligned}$$

Thus $|a_p x(p)| \leq r^{h(p)}$, where $r := e^{-\epsilon} < 1$. By (2.2.8), $\{p : h(p) = i\}$ has cardinality less than Ci^m for some C and m, so the set of partial sums of the series $\sum_p |a_p x(p)|$ is bounded by the set of partial sums of the series $\sum_i Ci^m r^i$. Since this latter series converges, so does the former.

Suppose on the other hand that $\alpha := \sum a_p e^p$ and $\{h(p)^{-1} \log |a_p| : p \in P^+\}$ is unbounded. For $c \in \mathbf{R}^+$, define $x_c \colon P \to \mathbf{C}$ by $x_c(p) := c^{-h(p)}$. Then $x_c \in \underline{\mathsf{A}}_{\mathsf{P}}(\mathbf{C})$, and if $\delta > 0$ and c is chosen large enough so that $\log c > (h(t))^{-1}(-\log \delta)$ for all $t \in T$, then $x_c \in U_\delta$. For every such c, there are infinitely many $p \in P^+$ such that $|a_p| > (c+1)^{h(p)}$. For such a p,

$$|a_p x(p)| \geq (1+c)^{h(p)} c^{-h(p)} = (1 + 1/c)^{h(p} \geq 1,$$

so the series $\sum_p a_p x(p)$ cannot converge. $\square$

The space X_{log} is in a natural way the domain of functions which correspond to logarithms of the sections of $M^{gp}_{X,x}$. (These functions would be multi-valued on X_{an}.) In general, if f is a section of $\mathcal{O}^{an}_X$ on some open set U, then $\exp \circ f$ is a section of $\mathcal{O}^{an*}_X$, and we have an exact seqeunce of sheaves of abelian sheaves:

$$0 \to \mathbf{Z}(1) \to \mathcal{O}^{an}_X \to \mathcal{O}^{an*}_X \to 0.$$

If Y and Z are topological spaces, let us write Z_Y for the sheaf which to every open set V of Y assigns the set of continuous functions $V \to Z$. (We sometimes omit the subscript if no confusion seems likely to result, and indeed we have already used this notation several times.) For any Y, there is an exact sequence of abelian sheaves:

$$0 \longrightarrow \mathbf{Z}(1)_Y \longrightarrow \mathbf{R}(1)_Y \xrightarrow{\exp} \mathbf{S}^1_Y \longrightarrow 0.$$

Definition 3.4.2 *Let X be a log scheme over $\mathbf{C}$, and let*

$$\sigma\colon \tau_X^{-1}(M^{gp}_{X,x}) \to \mathbf{S}^1_{X_{log}}$$

be the map sending a section m of M^{gp}_X to $\sigma(m) \in \mathbf{S}^1_X$.

1. *$\mathcal{L}_X$ is the fiber product, in the category of abelian sheaves on X_{log}, in the diagram below:*

$$\begin{tikzcd} \mathcal{L}_X \arrow[r] \arrow[d] & \tau_X^{-1}(M^{gp}_X) \arrow[d] \\ \mathbf{R}(1)_{X_{log}} \arrow[r, "\exp"] & \mathbf{S}^1_{X_{log}} \end{tikzcd}$$

2. *$\epsilon\colon \tau_X^{-1}(\mathcal{O}_X) \to \mathcal{L}_X$ is the map induced by the maps*

$$\mathcal{O}_X \xrightarrow{\mathrm{Im}} \mathbf{R}(1) \quad \textit{and} \quad \lambda \circ \exp\colon \mathcal{O}_X \to M_X,$$

where $\mathrm{Im}\colon \mathcal{O}_X \to \mathbf{R}(1)$ means “imaginary part,” and $\lambda\colon \mathcal{O}^*_X \to M_X$ is the canonical inclusion.

3. *$\mathcal{O}_X^{log}$ is the universal $\tau_X^{-1}(\mathcal{O}_X)$-algebra equipped with a map $\mathcal{L}_X \to \mathcal{O}_X^{log}$ compatible with the map $\epsilon\colon \tau^{-1}(\mathcal{O}_X) \to \mathcal{L}_X$.*

The map in (2) makes sense because for any section f of $\mathcal{O}_X$,

$$\sigma(\lambda(\exp f)) = \arg\exp(f) = \exp\mathrm{Im}(f).$$

The algebra $\mathcal{O}_X^{log}$ may be constructed explicitly as the quotient of $\tau_X^{-1}(\mathcal{O}_X)\otimes_{\mathbf{Z}} S^{\cdot}\mathcal{L}_X$ by the ideal generated by all the sections of the form $\epsilon(f) - 1f$, for f a local section of $\tau_X^{-1}(\mathcal{O}_X)$.

Proposition 3.4.3 *With the notation above, there is an exact sequence*

$$0 \to \tau_X^{-1}(\mathcal{O}_X) \to \mathcal{L}_X \to \overline{M}_X^{gp} \to 0.$$

Let $Fil^{-p}\mathcal{O}_X^{log}$ be the image of the map

$$\oplus_{j=0}^{p} S^j(\tau_X^{-1}\mathcal{O}_X \otimes \mathcal{L}) \to \mathcal{O}_X^{log}.$$

The the natural map

$$\mathrm{Gr}^{-p}\,\mathcal{O}_X^{log} \to \tau^{-1}\mathcal{O}_X \otimes_{\mathbf{Z}} \overline{M}_X^{gp}$$

is an isomoprhism.

Proof: A diagram chase shows that the sequence

$$0 \to \tau_X^{-1}\mathcal{O}_X \to \mathcal{L}_X \to \overline{M}_X^{gp} \to 0$$

is exact. Here is an alternative construction of $\tau_X^{-1}\mathcal{O}_X$, assuming that X is saturated. Then $\overline{M}_X^{gp}$ is torsion free, and so the sequence remains exact when tensored over $\mathbf{Z}$ with $\tau_X^{-1}\mathcal{O}_X$. The pushout of the resulting sequence via the multiplication map $\tau_X^{-1}\mathcal{O}_X \otimes_{\mathbf{Z}} \tau_X^{-1}\mathcal{O}_X \to \tau_X^{-1}\mathcal{O}_X$ is an exact sequence of $\tau_X^{-1}\mathcal{O}_X$-modules:

$$0 \to \tau_X^{-1}\mathcal{O}_X \to \mathcal{E}_X \to \tau_X^{-1}\mathcal{O}_X \otimes \overline{M}_X^{gp} \to 0.$$

For each n, the map $\tau_X^{-1}\mathcal{O}_X \to \mathcal{E}_X$ induces an injective map

$$S^{n-1}\mathcal{E}_X \to S^n\mathcal{E}_X$$

where these symmsetric products are computed in the category of $\tau_X^{-1}\mathcal{O}_X$-modules. Then $\mathcal{O}_X^{log}$ is the direct limit over this family of maps. Then $Fil^{-n}\mathcal{O}_X^{log} \cong S^n\mathcal{E}_{\mathcal{X}}$, and the proposition follows easily. □

Chapter III

Morphisms of log schemes

1 Exact morphisms, exactification

Definition 1.0.4 *A morphism $f\colon X \to Y$ of integral log schemes is exact if for every $x \in X$, the map $f^\flat\colon M_{Y,f(x)} \to M_{X,x}$ is an exact morphism of monoids (**??**).*

Thanks to (**??**), we see that f is exact if and only if each map $\overline{f}^\flat\colon \overline{M}_{Y,f(x)} \to \overline{M}_{X,x}$ is exact, and this is true if and only if each $f^*M_{Y,f(x)} \to M_{X,x}$ is exact.

Proposition 1.0.5 *The composition of two exact morphisms of log schemes is exact. The family of exact morphisms is stable under base change in the category of fine log schemes.*

Proposition 1.0.6 *If $f\colon X \to Y$ is exact, then the map $f\overline{M}_Y \to \overline{M}_X$ is injective.*

2 Integral morphisms

3 Weakly inseparable maps, Frobenius

4 Saturated morphisms

Chapter IV

Differentials and smoothness

1 Derivations and differentials

1.1 Basic definitions

Although log schemes are the focus of our study, it is convenient to define derivations and differentials for prelog schemes as well.

Definition 1.1.1 *Let $f\colon X \to Y$ be a morphism of prelog schemes and let E be a sheaf of $\mathcal{O}_X$-modules. A* derivation *(or, for emphasis,* log derivation*) of X/Y with values in E is a pair (D, δ), where $D\colon \mathcal{O}_X \to E$ is a homomorphism of abelian sheaves and $\delta\colon M_X \to E$ is a homomorphism of sheaves of monoids such that the following conditions are satisfied:*

1. *For every local section m of M_X, $D(\alpha_X(m)) = \alpha_X(m)\delta(m)$.*
2. *For every local section n of $f^{-1}(M_Y)$, $\delta(f^\flat(n)) = 0$.*
3. *For any two local sections a and b of $\mathcal{O}_X$, $D(ab) = aD(b) + bD(a)$.*
4. *For every local section c of $f^{-1}(\mathcal{O}_Y)$, $D(f^\sharp(c)) = 0$.*

We denote by $\mathrm{Der}_{X/Y}(E)$ the presheaf which to every $U \to X$ assigns the set of derivations of U/Y with values in $E_{|_U}$. In fact, since all the presheaves in the definition above are sheaves, $\mathrm{Der}_{X/Y}(E)$ is also a sheaf. Furthermore, if (D_1, δ_1) and (D_2, δ_2) are sections of $\mathrm{Der}_{X/Y}(E)$, so is $(D_1 + D_2, \delta_1 + \delta_2)$, and if a is a section of $\mathcal{O}_X$ and (D, δ) an element of $\mathrm{Der}_{X/Y}(E)$, then $(aD, a\delta)$

also belongs to $\mathrm{Der}_{X/Y}(E)$. Thus $\mathrm{Der}_{X/Y}(E)$ has a natural structure of a sheaf of $\mathcal{O}_X$-modules.

Formation of $\mathrm{Der}_{X/Y}$ is functorial in E: an $\mathcal{O}_X$-linear map $h\colon E \to E'$ induces a homomorphism

$$\mathrm{Der}_{X/Y}(h)\colon \mathrm{Der}_{X/Y}(E) \to \mathrm{Der}_{X/Y}(E') \quad (D, \delta) \mapsto (h \circ D, h \circ \delta).$$

The following proposition explains how Der is also functorial in X/Y.

Proposition 1.1.2 *Let*

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "h"] & Y \end{tikzcd}$$

be a commutative diagram of prelog schemes.

1. *Composition with $g^\sharp$ and $g^\flat$ induces a morphism of functors*

$$g_* \circ \mathrm{Der}_{X'/Y'} \to \mathrm{Der}_{X/Y} \circ g_*$$

which for any $\mathcal{O}'_X$-module E' is the map

$$g_* \mathrm{Der}_{X'/Y'}(E') \to \mathrm{Der}_{X/Y}(g_*(E'))\colon (D', \delta') \mapsto (D' \circ g^\sharp, \delta' \circ g^\flat).$$

2. *The functoriality morphism above is an isomorphism in the following cases:*

   (a) *$f'\colon X' \to Y'$ is the morphism of log schemes associated to the morphism f of prelog schemes;*

   (b) *the diagram is Cartesian in the category of prelog schemes;*

   (c) *the diagram is Cartesian in the category of log schemes.*

Proof: The verification that composition with $g^\sharp$ and $g^\flat$ takes derivations to derivations is immediate. To prove (2a), recall from (1.1.5) that the log structure $M^a_X \to \mathcal{O}_X$ associated to the prelog structure $M_X \to \mathcal{O}_X$ is obtained from the cocartesian square in the following diagram

$$\begin{tikzcd}
\alpha_X^{-1}(\mathcal{O}_X^*) \arrow[r, "\alpha_X"] \arrow[d] & \mathcal{O}_X^* \arrow[d] \arrow[dr] & \\
M_X \arrow[r] & M_X^a \arrow[r] & \mathcal{O}_X
\end{tikzcd}$$

Thus the monoid M^a_X is generated by the images of M_X and $\mathcal{O}^*_X$, and it follows that the map in (1) is injective. Conversely, if $(D,\delta) \in \mathrm{Der}_{X/Y}(E)$, define

$$\partial\colon \mathcal{O}^*_X \to E : \partial(u) := u^{-1}Du.$$

Then

$$\begin{aligned}
\partial(uv) &= u^{-1}v^{-1}D(uv) = u^{-1}v^{-1}(uDv + vDu)\\
&= v^{-1}D(v) + u^{-1}D(u) = \partial(u) + \partial(v).
\end{aligned}$$

Thus ∂ is a homomorphism $\mathcal{O}^*_X \to E$. Furthermore, if m is a section of $\alpha_X^{-1}(\mathcal{O}^*_X)$, then

$$\begin{aligned}
\partial(\alpha_X(m)) &= \alpha_X(m)^{-1}D(\alpha_X(m)) = \alpha_X(m)^{-1}\alpha_X(m)\delta(m)\\
&= \delta(m)
\end{aligned}$$

Since M^a is the pushout in the diagram above and δ and ∂ agree on $\alpha^{-1}(\mathcal{O}^*_X)$, there is a unique $\delta^a\colon M^a_X \to E$ which agrees with δ on M_X and with ∂ on $\mathcal{O}^*_X$. It follows from the fact that M^a_X is generated by M_X and $\mathcal{O}^*_X$ that $D\alpha_X(m) = \alpha_X(m)D(m)$ for any section of m of M^a_X. Furthermore, since M^a_Y is generated by $\mathcal{O}^*_Y$ and M_Y, it also follows that δ^a annihilates the image of $f^{-1}(M^a_Y)$. Thus (D,δ^a) is a section of $\mathrm{Der}_{X^a/Y^a}(E)$ which restricts to (D,δ). This shows that the functoriality map is also surjective and completes the proof of statement (2a). In case (2b), let $p := h \circ f' = f \circ g$. Then since the underlying diagram of schemes is Cartesian, the map

$$f'^{-1}\mathcal{O}_{Y'} \otimes_{p^{-1}\mathcal{O}_Y} g^{-1}\mathcal{O}_X \to \mathcal{O}_{X'}$$

is an isomorphism. Now if (D,δ) is a section of $\mathrm{Der}_{X/Y}(g_*E')$, $D\colon \mathcal{O}_X \to g_*E$ is $f^{-1}\mathcal{O}_Y$-linear, and by adjunction induces an $f^{-1}(\mathcal{O}_{Y'})$-linear map $\mathcal{O}_{X'} \to E'$, which satisfies conditions (3) and (4) of Definition 1.1.1. Since the diagram is Cartesian in the category of prelog schemes, the map

$$f'^{-1}M_{Y'} \oplus_{p^{-1}\mathcal{O}_Y} g^{-1}M_X \to M_{X'}$$

is also an isomorphism, and the map $\delta\colon M_X \to E$ induces a unique map $M_{X'} \to E$ which annihilates $f'^{-1}M_{Y'}$. It follows that (D',δ') satisfies conditions (1) and (2) of Definition 1.1.1 as well, and this completes the proof of (2b). Finally, we observe that (2c) is a consequence of (2a) and (2b), since the log structure of the fiber product in the category of log schemes is the log structure associated to prelog structure of the fiber product in the category of prelog schemes. □

Proposition 1.1.3 *Suppose that $f\colon X \to Y$ is a morphism of log schemes, E is a sheaf of $\mathcal{O}_X$-modules, and (D,δ) is a pair of homomorphisms of sheaves of monoids satisfying conditions (1) and (2) of Definition 1.1.1. Then D is uniquely determined by δ, and necessarily satisfies conditions (3) and (4) as well.*

Proof: The following simple lemma has been used before.

Lemma 1.1.4 *If X is any scheme, the image of $\mathcal{O}_X^* \to \mathcal{O}_X$ generates $\mathcal{O}_X$ as sheaf of additive monoids. That is, any local section of $\mathcal{O}_X$ can locally be written as a sum of sections of $\mathcal{O}_X^*$. In particular, if X is a log scheme, $\mathcal{O}_X$ is generated, as a sheaf of additive monoids, by the image of $\alpha_X\colon M_X \to \mathcal{O}_X$.*

Proof: Let a be a local section of $\mathcal{O}_X$ and let x be a point of X. If a maps to a unit in the local ring $\mathcal{O}_{X,x}$, then a is a unit in some neighborhood of x, and hence a is locally in the image of $\mathcal{O}_X^*$. If a maps to an element of the maximal ideal of $\mathcal{O}_{X,x}$, then $a-1$ maps to a unit, and then $a = 1+(a-1)$ is locally the sum of two units. □

The lemma evidently implies that D is uniquely determined by δ, when X is a log scheme. If Y is also a log scheme, condition (4) follows from condition (2). To check (3), observe that if m and n are sections of M_X and $a := \alpha_X(m)$, $b := \alpha_X(n)$, then

$$\begin{aligned} D(ab) &= D(\alpha_X(m)\alpha_X(n)) \\ &= D(\alpha_X(m+n)) \\ &= \alpha_X(m+n)\delta(m+n) \\ &= \alpha_X(m)\alpha_X(n)(\delta(n)+\delta(m)) \\ &= \alpha_X(m)\alpha_X(n)\delta(n)+\alpha_X(m)\alpha_X(n)\delta(m) \\ &= aD(b)+bD(a). \end{aligned}$$

More generally, if $a_i = \alpha_X(m_i)$ and $a = a_1 + a_2$, then again

$$D(ab) = D(a_1b+a_2b) = a_1D(b)+bD(a_1)+a_2D(b)+bD(a_2) = aD(b)+bD(a).$$

A similar argument with b, together with an application of Lemma 1.1.4, shows that (2) holds for any sections a and b of $\mathcal{O}_X$. $\square$

Corollary 1.1.5 *Let $f: X \to Y$ be a morphism of schemes with trivial log structure and let E be a sheaf of $\mathcal{O}_X$-modules. Then $\mathrm{Der}_{X/Y}(E)$ can be identified with the usual sheaf of derivations of $\underline{X}/\underline{Y}$ with values in E, i.e., with the sheaf of homomorphisms of abelian groups $D: \mathcal{O}_X \to E$ satisfying conditions (3) and (4) of Definition 1.1.1.*

Proof: Let X_0 be X with the prelog structure $0 \to \mathcal{O}_X$ and let Y_0 be defined analogously. The morphism $\underline{f}$ defines a morphism of prelog schemes $f_0: X_0 \to Y_0$. It is clear from the definition that $\mathrm{Der}_{X_0/Y_0}(E) \cong \mathrm{Der}_{\underline{X}/\underline{Y}}(E)$. Proposition 1.1.2 implies that $\mathrm{Der}_{X/Y}(E) \cong \mathrm{Der}_{X_0/Y_0}$, and this completes the proof. $\square$

Proposition 1.1.6 *Let $f: X \to Y$ be a morphism of log schemes. Then the functor $E \mapsto \mathrm{Der}_{X/Y}(E)$ is representable by a universal object*

$$\mathcal{O}_X \xrightarrow{d} \Omega^1_{X/Y}, \quad M_X \xrightarrow{d} \Omega^1_{X/Y} \quad \left(\text{or} \quad M_X \xrightarrow{dlog} \Omega^1_{X/Y}\right)$$

Here

$$\Omega^1_{X/Y} := \left(\mathcal{O}_X \otimes M_X^{gp}\right)/(R_1 + R_2),$$

where R_1 and R_2 are described below, and

$$d\colon M_X \to \Omega^1_{X/Y} \quad : \quad m \mapsto 1 \otimes m \pmod{R_1 + R_2}.$$

Here R_2 is the image of the map

$$\mathcal{O}_X \otimes f^{-1}M_Y^{gp} \to \mathcal{O}_X \otimes M_X^{gp}$$

and $R_1 \subseteq \mathcal{O}_X \otimes M_X^{gp}$ is the subsheaf of sections locally of the form

$$\sum_i \alpha_X(m_i) \otimes m_i - \sum_i \alpha_X(m'_i) \otimes m'_i,$$

where $(m_1, \ldots m_k)$ and $(m'_1, \ldots m'_{k'})$ are sequences of local sections of M_X such that $\sum_i \alpha_X(m_i) = \sum_i \alpha_X(m'_i)$.

Proof: It is clear that R_2 is a sub-$\mathcal{O}_X$-module of $\mathcal{O}_X \otimes M_X^{gp}$; we claim that the same is true of R_1. For a sequence $\mathbf{m} := (m_1, \ldots m_k)$ of sections of M_X, let

$$\begin{aligned} s(\mathbf{m}) &:= \sum_i \alpha_X(m_i) \in \mathcal{O}_X \\ r(\mathbf{m}) &:= \sum_i \alpha_X(m_i) \otimes m_i \in \mathcal{O}_X \otimes M_X^{gp} \end{aligned}$$

Let S be the sheaf of pairs $(\mathbf{m}, \mathbf{m}')$ of finite sequences of sections of M_X such that $s(\mathbf{m}) = s(\mathbf{m}')$. Then R_1 is the subsheaf of sections of $\mathcal{O}_X \otimes M_X^{gp}$ which are locally of the form $r(\mathbf{m}) - r(\mathbf{m}')$ where $(\mathbf{m}, \mathbf{m}')$ is a local section of S. Note that the pair $(0, 0) \in S$ and $r(0) - r(0) = 0$, so that $0 \in R_1$. Since $(\mathbf{m}', \mathbf{m}) \in S$ if $(\mathbf{m}, \mathbf{m}') \in S$, it follows that $-r \in R$ whenever $r \in R$. If $(\mathbf{m}\,\mathbf{m}')$ and $(\mathbf{n}, \mathbf{n}') \in S$, let $\mathbf{p}$ (resp. $\mathbf{p}'$) denote the concatenation of $\mathbf{m}$ and $\mathbf{n}$ and (resp. of $\mathbf{m}'$ and $\mathbf{n}'$.). Then $(\mathbf{p}, \mathbf{p}') \in S$ and

$$r(\mathbf{p}) - r(\mathbf{p}') = r(\mathbf{m}) - r(\mathbf{m}') + r(\mathbf{n}) - r(\mathbf{n}').$$

Thus R is an abelian subsheaf of $\mathcal{O}_X \otimes M_X^{gp}$.

It remains to check that R is stable under multiplication by sections a of $\mathcal{O}_X$, and Lemma 1.1.4 shows that it suffices to check this for $a = \alpha_X(n)$, with n a section of M_X.

Let us first observe that S is stable under the action of M_X by translation. Thus, if $\mathbf{m} = (m_1, \dots m_k)$ is a sequence of sections of M_X and n is a section let $\mathbf{m} + n := (m_1 + n, \dots m_k + n)$. If $(\mathbf{m}, \mathbf{m}') \in S$,

$$\begin{aligned} s(\mathbf{m} + n) &= \sum_i \alpha_X(m_i + n) \\ &= \alpha_X(n) \sum_i \alpha_X(m_i) \\ &= \alpha_X(n) \sum \alpha_X(m'_i) \\ &= \sum \alpha_X(m'_i + n). \\ &= s(\mathbf{m}' + n) \end{aligned}$$

so that $(\mathbf{m} + n, \mathbf{m}' + n) \in S$. Next, we compute

$$\begin{aligned} r(\mathbf{m} + n) &= \sum_i \alpha_X(m_i + n) \otimes (m_i + n) \\ &= \alpha_X(n) \sum_i \alpha_X(m_i) \otimes m_i + \alpha_X(n) \sum_i \alpha_X(m_i) \otimes n \\ &= \alpha_X(n) r(\mathbf{m}) + \alpha_X(n) s(\mathbf{m}) \otimes n \end{aligned}$$

Hence if $(\mathbf{m}, \mathbf{m}') \in S$, $(\mathbf{m} + n, \mathbf{m}' + n) \in S$ and

$$\begin{aligned} r(\mathbf{m} + n) - r(\mathbf{m}' + n) &= \alpha_X(n) r(\mathbf{m}) - \alpha_X(n) r(\mathbf{m}') + \\ &\quad\ \alpha_X(n) s(\mathbf{m}) \otimes n - \alpha_X(n) s(\mathbf{m}') \otimes n \\ &= \alpha_X(n)(r(\mathbf{m}) - r(\mathbf{m}')). \end{aligned}$$

Thus R_1 indeed an $\mathcal{O}_X$-submodule of $\mathcal{O}_X \otimes M_X^{gp}$, as required.

Let $d \colon M_X \to \Omega^1_{X/Y}$ be the map described in the statement. As we have seen, $d \colon \mathcal{O}_X \to \Omega^1_{X/Y}$ is unique if it exists. If a is any section of $\mathcal{O}_X$, choose a sequence $\mathbf{m}$ of local sections of M_X with $s(\mathbf{m}) = a$. Then it follows from the definition of R_1 that the image of $r(\mathbf{m})$ in $\Omega^1_{X/Y}$ is independent of the choice of m. Let $d \colon \mathcal{O}_X \to \Omega_{X/Y}$ be the map of abelian sheaves such that $ds(\mathbf{m})$ is the class of $r(\mathbf{m})$ for every sequence $\mathbf{m}$. In particular, if m is a section of M_X and $\mathbf{m} := (m)$ then $\alpha_X(m) = s(\mathbf{m})$ and so $d\alpha_X(m)$ is the class of $r(\mathbf{m}) = \alpha_X(m) \otimes m$. Thus, $d\alpha_X(m) = \alpha_X(m) dm$, and the pair (d, d) satisfies (1) and (2), hence also (3) and (4), of Definition 1.1.1.

To check that (d, d) is universal, suppose that E is a sheaf of $\mathcal{O}_X$-modules and $(D, \delta) \in \mathrm{Der}_{X/Y}(X, E)$. Since E is a sheaf of abelian groups, δ factors uniquely through M_X^{gp}, and since E is a sheaf of $\mathcal{O}_X$-modules, it factors through a unique $\mathcal{O}_X$-linear map $\theta\colon \mathcal{O}_X \otimes M_X^{gp} \to E$. Property (2) of the definition implies that θ annihilates R_2. If $\mathbf{m}$ is a sequence of sections of M_X,

$$\begin{aligned}
\theta(r(\mathbf{m})) &= \theta\Big(\sum_i \alpha_X(m_i) \otimes m_i\Big) \\
&= \sum_i \Big(\alpha_X(m_i)\delta(m_i)\Big) \\
&= \sum_i \Big(D(\alpha_X(m_i)\Big) \\
&= D\Big(\sum_i \alpha_X(m_i)\Big) \\
&= D(s(\mathbf{m}))
\end{aligned}$$

Consequently if $(\mathbf{m}, \mathbf{m}') \in S$, $\theta(\rho(\mathbf{m})) = \theta(\rho(\mathbf{m}'))$, so θ factors uniquely through an $\mathcal{O}_X$-linear map $h\colon \Omega^1_{X/Y} \to E$. This is the unique unique homomorphism such that $hd(m) = \delta(m)$ for every local section m of M. It follows that $hd(a) = D(a)$ for every local section a of $\mathcal{O}_X$. $\square$

Remark 1.1.7 When using additive notation for M_X, it seems sensible to write d for the map $M_X \to \Omega^1_{X/Y}$. Then $\alpha_X\colon M_X \to \mathcal{O}_X$ behaves like an exponential map, which is consistent with the equation $d\alpha_X(m) = \alpha_X(m)dm$. In this case, the canonical injection $\mathcal{O}_X^* \to M_X$ needs a symbol λ, which should be regarded as a logarithm map, and one has $d\lambda(u) = u^{-1}du$, as expected. When the monoid law on M_X is written multiplicatively and $\mathcal{O}_X^*$ is viewed as a submonoid of M_X, it is more natural (and more usual) to write $dlog$ for the universal map $M_X \to \Omega^1_{X/Y}$, since $dlog\,(mn) = dlog\,(m) + dlog\,(n)$ and since $dlog\,(u) = u^{-1}du$ if $u \in \mathcal{O}_X^* \subseteq M_X$. For example if $j\colon U \to X$ is an open immersion and $\alpha_{U/X}\colon M_X \to \mathcal{O}_X$ is the direct image of the triviallog structure on X, α_X is an injection, and a section m of M_X is a function on X whose restriction to U is invertible. The multiplicative notation is more natural in this case, and for each section m of M_X, $\alpha_X(m) = m$, so $m dlog\, m = dm$, as expected. For example, if the underlying scheme $\underline{X}$ is smooth over Y (with trivial log structure) and U is the complement of a divisor D with normal crossings, we shall see in (3.1.20) that $\Omega^1_{X/Y}$ agrees with the classically considered sheaf of "differentials with log poles along D" [4].

Remark 1.1.8 It is possibly, and perhaps simpler, to give a more familiar looking construction of $\Omega^1_{X/Y}$, using generators and relations. This is fairly straightforward, but is sometimes cumbersome in applications. If one is willing to use the standard construction of $\Omega^1_{X/Y}$ for schemes, one can also use the following construction. For any morphism $f\colon X \to Y$ of prelog schemes,

$$\Omega^1_{X/Y} = \left(\Omega^1_{\underline{X}/\underline{Y}} \oplus \mathcal{O}_X \otimes M_X^{gp}\right)/R,$$

where R is the sub $\mathcal{O}_X$-module generated by sections of the form

$$(d\alpha_X(m), -\alpha_X(m)\otimes m) \quad \text{for} \quad m \in M_X, \quad (0, 1\otimes f^\flat(n)) \quad \text{for} \quad n \in f^{-1}(M_Y).$$

Definition 1.1.9 *Let θ be a morphism of log rings:*

$$\begin{tikzcd} P \arrow[r, "\alpha"] & A \\ Q \arrow[u, "\theta^\flat"] \arrow[r, "\beta"] & B \arrow[u, "\theta^\sharp"'] \end{tikzcd}$$

and let E be an A-module. Then a (log) derivation of $(A,P)/(B,Q)$ with values in E is a pair (D,δ), where $D\colon A \to E$ is a homomorphism of abelian groups and $\delta\colon P \to E$ is a homomorphism of monoids, such that

1. *For every $p \in P$, $D(\alpha(p)) = \alpha(p)\delta(p)$.*
2. *For every $q \in Q$, $\delta(\theta^\flat(q)) = 0$.*
3. *For any two elements b and b' of B, $D(bb') = bD(b') + b'D(b)$.*
4. *For any $b \in B$, $D(\theta^\sharp(b)) = 0$.*

Lemma 1.1.10 *Let $f\colon X \to Y$ be the morphism of log schemes or prelog schemes corresponding to a morphism $\theta\colon (B,Q) \to (A,P)$ of log rings, and let E be a sheaf of $\mathcal{O}_X$-modules. Then the natural map*

$$\mathrm{Der}_{X/Y}(E) \to \mathrm{Der}_{(A,P)/(B,Q)}(\Gamma(X,E))$$

is an isomorphism.

Proof: Thanks to (1.1.2), it suffices to treat the case of prelog schemes. Let $X := \operatorname{Spec} A$ with the prelog structure

$$\alpha: P \to A,$$

and similarly for Y. It is standard (and straightforward in the case of the Zariski topology) that any $D: A \to \Gamma(X, E)$ satisfying (3) and (4) of (1.1.9) extends uniquely to a $\tilde{D}: \mathcal{O}_X \to E$ satisfying (3) and (4) of (1.1.1). If (D, δ) is a log derivation of $(A, P)/(B, Q)$ with values in $\Gamma(X, E)$, then corresponding to the homomorphism of monoids $\delta: Q \to \Gamma(X, E)$ is a homomorphism of sheaves of monoids $Q \to E$, and it follows immediately that $(\tilde{D}, \tilde{\delta})$ is the unique element of $\mathrm{Der}_{X/Y}(E)$ corresponding to (D, δ). □

The following corollary is an immediate consequence of (1.1.10).

Corollary 1.1.11 *Let $f: X \to Y$ be a morphism of log schemes which is given by a morphism of log rings θ as in (1.1.9). Then $\Omega^1_{X/Y}$ is the quasi-coherent sheaf associated to the A-module obtained by dividing $\Omega^1_{A/B} \oplus A \otimes P^{gp}$ by the submodule generated by elements of the form $(d\alpha(p), -\alpha(p) \otimes p)$ for $p \in P$ and $(0, 1 \otimes \theta^\flat(q)$ for $q \in Q$.*

Corollary 1.1.12 *If $f: X \to Y$ is a morphism of coherent log schemes, $\Omega^1_{X/Y}$ is quasi-coherent, and it is of finite type (resp. of finite presentation) if $\underline{f}$ is of finite type (resp. of finite presentation).*

Proof: This assertion is of a local nature on X, so we may assume that X and Y are affine, and by (II, 2.2.3), that f admits a coherent chart. Then f comes from a morphism of log rings, and hence by (1.1.11) $\Omega^1_{X/Y}$ is quasi-coherent. Since $\Omega^1_{\underline{X}/\underline{Y}}$ is of finite type (resp. of finite presentation) if f is, and since P^{gp}/Q^{gp} is a finitely generated abelian group, $\Omega^1_{X/Y}$ is of finite type (resp. of finite presentation) if $\underline{f}$ is. □

The following result shows that formation of the sheaf of differentials is almost unchanged when passing to X^{int} or X^{sat}.

Proposition 1.1.13 *Let $f: X \to Y$ be a morphism of coherent (resp. fine) log schemes, and let $X' := X^{\mathrm{int}}$ and $Y' := Y^{\mathrm{int}}$ (resp. X^{sat} and $Y' := Y^{\mathrm{sat}}$). Then the natural maps $g: X' \to X$ and $h: Y' \to Y$ induce isomorphisms:*

$$g^*\Omega^1_{X/Y} \to \Omega^1_{X'/Y} \quad \text{and} \quad \Omega^1_{X'/Y} \to \Omega^1_{X'/Y'}.$$

Proof: This assertion is of a local nature on X, so we may by (2.2.3) assume that X and Y are affine and that there exists a chart for f subordinate to a morphism of finitely generated monoids. Then f is induced by a morphism $\theta\colon (B,Q) \to (A,P)$ of log rings. Let $P' := P^{\mathrm{int}}$ (resp. P^{sat}) and let $A' := A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[P']$, so that $X' = \operatorname{Spec}(P' \to A')$ (2.4.5), and analogously for Y and Y'. Since in all cases the maps P^{gp} to P'^{gp} and $Q \to Q^{gp}$ are isomorphisms, it will suffice to prove the following lemma. □

Lemma 1.1.14 *Let $\theta\colon (B,Q) \to (A,P)$ be a homomorphism of log rings, and let $\theta'^{\flat}\colon Q' \to P'$ be an extension of $\theta^{\flat}$ such that the corresponding group homomorphisms $Q^{gp} \to Q'^{gp}$ and $P^{gp} \to P'^{gp}$ are isomorphisms. Let $A' := A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[P']$ and $B' := B \otimes_{\mathbf{Z}[Q]} \mathbf{Z}[Q']$. Then the natural maps*

$$A' \otimes \Omega^1_{(A,P)/(B,Q)} \to \Omega^1_{(A',P')/(B,Q)} \quad \text{and} \quad \Omega^1_{(A',P')/(B,Q)} \to \Omega^1_{(A',P')/(B',Q')}$$

are isomorphisms.

Proof: To prove that the second arrow is an isomorphism, we must prove that for any A'-module E', the natural map

$$\operatorname{Der}_{(A',P')/(B',Q')}(E') \to \operatorname{Der}_{(A',P')/(B,Q)}(E')$$

is an isomorphism. This map is obviously injective. Suppose that $(D,\delta) \in \operatorname{Der}_{(A',P')/(B,Q)}(E')$. Since $Q^{gp} \to Q'^{gp}$ is an isomorphism, δ annihilates the image of Q'. It remains only to prove that D also annihilates the image of B'. For $q' \in Q$,

$$D(\theta^{\sharp}(e^{q'})) = D(\alpha(\theta^{\flat}(q')) = \alpha(\theta^{\flat}(q'))\delta(\theta^{\flat}(q') = 0.$$

Since B' is generated by B and $\mathbf{Z}[Q']$ and D annihilates B, it annihilates all of B', and this completes the proof.

For the first arrow, it suffices to prove that for every A'-module E', the map

$$\operatorname{Der}_{(A',P')/(B,Q)}(E') \to \operatorname{Der}_{(A,P)/(B,Q)}(E').$$

The injectivity follows from the fact that A' is generated by A and $\mathbf{Z}[P']$ and the fact that $P^{gp} \to P'^{gp}$ is an isomorphism. Suppose that (D,δ) is an E'-valued log derivation of $(A,P)/(B,Q)$. Then δ factors through a group homomorphism $P^{gp}/Q^{gp} \to E'$ and hence also through an A'-linear homomorphism $A' \otimes P^g/Q^{gp} \to E'$. Since $P^{gp} \to P'^{gp}$ is an isomorphism, it also factors through an A'-linear map

$$\tilde{\delta}: A' \otimes (P'^{gp}/Q^{gp}) \to E'.$$

Let $\pi: P' \to P'^{gp}/Q^{gp}$ be the natural map and for each $p' \in P'$, let

$$\delta'(p') := \tilde{\delta}(1 \otimes \pi(p')).$$

Then $\delta': P' \to E'$ is a monoid homomorphism annihilating the image of Q. Let

$$\beta: \mathbf{Z}[P'] \times A \to E'$$

be the unique biadditve mapping such that $\beta(e^{p'}, a) := \alpha'(p')(a\delta'(p') + Da)$. View $\mathbf{Z}[P']$ as a $\mathbf{Z}[P]$-module via the homomorphism $\mathbf{Z}[P] \to \mathbf{Z}[P']$ induced by the map $\phi: P \to P'$. Then if $p \in P$,

$$\begin{aligned}
\beta(e^p e^{p'}, a) &= \beta(e^{\phi(p)+p'}, a) \\
&= \alpha'(\phi(p) + p')(a\delta'(\phi(p) + p') + Da) \\
&= \alpha(p)\alpha'(p')\Big(a\delta(p) + a\delta'(p') + Da\Big) \\
&= \alpha'(p')\Big(\alpha(p)a\delta'(p') + a\alpha(p)\delta(p) + \alpha(p)Da\Big) \\
&= \alpha'(p')\Big(\alpha(p)a\delta'(p') + aD\alpha(p) + \alpha(p)Da\Big) \\
&= \alpha'(p')\Big(a\alpha(p)\delta'(p') + D(a\alpha(p))\Big) \\
&= \beta(e^{p'}, e^p a)
\end{aligned}$$

Thus the pairing is bilinear over $\mathbf{Z}[P]$ and induces a homomorphism of abelian groups

$$D': A' := \mathbf{Z}[P'] \otimes_{\mathbf{Z}[P]} A \to E'.$$

Then $(D', \delta') \in \mathrm{Der}_{(A',P')/(B,Q)}(E')$ and is the desired extension of (D, δ). □

Remark 1.1.15 Derivations for idealized log schemes are defined in exactly the same way as in (1.1.1). Thus, if $f: X \to Y$ is a morphism of idealized log schemes, and $(D, \delta) \in \mathrm{Der}_{X/Y}(E)$, we do not require that $\delta(k) = 0$ for $k \in K_X$, and $\Omega^1_{X/Y} = \Omega^1_{(X,\emptyset)/(Y,\emptyset)}$. The reason for this definition will become apparent from the geometric interpretation of log derivations in (2.2.2).

1.2 Examples

The sheaf of log differentials has an especially simple description in the case of monoid algebras.

Proposition 1.2.1 *Let $f\colon X \to Y$ be the morphism of log monoid schemes given by a homomorphism of monoids $\theta\colon P \to Q$. For $p \in P$, let $\pi(p)$ denote the class of p in $\operatorname{Cok}(\theta^{gp})$, and let $d\colon \mathbf{Z}[P] \to \mathbf{Z}[P] \otimes P^{gp}/Q^{gp}$ be the homomorphism of abelian groups sending e^p to $e^p \otimes \pi(p)$. Then (d, π) is the universal log derivation of X/Y, and $\Omega^1_{X/Y}$ is the quasi-coherent sheaf associated to $\mathbf{Z}[P] \otimes \operatorname{Cok}(\theta^{gp})$.*

Proof: By (1.1.10) we can work with derivations on the level of rings and modules. If p and p' are elements of P,

$$\begin{aligned} d(e^p)e(p')) &= d(e^{p+p'}) \\ &= e^{p+p'} \otimes (\pi(p+p')) \\ &= e^p e^{p'} \otimes (\pi(p) + \pi(p')) \\ &= e^p e^{p'} \otimes \pi(p') + e^p e^{p'} \otimes \pi(p) \\ &= e^p de^{p'} + e^{p'} de^p \end{aligned}$$

It follows that $d(aa') = ad(a') + a'd(a)$ for any $a, a' \in \mathbf{Z}[P]$. Since $de^p = e^p \otimes \pi(p)$ by definition, and since

$$de^{\theta(q)} = e^{\theta(q)}\pi(\theta(q)) = 0 - \pi(\theta(q)),$$

(d, π) is a log derivation of $(\mathbf{Z}[P], P)$ over $(\mathbf{Z}[Q], Q)$. If (D, δ) is log derivation with values in E, then since δ annihilates θ, δ factors through π, and since $De^p = e^p\delta(p)$, D factors through d. $\square$

Corollary 1.2.2 *Let G be an abelian group and let $X := \operatorname{Spec} R[G]$. Then there is a unique isomorphism:*

$$R[G] \otimes_{\mathbf{Z}} G \to \Gamma(X, \Omega^1_{X/R})$$

mapping $e^g \otimes g$ to de^g for each $g \in G$.

Proof: The previous result shows that this is the case when X is replaced by the log scheme $\operatorname{Spec} G \to R[G]$. However, since G is a group, the log structure on this log scheme is trivial, so the log differentials agree with the usual differentials of the underlying scheme, by (1.1.5).

It seems worthwhile to give an alternative direct proof of the corollary. Since $R[G]$ is the free R-module with basis $e : G \to R[G]$, there is a unique R-linear map R-linear map

$$d\colon R[G] \to R[G] \otimes_{\mathbf{Z}} G \quad \text{such that} \quad e^g \mapsto e^g \otimes g.$$

Then

$$d(e^g e^h) = D(e^{g+h}) = e^{g+h} \otimes (g+h) = e^g e^h \otimes g + e^g e^h \otimes h = e^h D(g) + e^h \otimes d(g).$$

Thus d is a derivation, and the corollary will be proved if we show that d is universal. Let $D\colon R[G] \to E$ be any derivation, and define $\delta\colon G \to E$ by $\delta(g) := e^{-g} D(e^g)$. Then

$$\delta(g+h) = e^{-g-h} D(e^{g+h}) = e^{-g} e^{-h} \left(e^h D(e^g) + e^{-g} D(e^h) \right) = \delta(g) + \delta(h).$$

Thus δ is a homomorphism of abelian groups and induces by adjunction an $R[G]$-linear $\tilde{\delta}\colon R[G] \otimes G \to E$ such that $\tilde{\delta}(1 \otimes g) = e^{-g} D(e^g)$ for all g. But then

$$D(e^g) = e^g \tilde{\delta}(1 \otimes g) = \tilde{\delta}(e^g \otimes g) = \tilde{\delta}(d(e^{gp})).$$

In other words, $D = \tilde{\delta} \circ d$, proving the required universality of d. $\square$

Corollary 1.2.3 *Let $0 \to G' \to G \to G'' \to 0$ be an exact sequence of abelian groups and let $I \subseteq R[G]$ be the kernel of the corresponding homomorphism $R[G] \to R[G'']$. Then there is a unique isomorphism*

$$R[G''] \otimes_{\mathbf{Z}} G' \cong I/I^2 \quad \textit{such that} \quad g' \mapsto (e^{g'} - 1) \pmod{I^2} \quad \textit{for all } g' \in G'.$$

Proof: If $g' \in G'$, then $e^{g'} - 1 \in I$. If also $h' \in G'$,

$$\begin{aligned} e^{g'+h'} - 1 &= e^{g'} e^{h'} - 1 \\ &= (e^{g'} - 1)(e^{h'} - 1) + (e^{g'} - 1) + (e^{h'} - 1) \\ &= (e^{g'} - 1) + (e^{h'} - 1) + \pmod{I^2}. \end{aligned}$$

Thus the map $G' \to I/I^2$ sending g' to the class of $e^{g'} - 1$ is a group homomorphism. Since I/I^2 is an $R[G]/I \cong R[G'']$-module, this homomorphism induces by adjunction an $R[G'']$-linear map as in the statement of the corollary.

On the other hand, we constructed in the Corollary 1.2.2 above a derivation $D: G \to R[G] \otimes G$ sending each e^g to $e^g \otimes g$. Consider the composite

$$I \xrightarrow{D} R[G] \otimes G \to R[G]/I \otimes G \cong R[G''] \otimes G.$$

Since the last of these $R[G]$-modules is annihilated by I and D is a derivation, it follows that the above map is in fact $R[G]$-linear, and in particular that it annihilates I^2. Furthermore, the ideal I is generated as an ideal by the set of all elements of the form $e^{g'} - 1$ with $g' \in G'$, and for any such element $D(e^{g'} - 1) = e^{g'} \otimes g'$. This shows that the image of the map is contained in $R[G''] \otimes G'$, and in fact our map factors through an $R[G]$-linear map $I/I^2 \to R[G''] \otimes G'$. One sees by checking on generators that this map is inverse to the map in the statement of the corollary. $\square$

Example 1.2.4 In the category of schemes, the sheaf of differentials $\Omega^1_{X/Y}$ can be identified with the conormal sheaf of the diagonal embedding $X \to X \times_Y X$. The logarithmic version of this useful interpretation is not straightforward, because in general the diagonal embedding is not strict, and the notion of the conormal sheaf requires some preparation; see **??**). Let us explain here how this works when X is the log scheme associated to a fine monoid P and $Y := \operatorname{Spec} R$ (with trivial log structure). The product $X \times_Y X$ is just $\mathsf{A}_{\mathsf{P}\oplus\mathsf{P}}$ in the category of log schemes over Y and the diagonal mapping $\Delta_X: X \to X \times_Y X$ corresponds to the morphism of monoids $s: P \oplus P \to P$ sending a pair (p_1, p_2) to $p_1 + p_2$. This map is not strict, but in this case this difficulty can be remedied in a fairly canonical way. Let $(P \oplus P)^e := \{(x_1, x_2) \in P^{gp} \oplus P^{gp} : x_1 + x_2 \in P$, so that s factors

or morphism of fine monoids?

$P \oplus P \to (P \oplus P)^e \to P$. In fact, there is a commutative diagram

$$\begin{tikzcd}
P \oplus P \arrow[d, "g^\flat"'] \arrow[dr, "h^\flat"] & \\
(P \oplus P)^e \arrow[r, "\cong"] \arrow[d, "s'^\flat"'] & P \oplus P^{gp} \arrow[dl, "t^\flat"] \\
P &
\end{tikzcd}$$

The horizontal arrow in the diagram sends a pair (x_1, x_2) to $(x_1 + x_2, x_2)$, h sends (p_1, p_2) to $(p_1 + p_2, p_2)$, and t sends (p, x) to p. The corresponding diagram in the category of log schemes is the following:

$$\begin{tikzcd}
& X \times X^* \arrow[d, "\cong"] \\
X \arrow[ur, "t"] \arrow[r, "s'"] \arrow[dr, "\Delta"'] & (X \times X)^e \arrow[d, "g"] \\
& X \times X
\end{tikzcd}$$

Note that s' and t are strict closed immersions, but Δ is in general not strict. The map g is a part of a blowing up, and the modified diagonal t is just $(\mathrm{id}_X, 1_{X^*})$. The diagram shows that the conormal sheaf I_t/I_t^2 of t can be identified with the pullback of the conormal sheaf of the identity section 1_{X^*} of the group scheme $X^* = \operatorname{Spec} R[P^{gp}]$. By Corollary 1.2.2, the latter is canonically isomorphic to $R \otimes_{\mathbf{Z}} P^{gp}$. Thus

$$I_t/I_t^2 \cong R[P] \otimes_{\mathbf{Z}} P^{gp} \cong \Omega^1_{X/R}.$$

Example 1.2.5 Let $f\colon X \to Y$ be a morphism of log points, with underlying schemes $x := \underline{X}$ and $y := \underline{Y}$, and recall that $M^{gp}_{X/Y}$ is the cokernel of the map $f^*M^{gp}_Y \to M^{gp}_X$ (1.2.8). Let $Q_{X/Y}$ be the cokernel of the map $f^{-1}M^{gp}_Y \to M^{gp}_X$, so that there is an an exact sequence:

$$0 \to k(x)^*/k(y)^* \to Q_{X/Y} \to M^{gp}_{X/Y} \to 0.$$

If $m \in M^+_X$, $\alpha_X(m) = 0$ in $k(x)$, and if $m \in M^*_X$, its image $\pi(m)$ in $M^{gp}_{X/Y}$ is zero. Thus in any case $\alpha_X(m) \otimes \pi(m) = 0$ in $k(x) \otimes M^{gp}_{X/Y}$, so $(0, \pi)$ is a log derivation of X/Y, and by the universal property of $\Omega^1_{X/Y}$, the obvious map $k(x) \otimes Q_{X/Y} \to k(x) \otimes M^{gp}_{X/Y}$ factors through $\Omega^1_{X/Y}$. Thus there is a commutative diagram with exact rows:

$$\begin{tikzcd}
k(x) \otimes k(x)^*/k(y)^* \arrow[r] \arrow[d, "dlog"'] & k(x) \otimes Q_{X/Y} \arrow[r] \arrow[d] & k(x) \otimes M^{gp}_{X/Y} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
\Omega^1_{k(x)/k(y)} \arrow[r] & \Omega^1_{X/Y} \arrow[r, "\rho"] & k(x) \otimes M^{gp}_{X/Y} \arrow[r] & 0.
\end{tikzcd}$$

The map ρ can be thought of as a kind of Poincaré residue. In particular, if $x = y$, $\Omega^1_{X/Y} \cong k(x) \otimes M^{gp}_{X/Y}$.

The difference between classical and log differentials is revealed (as we saw in the case of log points (1.2.5)), by the Poincaré residue. We describe a generalization, first for log rings, then for log schemes.

Example 1.2.6 Let $\theta\colon (B, Q) \to (A, P)$ be a morphism of log rings, let F be a face of P which contains the image of Q and let I be the ideal of A generated by $\alpha(\mathfrak{p}_F)$. Define $\delta\colon P \to A/I \otimes (P/F)^{gp}$ to be the homomorphism sending p to $1 \otimes \pi_F(p)$, where $\pi_F(p)$ is the image of p in $(P/F)^{gp}$. Then if $p \in P$, $\alpha(p)\delta(p) = 0$ in $A/I \otimes (P/F)^{gp}$, and so $(0, \delta)$ is a log derivation of $(A, P)/(B, Q)$ with values in $(A/I \otimes P/F)^{gp}$. It follows that there is unique A-linear homomorphism

$$\rho_F\colon \Omega^1_{(A/P)/(B/Q)} \to A/I \otimes (P/F)^{gp}$$

such that $\rho_F(dp) = 1 \otimes \pi_F(p)$ for $p \in P$ and $\rho_F(da) = 0$ for $a \in A$.

Proposition 1.2.7 *Let $f\colon X \to Y$ be a morphism of idealized log schemes and let $F \subseteq M_X$ be a sheaf of faces containing the image of $f^{-1}(M_Y)$ and such that the sheaf-theoretic union of F and K_X is M_X. Then there is a unique $\mathcal{O}_X$-linear map ρ_F making the following diagram commute:*

$$\begin{tikzcd}
M_X \arrow[d, "dlog"'] \arrow[dr, "\delta"] & \\
\Omega^1_{X/Y} \arrow[r, "\rho_F"] & \mathcal{O}_X \otimes (M_X/F)^{gp},
\end{tikzcd}$$

where $\delta(m) := 1 \otimes \pi_F(m)$ and $\pi_F(m)$ is the image of m in $(M_X/F)^{gp}$ for every $m \in M_X$. The map ρ_F is called the Poincaré residue *along the face F, and $\rho_F(da) = 0$ for every $a \in \mathcal{O}_X$.*

Proof: If $m \in K_X$, $\alpha_X(m) = 0$, and if $m \in F$, $\pi_F(m) = 0$. If m is any section of M_X, m is locally either in F or in K_X, and hence $\alpha_X(m) \otimes \delta(m) = 0$. Then $(0, \delta)$ is a log derivation of X/Y with values in $\mathcal{O}_X \otimes (M_X/F)^{gp}$, and the existence and uniqueness of ρ_F follow from the universal mapping property of $\Omega^1_{X/Y}$. $\square$

1.3 Functoriality

Most of the results about differentials and derivations for schemes carry over to log schemes, so we provide only a sketch.

Proposition 1.3.1 *Let*

$$\begin{tikzcd}
X' \arrow[r, "g"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "h"] & Y
\end{tikzcd}$$

be a commutative diagram of prelog schemes. Then there is a unique homomorphism

$$g^*\Omega^1_{X/Y} \to \Omega^1_{X'/Y'},$$

sending $1 \otimes da$ to $dg^\sharp(a)$ for every section a of $g^{-1}(\mathcal{O}_X)$ and $1 \otimes d\log(m)$ to $d\log g^\flat(m)$ for every section m of $g^{-1}(M_X)$. This morphism is an isomorphism in the following cases:

1. *f' is the morphism of log schemes associated to to the morphism f of prelog schemes.*
2. *The diagram is Cartesian in the diagram of prelog schemes.*
3. *The diagram is Cartesian in the diagram of log schemes.*
4. *The diagram is Cartesian in the diagram of fine or fine saturated log schemes.*

Proof: Let E' be any sheaf of $\mathcal{O}_{X'}$-modules and let (D, δ) be an element of $\mathrm{Der}_{X'/Y'}(E')$. As we have seen in (1.1.2), there is a natural homomorphism

$$g_* \mathrm{Der}_{X'/Y'}(E') \to \mathrm{Der}_{X/Y}(g_* E').$$

The existence of and uniqueness of the map on differentials follows from their defining universal property. The fact that the maps are isomorphisms follows from the corresponding statements in (1.1.2) in cases (1), (2), and (3), and (4) follows from (1.1.13). □

Example 1.3.2 Associated to any morphism $f\colon X \to Y$ of prelog schemes is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \underline{X} \arrow[d, "\underline{f}"] \\ Y \arrow[r] & \underline{Y}, \end{tikzcd}$$

hence a canonical homomorphism

$$\Omega^1_{\underline{X}/\underline{Y}} \to \Omega^1_{X/Y},$$

sending (D, δ) to D. If f is strict, the diagram is Cartesian, and this homomorphism is an isomorphism by (1.3.1).

Corollary 1.3.3 *Let $f\colon X \to Y$ be a morphism and let $X \to X_Y \to Y$ be its canonical factorization, with $X \to X_Y$ an isomorphism of underlying schemes and $X_Y \to Y$ strict. Then the map*

$$\Omega^1_{\underline{X}/\underline{Y}} \to \Omega^1_{X_Y/Y}$$

is an isomorphism.

Proof: In fact, the diagram

$$\begin{tikzcd} X_Y \arrow[r] \arrow[d] & \underline{X} \arrow[d] \\ Y \arrow[r] & \underline{Y} \end{tikzcd}$$

is Cartesian, and so it suffices to apply (1.3.1). $\square$

Corollary 1.3.4 *If the square in (1.3.1) is Cartesian in the category of coherent (resp. fine, resp. saturated), then the induced homomorphism*

$$f'^*\Omega^1_{Y'/Y} \oplus g^*\Omega^1_{X/Y} \to \Omega^1_{X'/Y}$$

is an isomorphism.

Proof: As we have seen, the fact that the diagram is Cartesian implies that the map $g^*\Omega^1_{X/Y} \to \Omega^1_{X'/Y'}$ is Cartesian. Then the map $g^*\Omega^1_{X/Y} \to \Omega^1_{X'/Y}$ provides a splitting of the map $\Omega^1_{X'/Y} \to \Omega^1_{X'/Y'}$. By the same token, the map $f'^*\Omega^1_{Y'/Y} \to \Omega^1_{X'/X}$ is an isomorphism, and the map $f'^*\Omega^1_{Y'/Y} \to \Omega^1_{X'/Y}$ provides a splitting of the map $\Omega^1_{X'/Y} \to \Omega^1_{X'/S}$. $\square$

2 Thickenings and deformations

2.1 Thickenings and extensions

Definition 2.1.1 *A log thickening is a strict closed immersion $i\colon S \to T$ of log schemes such that:*

1. *the ideal I of S in T is a nil ideal, and*
2. *the subgroup $1 + I$ of $\mathcal{O}_T^* \cong M_T^*$ operates freely on M_T.*

A log thickening of order n is a log thickening such that $I^{n+1} = 0$.

If T is quasi-integral, condition (2) in (2.1.1) is automatic. A thickening $i\colon S \to T$ induces a homeomorphism of the underlying topological spaces of S and T, and it is common to identify them. An idealized log thickening is defined in the same way, and in particular the map $\overline{K}_T \to \overline{K}_S$ is required to be an isomorphism.

Proposition 2.1.2 *Let $i\colon S \to T$ be a log thickening, with ideal I.*

1. *The commutative square*

$$\begin{tikzcd} \mathcal{O}_T^* \arrow[r] \arrow[d] & M_T \arrow[d, "i^\flat"] \\ \mathcal{O}_S^* \arrow[r] & M_S \end{tikzcd}$$

   is Cartesian and cocartesian (i.e., $\mathcal{O}_T^*$ is the inverse image of $\mathcal{O}_S^*$ in $\mathcal{O}_T$, and M_S is the amalgamated sum of $\mathcal{O}_S^*$ and M_T.)

2. $\operatorname{Ker}\Big(\mathcal{O}_T^* \to \mathcal{O}_S^*\Big) = \operatorname{Ker}\Big(M_T^{gp} \to M_S^{gp}\Big) = 1 + I.$

3. *The action of $1 + I$ on M_T (resp. on M_T^{gp}) makes it a torsor over M_S (resp. over M_S^{gp}). That is, the maps*

$$(1+I) \times M_T \to M_T \times_{M_S} M_T \quad \text{and} \quad (1+I) \times M_T^{g} \to M_T^{gp} \times_{M_S^{gp}} M_T^{gp}$$

$$(u, m) \mapsto (m, um)$$

   are isomorphisms.

4. *The square*

$$\begin{tikzcd} M_T \arrow[r] \arrow[d] & M_T^{gp} \arrow[d] \\ M_S \arrow[r] & M_S^{gp} \end{tikzcd}$$

is Cartesian.

Proof: The fact that the square in (1) is Cartesian is just the statement that the homomorphism $i^\flat$ is local, which is always the case (**??**). The fact that the diagram is cocartesian comes from the fact that i is strict, so that M_S is the log structure associated to the prelog structure $M_T \to \mathcal{O}_S$. Since I is a nilideal, any local section a of I is locally nilpotent, and hence $1 + a$ is a unit of $\mathcal{O}_T$. In fact it is clear that $1 + I$ is exactly the kernel of the homomorphism $\mathcal{O}_T^* \to \mathcal{O}_S^*$. Since $M_T \to \mathcal{O}_T$ is a log structure, $M_T^* = \mathcal{O}_T^*$, and since the action of $1+I$ on M_T is free, the map $1+I \to M_T^{gp}$ is injective, and evidently is contained in the kernel of the map $M_T^{gp} \to M_S^{gp}$. Conversely, if is any local section x of $M_T^{;}t$ is the class of $m' - m$ for two sections of M_T, and if x maps to zero in M_S^{gp}, there exists a local section n of M_S such that $i^\flat(m')+n = i^\flat(m)+n$. Locally n lifts to a section m'' of M_T, and the equation then becomes $i^\flat(m' + m'') = i^\flat(m + m'')$. Then there exists a $uin1 + I$ such that $m' + m" = u + m + m"$, and hence $m' - m = u$ in M_T^{gp}. This shows that $x \in 1 + I$ and completes the proof of (2). These same arguments also prove (3).

The last statement is trivial when M_T and M_S are integral; let us check it in the general case as well. Let (m, x) be a local section of $M_S \times_{M_S^{gp}} M_T^{gp}$. We may write $m = i^\flat(m')$ for a local section of M_T and let x be the class of $m_2 - m_1$ for local sections m_i of M_T. Since m' and x have the same image in M_S^{gp}, there exists a local section m' of M_T such that

$$i^\flat(m') + i^\flat(m) + i^\flat(m_1) = i^\flat(m') + i^\flat(m_2).$$

Then there is a local section u of $1+I$ such that $u+m'+m+m_1 = m'+m_2$ in M_T. Then $u+m$ is a section of M_T mapping to (m, x). Suppose on the other hand that m and m' are sections of M_T with the same image in $M_S \times_{M_S^{gp}} M_T^{gp}$. Since the images in M_S of m and m' are the same, $m' = u + m$ for some section u of $1+I$, and since the images of m and m' in M_T^{gp} are the same, u maps to 0 in M_T^{gp}. But this implies that $u = 0$, so $m' = m$, completing the proof. □

Corollary 2.1.3 *Let $i: S \to T$ be a log thickening.*

1. *T is coherent (resp. integral, fine, saturated) if and only if S is.*

2. *Let $\beta: P \to M_T$ be a homomorphism from a constant monoid P to M_T. Then if β is a chart for T, $i^\flat \circ \beta$ is a chart for M_S, and conversely if S is quasi-integral.*

Remark 2.1.4 Let $i: S \to T$ be a log thickening of S. Since i is defined by a nilideal, i induces a homeomorphism on the underlying topological spaces (*i.e.*, with the Zariski topologies). If U is a Zariski open subset of S, the restriction of T to U is a log thickening of U. Thus the category Thick of log thickenings can be viewed as a fibered category over the category S_{zar} of Zariski open subsets of S. If T_1 and T_2 are log thickenings of U_1 and U_2 respectively, then the functor which to every open set V of U_1 assigns the set of morphisms $T_{1|_V} \to T_2$ forms a sheaf on U_1. Moreover, log thickenings can be described locally and glued: given an open covering $\{U_i\}$ of an open $U \subseteq S$, a collection of thickenings $U_i \to T_i$, and a collection of isomorphisms (*descent data*)

$$\epsilon_{ij}: T_{i|_{U_i \cap U_j}} \cong T_{j|_{U_i \cap U_j}}$$

satisfying the cocycle condition [], there is a unique thickening $U \to T$, together with isomorphisms $T_{|_{U_i}} \cong T_i$ compatible with the descent data. These conditions mean that Thick forms a *stack* for the Zariski topology of S.

For log thickenings of finite order n, an analogous statement holds for the étale topology. Explain this more

Definition 2.1.5 *Let $f: X \to Y$ be a morphism of log schemes and let I be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. A Y-extension of X by I is a* *commutative diagram*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & T \arrow[dl] \\ Y & \end{tikzcd}$$

where i is a thickening of order one with $I = \ker(i^\sharp)$. If $u\colon J \to I$ is a homomorphism of quasi-coherent sheaves of $\mathcal{O}_X$-modules, and $i\colon X \to S$ (resp. $j\colon X \to T$) is a Y-extension of X by I (resp. by J), then a morphism of Y-extensions over u *is a Y-morphism $g\colon S \to T$ such that $g \circ i = j$ and $g^\sharp$ acts as u on J. When $I = J$ and $u = \mathrm{id}$, one says simply that g is a* morphism of Y-extensions.

If $g\colon i \to j$ is a morphism of Y-extensions over u, then $g^\flat$ fits into a diagram

$$\begin{tikzcd} M_T \arrow[r, "g^\flat"] \arrow[d] & M_S \arrow[dl] \\ M_X & \end{tikzcd}$$

and $g^\flat$ is a morphism of torsors over M_X associated to $1 + u\colon 1 + J \to 1 + I$. The category of Y-extensions of X with a fixed I (with morphisms over id_I) is a groupoid: any morphism is an isomorphism.

Example 2.1.6 If E is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, the *trivial Y-extension of X by E*, denoted $X \oplus E$, is the log scheme T defined by $\mathcal{O}_T := \mathcal{O}_X \oplus E$ with $(a, b)(a', b') := (aa', ab' + ba')$, with $M_T := M_X \oplus E$, and $\alpha_T(m, e) := (\alpha_X(m), \alpha_X(m)e)$ if $m \in M_X$ and $e \in e$. The kernel of $\mathcal{O}_T \to \mathcal{O}_X$ is the ideal $(0, E) \subseteq \mathcal{O}_T$, which acts freely on M_T, so that (id_X, i) is a first-order thickening. Furthermore, we have an evident retraction $T \to X$. Conversely, a Y-extension is trivial (isomorphic to $X \oplus I$) if and only if i admits a Y-retraction $r\colon T \to X$.

Example 2.1.7 Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of log schemes such that the underlying morphism of schemes $\underline{f}$ is affine, and let $i\colon X \to S$ be a Z-extension of X by a quasi-coherent $\mathcal{O}_X$-module I. Then f_*I is quasi-coherent on Y, and we can construct a Z-extension $f_*(i) := j\colon Y \to T$ of Y by f_*I and a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & S \arrow[d] \\ Y \arrow[r, "j"] & T \end{tikzcd}$$

as follows. Since I is quasi-coherent and f is affine, there is an exact sequence of sheaves:

$$0 \to f_*I \to f_*\mathcal{O}_S \to f_*\mathcal{O}_X \to 0$$

on Y. Let $\mathcal{O}_T$ be the fiber product of $f_*\mathcal{O}_S$ and $\mathcal{O}_Y$ over $f_*\mathcal{O}_X$, which fits into an exact sequence:

$$0 \to f_*I \to \mathcal{O}_T \to \mathcal{O}_Y \to 0.$$

Since $J := f_*I$ is quasi-coherent, there is a closed immersion $j\colon Y \to T$ with square-zero ideal J corresponding to this exact sequence. Since i is a thickening, $1+I$ acts freely on M_S, and the sequence

$$0 \to 1+I \to M_S \to M_X \to 0$$

is exact. Moreover, I is a square zero ideal, so as an abelian sheaf $1+I \cong I$, and consequently the sequence

$$0 \to f_*(1+I) \to f_*(M_S) \to f_*(M_X) \to 0$$

is also exact. Let M_T be the fiber product of $f_*(M_S)$ and M_Y over $f_*(M_X)$:

$$0 \to 1+J \to M_T \to M_Y \to 0.$$

Then the map $\alpha_T\colon M_T \to \mathcal{O}_T$ induced by α_S is a log structure, and $j\colon Y \to T$ is the desired extension.

If $g: S \to T$ is a morphism of Y-extensions of X over $u: J \to I$, then the sequence

$$0 \to I \to \mathcal{O}_S \to \mathcal{O}_X \to 0$$

is obtained by pushout of the sequence

$$0 \to J \to \mathcal{O}_T \to \mathcal{O}_X \to 0$$

along $u: J \to I$, and the sequence

$$1 \to 1 + I \to M_S \to M_X \to 0$$

is obtained by pushout of the sequence

$$1 \to 1 + J \to M_T \to M_X \to 0$$

along $1 + u: 1 + J \to 1 + I$. As in the classical case [], the Y-extensions of X by a variable module I form an $\mathcal{O}_X$-linear cofibered category over the category of quasi-coherent sheaves on X. For example, one can endow the set $\mathrm{Ext}_Y(X, I)$ of isomorphism classes of Y-extensions of X by I with an abelian group structure in a natural way. If $i: X \to S$ and $j: X \to T$ are Y-extensions of X by I, then the sum of the classes of i and j in $\mathrm{Ext}_Y(X,)$ is formed by first taking the Y-extension of X by $I \oplus I$ given by the fibered products $\mathcal{O}_S \times_{\mathcal{O}_X} \mathcal{O}_T$ and $M_S \times_{M_X} M_T$, and then taking the class of the pushout along the additional law $I \oplus I \to I$. The identity element of $\mathrm{Ext}_Y(X,)$ is the class of $X \oplus I$. If a is a section of $\mathcal{O}_X$ and T is an object of $\mathrm{Ext}_Y(X, E)$, then pushout along the endomorphism of E defined by a defines the class of aT in $\mathrm{Ext}_Y(X, E)$.

2.2 Differentials and deformations

The geometric motivation for the definition of log derivations lies in the study of extensions of morphisms to thickenings.

Definition 2.2.1 *Let $f: X \to Y$ be a morphism of log schemes. A log thickening over X/Y is a commutative diagram*

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[d, "g"'] & T \arrow[d, "h"] \\ X \arrow[r, "f"'] & Y, \end{tikzcd}$$

where i is a log thickening (2.1.1). A deformation of g to T is a section of

$$\mathrm{Def}_{X/Y}(g,T) := \{\tilde{g}: T \to X : \tilde{g} \circ i = g, f \circ \tilde{g} = h\}.$$

In the definition above, i is a homeomorphism, and we have identified the underlying topological spaces of S and T. If i has finite order, the étale topologies of S and T can also be identified, as we explained in (2.1.4). Thus $\mathrm{Def}_{X/Y}(g,T)$ forms a sheaf on S, and we can identify $\tilde{g}_*$ with g_*. Then a deformation of g to T amounts to a pair of homomorphisms:

$$\tilde{g}^\sharp : \mathcal{O}_X \to g_*\mathcal{O}_T \quad \text{and} \quad \tilde{g}^\flat : M_X \to g_*M_T,$$

such that $\alpha_T \circ g^\flat = g^\sharp \circ \alpha_X$, compatible with h and f.

Theorem 2.2.2 *Let $i: S \to T$ be a first-order log thickening of X/Y. Then there is an action of $\mathrm{Der}_{X/Y}(g_*I_T)$ on $g_*\mathrm{Def}_{X/Y}(g,T)$, with respect to which $g_*\mathrm{Def}_{X/Y}(g,T)$ becomes a pseudo-torsor under $\mathrm{Der}_{X/Y}(g_*I_T)$. With multiplicative notation for the monoid law of M_T, the action is given explicitly as follows: if $(D,\delta) \in \mathrm{Def}_{X/Y}(g_*I_T)$ and g_1 is a section of $g_*\mathrm{Def}_{X/Y}(g,T)$,*

$$(D,\delta)g_1 := (g_1^\sharp + D, (1+\delta)g_1^\flat).$$

Proof: Let g_1 be a deformation of g to T, and let (D,δ) be an element of $\mathrm{Der}_{X/Y}(g'_*I)$. If $g_2 := (D,\delta)g_1$ is given by the formulas above, then for $a \in \mathcal{O}_X$ and $m \in M_X$,

$$g_2^\sharp(a) := g_1^\sharp(a) + Da \quad \text{and} \quad g_2^\flat(m) := g_1^\flat(m)(1+\delta(m)).$$

We claim that g_2 is another deformation of g to T. It is standard and immediate to verify that $g_2^\sharp$ is a homomorphism of sheaves of $f^{-1}(\mathcal{O}_Y)$ algebras, because D is a derivation relative to Y and $I^2 = 0$. Moreover, since $I^2 = 0$, the map $I \to \mathcal{O}_T^* \subseteq M_T$ sending b to $1+b$ is a homomorphism of sheaves of monoids, and hence $g_2^\flat$ is also. Since $\delta \circ f^\flat = 0$, it still the case that $g_2^\flat \circ f^\flat = h^\flat$. Furthermore, if $m \in g^{-1}(M_X)$,

$$\begin{aligned}
g_2^\sharp(\alpha_X(m)) &= g_1^\sharp(\alpha_X(m)) + D\alpha_X(m) \\
&= g_1^\sharp(\alpha_X(m)) + \alpha_X(m)\delta(m) \\
&= g_1^\sharp(\alpha_X(m))(1+\delta(m)) \\
&= \alpha_T(g_1^\flat(m))(1+\delta(m)) \\
&= \alpha_T\Big((1+\delta(m))(g_1^\flat(m)\Big) \\
&= \alpha_T(g_2^\flat(m))
\end{aligned}$$

Thus g_2 really is a morphism of log schemes. Furthermore, $g_2 \circ i = g'$ because it takes values in I.

The calculations above show that the formulas above determine a mapping

$$\mathrm{Der}_{X/Y}(g_*I) \times g_*\mathrm{Def}_{X/Y}(g,T) \to g_*\mathrm{Def}_{X/Y}(g,T).$$

It is immediate from the formulas that this mapping is a group action. To see that this action of $\mathrm{Der}_{X/Y}(g_*I)$ makes $\mathrm{Def}_{X/Y}(g,T)$ a pseudo-torsor, we have to check that the map

$$\begin{array}{rcl} \mathrm{Der}_{X/Y}(g_*I) \times g_*\mathrm{Def}_{X/Y}(g,T) & \to & g_*\mathrm{Def}_{X/Y}(g,T) \times g_*\mathrm{Def}_{X/Y}(g,T) \\ ((D,\delta), g_1) & \mapsto & (g_1, (D,\delta) + g_1) \end{array}$$

is an isomorphism. If g_1 and g_2 are deformations of g to T, $(g_1^\flat, g_2^\flat)$ defines a homomorphism of sheaves of monoids

$$\delta\colon g^{-1}M_X \to M_T \times_{M_S} M_T \xrightarrow{\epsilon} (1+I) \times M_T \xrightarrow{pr} 1 + I \to I$$

where ϵ is the inverse of the isomorphism $(u, m_1) \mapsto (m_1, um_2)$ of (2.1.2.3) and the last map is the first-order logarithm homomorphism $u \mapsto b - 1$. Since $f \circ g_2 = f \circ g_1$, it follows that δ annihilates the image of M_Y. Moreover, $D\colon g_2^\sharp - g_1^\sharp$ defines a derivation $\mathcal{O}_X \to g_*I$, and reversing the calculation above shows that for $m \in g^{-1}(M_X)$, $\alpha_X(m)\delta(m) = D(\alpha_X(m))$. Thus (D,δ) is a derivation of X/Y with values in g_*I. □

more here

Corollary 2.2.3 *If $i\colon X \to T$ is a Y-extension of the log scheme X with ideal I, then* $\mathrm{Aut}(i) \cong \mathrm{Der}_{X/Y}(I)$.

2.3 Fundamental exact sequences

In most cases, standard arguments from classical algebraic geometry carry over to the log case to produce the familiar exact sequences showing the effect of closed immersions and compositions on differentials.

Proposition 2.3.1 *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of log schemes. The the functoriality maps fit into an exact sequence of sheaves of $\mathcal{O}_X$-modules:*

$$f^*\Omega^1_{Y/Z} \to \Omega^1_{X/Z} \to \Omega^1_{X/Y} \to 0$$

Proof: This is proved just as in the classical case: the morphisms in the sequence are induced by the commutative squares:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f \circ g"'] & Y \arrow[d, "g"] \\ Z \arrow[r] & Z \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

and once checks from the definitions that for any $\mathcal{O}_X$-module E, the sequence

$$0 \to \mathrm{Der}_{X/Y}(E) \to \mathrm{Der}_{X/Z}(E) \to \mathrm{Der}_{Y/Z}(f_*E)$$

is exact. The exactness of the sequence of differentials then follows from the universal properties. □

Proposition 2.3.2 *Let $f\colon X \to Y$ be a morphism of log schemes, let $i\colon Z \to X$ be a strict closed immersion of quasi-integral log schemes, with ideal sheaf I. Then there is an exact sequence of sheaves of $\mathcal{O}_Z$-modules*

$$I/I^2 \xrightarrow{\overline{d}} i^*(\Omega^1_{X/Y}) \to \Omega^1_{Z/Y} \to 0,$$

where the map $\overline{d}$ sends the class of an element a of I to the image of da in $i^*(\Omega^1_{X/Y})$. If the first infinitesimal neighborhood T of Z in X admits a Y-retraction, then d is injective and split.

Proof: Although $d\colon I \to \Omega^1_{X/Y}$ is not $\mathcal{O}_X$-linear, one verifies immediately that its compostion with the map $\Omega^1_{X/Y} \to i^*(\Omega^1_{X/Y})$ is, and hence that this composition factors through the map $\overline{d}$ as claimed. To prove the exactness of the sequence, it suffices to prove that for every sheaf E of $\mathcal{O}_Z$-modules, the sequence obtained by applying $\mathrm{Hom}(\ ,E)$ is exact. By by the universal mapping property of the sheaf of differentials, this amounts to verifying that the sequence

$$0 \to \mathrm{Der}_{Z/Y}(E) \to \mathrm{Der}_{X/Y}(i_*E) \to \mathrm{Hom}(I, i_*E)$$

is exact. The injectivity of the map $\mathrm{Der}_{Z/Y}(E) \to \mathrm{Der}_{X/Y}(E)$ follows from the fact that $i^\flat\colon M_X \to M_Y$ is surjective. Let (D, δ) be a derivation of X/Y

with values in i_*E such that $Da = 0$ for every section a of I. Then D factors through $i_*\mathcal{O}_Z$; we must also check that δ factors through $i_*(M_Z)$. Since i is strict, if m_1 and m_2 are two sections of $i^{-1}M_X$ with the same image in M_Z, then $m_2 = um_1$ for some $u \in 1 + I$. Hence $\delta(m_2) = \delta(u) + \delta(m_1) = u^{-1}Du + \delta(m_1)$, and $Du = 0$ since D annihilates I. Hence $\delta(m_2) = \delta(m_1)$, as required. Let $j\colon T \to X$ be the first infinitesimal neighborhood of Z in X, *i.e.*, the strict closed subscheme defined by I^2. Since M_X is quasi-integral, $i^{-1}(1+I)$ acts freely on $i^{-1}(M_T)$, and $i'\colon Z \to T$ is a first-order log thickening over X/Y. Suppose that $r\colon T \to Z$ is a map such that $r \circ i' = \mathrm{id}_Z$. Then j and ir are two deformations of i to T, and by (2.2.2) there is a unique $h\colon \Omega^1_{X/Y} \to I/I^2$ such that $h(dm) = j^\flat(m) - (ir)^\flat(m)$ for every local section m of M_X. Taking $m = 1 + a$ with $a \in I$, we see that $h(da) = j^\sharp(a)$, *i.e.*, the image of a in I/I^2. □

Corollary 2.3.3 *Let $f\colon X \to Y$ be a morphism of coherent log schemes, $K \subseteq M_X$ be a coherent sheaf of ideals, and let $i\colon Z \to X$ be the strict closed immersion of log schemes defined by K. Then there is a natural isomorphism $i^*(\Omega^1_{X/Y}) \cong \Omega^1_{Z/Y}$.*

Proof: The ideal I of Z in X is generated by $\alpha_X(K)$ as an abelian subsheaf of $\mathcal{O}_X$. If k is a local section of K, $d\alpha_X(k) = \alpha_X(k)dk$ which maps to zero in $i^*(\Omega^1_{X/Y})$. Thus the corollary follows from (2.3.2). □

Note that by (1.1.15), the same result holds if Z is regarded as an idealized log scheme.

Proposition 2.3.4 *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of log schemes and let I be an quasi-coherent $\mathcal{O}_X$-module. If $\partial \in \mathrm{Der}_{Y/Z}(f_*I)$, let $X \oplus_\partial I$ denote the Y-extension of X by I obtained by applying ∂ to $f \circ r$, where $r\colon X \oplus I \to X$ is the canonical retraction of the trivial extension (2.1.6) of X by E, using the action (2.2.2) of $\mathrm{Der}_{Y/Z}(f_*I)$ on $Y/Z(X \oplus I)$.*

1. *There is an exact sequence*

$$0 \to \mathrm{Der}_{X/Y}(I) \to \mathrm{Der}_{X/Z}(I) \to \mathrm{Der}_{Y/Z}(f_*I) \to \mathit{Ext}_Y(X, I) \to \mathit{Ext}_Z(X, I),$$

where $\mathrm{Der}_{Y/Z}(f_*I) \to \mathit{Ext}_Y(X, I)$ is the map sending ∂ to the isomorphism class of $X \oplus_\partial I$.

2. *If f is affine, the sequence prolongs to an exact sequence including the sequence:*

$$\mathrm{Der}_{Y/Z}(f_*I) \xrightarrow{\partial} \mathrm{Ext}_Y(X, I) \to \mathrm{Ext}_Z(X, I) \xrightarrow{f_*} \mathrm{Ext}_Z(Y, f_*I),$$

where f_* is the map of extension classes induced by the construction (2.1.7). I should write the proof.

Corollary 2.3.5 *If $f\colon X \to Y$ and $g\colon Y \to Z$ a morphisms of log schemes, the natural maps fit into an exact sequence:*

$$f^*(\Omega^1_{Y/Z}) \to \Omega^1_{X/Z} \to \Omega^1_{X/Y} \to 0$$

Proposition 2.3.6 *Suppose that $f\colon X \to Y$ is a morphism of log schemes and $x \in X$. Then there is a commutative diagram* simplify this statement; ρ is an isomorphism if $\underline{f}$ is an isomorphism.

$$\begin{tikzcd}
 & M^{gp}_{X,x} \arrow[r] \arrow[d, "dlog"'] \arrow[dr, "\pi"] & M^{gp}_{X/Y,x} \arrow[d, "\overline{\pi}"] \\
\Omega^1_{\underline{X}/\underline{Y}}(x) \arrow[r] & \Omega^1_{X/Y}(x) \arrow[r, "\rho_{X/Y,x}"] & k(x) \otimes M^{gp}_{X/Y,x},
\end{tikzcd}$$

*where $\overline{\pi}(m)$ sends a section m of $M^{gp}_{X/Y,x}$ to $1 \otimes m$ and the bottom row is exact. (*The map $\rho_{X/Y,x}$ is sometimes called the Poincaré residue mapping at *x.)*

Proof: Consider the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[dr] & X_Y \arrow[r] \arrow[d] & \underline{X} \arrow[d] \\
 & Y \arrow[r] & \underline{Y}
\end{tikzcd}$$

in which the square is Cartesian. In fact, X_Y is just $\underline{X}$ with the log structure induced from Y. Since the map $X_Y \to \underline{X}$ is an isomorphism on underlying schemes, the base change formula for differentials induces an isomorphism $\Omega^1_{\underline{X}/\underline{Y}} \to \Omega^1_{X_Y/Y}$, and we get from (3.2.3.1) an exact sequence:

$$\Omega^1_{\underline{X}/\underline{Y}} \to \Omega^1_{X/Y} \to \Omega^1_{X/X_Y} \to 0.$$

We shall prove that the composite map

$$\theta \colon M_X^{gp} \to \Omega^1_{X/Y} \to \Omega^1_{X/X_Y}$$

induces an isomorphism $k(x) \otimes M^{gp}_{X/Y} \to \Omega^1_{X/X_Y}(x)$. The image of $M_{Y,y}$ in $\Omega^1_{X/Y}$ is zero by definition, and the image of $\mathcal{O}^*_{X,x}$ is zero in Ω^1_{X/X_Y}. Thus θ kills $f^*M_Y^{gp}$ and hence induces maps

$$M^{gp}_{X/Y} =: M_X^{gp}/f^*M_Y^{gp} \to \Omega^1_{X/X_Y} \qquad \text{and} \qquad \overline{\theta} \colon k(x) \otimes M^{gp}_{X/Y} \to \Omega^1_{X/X_Y}(x).$$

We know that Ω^1_{X/X_Y} is generated by the image of $M^{gp}_{X/Y}$, so $\overline{\theta}$ is clearly surjective. If $m \in M_X^*$, then the image of m in $M_{X/Y}$ is zero, and hence $\pi(m)$ is zero, and if $m \in M_X^+$, $\alpha_X(m)$ maps to zero in $k(x)$. Thus in any case $\alpha_X(m)\pi(m) = 0$, and the pair

$$(0, \pi) \colon \mathcal{O}_X \oplus M_X^{gp} \to k(x) \otimes M^{gp}_{X/Y,x}$$

is a logarithmic derivation. Thus there is a unique map $r \colon \Omega^1_X \to k(x) \otimes M^g_{X/Y,x}$ such that $r(dm) = \pi(m)$ for all $m \in M_X$. Evidently r kills $d\mathcal{O}^*_{X,x}$, hence also the image of $\Omega^1_{\underline{X}}$, as well as the image of Ω^1_Y. Consequently r factors through a map $\overline{r} \colon \Omega^1_{X/X_Y} \to k(x) \otimes M^{gp}_{X/Y,x}$. Then $\overline{r}$ is inverse to $\overline{\theta}$. $\square$

3 Logarithmic Smoothness

3.1 Definition and examples

The basic definitions are copied from Grothendieck's geometric functorial characterization.

Definition 3.1.1 *A morphism of log schemes $f: X \to Y$ is formally smooth (resp. unramified, resp. étale) if for every n and every nth order log thickening (2.1.1) of X/Y:*

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[d, "g"'] & T \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

locally on T there exists at least one (resp. at most one, resp. exactly one) deformation $\tilde{g}$ of g to T (2.2.1). We say that f is smooth (resp. étale) if it is formally smooth (resp. étale) and in addition M_X and M_Y are coherent and $\underline{f}$ is locally of finite presentation.

Since an nth order log thickening can be written as a succession of first order thickenings, it is enough to check the condition when $n = 1$. In this case, the sheaf $g_*\mathrm{Def}_{X/Y}(g, T)$ of deformations of g is a pseudo-torsor under $\mathrm{Der}_{X/Y}(g_* I_T) \cong \mathrm{Hom}(\Omega^1_{X/Y}, g_* I_T)$ by Theorem 2.2.2. Thus the formal smoothness condition says that this pseudo-torsor is locally nonempty, *i.e.*, is in fact a torsor.

Remark 3.1.2 The family of formally smooth (resp. étale) morphisms is stable under composition and base change in the category of log schemes. If $g: Y \to Z$ is étale, then a morphism $f: X \to Y$ is smooth if and only if $g \circ f$ is smooth. If $X \to Z$ and $Y \to Z$ are formally étale, then any Z-morphism $X \to Y$ is formally étale. These properties follow immediately from the definitions.

Proposition 3.1.3 *A morphism $f: X \to Y$ of coherent log schemes is formally unramified if and only if $\Omega^1_{X/Y} = 0$.*

Proof: If $i: S \to T$ is a log thickening over X/Y, the sheaf of deformation of $g: S \to X$ to T is a torsor under $\mathrm{Der}_{X/Y}(g_* I) \cong \mathrm{Hom}(\Omega^1_{X/Y}, g_* I)$. This vanishes if $\Omega^1_{X/Y}$ vanishes, and so deformations are unique when they exist. Thus X/Y is formally unramified. If X and Y are coherent, the sheaf $\Omega^1_{X/Y}$ is quasi-coherent (1.1.12), and we can form the trivial extension T of X/Y

by $\Omega^1_{X/Y}$ (2.1.6). Then the set of deformations of id_X is a torsor under $\mathrm{End}(\Omega^1_{X/Y})$. If X/Y is unramified, the retraction $T \to X$ is the unique such deformation, so $\Omega^1_{X/Y} = 0$. $\square$

Proposition 3.1.4 *A morphism $f\colon X \to Y$ of log schemes is formally smooth if and only if for every affine open subset U of X and every quasi-coherent $\mathcal{O}_U$-module I, every Y-extension of U by I is trivial (or, equivalently, locally trivial).*

Proof: It follows immediately from the definition that if f is formally smooth, any Y-extension $U \to T$ of an affine open subset U of X locally admits a section $U \to X$ and hence is locally trivial. Conversely, suppose that any such extension is locally trivial and that $i\colon S \to T$ is an X/Y-thickening of order one with ideal I. The thickening i defines an element of ξ of $\mathrm{Ext}_Y(S, I)$. Assuming without loss of generality that X and S are affine, we may form the direct image extension (2.1.7) $g_*(T)$ of X/Y by g_*I. By assumption, this extension is trivial, and hence by the exact sequence

$$\mathrm{Ext}_X(S, I) \to \mathrm{Ext}_Y(S, I) \to \mathrm{Ext}_Y(X, g_*I)$$

of *op. cit.*, ξ comes from an element of $\mathrm{Ext}_X(S, I)$. The means that there is a map $g\colon T \to X$ such that $g \circ i' = g'$ and $f \circ g = h$, as desired. $\square$

Corollary 3.1.5 *In the definition of smooth, (resp., unramifed, étale), it is sufficient to consider thickenings such that $g'\colon T' \to X$ is an open immersion.*

If $f\colon X \to Y$ is a morphism of schemes, and if X and Y are endowed with the trivial log structure, then f is formally (log) smooth (resp.…) if and only if $\underline{f}$ is. More generally:

Proposition 3.1.6 *Let $f\colon X \to Y$ be a strict morphism of coherent log schemes. If the underlying morphism of schemes $\underline{f}\colon \underline{X} \to \underline{Y}$ is formally log smooth (resp. étale, unramified), then the same is true of f. The converse holds if the log structure on Y is quasi-integral.*

Proof: If f is strict, the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & \underline{X} \arrow[d, "\underline{f}"] \\
Y \arrow[r] & \underline{Y}
\end{tikzcd}$$

is Cartesian. Thus if $\underline{f}$ is smooth, the same is true of f. To prove the converse, suppose that $\underline{S} \to \underline{T}$ is a log thickening over $\underline{X}/\underline{Y}$. Endow $\underline{T}$ with the inverse image of the log structure on Y. Then $S \to T$ is a log thickening over X/Y. Any deformation of S/X to T gives a deformation of $\underline{S}/\underline{X}$ to $\underline{T}$. Thus if f is smooth, so is $\underline{f}$. Furthermore, $\Omega^1_{X/Y} \cong \Omega^1_{\underline{X}/\underline{Y}}$, so if f is unramified, so is $\underline{f}$.

$\square$

The next results explain when the morphisms of log schemes modeled on morphisms of monoids are, unramified, smooth, or étale

Theorem 3.1.7 *Let $\theta: Q \to P$ be a morphism of finitely generated monoids and let $f: Q \to P$ be the corresponding morphism of log schemes over a base ring R. Then the following conditions are equivalent:*

1. *The order of the torsion part of the cokernel of θ^{gp} is invertible in R.*
2. *The morphism of log schemes $f: \mathsf{A}_\mathsf{P} \to \mathsf{A}_\mathsf{Q}$ is unramified.*
3. *The morphism of group schemes $f^*: \underline{\mathsf{A}}^*_\mathsf{P} \to \underline{\mathsf{A}}^*_\mathsf{Q}$ is unramified.*

Proof: If (1) holds, then $R \otimes \mathrm{Cok}(\theta^{gp}) = 0$. By (1.2.1), $\Omega^1_{\mathsf{A}_\mathsf{P}/\mathsf{A}_\mathsf{Q}}$ is the quasi-coherent sheaf associated to $R[P] \otimes \mathrm{Cok}(\theta^{gp})$, and hence $\Omega^1_{X/Y} = 0$ and f is formally unramified, hence unramified. The implication of (3) by (2) is immediate. Finally, if f^* is unramified, $\Omega^1_{\underline{\mathsf{A}}^*_\mathsf{P}/\underline{\mathsf{A}}^*_\mathsf{Q}} = 0$, hence $R[P^{gp}] \otimes \mathrm{Cok}(\theta^{gp}) = 0$, hence $R \otimes \mathrm{Cok}(\theta^{gp}) = 0$. $\square$

Theorem 3.1.8 *Let $\theta: Q \to P$ be a morphism of finitely generated monoids. and let $f: \mathsf{A}_\mathsf{P} \to \mathsf{A}_\mathsf{Q}$ be the corresponding morphism of log schemes over a base ring R. Then the following conditions are equivalent:*

1. *The kernel and the torsion part of the cokernel of θ^{gp} are finite groups whose order is invertible in R.*

2. *The morphism of log schemes $f\colon \mathsf{A_P} \to \mathsf{A_Q}$ is smooth.*

3. *The morphism of group schemes $f^*\colon \underline{\mathsf{A}}^*_{\mathsf{P}} \to \underline{\mathsf{A}}^*_{\mathsf{Q}}$ is smooth.*

Corollary 3.1.9 *Let $\theta\colon Q \to P$ be a morphism of finitely generated monoids. and let $f\colon \mathsf{A_P} \to \mathsf{A_Q}$ be the corresponding morphism of log schemes over a base ring R. Then the following conditions are equivalent:*

1. *The kernel and cokernel of θ^{gp} are finite groups whose order is invertible in R.*

2. *The morphism of log schemes $f\colon \mathsf{A_P} \to \mathsf{A_Q}$ is étale.*

3. *The morphism of group schemes $f^*\colon \underline{\mathsf{A}}^*_{\mathsf{P}} \to \underline{\mathsf{A}}^*_{\mathsf{Q}}$ is étale.*

Proof of Theorem 3.1.8 Suppose that (1) holds. Recall from (1.1.9) that for any log scheme T, the set of morphisms $T \to \mathsf{A_P}$ is identified with the set of morphisms of monoids $P \to \Gamma(T, M_T)$. Thus a log thickening $i\colon S \to T$ over f can be thought of as commutative diagram

$$\begin{tikzcd}
Q \arrow[r, "\theta"] \arrow[d, "h"'] & P \arrow[d, "g"] \arrow[dl, dotted, "\tilde{g}"'] \\
\Gamma(T, M_T) \arrow[r, "i"] & \Gamma(S, M_S)
\end{tikzcd}$$

We must show that, locally on T, there is a map $\tilde{g}\colon P \to \Gamma(T, M_T)$ such that $i \circ \tilde{g} = g$ and $\tilde{h} \circ \theta = h$. Recall from (2.1.2.4) that the natural map

$$M_T \to M_T^{gp} \times_{M_S} M_S^{gp}$$

is an isomorphism. Thus it will suffice to find a corresponding map in the diagram:

$$\begin{tikzcd}
Q^{gp} \arrow[r, "\theta"] \arrow[d, "h"'] & P^{gp} \arrow[d, "g"] \arrow[dl, dotted, "\tilde{g}"'] \\
\Gamma(T, M_T^{gp}) \arrow[r, "i"] & \Gamma(S, M_S^{gp})
\end{tikzcd}$$

Since the question is local on T, we may assume without loss of generality that T is affine. By (2.1.2.2), the kernel of the surjection $M_T^{gp} \to M_S^{gp}$ is $1+I$, and since $I^2 = 0$, the sheaf of groups is isomorphic to I. Since I is quasi-coherent, $H^1(T, I) = 0$, and the sequence

$$0 \to \Gamma(I) \to \Gamma(M_T^{gp}) \to \Gamma(S, M_S^{gp}) \to 0$$

is exact. The pullback of the sequence along the map g fits into the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(\theta^{gp}) \arrow[r] \arrow[d] & Q^{gp} \arrow[d, "\phi"] \arrow[dr, "\theta^{gp}"] & & \\
0 \arrow[r] & \Gamma(I) \arrow[r] \arrow[d] & E \arrow[r, "\pi"] \arrow[d] & P^{gp} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(I) \arrow[r] & \mathrm{Cok}(\phi) \arrow[r] & \mathrm{Cok}(\theta^{gp}) \arrow[r] & 0
\end{tikzcd}$$

By construction $E = \Gamma(M_T^{gp}) \times_{\Gamma(M_S^{gp})} P^{gp}$, and the middle row is exact. The bottom row is also exact, except possibly at $\Gamma(I)$. We must find a section $\sigma\colon P^{gp} \to E$ of the map π such that $\sigma \circ \theta^{gp} = \phi$. Now $\Gamma(I)$ is an R-module and $\mathrm{Ker}(\theta^{gp})$ is a finite group whose order is invertible in R. It follows that the map $\mathrm{Ker}(\theta^{gp}) \to \Gamma(I)$ vanishes. This implies that the bottom row is also exact, which in turn implies that the middle row is the pullback of the bottom row, *i.e.*, that the square on the bottom right is Cartesian. Now $\Gamma(I)$ is an R-module and since the order of the torsion part of $\mathrm{Cok}(\theta^{gp})$ is invertible in R, the sequence on the bottom splits. Since the square on the lower right is Cartesian, such a splitting also defines a map $P^{gp} \to E$, which necessarily agrees with the given map $Q^{gp} \to E$. This map gives the desired deformation of g and completes the proof that (1) implies (2).

It is apparent from the definitions that the restriction of a smooth map to any open subset is smooth, and it follows that (2) implies (3). Thus it remains only to prove that (3) implies (1). For this implication we may as well replace Q by Q^{gp} and P by P^g. Thus we may and shall assume that Q and P are finitely generated abelian groups. Let Q' be the image of Q in P, so that the map θ factors

$$\theta = Q \xrightarrow{\phi} Q' \xrightarrow{\theta'} P,$$

where ϕ is surjective and θ' is injective. The corresponding maps of group schemes are

$$f = \underline{\mathsf{A}}_{\mathsf{P}} \xrightarrow{f'} \underline{\mathsf{A}}_{\mathsf{Q'}} \xrightarrow{g} \underline{\mathsf{A}}_{\mathsf{Q}},$$

where g is a closed immersion and f' is dominant. In fact more is true. Observe that the group homomorphism θ' makes P into a Q'-set, and since θ' is injective, each Q' orbit is isomorphic to Q'. Thus P is a free Q'-set, $R[P]$ is a free $R[Q']$-module, and f' is faithfully flat. Let x be a point of $\underline{\mathsf{A}}_{\mathsf{P}}$, let $y := f'(x) \in \underline{\mathsf{A}}_{\mathsf{Q'}} \subseteq \underline{\mathsf{A}}_{\mathsf{Q}}$, and let x be the image of x in $\operatorname{Spec} R$. Then y lies in the inverse image of s in

$$Y'_s := \operatorname{Spec} k(s) \times_{\operatorname{Spec} R} \underline{\mathsf{A}}_{\mathsf{Q'}} = \operatorname{Spec} k(s)[Q'],$$

and the fiber of y in $\underline{\mathsf{A}}_{\mathsf{P}}$ can be identifed with its fiber in

$$X_s := \operatorname{Spec} k(s) \times_{\operatorname{Spec} R} \underline{\mathsf{A}}_{\mathsf{P}} = \operatorname{Spec} k(s)[P],$$

Now the dimension of X_s is the rank of the abelian group P, the dimension of Y_s is the rank of Q', and the morphism $f'_s \colon X_s \to Y'_s$ is faithfully flat. It follows that all the fibers of f'_s have dimension equal to the rank of P minus the rank of Q', *i.e.*, the rank of P/Q'. Since f is smooth, its sheaf of relative differentials is locally free, and its rank at any point x is the dimension of the fiber containing it [,], *i.e.*the rank of P/Q'. By (1.2.1,), $\Omega^1_{\underline{\mathsf{A}}_{\mathsf{P}}/\underline{\mathsf{A}}_{\mathsf{Q}}}$ is the sheaf associated to the module $R[P] \otimes_{\mathbf{Z}} \operatorname{Cok}(\theta)$. Write $\operatorname{Cok}(\theta)$ as a direct sum of a free group F and a finite group T. Then $R \otimes F \oplus R \otimes T$ is a free R-module of rank equal to the rank of F, and hence that $R \otimes_{\mathbf{Z}} T = 0$. This implies that the order of T is invertible in R.

It remains to prove that $\operatorname{Ker}(\theta)$ is a finite group whose order is invertible in R. Since f is smooth, it is also flat, and since f' is faithfully flat, it follows that the closed immersion g is also flat. Then the result follows from the follow lemma. $\square$

Lemma 3.1.10 *Let $\phi \colon Q \to Q'$ be a surjective homomorphism of finitely generated abelian groups with kernel K. Then the corresponding homomorphism $R[Q] \to R[Q']$ is flat if and only if $R \otimes K = 0$.*

Proof: Let $I \subseteq R[Q]$ be the ideal of $R[\phi]$. If $R[Q] \to R[Q]/I$ is flat, $I^2 = I$. But (1.2.3) gives an isomorphism of $R[Q']$-modules

$$I/I^2 \cong R[Q'] \otimes \operatorname{Ker}(\theta).$$

Since $I/I^2 = 0$, it follows that $R \otimes \operatorname{Ker}(\theta) = 0$, and this implies that $\operatorname{Ker}(\theta)$ is a finite group whose order is invertible in R. Conversely, if $R \otimes \operatorname{Ker}(\theta) = 0$, then $I = I^2$. Since I is finitely generated, Nakayama's lemma implies that, at each point x of $\underline{\mathsf{A}}_{\mathsf{Q}}$, either $I_x = \mathcal{O}_{X,x}$ or $I_x = 0$. Thus the map $\underline{\mathsf{A}}_{\mathsf{Q'}} \to \underline{\mathsf{A}}_{\mathsf{Q}}$ is an open immersion, hence flat. $\square$

Corollary 3.1.11 *Let P be a finitely generated monoid, let $\mathsf{A_P} := \operatorname{Spec} P \to R[P]$, and let $S := \operatorname{Spec} R$ (with trivial log structure). Then the following conditions are equivalent:*

1. *The order of the torsion subgroup of P^{gp} is invertible in R.*
2. *The morphism of log schemes $\mathsf{A_p} \to S$ is smooth.*
3. *The group scheme $\underline{\mathsf{A}}^*_{\mathsf{P}} := \operatorname{Spec} R[P^{gp}]$ is smooth over S.*

Corollary 3.1.12 *If X is a coherent log scheme, the canonical maps $X^{\rm int} \to X$ and $X^{\rm sat} \to X^{\rm int}$ are log étale.*

Corollary 3.1.13 *Let $f\colon X \to Y$ be a morphism of coherent log schemes with X fine. Then f is smooth if and only if the canonical factorization $\tilde{f}\colon X \to Y^{\rm int}$ is smooth, and the same holds with $f^{\rm sat}$ in place of $f^{\rm int}$.*

Proof: Let $\zeta\colon Y^{\rm int} \to Y$ be the canonical map, and consider the following diagram, in which the square is Cartesian:

$$\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[dr, "\tilde{f}"'] & X \times_Y Y^{\rm int} \arrow[r, "\zeta'"] \arrow[d, "pr"] & X \arrow[d, "f"] \\
& Y^{\rm int} \arrow[r, "\zeta"] & Y
\end{tikzcd}$$

Since ζ is smooth, so is ζ', and since $\zeta' \circ \eta = \mathrm{id}$ is étale, η is also étale. If f is smooth, pr is smooth, and hence $\tilde{f} = pr \circ \eta$ is also smooth. If $\tilde{f}$ is smooth, then $f = \zeta \circ \tilde{f}$ is also smooth. $\square$

Proposition 3.1.14 *Let $f\colon X \to Y$ be the morphism of log schemes admitting a coherent chart $\theta\colon Q \to P$ and let x be a point of X. Assume that Q and P are finitely generated and that*

1. *$k(x) \otimes \operatorname{Ker}(\theta^{gp}) = 0$ and $k(x) \otimes \operatorname{Cok}(\theta^{gp})_t = 0$ (resp. $k(x) \otimes \operatorname{Cok}(\theta^{gp}) = 0$.)*
2. *The map $\underline{X} \to \underline{X}' := \underline{Y} \times_{\underline{\mathsf{A}}_{\mathsf{Q}}} \underline{\mathsf{A}}_{\mathsf{P}}$ is smooth (resp. étale) in some neighborhood of x.*

Then $f\colon X \to Y$ is smooth (resp. étale) in some neighborhood of x.

Proof: Consider the commutative diagram of log schemes: Let n be the order of $\operatorname{Ker}(\theta^{gp})$. Condition (1) implies that n is a unit in $k(x)$ and hence also in $k(y)$, where $y = f(x)$. It follows that n is a unit in the local ring of y in Y, and so, after replacing Y by an open neighborhood of y, we may assume that Y is a scheme over $\mathbf{Z}[1/n]$. The same argument with $\operatorname{Cok}(\theta^{gp})$ shows that there is localization R of $\mathbf{Z}$ such that the orders of $\operatorname{Cok}(\theta)^{gp})_t$ and $\ker(\theta^{gp})$ are invertible in R, and (perhaps after a further localization) that Y is an R-scheme. We work over R from now on.

$$\begin{tikzcd}
X \arrow[r, "f'"] \arrow[dr, "f"'] & Y \times_{\mathsf{A}_{\mathsf{Q}}} \mathsf{A}_{\mathsf{P}} \arrow[r] \arrow[d, "g'"'] & \mathsf{A}_{\mathsf{P}} \arrow[d, "g"] \\
 & Y \arrow[r] & \mathsf{A}_{\mathsf{Q}}
\end{tikzcd}$$

Then by Theorem 3.1.8 (resp. Corollary 3.1.9) the map g is smooth (resp. étale), and the same holds for g' by base change. Since $X \to \mathsf{A}_{\mathsf{P}}$ is a chart for X, the map f' is strict. Since $\underline{X} \to \underline{X}'$ is smooth (resp. étale), it follows from (3.1.6) that the same is true for $X \to X'$. Since the family of smooth (resp. étale) maps is closed under composition, this completes the proof. $\square$

consolidate idealized stuff somewher

Example 3.1.15 If X is a fine log scheme and K is a coherent sheaf of ideals in M_X, let X_K be the closed subscheme defined by $\alpha_X(K)\mathcal{O}_X$ with the induced log structure. Then $j\colon (X_K, j^*K) \to (X, \emptyset_X)$ is ideally étale.

Proof: Suppose (g', i) is an idealized log thickening over X_K/X as in (**??**). Then the map $g'^* M_X \to M_{T'}$ sends $g'^* j^* K$ to $K_{T'}$. Since i is a homeomorphism and i is ideally strict, it follows that $h^* K$ maps to K_T and also that $\alpha_X(K)$ maps to zero in $\mathcal{O}_T$. This implies that $\underline{h}$ factors through $\underline{X}_K$. Finally we have to check that the induced map on monoids $h^{-1} M_X \to M_T$ factors through $h^{-1} M_{X_K}$. But M_{X_K} is the quotient of $j^{-1} M_X$ by the action of $1 + j^{-1}\alpha_X(K)$, which acts trivially on M_T because $\alpha_X(K)$ maps to zero in $\mathcal{O}_T$. □

For example, if P is a fine monoid with $P^* = 0$, $X_P =: \mathrm{Spec}(P \to \mathbf{Z}[P])$ is smooth over $\mathbf{Z}$. The closed log subscheme ξ_P defined by P^+ is $\mathrm{Spec}(P \to \mathbf{Z})$ (where the map sends every element of P^+ to 0), and the map of idealized log schemes $(\xi_P, P^+) \to (X_P, \emptyset)$ is étale. It follows that (ξ_P, P^+) is smooth over $\mathbf{Z}$ in the category of idealized log schemes, although ξ_P is not smooth over $\mathbf{Z}$ in the category of log schemes.

Example 3.1.16 Let P be a fine monoid and let k be a field such that the order of the torsion of P^{gp} is invertible in k. Then $\mathsf{A}_{\mathsf{P}k} \to \mathrm{Spec}\, k$ is log smooth. If X is a log scheme and $X \to \mathsf{A}_{\mathsf{P}}$ is a chart such that $\underline{X} \to \underline{\mathsf{A}}_{\mathsf{P}k}$ is étale, then $X \to \mathrm{Spec}\, k$ is log smooth.

Example 3.1.17 Let n be an integer and let $\theta \colon \mathbf{N} \to \mathbf{N}$ be mulitplication by n. Then the corresponding morphism $f \colon \mathsf{A}_{\mathbf{N}} \to \mathsf{A}_{\mathbf{N}}$ is étale if and only if n is invertible in the base ring R. The map $\underline{f}$ on underlying schemes is a finite covering, tamely and totally ramified over the origin. This is a simple example of a Kummer covering. More generally....

Example 3.1.18 Let $\theta \colon Q \to P$ be a homomorphism of monoids such that θ^g is an isomorphism. Then $\mathsf{A}_\theta \colon \mathsf{A}_{\mathsf{P}} \to \mathsf{A}_{\mathsf{Q}}$ is étale. For example, let r be a positive integer and let

$$\theta \colon \mathbf{N}^r \to \mathbf{N}^r \quad \text{by} \quad (a_1, a_2, \ldots, a_n) \mapsto (a_1, a_2 + a_1, \ldots, a_n + a_1).$$

Then the corresponding map θ^{gp} is an isomorphism and A_θ is étale. However the underlying map on schemes $\underline{\mathsf{A}}_\theta \colon \mathbf{A}^n \to \mathbf{A}^n$ is an affine piece of a blowing up, and is not even flat.

Example 3.1.19 Let r be a positive integer and let $\phi\colon \mathbf{N} \to \mathbf{N}^r$ be the map sending a to $(a, a, \ldots, a)$. Then the correspoding morphism of log schemes $\mathsf{A}_{\mathbf{N}^r} \to \mathsf{A}_{\mathbf{N}}$ is smooth. The map of underlying schemes sends a point $(x_1, \ldots x_r)$ to the point $x_1 x_2 \cdots x_r$, and is the standard model of stable reduction. Notice that there are commutative diagrams

$$\begin{tikzcd} \mathbf{N}^r & \mathbf{N}^r \arrow[l, "\theta"'] \\ & \mathbf{N} \arrow[ul, "\phi"] \arrow[u, "\pi"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{A}_{\mathbf{N}^r} \arrow[r] \arrow[dr] & \mathsf{A}_{\mathbf{N}^r} \arrow[d] \\ & \mathsf{A}_{\mathbf{N}}, \end{tikzcd}$$

where θ is the map in Example 3.1.18 corresponding to a blowup and $\pi(a) := (a, 0, \cdots, 0)$ corresponds to a projection. Thus in the log world, a semitable map can be factored as an étale map followed by a standard projection.

More generally, if $(m_1, m_2, \ldots, m_r)$ is a sequence of positive integers, the on log schemes corresponding to the map

$$\mathbf{N} \to \mathbf{N}^r \quad \text{given by} \quad a \mapsto (m_1 a, m_2 a, \ldots m_r a)$$

is smooth if and only if the greatest common divisor of $(m_1, m_2, \ldots m_r)$ is invertible in the base ring R.

Examples 3.1.20 Let $\underline{X}/k$ be a smooth scheme of dimension n over a field k and let $D \subseteq \underline{X}$ be a divisor with normal crossings. By definition, this means that locally on $\underline{X}$ there exists a system of local coordinates for X adapted to D, *i.e.*, an étale map $g\colon \underline{X} \to \mathbf{A}^n/k := \operatorname{Spec} k[t_1, \ldots t_n]$ and an integer $r \leq n$ such that D is the divisor defined by $g^\sharp(t_1 \ldots t_r)$. Let $\alpha_X\colon M_X \to \mathcal{O}_X$ be the direct image (II,1.2) of the trivial log structure on $U := \underline{X} \setminus D$, and let X be the corresponding log scheme. For each $i \leq r$, $x_i := g^\sharp(t_i)$ is a unit on U, and hence there is a unique section m_i of M_X with $\alpha_X(m_i) = x_i$. Since $\underline{X}$ is smooth, it is locally factorial, and by (2.1.9) the map $\mathbf{N}^m \to M_X$ sending the i^{th} standard basis vector e_i to m_i is a chart for M_X. If $i \leq r$, $dx_i = d\alpha_X(m_i) = \alpha_X(m_i) dlog\, m_i = x_i dlog\, m_i$, *i.e.*, $\mathrm{dlog}\, m_i = x_i{-}1 dx_i$. As we shall see in (**??**), $\Omega^1_{X/k}$ is locally free of rank n, with a local basis $(dlog\, m_1, \ldots dlog\, m_r, dx_{r+1}, \ldots dx_n)$, Thus $\underline{\Omega}^1_{X/k}(M_X)$ can be identified with the classically considered set of differential one forms with log poles along D. Now suppose that S is a smooth scheme of dimension one over k, y is a point of S, and $f: X \to S$ is a morphism such that $f-1(y) = D$. (This is an example of a semistable reduction.) Endow S with the log strucure induced from the open embeding $S \setminus \{y\} \to S$, and let s be a local coordinate at s. Then (after a change of coordinates) $f^\sharp(s) = \prod_1^r x_i$, and $\Omega^1_{X/Y}$ is given by generators $dlog\, m_i$ for $i \leq r$ and dx_i for $i > r$, with $\sum dlog\, m_i = 0$. Write the proof

3.2 Differential criteria for smoothness

The next set of results follow the standard pattern from algebraic geometry.

Proposition 3.2.1 *If $f: X \to S$ is a smooth map of idealized log schemes, then $\Omega^1_{X/S}$ is locally free of finite type.*

Proof: For any quasi-coherent E, the set of retractions $X \oplus E \to X$ is bijective with $\mathrm{Hom}(\Omega^1_{X/Y}, E)$. Now if $E \to E''$ is a surjective map of quasi-coherent $\mathcal{O}_X$-modules, we get another first order thickening $X \oplus E'' \to X \oplus E$, and by the smoothness of X/S, every retraction $X \oplus E'' \to X$ lifts locally to $X \oplus E$. This says that the map

$$\mathrm{Hom}(\Omega^1_{X/Y}, E) \to \mathrm{Hom}(\Omega^1_{X/Y}, E'')$$

is locally surjective. Since $\Omega^1_{X/Y}$ is of finite presentation, it follows that it is locally free. □

Theorem 3.2.2 *Let $f: X \to Y$ be a smooth morphism of coherent and quasi-integral log schemes and let $i: Z \to X$ be a strict closed immersion defined by an ideal I of $\mathcal{O}_X$. Then $Z \to Y$ is smooth if and only if the map $\overline{d}$ in the sequence (2.3.2)*

$$0 \to I/I^2 \to i^*(\Omega^1_{X/Y}) \to \Omega^1_{Z/Y} \to 0$$

is injective and locally split.

Proof: The proof is standard; we recall the main outline for the convenience of the reader. Let $j: Z \to T$ be the first infinitesimal neighborhood of Z in X. If Y/Z is smooth, then locally on Z there exists a retraction $T \to Z$, and hence by (2.3.2), the sequence is locally split. Suppose that the sequence is locally split, let $S \to T$ be a first order log thickening over Y, and let $g: S \to Z$ be a Y-morphism. Since X/Y is smooth, locally on X there exists a deformation $\tilde{h}$ of ig to T. Then $\tilde{h}$ induces a map I_Z/I_Z^2 to $g_* I_T$. Since the map $\overline{d}$ is locally split, this map can locally be extended to a map $\Omega^1_{X/Y} \to g_* I_T$. Such a map corresponds to a section ξ of $\mathrm{Der}_{X/Y}(g_* I_T)$. Then the deformation $-\xi\tilde{h}$ of $\tilde{h}$ factors through $i: Z \to X$. This proves that Z/Y is smooth. □

Theorem 3.2.3 *Let $f: X \to Y$ and $g: Y \to Z$ be morphisms of fine idealized log schemes, and consider the resulting exact sequence (2.3.5).*

$$f^*\Omega^1_{Y/Z} \xrightarrow{s} \Omega^1_{X/Z} \xrightarrow{t} \Omega^1_{X/Y} \to 0.$$

1. *If f is log smooth, the map s above is injective and locally split.*
2. *If $g \circ f$ is log smooth and s is injective and locally split, then f is log smooth.*

I should write the proof.

Proof: This follows from 2.3.4. □

Theorem 3.2.4 *Let $g: X \to Z$ be a smooth morphism of coherent log schemes and $\overline{x}$ is a geometric point of X. Then in an étale neighborhood of x, there exists a diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & Z \times \mathsf{A}_{\mathbf{N}^r} \arrow[d, "pr_Z"] \\ & Z \end{tikzcd}$$

in which f is étale.

Proof: Recall that the map $\mathcal{O}_X \otimes M_X^{gp} \to \Omega^1_{X/Z}$ is surjective. It follows that the fiber $\Omega^1_{X/Z}(\overline{x})$ of $\Omega^1_{X/Z}$ at $\overline{x}$ is spanned as a $k(\overline{x})$-vector space by the image of the map $\mathrm{dlog}: M_{X,\overline{x}} \to \Omega^1_{X/Z}(\overline{x})$. Thus there exists a fine sequence

$(m_1, m_2, \dots m_r)$ of local sections of M_X whose images in the vector space $\Omega^1_{X/Z}(\overline{x})$ form a basis. Restricting to some étale neighborhood of $\overline{x}$, we may assume that the m_i are global sections and then define a map $\mathbf{m}$ of log schemes $X \to \mathsf{A}_{\mathbf{N}^r}$. Let $Y := Z \times \mathsf{A}_{\mathbf{N}^r}$, let $f\colon X \to Y$ be the map $(f, \mathbf{m})$, and let $g\colon Y \to Z$ be the projection. Consider the sequence

$$0 \to f^*\Omega^1_{Y/Z} \xrightarrow{s} \Omega^1_{X/Z} \to \Omega^1_{X/Y} \to 0.$$

The sequence $(d\log m_1, d\log m_2, \dots d\log m_r)$ forms a basis for $\Omega^1_{Y/Z,\overline{x}}$, and s takes this sequence to a basis for $\Omega^1_{X/Z,\overline{x}}$. It follows that s induces an isomorphism on the stalks at $\overline{x}$, hence in some neighborhood of $\overline{x}$. Replacing X by such a neighborhood, we find that $\Omega^1_{X/Y} = 0$ and s is an isomorphism. Then it follows from Theorem 3.2.3 that $X \to Y$ is smooth and unramified, hence étale. $\square$

3.3 Charts for smooth morphisms

The following theorem shows the local structure of a smooth morphism of idealized log schemes.

Theorem 3.3.1 *Let $f\colon X \to Y$ be a smooth (resp. étale) morphism of fine log schemes and let $\gamma\colon Y \to \mathsf{A}_Q$ be a chart for Y. Then étale locally on $\underline{X}$, γ fits in a chart for f*

$$\begin{tikzcd} X \arrow[r, "\beta"] \arrow[d, "f"'] & \mathsf{A}_P \arrow[d, "\mathsf{A}_\theta"] \\ Y \arrow[r, "\gamma"] & \mathsf{A}_Q \end{tikzcd}$$

with the following properties:

1. *θ^{gp} is injective, and the order of $(P^{gp}/Q^{gp})_t$ is invertible in $\mathcal{O}_X$ (resp. and P^{gp}/Q^{gp} is finite of order invertible in $\mathcal{O}_X$).*
2. *The map $h\colon X \to Y \times_{\mathsf{A}_Q} \mathsf{A}_P$ induced from the above diagram is étale and strict.*

Proof: First suppose that f is étale. Let $\overline{x}$ be a geometric point of X. Then $\Omega^1_{X/Y}(\overline{x}) = 0$, and it follows from (2.3.6) that $k(\overline{x})\otimes M^{gp}_{X/Y,\overline{x}} = 0$. Thus $M^{gp}_{X/Y,\overline{x}}$ is a finite abelian group whose order is invertible in $k(\overline{x})$. Localizing X, we may assume that this order is invertible in $\mathcal{O}_X$. Now Theorem 2.2.18 tells us that γ can be embedded in a chart for f which is neat at x, subordinate to a morphism $\theta\colon Q \to P$. In particular, θ^{gp} is injective, and the map $P^{gp}/Q^{gp} \to M_{X/Y,\overline{x}}$ induced by β is bijective. Thus property (1) is certainly satisfied, and it remains only to prove that the map

$$\underline{h}\colon \underline{X} \to \underline{X}' =: \underline{Y} \times_{\underline{\mathsf{A}}_\mathsf{Q}} \underline{\mathsf{A}}_\mathsf{P}$$

is étale. By (3.1.9), the map $\mathsf{A}_\mathsf{P} \to \mathsf{A}_\mathsf{Q}$ is étale, and hence by (**??**), the base changed map $g\colon X' \to Y$ is étale. Since $f = gh$ is étale, if follows from from (3.1.2) that h is also étale. Since h is strict, $\underline{h}$ is also étale, by (3.1.6)

Now suppose that f is only smooth. Let us apply Theorem 3.2.4 to find, after a localization, a diagram

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[dr, "f"'] & Y \times \mathsf{A}_{\mathbf{N}^r} \arrow[d, "p"] \\ & Y \end{tikzcd}$$

in which f' is étale. Since $\gamma\colon Y \to \mathsf{A}_\mathsf{Q}$ is a chart for Y,

$$\gamma' := \gamma \times \mathrm{id}\colon Y' := Y \times \mathsf{A}_{\mathbf{N}^r} \to \mathsf{A}_\mathsf{Q} \times \mathsf{A}_{\mathbf{N}^r} \cong \mathsf{A}_{\mathsf{Q}\oplus\mathbf{N}^r}$$

is a chart for Y'. Now let us apply the case we have already proved to find a chart for f' subordinate to a morphism $\theta'\colon Q \oplus \mathbf{N}^r \to P$ satisfying conditions (1) and (2). Let $\theta\colon Q \to P$ be the composite of θ' with the inclusion $Q \to Q \oplus \mathbf{N}^r$. Then θ^{gp} is injective, and there is an exact sequence:

$$0 \to \mathbf{Z}^r \to P^{gp}/Q^{gp} \to P^{gp}/(\mathbf{Z}^g \oplus Q^{gp}) \to 0.$$

Then the torsion subgroup of P^{gp}/Q^g injects in the torsion subgroup of $P^g/(\mathbf{Z}^g \oplus Q^{gp})$, and hence has order invertible in $\mathcal{O}_X$. Finally, observe that the two squares in the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, "f'"] \arrow[ddr, "f"'] & X' \arrow[r] \arrow[d] & \mathsf{A}_{\mathsf{P}} \arrow[d] \\
& Y' \arrow[r] \arrow[d] & \mathsf{A}_{\mathbf{Q}\oplus\mathbf{N}^r} \arrow[d] \\
& Y \arrow[r] & \mathsf{A}_{\mathsf{Q}}
\end{tikzcd}$$

are Cartesian, and hence so is the rectangle. Since $\underline{X} \to \underline{X}'$ is étale, (2) is also satsified, and the proof is complete. $\square$

Remark 3.3.2 The chart constructed in the smooth case of Theorem 3.3.1 may not be neat. Indeed, if f is smooth, it can happen that $M^{gp}_{X/Y}$ can have torsion which is not invertible in $\mathcal{O}_X$, and that a flat (not étale) localization can be required before a neat chart exists. Should I give Kato's example?

Corollary 3.3.3 *Suppose that $f\colon X \to Y$ is an ideally smooth morphism of idealized log schemes. Then étale locally on $\underline{X}$, f factors as a composite $X = \tilde{Y}_K \to \tilde{Y} \to Y$, where $\tilde{Y} \to Y$ is ideally strict and log smooth and $\tilde{Y}_K \to Y$ is a closed immersion defined by a coherent sheaf of ideals K in $\tilde{Y}$.*

Proof: We may suppose that there exists a chart for f as in (3.3.1), and we use the notation there. Let J' be the ideal of P generated by J. Then the map $X_{P,J'} \to X_{Q,J}$ is ideally strict and log smooth, and hence the same is true of the map $Y' \to Y$ obtained from $X_{P,J'} \to X_{Q,J}$ by base change with the map $Y \to X_{Q,J}$. Let I' be the ideal of $M_{Y'}$ generated by I via the map $P \to M_{Y'}$. Then the map $X \to Y'$ factors through a strict map $X \to Y'_{I'}$ which by (3.2.3) is étale. Hence this map is classically étale, and it is well-known that we can Zariski locally find a classically étale map $\tilde{Y} \to Y'$ whose restriction to $Y'_{I'}$ is $X \to Y'_{I'}$. If we endow $\tilde{Y}$ with the idealized log structure induced from Y', we see that $X \to \tilde{Y} \to Y$ is the desired factorization. $\square$

3.4 Unramified morphisms and the conormal sheaf

Log étale morphisms and log immersions (**??**) are log unramified. A strict morphism f is formally unramified if and only if $\underline{f}$ is, since $\Omega^1_{X/Y} \cong \Omega^1_{\underline{X}/\underline{Y}}$.

Lemma 3.4.1 *Let $f\colon X \to Y$ be an unramified morphism of log schemes. Then f is small, and $M^{gp}_{X/Y}$ is locally on X annihilated by an integer invertible in $\mathcal{O}_X$.*

Proof: If x is a point of X, (2.3.6) says that $k(x) \otimes M^{gp}_{X/Y,}$ is a quotient of $\Omega^1_{X/Y}(x)$, and hence vanishes. Since $M^{gp}_{X/Y,x}$ is a finitely generated abelian group, its free part must vanish and it is finite and of order prime to the characteristic of $k(x)$. Since this holds for every x and $M^{gp}_{X/Y}$ is quasi-constructible (**??**), the same holds in a neighborhood of x. □

Theorem 3.4.2 *Let $f\colon X \to Y$ be a log unramified morphism of fine log schemes. Then étale locally on X, there exists a factorization $f = g \circ i$ where g is log étale and i is an exact closed immersion.*

Proof: The proof is analogous to the proof (**??**) of the structure theorem for smooth morphisms. It follows from the previous lemma and (I,**??**) that, in an étale neighborhood of any point x of X, f admits a neat chart

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathsf{A}_{\mathsf{P}} \arrow[d] \\ Y \arrow[r] & \mathsf{A}_{\mathsf{Q}} \end{tikzcd}$$

Let $X' := Y \times_{\mathsf{A}_{\mathsf{Q}}} \mathsf{A}_{\mathsf{P}}$, let $g\colon X \to X'$ be the morphism induced by f and $X \to \mathsf{A}_{\mathsf{P}}$, and let $x' := g(x)$. Then

$$\Omega^1_{X'/Y,x'} \cong \mathcal{O}_{X',x'} \otimes P^{gp}/Q^{gp} \cong \mathcal{O}_{X',x'} \otimes M^{gp}_{X/Y,x} = 0.$$

Since $\Omega^1_{X'/Y}$ is of finite type, it vanishes in some neighborhood of x', in which $X' \to Y$ is étale. Since $X \to Y$ is unramified, the same it true of the map $g\colon X \to X'$, and since it is also strict, $\underline{g}$ is unramified. Then by the structure theorem for unramified morphism [], $\underline{g}$ can, étale locally on $\underline{X}$ be written as a composite of a closed immersion and an étale map. The conclusion follows. □

The previous result can be used to construct strict infinitesimal neighborhoods of a closed immersion, or, more generally, of an unramified morphism.

Theorem 3.4.3 *Let* Lognet *denote the category of log unramified morphisms* f *of fine log schemes, with morphisms* $f \to f'$ *given by commutative squares For* $n \in \mathbf{N}$, *let* Thick_n *be the full subcategory of* Lognet *whose objects are the log thickenings of order less than or equal to* n *(**??**). Then the inclusion functor* $\mathsf{Thick}_n \to \mathsf{Lognet}$ *admits a left adjoint* $(f\colon X \to Y) \mapsto f_n\colon X \to Y_n$ *(so that* f *and* f_n *have the same source).*

Proof: We will need to use the fact that the notion of log thickening is local for the étale topology, as we now explain. If $i\colon X \to Y$ is a log thickening of order n and $f\colon X' \to X$ is strict and étale, then by [], there is a Cartesian square

$$\begin{tikzcd} X' \arrow[r, "i'"] \arrow[d, "f"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

in which g is strict and étale. Then i' is a log thickening of order n and is unique up to unique isomorphism. The log thickenings of order n thus form a fibered category $\mathsf{Thick}_{n/X}$ on the étale site of $\underline{X}$, of which the fiber on an étale $X' \to X$ is the category of log thickenings $X' \to Y'$ of order n, with morphisms the morphisms of thickenings inducing the identity on X'.

Lemma 3.4.4 *Let* X *be a fine log scheme and let* n *be a natural number. Then the fibered category* Thick_n *on the étale site of* X *is a stack [].*

Proof: We have to prove that if $f\colon X' \to X$ is strict, étale, and surjective, then the inverse image functor

$$f^*\colon \mathsf{Thick}_n(X) \to \mathsf{Thick}_n(X'/X)$$

from the category of log thickenings of order n of X to the category of log thickenings of X' endowed with descent data relative to f is an equivalence of categories. The case of a Zariski affine open covering being immediate, one reduces to the case in which X and X' are affine, with rings A and A'.

□

□

Let $i: X \to Y$ be a log unramified morphism of fine log schemes, and let $i_1: X \to Y_1$ be its first strict infinitesimal neighborhood. The ideal of X in Y_1 is a square zero ideal, hence an $\mathcal{O}_X$-module, called the *conormal sheaf* of X in Y and denoted by $\mathcal{N}_{X/Y}$. It depends functorially on i: a morphism from $i': X' \to Y'$ to $i: X \to Y$ given by $f: X' \to X$ and $g: Y' \to Y$ induces a morphism of thickenings $i'_1 \to i_1$ and hence a morphism $f^*\mathcal{N}_{X/Y} \to \mathcal{N}_{X'/Y'}$.

If $i: X \to Y$ is a strict closed immersion with ideal I, then $\mathcal{N}_{X/Y}$ is the usual conomoral sheaf I/I^2. It is also possible to describe $\mathcal{N}_{X/Y}$ fairly explicitly if i is a closed immersion of fine log schemes, not necesarily strict.

Proposition 3.4.5 *Let $i: X \to Y$ be a closed immersion of fine log schemes, let*

$$K := \operatorname{Ker}\Big(i^{-1}(M_Y^{gp}) \to M_X^{gp}\Big)$$

and let

$$I := \operatorname{Ker}\Big(i^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X\Big).$$

Let $R \subseteq \mathcal{O}_X \otimes K$ be the abelian subsheaf generated by the set of all elements of the form $\alpha_X i^\flat(b)) \otimes (a/b)$ where (a,b) is a pair of sections of $i^{-1}M_Y$ with $i^\flat(a) = i^\flat(b) \in M_X$ and $\alpha_Y(b) - \alpha_Y(a) \in I^2$. Then R is in fact an $\mathcal{O}_X$-submodule of $\mathcal{O}_X \otimes K$, and there is an isomorphism

$$(\mathcal{O}_X \otimes K)/R \to \mathcal{N}_{X/Y}$$

sending $1 \otimes k$ to the class of $\alpha_{Y_1}(f_1^\flat(k))$ for any section k of K, where $f_1: Y_1 \to Y$ is the the canonical map.

To see that R is an $\mathcal{O}_X$-submodule of $\mathcal{O}_X \otimes K$, it suffices to check that the described set of generators is stable under multiplication by elements of $\mathcal{O}_X$, and since every element of $\mathcal{O}_X$ is locally the sum of units, it suffices to check stability by elements in the image of α_X. Since $i^\flat$ is surjective, we may locally write such an element as $\alpha_X i^\flat(c)$ for some $c \in i^{-1}(\mathcal{O}_Y)$. Then if (a,b) is a pair of sections of $i^{-1}(M_Y)$ satisfying the above conditions, (ac, bc) is another, and

$$\alpha_X i^\flat(c) \alpha_X(i^\flat(b) \otimes (a/b) = \alpha_X(i^\flat(bc) \otimes (ac/bc).$$

Since $\mathcal{N}_{X/Y}$ is a square-zero ideal, $1 + \mathcal{N}_{X/Y} \cong \mathcal{N}_{X/Y}$. Define $\delta(k)$ to be $\alpha_1 \pi^\flat(k) - 1 \in \mathcal{N}_{X/Y}$. If a and b are elements of $i^{-1}(M_Y)$ with the same image in M_X, then $\alpha_Y(a)$ and $\alpha_Y(b)$ have the same image in $\mathcal{O}_X$, and $a/b \in K$. Furthermore,

$$\alpha_X(b)(\delta(a/b) = \alpha_{Y_1}(\pi^\flat(b)(\alpha_{Y_1}\pi^\flat(a/b) - 1) = \alpha_X(a) - \alpha_Y(b).$$

In particular, if $\alpha_X(a) - \alpha_Y(b) \in I^2$, then $\alpha_X(b)\delta(a/b) = 0$ in $\mathcal{N}_{X/Y}$.

4 More on smooth maps

4.1 Kummer maps

4.2 Log blowups

Chapter V

De Rham and Betti cohomology

One of the most important historical inspirations for log geometry is the theory of differential forms with log poles. These have been used for a long time to study the de Rham cohomology of an open subset U whose complement in a smooth proper scheme is a divisor with normal crossings. This method was used, for example, by Grothendieck in his original proof [] of the comparison theorem between Betti cohomology and algebraic de Rham comohology, and also by Deligne in his treatment [] of differential equations with regular singularities. It is no surprise then that logarithmic de Rham cohomology is quite well developed, and that it gives a good idea of the geometric meaning of log geometry.

By way of motivation, let us explain here the main results for a saturated log scheme X which is smooth, separated, and of finite type over the complex numbers. Our first task is to show that the universal log derivation $d\colon \mathcal{O}_X \to \Omega^1_{X/\mathbf{C}}$ (1.1.6) fits into a complex $\Omega^{\cdot}_{X/\mathbf{C}}$ of coherent sheaves on X as well on its analytic realization X_{an}. When the log structure on X is trivial, the classical Poincaré lemma [] asserts that the corresponding complex $\Omega^{\cdot an}_X$ on the analytic space X_{an} associated to X is a resolution of the constant sheaf $\mathbf{C}$. This is no longer true if the log structure is not trivial. As a subsitute, one constructs a de Rham complex $\Omega^{\cdot log}_X$ on the Betti realization X_{log} (3.1.1) of X_{log}, where the Poincaré Lemma does hold. The following statement summarizes the main results.

Theorem 0.2.1 *Let $X/\mathbf{C}$ be a saturated log scheme, smooth and of finite* *type over the complex numbers, and et $X^* \subseteq X$ be the open set of X where the log structure is trivial. Then one has a commutative diagram of isomorphisms:*

$$\begin{tikzcd}
H^\cdot(X, \Omega^\cdot_{X/\mathbf{C}}) \arrow[r, "a"] \arrow[d, "e"'] & H^\cdot(X_{an}, \Omega^{\cdot an}_X) \arrow[r, "b"] \arrow[d, "f"'] & H^\cdot(X_{log}, \Omega^{\cdot log}_X) \arrow[d, "g"'] & H^\cdot(X_{log}, \mathbf{C}) \arrow[l, "c"'] \arrow[d, "h"'] \\
H^\cdot(X^*, \Omega^\cdot_{X^*/\mathbf{C}}) \arrow[r, "a^*"] & H^\cdot(X^*_{an}, \Omega^{\cdot an}_X) \arrow[r, "b^*"] & H^\cdot(X^*_{log}, \Omega^{\cdot log}_X) & H^\cdot(X^*_{log}, \mathbf{C}) \arrow[l, "c^*"']
\end{tikzcd}$$

Our strategy will be the following. We prove that e, b, c, and c^* are isomorphisms by local calculations. We deduce that h is a an isomorphism from (), and b^* is trivially an isomorphism. It follows that g and f are isomorphisms, and then that a is an isomorphism if and only if a^* is. If X is proper, Serre's GAGA theorem [] implies that a is an isomorphism, and hence so is a. On the other hand, if X^* is separated, it can be embedded as a dense open subset in some projective smooth $Y/\mathbf{C}$ such that the complement is a divisor with normal crossings. Then the compactification log structure on Y coming from the embedding $X^* \to Y$ makes $Y/\mathbf{C}$ a smooth log scheme, and the same diagram works for $Y/\mathbf{C}$. Since $Y/\mathbf{C}$ is proper, the map a for Y is an isomorphism, hence so is a^*, and hence so is the morphism a for the log scheme X.

1 The De Rham complex

1.1 Exterior differentiation and Lie bracket

Proposition 1.1.1 *Let $f\colon X \to Y$ be a morphism of coherent log schemes and for each i let $\Omega^i_{X/Y}$ be the i^{th} exterior power of $\Omega^1_{X/Y}$. Then there is a unique collection of homomorphisms of sheaves of abelian groups, callaed the exterior derivative:*

$$\{d^i\colon \Omega^i_{X/Y} \to \Omega^{i+1}_{X/Y} : i \in \mathbf{N}\}$$

such that

1. *$d^i d^{i-1}\omega = 0$ if ω is any section of $\Omega^{i-1}_{X/Y}$, and $d^1 d\log m = 0$ if m is any section of M_X.*

2. $d^{i+j}(\omega\wedge\omega') = (d^i\omega)\wedge\omega' + (-1)^i\omega\wedge(d^j\omega')$ *if* $\omega\in\Omega^i_{X/Y}$ *and* $\omega'\in\Omega^j_{X/Y}$.

Proof: By Proposition (1.1.13) we may without loss of generality assume that M_X is integral, and we identify a section of M_X with its image in M_X^{gp}. The main point is the existence of $d^1\colon\Omega^1_{X/Y}\to\Omega^2_{X/Y}$. Classically, this is proved by checking compatibility with all the relations used in the construction of $\Omega^1_{X/Y}$; this is somewhat tedious since d is not $\mathcal{O}_X$-linear [2, II, §3]. It is more convenient to use the description (1.1.6) of $\Omega^1_{X/Y}$ as a quotient of $\mathcal{O}_X\otimes M_X^g$ by the abelian subsheaf R_1+R_2. The map $:\mathcal{O}_X\times M_X^{gp}\to\Omega^2_{X/Y}$ sending $(a\times m)$ to $da\wedge dlog\, m$ is evidently bilinear, and hence induces a map of abelian sheaves

$$\phi\colon\mathcal{O}_X\otimes M_X^{gp}\to\Omega^2_{X/Y}.$$

If m is any section of M_X,

$$\phi(\alpha_X(m)\otimes m) = d\alpha_X(m)\wedge dlog\, m = \alpha_X(m)dlog\,(m)\wedge dlog\, m = 0,$$

and if n is any section of $f^{-1}M_Y^{gp}$ $\phi(a\otimes n) = da\wedge dlog\, n = 0$. It follows that ϕ annihilates all the elements in R_1+R_2, and hence that it factors through a homomorphism of abelian groups $d^1\colon\Omega^1_{X/Y}\to\Omega^2_{X/Y}$. Then $d(adlog\, m) = da\wedge dlog\, m$ for $a\in\mathcal{O}_X$. In particular, $d(dlog\, m)=0$ and if $a=\alpha_X(m)$, $dda = d\alpha_X(m)\wedge dlog\, m = 0$. It follows that $dda=0$ for any local section of $\mathcal{O}_X$, so (1) is satisfied for $i=1$. Furthermore, $\Omega^1_{X/Y}$ is locally generated as an abelian sheaf by sections of the form $\omega = bdlog\, m$, where b is a section of $\mathcal{O}_X$ and m a section of M_X. If a is another section of $\mathcal{O}_X$,

$$d(a\omega) = d(abdlog\, m) = (dab)\wedge dlog\, m = (bda+adb)\wedge dlog\, m = da\wedge\omega + a\wedge d\omega.$$

Hence (2) holds when $i=0$ and $\omega'\in\Omega^1_{X/Y}$. Thus we have constructed d^0 and d^1 satisfying conditions (1) and (2). For $i>1$ consider the map

$$\begin{array}{rcl}\Omega^1_{X/Y}\times\Omega^1_{X/Y}\times\cdots\Omega^1_{X/Y} & \to & \Omega^{i+1}_{X/Y}\\ (\omega_1,\omega_2,\ldots\omega_i) & \mapsto & \sum_j(-1)^{j+1}\omega_1\wedge\cdots d\omega_j\wedge\cdots\omega_i\end{array}$$

If a is a local section of $\mathcal{O}_X$, $(a\omega_1,\omega_2,\ldots\omega_i)$ maps to

$$\sum_j(-1)^{j+1}a\omega_1\wedge\cdots d\omega_j\wedge\cdots\omega_i + da\wedge\omega_1\cdots\omega_i,$$

and for any k, $(\omega_1, \omega_2, \ldots, a\omega_k, \ldots \omega_i)$ maps to

$$\sum_j (-1)^{j+1} a\omega_1 \wedge \cdots d\omega_j \wedge \cdots \omega_i + (-1)^{k+1}\omega_1 \wedge \omega_2 \wedge \cdots da \wedge \omega_k \wedge \cdots \omega_i.$$

In the last term above, the da is in the kth place, so this term is equal to

$$da\omega_1 \wedge \omega_2 \wedge \cdots \omega_i.$$

Thus the map above is $\mathcal{O}_X$-multi-near. Since it clearly annihilates any i-tuple with a repeated factor, it factors through a map $d^i \colon \Omega^i_{X/Y} \to \Omega^{i+1}$. It is easy to check that this map has the desired properties. $\square$

In the classical case, the exterior derivative $d \colon \Omega^1_{X/Y} \to \Omega^2_{X/Y}$ corresponds to a Lie-algebra structure on the dual $T_{X/Y}$. Let us verify that the same holds here.

Proposition 1.1.2 *Let $f \colon X \to Y$ be a morphism of coherent log schemes and let $T_{X/Y} := Der_{X/Y}(\mathcal{O}_X)$. Then $T_{X/Y}$ has a structure of a Lie algebra over $f^{-1}\mathcal{O}_Y$, with Lie bracket defined by*

$$[(D_1, \delta_1), (D_2, \delta_2)] =: ([D_1, D_2], D_1\delta_2 - D_2\delta_1).$$

If $\omega \in \Omega^1_{X/Y}$ and ∂_1,$\partial_2 \in T_{X/Y}$, then

$$\langle d\omega, \partial_1 \wedge \partial_2 \rangle = \partial_1 \langle \omega, \partial_2 \rangle - \partial_2 \langle \omega, \partial_1 \rangle - \langle \omega, [\partial_1, \partial_2] \rangle.$$

Write the proof

Proof: $\square$

1.2 De Rham complexes of monoid algebras

Since smooth morphisms of log schemes are locally modeled by morphisms of monoid schemes, it is both useful and instructive to have a good picture of the de Rham complexes of arising from morphisms of monoids. Since the de Rham complex in this case is invariant under the group action, it is equipped with a canonical grading. Although the group action and grading are destroyed by localization and do not exist even in the local models, they are extremely revealing and useful.

Let $\theta\colon Q \to P$ be a morphism of fine monoids and let R be a fixed ring. We suppose for simplicity of notation that θ^{gp} is injective, and write

$$\pi\colon P^{gp} \to P^{gp}/Q^{gp}$$

for the natural projection. Then θ induces a morphism of prelog R-algebras

$$\bigl(Q \to R[Q]\bigr) \to \bigl(P \to R[P])\bigr),$$

and hence a corresponding map of log schemes

$$X \to Y := \mathsf{A_P} \to \mathsf{A_Q}\,.$$

According to Theorem 3.1.8, $X \to Y$ is smooth if and only if the order of the torsion part of $\mathrm{Cok}(\theta^{gp})$ is invertible in R. Let us also assume this from now on. In particular, $R \otimes_{\mathbf{Z}} P^{gp}/Q^{gp}$ is a free R-module of finite rank. As we saw in (1.2.1), the sheaf of Kahler differentials $\Omega^1_{X/Y}$ is the quasi-coherent sheaf of $\mathcal{O}_X$-modules associated to

$$\Omega^1_{P/Q} := R[P] \otimes P^{gp}/Q^{gp},$$

and $d\colon \mathcal{O}_X \to \Omega^1_{X/Y}$ on global section is given by

$$d\colon e^p \in R[P] \mapsto e^p \otimes \pi(p) \in R[P] \otimes P^{gp}/Q^{g}.$$

In particular, if $\Omega^1_{P/Q}$ is endowed with the P^{gp}-grading in which P^{gp}/Q^{gp} is assigned degree zero, the map d preserves degrees. In the same way, $\Omega^i_{X/Y}$ is the quasi-coherent sheaf associated to the P^{gp}-graded $R[P]$-modules

$$\Omega^i_{P/Q} := R[P] \otimes \Lambda^i P^{gp}/Q^{gp},$$

and we find the following:

Proposition 1.2.1 *Let $\theta\colon Q \to P$ be a homomorphism of fine monoids such that θ^{gp} is injective and the torsion part of* $\mathrm{Cok}(\theta^{gp})$ *is invertible in R. Then the De Rham complex $\Omega^{\cdot}_{P/Q}$ associated to $\theta\colon Q \to P$ is a P-graded complex of free graded $R[P]$-modules, generated in degree 0. In fact, it admits a direct sum decomposition*

$$\Omega^{\cdot}_{P/Q} \cong \bigoplus_{p \in P} \Bigl(\Lambda^{\cdot} P^{gp}/Q^{gp}, \pi(p)\wedge\Bigr),$$

where $(\Lambda^{\cdot} P^{gp}/Q^{gp}, dp\wedge)$ is the exterior algebra on P^{gp}/Q^{gp} with differential given by exterior multiplication by the image $\pi(p)$ of p in P^{gp}/Q^{gp}. Furthermore, the exterior multiplication

$$\Omega^i_{P/Q} \otimes \Omega^j_{P/Q} \to \Omega^{i+j}_{P/Q}$$

is compatible with the grading.

Proof: Each element ω of $\Omega^i_{P/Q}$ can be written uniquely as a sum

$$\omega = \sum_{p \in P} e^p \otimes \omega_p \quad \text{where} \quad \omega_p \in \Lambda^i(P^{gp}/Q^{gp})$$

Thus ω_p is the homogeneous component of degree p of ω. Since the elements of P^{gp}/Q^{gp} are all closed, so is each ω_p. Hence

$$d(e^p\omega_p) = de^p \wedge \omega_p = e^p\pi(p) \wedge \omega_p.$$

□

Since the differentials of the de Rham complex preserve the grading, the cohomology modules are also graded. The proposition shows that the differential in degree zero vanishes, and so the cohomology in degree zero is easy to describe.

Corollary 1.2.2 *With the hypotheses above, there is an injection:*

$$\sigma\colon R \otimes \Lambda^{\cdot} P^{gp}/Q^{gp} \to H^{\cdot}_{dR}(P/Q) := H^{\cdot}(\Omega^{\cdot}_{P/Q}),$$

induced by the natural map $P^{gp}/Q^{gp} \to \Omega^1_{P/Q}$ and compatible with the algebra structures on both sides. In fact, σ is an isomorphism onto the homogeneous component of $H^{\cdot}(\Omega^{\cdot}_{P/Q})$ of degree zero.

Suppose now that R contains a field, so that it make sense to speak of the characteristic of R. In this case it is easy to compute $H^{\cdot}_{DR}(P/Q)$ explicitly. Indeed, if k is the prime field contained in R (either $\mathbf{Q}$ or $\mathbf{F}_p$), then morphism $\mathsf{A_P} \to \mathsf{A_Q}$ over R is obtained by base change from the corresponding morphism over k. Since the differentials of the de Rham complex $\Omega^{\cdot}_{X/Y}$ are k-linear and k is a field, its cohomology over R is obtained by base change from its cohomology over k. then it is clear that $H^i_{DR}(P/Q)$ is obtained from the cohomology of the corresponding objects over k by base change $k \to R$. In particular, the basis element e^p for the degree p component of $R[P]$ lies in $H^0_{dR}(P/Q)$ if and only if $\pi(p)$ maps to zero in in $k \otimes P^{gp}/Q^{gp}$. In the characteristic of R is zero, $k = \mathbf{Q}$, and $1 \otimes \pi(p) = 0$ if and only if some positive multiple of m lies in Q^{gp}. On the other hand, if the characteristic of R is p, $1 \otimes \pi(p) = 0$ if and only if $p \in pP^{gp} + Q^{gp}$. This leads to the following result.

Proposition 1.2.3 *With the hypotheses of (1.2.1), let*

$$(P/Q)_{st} := \{p \in P : \exists n > 0 : np \in Q^{gp}\},$$

and if p is a prime number, let

$$(P/Q)_p := P \cap (pP^g + Q^{gp}).$$

If R has characteristic zero, the map σ and the canonical inclusions induce isomorphisms:

$$R[(P/Q)_{st}] \otimes \Lambda^{\cdot} P^{gp}/Q^{gp} \to H^{\cdot}_{DR}(P/Q).$$

If R has characteristic $p > 0$, σ and the inclusions induce isomorphisms:

$$R[(P/Q)_p] \otimes \Lambda^{\cdot} P^{gp}/Q^{gp} \to H^{\cdot}_{DR}(P/Q).$$

Proof: Let us write $\tilde{Q}$ for the monoid $(P/Q)_{st}$ if the characteristic is zero and for $(P/Q)_p$ if the characteristic is p. Thus $R[\tilde{Q}] = H^0_{DR}(P/Q)$ in both cases, and the cohomology groups are modules over this ring. This explains the existence of the arrow. We check that it is an isomorphism degree by degree. If $p \in \tilde{Q}$, then $\pi(p) =\in R \otimes P^{gp}/Q^{gp}$ and the differential of the complex in degree p vanishes, so the map is an isomorphism. On the other hand, if $p \notin \tilde{Q}$, we claim that the degree p term of the complex $\Omega^{\cdot}_{P/Q}$ is acyclic. It suffices to prove this when R is a prime field. If $p \notin \tilde{Q}$, $\pi(p)$ is not zero in the R-vector space $V := R \otimes P^{gp}/Q^{gp}$, and hence is part of a basis for V. The degree p term of the complex is just the exterior algebra $\Lambda^{\cdot} V$, with differential multiplication by v. This complex is well-known to be acyclic, but it is valuable to have an explicit proof. Since v is part of a basis for V, there exists a homomorphism $\partial\colon V \to k$ such that $\partial(v) = 1$. Then interior multiplication by ∂ defines a map of degree -1

$$s\colon \Lambda^{\cdot} V \to \Lambda^{\cdot} V.$$

Then for $\omega \in \Lambda^{\cdot} V$,

$$\begin{aligned}(ds + sd)(\omega) &= v \wedge s(\omega) + s(v \wedge \omega)\\ &= v \wedge s(\omega) + s(v)\omega - v \wedge s(\omega)\\ &= \omega.\end{aligned}$$

Thus the identity map of the complex V is homotopic to zero, and hence V is acyclic. □

Corollary 1.2.4 *Suppose that R has characteristic zero and P is a fine monoid such that the order of the torsion group of P^{gp} is invertible in R. Then the natural maps:*

$$\begin{aligned} R[P^*_t] \otimes \Lambda^{\cdot}(P^{gp}) &\longrightarrow \Omega^{\cdot}_{P/R}\\ R[P^{gp}_t] \otimes \Lambda^{\cdot}(P^{gp}) &\longrightarrow \Omega^{\cdot}_{P^{gp}/R}\end{aligned}$$

are quasi-isomorphisms. In particular, if P is saturated, the map

$$\Omega^{\cdot}_{P/R} \to \Omega^{\cdot}_{P^{gp}/R}$$

is a quasi-isomorphism.

Proof: The statement for P is a special case of (1.2.3), and the second statement follows, after replacing P by P^g. When P is saturated, the map $P \to P^{gp}$ induces an isomorphism on torsion subgroups $P^*_t \to P^{gp}_t$ (1.2.3), and so the map $\Omega^{\cdot}_{P/R} \to \Omega^{\cdot}_{P^{gp}/R}$ is a quasi-isomorphism. □

Remark 1.2.5 The corollary can be interpreted geometrically as follows. The morphism $P^* \to P^{gp}$ of finitely generated abelian groups induces a commutative diagram of group schemes:

$$\begin{tikzcd}
\underline{\mathsf{A}}_{\mathsf{P}^{\mathsf{gp}}} \arrow[r] \arrow[d] & \underline{\mathsf{A}}_{\mathsf{P}_{\mathsf{t}}^{\mathsf{gp}}} \arrow[d] \\
\underline{\mathsf{A}}_{\mathsf{P}^*} \arrow[r] & \underline{\mathsf{A}}_{\mathsf{P}_{\mathsf{t}}^*}
\end{tikzcd}$$

The groups on the right are just the groups of connected components of the corresponding group schemes. Thus the corollary says that the map on de Rham cohomology is an isomorphism if and only if these two group schemes have the same connected components. For example, this is not the case for the monoid given by generators x and y and relations $2x = 2y$.

Corollary 1.2.6 *Suppose that R has characteristic p and P is a fine monoid such that the torsion subgroup of P^{gp} has order prime to p. Then the map*

$$R[P \cap pP^{gp}] \otimes \Lambda^{\cdot} P^{gp} \to \Omega^{\cdot}_{P/R}$$

is a quasi-isomorphism. In particular, if P is saturated, then the pth power map $P \to P$ induces a quasi-isomorphism:

$$R[P] \otimes \Lambda^{\cdot} P^{gp} \to \Omega^{\cdot}_{P/R}.$$

□

The calculation of the cohomology given in the proof of Proposition 1.2.3 was done homogeneous degree by homogeneous degree. Since the grading on the monoid algebra $R[P]$ is destroyed by localization, it will be important to give a variation of the method that is more geometric.

Definition 1.2.7 *Let $\theta\colon Q \to P$ be a morphism of integral monoids. A homogeneous flow over θ is a homomorphism of monoids $h\colon P \to \mathbf{N}$ such that $\partial(q) = 0$ for all $q \in Q$. A homogeneous vector field homogeneous vector field over θ is a group homomorphism $\partial\colon P^{gp} \to \mathbf{Z}$ such that $\partial \circ \theta^{gp} = 0$.*

Remark 1.2.8 It is clear that the set $H_\theta(P)$ of homogeneous flows over θ forms a submonoid of the dual monoid $H(P)$ (2.2.1) of P. There is an evident homomorphism from $H_\theta(P)$ to the the group T_θ of homogeneous vector fields over θ. Note that if $h \in H_\theta(P)$, then then $h(p) = 0$ for all p belonging to the face F of P generated by the image of θ, so h factors through P/F. On the other hand, if P is fine, then it follows from (2.2.4) that $H(P/F)^{gp} \cong \mathrm{Hom}(P^{gp}/F^{gp}, \mathbf{Z}) \subseteq T_\theta$..

Let $\partial\colon P^{gp} \to \mathbf{Z}$ be a homogeneous flow over θ. Then

$$\mathrm{id} \otimes \partial\colon R[P] \otimes \mathrm{Cok}(\theta^{gp}) \cong \Omega^1_{Q/P} \to R[P]$$

is an $R[P]$-linear map which we also denote by ∂. Thus

$$\partial(e^p dq) = e^p \partial(q) \quad \text{for } p, q \in P$$

and is a vector field in the usual sense. Any two such homogeneous vector fields commute with each other under the bracket operation [16, 1.1.7] For any i, *interior multiplication* by ∂ is the unique $R[P]$-linear map

$$\xi\colon \Omega^i_{P/Q} \to \Omega^{i-1}_{P/Q} : \omega_1 \wedge \cdots \omega_i \mapsto \sum_j (-1)^{j-1} \partial(\omega_j) \omega_1 \wedge \cdots \hat{\omega}_i \wedge \cdots \omega_i.$$

Then $\xi\colon \Omega^{\cdot} \to \Omega^{\cdot -1}$ is a derivation of degree -1, *i.e.*, it satisfies

Classically, if X/S is a smooth morphism of schemes, a vector field on X/S is a section ∂ of the dual of $\Omega^1_{X/S}$ and induces a linear derivation

$$\xi\colon \Omega^{\cdot}_{X/Y} \to \Omega^{\cdot -1}_{X/Y}$$

as above. The *Lie derivative* with respect to ∂ is by definition the map

$$\kappa := d\xi + \xi d : \Omega^{\cdot}_{X/Y} \to \Omega^{\cdot}_{X/Y}.$$

Lemma 1.2.9 *Let $\partial\colon P^{gp}/Q^{gp} \to \mathbf{Z}$ be a homogeneous vector field over θ, let*

$$\xi\colon \Omega^{\cdot}_{P/Q} \to \Omega^{\cdot -1}_{P/Q}$$

be the corresponding $R[P]$-linear derivation of degree -1, and let

$$\kappa := d\xi + \xi d : \Omega^{\cdot}_{P/Q} \to \Omega^{\cdot}_{P/Q}.$$

1. *If $\alpha \in \Omega^a_{P/Q}$ and $\beta \in \Omega^b_{P/Q}$,*

   $$\xi(\alpha\wedge\beta) = \xi(\alpha)\wedge\beta+(-1)^{ab}\alpha\wedge\xi(\beta) \quad \textit{and} \quad \kappa(\alpha\wedge\beta) = \kappa(\alpha)\wedge\beta+\alpha\wedge\kappa(\beta),$$

   i.e., ξ and κ are derivations of degree -1 and 0, respectively.

2. *ξ, d, and κ preserve the P-grading of $\Omega^{\cdot}_{P/Q}$. In particular, κ is an endomorphism of the P-graded complex $\Omega^{\cdot}_{P/Q}$, and for $p \in P$,*

   $$\kappa_p \colon \Omega^i_{P/Q,p} \to \Omega^i_{P/Q,p} = \partial(p)\cdot,$$

   Moreover, κ induces zero on the cohomology modules of $\Omega^{\cdot}_{P/Q}$.

Proof: The formula (1) for ξ is an immediate consequence of the definition, and the formula (1) for κ follows by a computation which we leave to the reader to verify. Of course, κ is automatically a morphism of complexes and induces zero on cohomology since it is visibly homotopic to zero. To compute κ, let ω be any element of $R\otimes\Lambda^i \operatorname{Cok}(\theta^q) = \Omega^i_{P/Q_0}$. We have already observed that $d\omega = 0$. Since ∂ is homogeneous, $\xi(\omega) \in R \otimes \Lambda^{i-1}$, so $d\xi\omega = 0$. Since $\xi d\omega = 0$ as well, it follows that $\kappa(\omega) = 0$. This proves the formula when $p = 0$. On the other hand, if $i = 0$ and p is arbitrary,

$$\kappa(e^p) = d\xi e^p + \xi de^p = 0 + \xi(e^p \otimes \pi(p)) = e^p\partial(p),$$

which again is consistent with the formula in (2). For any p and i, $\Omega^i_{P/Q,p}$ is spanned as an R-module by elements of the form $e^p\omega$ with $\omega \in \Omega^i_{P/Q,0}$. Hence

$$\kappa(e^p\omega) = \kappa(e^p)\omega + e^p\kappa(\omega) = \partial(p)e^p\omega + 0.$$

This proves (2) in general. □

Corollary 1.2.10 *Suppose that in the above lemma, ∂ is induced by a homomorphism of monoid $h\colon P \to \mathbf{N}$, and let $\mathfrak{p} := h^{-1}(\mathbf{N}^+)$. Then the image of*

$$\kappa\colon \Omega^{\cdot}_{P/Q} \to \Omega^{\cdot}_{P/Q}$$

is contained in $\mathfrak{p}\Omega^{\cdot}_{P/Q}$.

We shall use the lemma above to prove the acyclicity of various subcomplexes of the de Rham complex $\Omega^{\cdot}_{P/Q}$ and eventually of their localizations in the étale topology. We illustrate this technique with the subcomplexes coming from ideals in the monoid P, or more generally, fractional ideals $K \subseteq P^{gp}$. (Recall that a fractional ideal is a subset of P^{gp} which is stable under the action of P; it is not necessarily a submonoid of P^{gp}).

Proposition 1.2.11 *Let $\theta\colon Q \to P$ be a morphism of monoids satisfying the hypothesis of 1.2.1 and let $K \subseteq P^{gp}$ be a fractional ideal. For each i, let*

$$K\Omega^i_{P/Q} \subseteq \Omega^i_{P^{gp}/Q^{gp}} \cong R[P^{gp}] \otimes \Lambda^i \operatorname{Cok}(P^{gp}/Q^{gp})$$

denote the R-submodule generated by the elements of the form $e^k\omega$ with $k \in K$ and $\omega \in R \otimes \Lambda^i P^{gp} Q^{gp}$. Then in fact $K\Omega^i_{P/Q}$ is an $R[P]$-submodule of $\Omega^i_{P^{gp}/Q^{gp}}$, and

$$K\Omega^{\cdot}_{P/Q} := \{K\Omega^i_{P/Q} : i \geq 0\}$$

is stable under d and under interior multiplication by any vector field in $T_{P/Q}$.

Proof: The fact that $K\Omega^i_{P/Q}$ is stable under multiplication by $R[P]$ follows from the fact that K is stable under translation by P. For any k, ω,

$$d(e^k\omega) = de^k \wedge \omega + e^k d\omega = e^k d\log k \wedge \omega,$$

and it follows that $K\Omega^{\cdot}_{P/Q}$ is stable under d. A vector field ξ induces an $R[P^{gp}]$-linear map $\xi\colon \Omega^1 \to R[P]$, and then (writing ξ also for interior multiplication by itself):

$$\xi(e^k\omega_1 \wedge \cdots \omega^i) = e^k\xi(\omega_1 \wedge \cdots \omega^i),$$

so $K\Omega^{\cdot}_{P/Q}$ is also stable under ξ. $\square$

Lemma 1.2.12 *Let $\theta\colon Q \to P$ be a morphism of fine monoids and let $K \subseteq P$ be an ideal. Then the following conditions are equivalent.*

1. *For each $k \in K$, there exists an $h \in H_\theta(p)$ such that $h(k) \neq 0$.*

2. *K is disjoint from the face F of P generated by the image of θ.*

3. *K is the inverse image of a proper ideal K' of the quotient monoid* $\mathrm{Cok}(\theta)$.

4. *There exists an $h \in H_\theta(P)$ such that $h(k) \neq 0$ for all $k \in K$.*

An ideal satisfying these conditions will be called horizontal.

Proof: If $h \in H_\theta(P)$, then $h(f) = 0$ for all $f \in F$, so (1) implies (2). If (2) is true, then K is the inverse image of the ideal it generates in the localization P_F of P by F, and hence by the ideal it generates in $\overline{P_F} = P/F$. Since the map $P \to P/F$ factors through $\mathrm{Cok}(\theta)$ of θ, K is also the inverse image of an ideal of $\mathrm{Cok}(\theta)$; this ideal must be proper since K is proper. Since P and Q are fine, $\mathrm{Cok}(\theta)$ is also fine (**??**), and hence by (2.2.2) there exist a local homomorphism $h'\colon \mathrm{Cok}(\theta) \to \mathbf{N}$. Since K' is proper, it contains no units of $\mathrm{Cok}(\theta)$. Then the composite h of h' with the projection $P \to \mathrm{Cok}(\theta)$ satisfies (4). The implication of (1) by (4) is trivial. $\square$

Corollary 1.2.13 *Let $\theta\colon Q \to P$ be a morphism of fine monoids, let $K \subseteq P$ be a horizontal ideal, and let $F\Omega^{\cdot}_{P/Q}$ be a P-graded subcomplex of $\Omega^{\cdot}_{P/Q}$ which is closed under interior multiplication by the vector fields coming from horizontal flows. Suppose that R has characteristic zero, and let $h \in H_\theta(P)$ be a horizontal flow with $h(k) \neq 0$ for all $k \in K$. Then the Lie derivative $\kappa := d\xi + \xi d$ corresponding to h induces an isomorphism of complexes*

$$\kappa\colon K\Omega^{\cdot}_{P/Q} \cap F\Omega^{\cdot}_{P/Q} \to K\Omega^{\cdot}_{P/Q} \cap F\Omega^{\cdot}_{P/Q}$$

In particular, $K\Omega^{\cdot}_{P/Q} \cap F\Omega^{\cdot}_{P/Q}$ is homotopic to zero and acyclic.

Proof: We have already seen that $K\Omega^{\cdot}_{P/Q}$ is stable under the exterior derivative d and by interior multiplication ξ. If $F\Omega^{\cdot}_{P/Q}$ is also stable under d and ξ, then the same is true of their intersection. Now Lemma 1.2.9 implies that κ is multiplication by $h(k)$ in degree k of the complex $K\Omega^{\cdot}_{P/Q}$ and hence also in degree k of the subcomplex $K\Omega^{\cdot}_{P/Q} \cap F\Omega^{\cdot}_{P/Q}$. Since $h(k) \neq 0 \in \mathbf{Q} \subseteq R$, κ is an isomorphism. This certainly implies that $K\Omega^{\cdot}_{P/Q} \cap F\Omega^{\cdot}_{P/Q}$ is acyclic; to see that it is even homotopic to zero, we can argue further as follows. Note that $\kappa\xi = \xi\kappa = \xi d\xi$, and let

$$\xi' := \xi\kappa^{-1} = \kappa^{-1}\xi : K\Omega^{\cdot} \cap F\Omega^{\cdot}_{P/Q} \to K\Omega^{\cdot -1} \cap F\Omega^{\cdot -1}_{P/Q}.$$

Then $d\xi' + \xi' d = \mathrm{id}$. $\square$

Example 1.2.14 For each $p \in P$, let

$$\underline{\Omega}^i_{P/Q,p} := \operatorname{Im}\Big(R \otimes \Lambda^i \langle p\rangle^{gp} \to R \otimes \Lambda^i \operatorname{Cok}(\theta^{gp})\Big) \subseteq \Omega^i_{P/Q,p},$$

where $\langle p\rangle$ is the face of P generated by p.

1.3 Algebraic de Rham cohomology

Our first goal is the proof that the arrow e in the diagram of Theorem 0.2.1 is an isomorphism. In fact we shall prove a more precise statement, using the language of derived categories. Our method will be to sheafify the techniques of the previous section, replacing the P^{gp}-grading of $R[P^{gp}]$ used there by filtrations by $R[P]$-submodules. For simplicity, we work over an affine base scheme $S = \operatorname{Spec} R$ with trivial log structure.

We begin with some preliminary remarks.

Lemma 1.3.1 *Let X/S be a fine log scheme, locally of finite type over S, and let ξ be a vector field on X/S, i.e., a homomorphism $\Omega^1_{X/S} \to \mathcal{O}_X$. Let $\underline{X}$ be the underlying scheme X with trivial log structure, and let*

$$I_\xi := \operatorname{Im}\Big(\Omega^1_{\underline{X}/S} \longrightarrow \Omega^1_{X/S} \xrightarrow{\ \xi\ } \mathcal{O}_X\Big).$$

Then I_ξ is a quasi-coherent ideal of $\mathcal{O}_X$-modules in the étale topology on X, and is the $\mathcal{O}_X$-ideal generated the image of the derivation $\xi \circ d\colon \mathcal{O}_X \to \mathcal{O}_X$.

Proof: Let $f\colon U \to X$ be an étale map. Since f is étale and strict, the map $f^*\Omega^1_{\underline{X}/S} \to \Omega^1_{\underline{U}/S}$ is an isomorphism, and since f is flat, it follows that the map $f^*I_{\xi,X} \to I_{\xi,U}$ is an isomorphism. This shows that I_ξ forms a quasi-coherent sheaf of ideals for the étale topology. Since $\Omega^1_{\underline{X}/S}$ is locally generated as an $\mathcal{O}_X$-module by sections of the form da, for a a section of $\mathcal{O}_X$, I_ξ is locally generated by sections of the form $\xi(da)$. □

do this for fr ideals in M^g?

Lemma 1.3.2 *Let P be a fine monoid, $X := \mathsf{A}_\mathsf{P}$, let $K \subseteq P^{gp}$ be a fractional ideal, and for each $q \in \mathbf{N}$, let $K\Omega^q_{X/S} \subseteq j_*\Omega^q_{X^*/S}$ be the quasi-coherent sheaf of $\mathcal{O}_X$-modules corresponding to the $R[P]$-module $K\Omega^q_P$ defined in (**??**).*

1. *The family $K\Omega^{\cdot}_{X/S}$ is closed under the exterior derivative d and under interior multiplication ξ by any vector field in $\Gamma(T_{X/S})$.*

2. *Let $J \subseteq K$ be a fractional ideal contained in K and let ξ be a vector field such that $I_\xi K \subseteq J$. Then $\kappa := d\xi + \xi d$ acts $\mathcal{O}_X$-linearly on the quotient $K\Omega^{\cdot}_{X/S}/J\Omega^{\cdot}_{X/S}$.*

Proof: Let $f\colon U \to X$ be an étale map. Since interior multiplication ξ is $\mathcal{O}_X$-linear, $K\Omega^{\cdot}_{U/S}$ is certainly stable under ξ. Every section of $K\Omega^q_{U/S}$ can locally be written as a sum of sections of the form $a\omega$, where a is a section of $\mathcal{O}_U$ and ω is a section of $f^{-1}(K\Omega^q_{K/S})$ Since $d(a\omega) = da \wedge \omega + ad\omega$, and since $K\Omega^{\cdot}_P$ is stable under d by (**??**), it follows that $d(a\omega)$ belongs to $K\Omega^{q+1}_{U/S}$. This proves (1). In this situation of (2), suppose that a is a section of $\mathcal{O}_X$ and ω is a section of $K\Omega^q_{X/S}$. Then $\kappa(a\omega) = \kappa(a)\omega + a\kappa(\omega)$. But $\kappa(a) = \xi(da) \in I_\xi$, so $\kappa(a)\omega \in J\Omega^q_{X/S}$, and $\kappa(a\omega) = a\kappa(\omega) \pmod{J\Omega^q_{X/S}}$. □

Theorem 1.3.3 *Let X/S be a smooth morphism of log schemes, where X is fine and saturated and S is a noetherian $\mathbf{Q}$-scheme (with trival log structure). Let $j\colon X^* \to X$ be the inclusion of the open set of triviality of the log structure of X. Then the natural maps*

$$\Omega^{\cdot}_{X/S} \longrightarrow j_*\Omega^{\cdot}_{X^*/S} \longrightarrow Rj_*\Omega^{\cdot}_{X^*/S}$$

are isomorphisms in the derived category of abelian sheaves on $X_{\acute{e}t}$.

Proof: Recall from (2.1.6) that the map $j\colon X^* \to X$ is a relatively affine open immersion. Then if E is any quasi-coherent sheaf on X, $R^q j_* E = 0$ for all $q > 0$. Thus the sheaves comprising the complex $\Omega^{\cdot}_{X^*/S}$ are acyclic for j_*, and it follows that the map $j_*\Omega^{\cdot}_{X^*/S} \longrightarrow Rj_*\Omega^{\cdot}_{X^*/S}$ is an isomorphism in the derived category [,].

Since X/S is smooth and the log structure of S is trivial, the structure theorem (3.3.1) says that locally on X there exists a chart for X/S subordinate to a fine saturated monoid P such that the map $X \to \mathsf{A}_\mathsf{P}$ is étale. explain why P is saturated
Since the statement we are trying to prove is local in the étale topology, we may as well assume that $X = \mathsf{A}_\mathsf{P}$. Then $X = \operatorname{Spec}(P \mapsto R[P])$, $\Omega^i_{X/S}$ is the coherent sheaf on X corresponding to the $R[P] \otimes_R R \otimes_{\mathbf{Z}} \Lambda^i P^{gp}$, and $j_*\Omega^i_{X/S}$ is the quasi-coherent sheaf on X corresponding to $R[P^{gp}] \otimes_R R \otimes_{\mathbf{Z}} \Lambda^i P^{gp}$.

By Theorem (2.2.1), the dual monoid $H(P)$ is finitely generated. Choose a finite sequence $(h_1, h_2, \cdots h_r)$ of elements of $H(P)$ which generate the cone $C_{\mathbf{Q}}(H(P))$. For each $I \in \mathbf{Z}^r$, let

$$K^I := \{p \in P^{gp} : h_i(p) \geq n_i \text{ for } i = 1, \ldots r\}.$$

Then $K^I \subset P^{gp}$ is a fractional ideal of P, $K^I + K^J \subseteq K^{I+J}$ for any I and J, and $K^I \subseteq K^J$ if $J \leq I$ in the order relation on $\mathbf{Z}^r$ corresponding to the submonoid $\mathbf{N}^r$. Furthermore, by Corollary 2.2.3,

$$K^0 := \{p \in P^{gp} : h(p) \geq 0 \text{ for all } h \in H(P)\} = P^{\text{sat}} = P,$$

since P is saturated.

Let $K^I\Omega^{\cdot}_P$ be the complex of submodules of $\Omega^q_{P^{gp}}$ defined by the fractional ideal K^I as explained in (1.2.11), and let $K^I\Omega^{\cdot}_{X/S}$ be the corresponding complex of quasi-coherent subsheaves of $j_*(\Omega^q_{X^*/S})$. Of course, the boundary maps of these complexes are only $f^{-1}(\mathcal{O}_S)$-linear.

Proposition 1.3.4 *If $J \leq I \leq (0, 0, \cdots 0)$, the map*

$$K^I\Omega^{\cdot}_{X/S} \to K^J\Omega^{\cdot}_{X/S}$$

is a quasi-isomorphism.

Proof: Suppose that $J \leq I' \leq I$. Since the composite of two quasi-isomorphisms is a quasi-isomorphism, if the proposition is true for the pairs (J, I') and (I', I), then it is also true for the pair (J, I). In this way we reduce to the case in which there is an i such that $I = J + \epsilon_i$, where $\epsilon_i := (0, \cdots, 1, \cdots, 0)$. Then $K^{\epsilon_i} := \{p \in P : h_i > 0\}$ is a prime ideal $\mathfrak{p}_i$ of P. The exact sequence of complexes:

$$0 \to K^I\Omega^{\cdot}_{X/S} \to K^J\Omega^{\cdot}_{X/S} \to K^J\Omega^{\cdot}_{X/S}/K^I\Omega^{\cdot}_{X/S} \to 0$$

induces a long exact sequence of cohomology sheaves, and so it suffices to prove that the quotient complex $Q^{\cdot}$ on the right is acylic.

Interior multiplication ξ by the vector field induced by h acts on all the complexes in the exact sequence above, and in particular on the quotient $Q^{\cdot}$. By Lemma 1.2.9, $d\xi e^p = h(p)e^p$ for all $p \in P$, and this is nonzero if and only if $p \in \mathfrak{p}_i$. Thus by Lemma 1.3.1, I_ξ is the quasi-coherent sheaf of ideals corresponding to $\mathfrak{p}_i R[P]$. Since $\mathfrak{p}_i K^J \subseteq K^I$, $\kappa := d\xi + \xi d$ acts $\mathcal{O}_X$-linearly on $Q^\cdot$, by (1.3.2). Hence κ agrees with the map obtained by base change from the corresponding operator κ on $K^J\Omega^\cdot_P/K^I\Omega^\cdot$. This complex is graded, and κ preserves the grading. For any $p \in K^J \setminus K^I$, $h_i(p) = I_i$, and so by (1.2.9) κ is just multiplication by $h_i(p)$ on $Q^\cdot$. Since R has characteristic zero and $I_i < 0$, κ is an isomorphism. Thus the complex $Q^\cdot$ is homotopic to zero, hence acyclic. □

A similar technique can be used to analyze the de Rham complexes coming from sheaves of ideals.

Lemma 1.3.5 *Let $f: X \to Y$ be a morphism of fine log schemes, let $K \subseteq M_X$ be a sheaf of ideals, and for each q let $K\Omega^q_{X/Y}$ the abelain subsheaf of $\Omega^q_{X/Y}$ generated by sections of the form $\alpha_X(k)\omega$, where k is a local section of K and ω is a local section of ω. Then $K\Omega^q_{X/Y}$ is an $\mathcal{O}_X$-submodule of $\Omega^q_{X/Y}$, and the exterior differential d maps $\Omega^q_{X/Y}$ to $\Omega^{q+1}_{X/Y}$.*

Proof: Recall (**??**) that every local section a of $\mathcal{O}_X$ can locally be written as a sum $\sum_i u_i$, where u_i is a section of $\mathcal{O}^*_X$. Then if k is a section of K and ω is a section of $\Omega^q_{X/Y}$,

$$a\alpha_X(k)\omega = \sum \alpha_X(u_i k)\omega.$$

Since the latter sum belongs to $K\Omega^q_{X/Y}$, so does any sum of elements of the form $a\alpha_X(k)\omega$. This shows that $K\Omega^q_{X/Y}$ is an $\mathcal{O}_X$-submodule of $\Omega^q_{X/Y}$. Furthermore,

$$d(\alpha_X(k)\omega) = \alpha_K(k)\,\mathrm{dlog}\,k \wedge \omega + \alpha_X(k)d\omega \in K\Omega^{q+1}_{X/Y}.$$

□

Recall that if $f: X \to Y$ is a morphism of log schemes, $M_{X/Y}$ is defined to be the cokernel (in the category of sheaves of monoids), of the map $f^\flat f^* M_Y \to M_X$. We shall say that a sheaf of ideals K of M_X is *horizontal* if it the inverse image of a sheaf of ideals of $M_{X/Y}$.

Theorem 1.3.6 *Let $f: X \to Y$ be a smooth morphism of fine log schemes in characteristic zero, let K be a horizontal and coherent sheaf of ideals of M_X, and let $\sqrt{K}$ be its radical. Then the natural map*

$$K\Omega^\cdot_{X/Y} \to \sqrt{K}\Omega^\cdot_{X/Y}$$

is a quasi-isomorphism.

Proof: Let $\overline{x}$ be a geometric point of X lying over a scheme-theoretic point x. It is enough to prove that the stalk of the map in the theorem is a quasi-isomorphism at each such point $\overline{x}$. $\square$

1.4 Analytic de Rham cohomology

Our first task is to describe the cohomology of the analytic stalks of the de Rham complex of a smooth log scheme X over $\mathbf{C}$. Notice first that the map $dlog\colon M_X^{gp} \to \Omega^1_{X/\mathbf{C}}$ factors through the sheaf $\mathcal{Z}^1_{X/\mathbf{C}}$ of closed one-forms.

Proposition 1.4.1 *Let $X/\mathbf{C}$ be a fine and smooth log scheme over $\mathbf{C}$. There is a unique family of isomorphisms of sheaves $\mathbf{C}$-vector spaces on X_{an}:*

$$\{\sigma\colon \mathbf{C} \otimes \Lambda^q \overline{M}^{gp} \to \mathcal{H}^q(\Omega^{\cdot}_{X/\mathbf{C}}) : q \in \mathbf{N}\}$$

satisfying the following conditions:

1. *When $q = 0$, the composite*

$$\sigma\colon \mathbf{C} \to \mathcal{H}^0(\Omega^{\cdot}_{X/\mathbf{C}}) \to \mathcal{O}_X$$

   is the standard inclusion.

2. *When $q = 1$, the diagram*

$$\begin{tikzcd} M_X^{gp} \arrow[r, "dlog"] \arrow[d] & \mathcal{Z}^1_{X/\mathbf{C}} \arrow[d] \\ \overline{M}_X^{gp} \arrow[r, "\sigma"] & \mathcal{H}^1(\Omega^{\cdot}_{X/\mathbf{C}}) \end{tikzcd}$$

   commutes.

3. *The family of maps σ is compatible with multiplication.*

Proof: First we must show that the family of maps σ is well-defined. This is apparent when $q = 0$. For $q = 1$, note that on X_{an} there is an exact sequence of abelian sheaves,

$$0 \longrightarrow \mathbf{Z}(1) \longrightarrow \mathcal{O}_X \xrightarrow{\exp} M_X^{gp} \to \overline{M}_X^{gp} \to 0.$$

and the map exp fits into a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r, "\exp"] \arrow[dr, "d"] & M_X^{gp} \arrow[r] \arrow[d, "dlog"] & \overline{M}_X^{gp} \arrow[d, "\sigma"] \\
& \mathcal{Z}^1_{X/\mathbf{C}} \arrow[r] & \mathcal{H}^1(\Omega^{\cdot}_{X/\mathbf{C}}).
\end{tikzcd}$$

Then it follows that if $q > 1$, there is a unique map $\Lambda^q \overline{M}_X^g \to \mathcal{H}^q(\Omega^{\cdot}_{X/\mathbf{C}})$ sending the class of $m_1 \wedge \cdots m_q$ to $dlog\, m_1 \wedge \cdots dlog\, m_q$ for any q-tuple of sections of M_X^{gp}. □

□

1.5 Filtrations on the De Rham complex

If F is a face of P, the filtration it induces (1.5.5) also admits a convenient graded description. If A is any P-graded R-algebra and E is an R-module, then $A \otimes_R E$ has a natural structure of a P-graded A-module: its component of degree p is just $A_p \otimes_R E$. Suppose we are given, for each $p \in P$, an R-submodule L_pE of E such that, for $p' \geq p$, $L_pE \subseteq L_{p'}E$. We call such a collection of submodules a "P-filtration of E." Then the image of

$$\bigoplus_p A_p \otimes L_pE \to A \otimes_R E$$

is a P-graded A-submodule. In our case, A will be sufficiently simple so that every submodule can be described in such a way. Namely, if A_p is free of rank zero or one for every p, and if $M \subseteq A \otimes_R E$ is a P-graded submodule, then for each $p \in P$, its component of degree p can be viewed as an R-submodule L_p of E. This gives us an equivalence between P-filtrations on E and graded P-submodules of $A \otimes_R E$.

Definition 1.5.1 *Suppose that F is a face of P. For $p \in P$, let $L^i_p(F)\Lambda^j(P^{gp}/Q^{gp})$ be the subgroup generated by all the elements of the form $dp_1 \wedge \cdots dp_j$ such that there exist $k \in \mathbf{N}$ and $f \in F$ such that $kp + f \geq p_1 + \cdots p_i$.*

It is clear that $L^i(F)$ defines a P-filtration on $\Lambda^j(P^{gp}/Q^{gp})$, and hence a P-graded submodule.

Note that for each $p \in P$, if $\langle p, F\rangle$ is the face of P generated by p and F, then $L^i_p(F)$ is just the image of the natural map

$$\Lambda^i\langle p, F\rangle^{gp} \otimes \Lambda^{j-i}P^{gp} \to \Lambda^j P^{gp}/Q^{gp}.$$

The proof of the following lemma is then straightforward.

Lemma 1.5.2 *With the above notation, if $\tilde{F}$ is the sheaf of faces on X corresponding to F, then the quasi-coherent sheaf on X associated to $L^i(F)\Omega^j_{X/Y}$ is $L^i(\tilde{F})\Omega^j_{X/Y}$. The differential d of $\Omega^j_{X/Y}$ send $L^i(F)$ into $L^{i+1}(F)$, and if ∂ is a homogeneous vector field for θ, interior multiplication by ∂ maps $L^i(F)$ to $L^{i-1}(F)$. Let $\tilde{L}(F)$ denote the décalé of the filtration $L(F)$. Then*

$$\tilde{L}(F)^i\Omega^j_{X/Y} = L(F)^{i+j}\Omega^j_{X/Y}$$

and interior multiplication by any homogeneous vector field over θ preserves the filtration $\tilde{L}(F)$.

Corollary 1.5.3 *Suppose that $S = \mathrm{Spec}(\mathbf{N}^r \to \mathbf{Z}[\mathbf{N}^r])$. Then the natural map $\Omega^i_{\underline{S}/\mathbf{Z}} \to \underline{\Omega}^i_{S/\mathbf{Z}}$ is an isomorphism.*

Proof: Since $S/\mathbf{Z}$ is smooth $\Omega^i_{\underline{S}/\mathbf{Z}}$ is locally free, and it follows that the map $\Omega^i_{\underline{S}/\mathbf{Z}} \to j_*\Omega^i_{S^*/\mathbf{Z}}$ is injective. Hence the map $\Omega^i_{\underline{S}/\mathbf{Z}} \to \underline{\Omega}^i_{S/\mathbf{Z}}$ is also injective. Let $(e_1, \cdots e_r)$ be the standard basis for $\mathbf{N}^r$. Then $F_i =: \{ne_i : n \geq 0\}$ is the face of $\mathbf{N}^r$ generated by e_i, and if $p = (p_1, \cdots p_r) \in \mathbf{N}^r$, the face $\langle p\rangle$ generated by p is $\sum\{F_j : p_j > 0\}$. Then

$$\Lambda^q\langle p\rangle = \bigoplus\{F_{J_1} \otimes \cdots F_{J_q} : p_{J_i} > 0 \forall i\},$$

which admits as a basis $\{de_J : p_{J_i} > 0 \forall i\}$. Then $\underline{\Omega}^i_{X/\mathbf{Z}}$ in degree has basis $\{de_J : p_{J_i} > 0 \forall i\}$. Write $x_i = \epsilon(e_i)$, and observe that

$$\epsilon(p)de_J = x_1^{p_1} \cdots x_r^{p_r} de_1 \wedge \cdots de_r = x_1^{p_1 - 1} \cdots x_r^{p_r - 1} dx_1 \wedge \cdots dx_r.$$

This proves that the map $\Omega^i_{\underline{S}/\mathbf{Z}} \to \underline{\Omega}^i_{S/\mathbf{Z}}$ is surjective. $\square$

Throughout this section we let $f\colon X \to Y$ be a morphism of log schemes; we assume f is quasi-compact and quasi-separated. Our goal is to show how the combinatorics of the toroidal geometry associated with the log structure is reflected in the De Rham complex of X/Y. These combinatorics manifest themselves through two sheaves of partially ordered sets: the sets of ideals and faces of M_X.

Proposition 1.5.4 *If $J \subseteq M_X$ is a sheaf of ideals of M_X, let $J\Omega^i_{X/Y}$ be the subsheaf of $\Omega^i_{X/Y}$ generated by all sections of the form $\alpha_X(m)\omega$ with $m \in J$ and $\omega \in \Omega^i_{X/Y}$. Then the exterior derivative maps $J\Omega^i_{X/Y}$ to $J\Omega^{i+1}_{X/Y}$, so that $J\Omega^{\cdot}_{X/Y}$ forms a subcomplex of $\Omega^{\cdot}_{X/Y}$. If the log stuctures M_X and M_X are coherent and J is a coherent sheaf of ideals, and f is of finite presentation, then each $J\Omega^i_{X/Y}$ is quasi-coherent.*

Proof: $\square$

The filtrations defined by sheaves of faces are more subtle. To motivate the constructions, consider first the case in which X is endowed with the log structure arising from a relative divisor with normal crossings on a smooth $\underline{X}$ over Y. Then $\underline{X} \to Y$ is also smooth, and we would like to understand the Leray spectral sequence of the map $j\colon X \to \underline{X}$. If j were also smooth, this could be done using the Koszul filtration associated with the morphism $\Omega^1_{\underline{X}} \to \Omega^1_{X/Y}$. Our construction is based on a modification of this construction.

It seems more convenient in the calculations which follow to use additive notation for the monoid law of M_X. Hence we write λ for the inclusion $\mathcal{O}^*_X \to M_X$ and d instead of $dlog$ for the map $M_X \to \Omega^1_{X/Y}$.

Definition 1.5.5 *Let $f\colon X \to Y$ be a morphism of log schemes and let F be a sheaf of faces in M_X. If m is a local section of M_X, let $F\langle m\rangle$ denote the sheaf of faces of M_X generated by F and m.*

1. *$\tilde{L}^i(F)\Omega^j_{X/S} \subseteq \Omega^j_{X/S}$ is the subsheaf of abelian groups generated by the local sections of the form $\alpha(m_0)dm_1 \wedge \cdots dm_j$ such that at least i of the elements $(m_1, \cdots m_j)$ belong to $F\langle m_0\rangle$.*
2. *$L^i(F)\Omega^j_{X/S} := \tilde{L}^{i+j}(F)\Omega^j_{X/S}$, and $L := L(\mathcal{O}^*_X)$;*
3. *$\underline{\Omega}^j_{X/S}(F) := L^0(F)\Omega^j_{X/S}$, and $\underline{\Omega}^j_{X/S} =: L^0\Omega^j_{X/S}$.*

Note that L is just the décalé [,] of the filtration $\tilde{L}$. We shall see that if X is the log scheme associated to a toric variety over a field (**??**), $\underline{\Omega}^i_{X/Y}$ is the sheaf of differentials defined by Danilov [].

Remark 1.5.6 In fact, $F\langle m_0\rangle$ is the set of sections m of M_X such that there exists $k \in \mathbf{N}$ and $f \in F$ with $km_0 + f \geq m$. Hence $\tilde{L}^i(F)\Omega^j_{X/S} \subseteq \Omega^j_{X/S}$ is the subsheaf of abelian groups generated by the local sections of the form $\alpha(m_0)dm_1 \wedge \cdots dm_j$ such that there exist a $k \in \mathbf{N}$ and $f \in F$ with $km + f \geq m_1 + \cdots m_i$;

Proposition 1.5.7 *Let F be a sheaf of faces in M_X.*

1. *$L^i(F)\Omega^j_{X/Y}$ (resp. $\tilde{L}^i(F)\Omega^j_{X/Y}$) is a sheaf of $\mathcal{O}_X$-submodules of $\Omega^j_{X/Y}$ containing the image of $\Omega^j_{\underline{X}/\underline{Y}}$ (resp, if $i \leq j$).*
2. *The exterior derivative maps $\tilde{L}^i(F)\Omega^j_{X/Y}$ to $\tilde{L}^{i+1}(F)\Omega^{j+1}_{X/Y}$ and $L^i(F)\Omega^j_{X/Y}$ to $L^i(F)\Omega^{j+1}_{X/Y}$.*
3. *The exterior product maps $\tilde{L}^i(F)\Omega^j_{X/Y} \times \tilde{L}^{i'}(F)\Omega^{j'}_{X/Y}$ to $\tilde{L}^{i+i'}(F)\Omega^{j+j'}_{X/Y}$ and $L^i(F)\Omega^j_{X/Y} \times L^{i'}(F)\Omega^{j'}_{X/Y}$ to $L^{i+i'}(F)\Omega^{j+j'}_{X/Y}$*
4. *Interior multiplication by an element of $T_{X/Y}$ maps $\tilde{L}^i(F)\Omega^j_{X/Y}$ to $\tilde{L}^{i-1}(F)\Omega^{j-1}_{X/Y}$ and $L^i(F)\Omega^j_{X/Y}$ to $L^i\Omega^{j-1}_{X/Y}$.*
5. *Suppose f is locally of finite presentation, that X and Y are fine, and that $F \subseteq M_X$ is relatively coheren. Then $L^i(F)\Omega^j_{X/Y}$ and $\tilde{L}^i(F)\Omega^j_{X/Y}$ are quasi-coherent.*

Proof: Any element of $\tilde{L}^i(F)\Omega^j_{X/Y}$ is a sum of elements of the form $\omega := \alpha_X(m_0)dm_1\wedge\cdots dm_j$, wheren $(m_0, m_1, \cdots m_j)$ is a sequence of sections of M_X such that there exist $k \in \mathbf{N}$ and $f \in F$ with $km_0+f \geq m_1+\cdots m_i$. If m is any section of M_X, $k(m_0+m)f \geq m_1 + \cdots m_i$, and hence $\alpha_X(m)\omega$ also belongs to $\tilde{L}^i(F)\Omega^j_{X/Y}$. In particular, $\tilde{L}^i(F)\Omega^j_{X/Y}$ is stable under multiplication by sections of $\mathcal{O}^*_X$. Since any section of $\mathcal{O}_X$ is a locally a sum of sections of $\mathcal{O}^*_X$ and $\tilde{L}^i(F)\Omega^j_{X/Y}$ is a subgroup of $\Omega^j_{X/Y}$, it follows that $\tilde{L}^i(F)\Omega^j_{X/Y}$ is stable under multiplication by $\mathcal{O}_X$, and *i.e.* is an $\mathcal{O}_X$-submodule. Furthermore,

$$d\omega = \alpha(m_0)dm_0 \wedge dm_1 \cdots \wedge dm_j,$$

and since $(k+1)m_0 + f \geq m_0 + \cdots m_i$, we see that $d\omega \in L^{i+1}(F)\Omega^{j+1}_{X/Y}$. If $(m'_0, m'_1, \cdots m'_{j'})$ is another sequence of sections and $k'm'_0 + f' \geq m'_1 + \cdots m'_{i'}$, then $\omega' =: \alpha_X(m'_0)dm_1 \wedge \cdots dm_{j'}$ is a typical element of $\tilde{L}^{i'}(F)\Omega^{j'}_{X/Y}$, and since $(k+k')(m_0 + m'_0) + f + f' \geq m_1 + \cdots m_i + m'_1 + \cdots m'_i$, we see that $\omega \wedge \omega' \in \tilde{L}^{i+i'}(F)\Omega^{j+j'}_{X/Y}$. Note that $\Omega^1_{\underline{X}/\underline{Y}}$ is generated by sections of the form $u^{-1}du = d\lambda(u)$ for $u \in \mathcal{O}^*_X$, and since $\lambda(u) \leq \lambda(1)$, the image of each of these in $\Omega^1_{X/Y}$ in fact belongs to $L^1(F)\Omega^1_{X/Y}$. It follows that $\tilde{L}^j(F)\Omega^j_{X/Y}$ contains the image of $\Omega^j_{\underline{X}/\underline{Y}} \to \Omega^j_{X/Y}$. If $\omega := \alpha_X(m_0)dm_1 \wedge \cdots dm_j$ with $km_0 + f \geq m_1 + \cdots m_i$, and if θ is a section of $\mathrm{Hom}(\Omega^1_{X/Y}, \mathcal{O}_X)$, then interior multiplication by θ takes ω to

$$\sum_r \alpha_X(m_0)(-1)^{r-1}\theta(dm_r)dm_1 \wedge \cdots d\hat{m}_r \wedge \cdots dm_j,$$

which evidently belongs to $L^{i-1}(F)\Omega^{j-1}_{X/Y}$.

Now suppose that X and Y are fine and F is relatively coherent. To prove that $L^i(F)\Omega^j_{X/Y}$ is quasi-coherent we may suppose that $X = \operatorname{Spec} A$ is affine, that $\beta\colon P \to M_X$ is a chart for M_X, and that $G \subseteq P$ is a relative chart for F. Let $\gamma =: \alpha_X \circ \beta$, let $E^{ij} =: \Gamma(X, L^i(F)\Omega^j_{X/Y})$, and let $\Omega^j =: \Gamma(X, \Omega^j_{X/Y})$. If $\tilde{E}^{ij}$ is the quasi-coherent sheaf associated with E^{ij}, we shall prove that the natural map $\tilde{E}^{ij} \to L^i(F)\Omega^j_{X/Y}$ is an isomorphism. Since $E^{ij} \subseteq \Omega^j$, $\tilde{E}^{ij} \subseteq \Omega^j_{X/Y}$, and so we need only prove the surjectivity. If $x \in X$, it will suffice to prove that the map $E^{ij} \otimes \mathcal{O}_{X,x} \to L^i(F)\Omega^j_{X/Y,x}$ is surjective. Suppose that $\omega = \alpha_X(m_0)dm_1 \wedge \cdots dm_j$, where the m_i's belong to $M_{X,x}$ and where $km_0 + f \geq m_1 + \cdots m_i$, with $f \in F_x$. For each $n = 1, \ldots j$ we can find a $u_n \in \mathcal{O}^*_{X,x}$ and a $p_n \in P$ such that $m_n = \beta(p_n) + \lambda(u_n)$, and we can also find $p' \in P$ and $v \in \mathcal{O}^*_{X,x}$ with $f = \lambda(v) + \beta(p')$. Since each u_n is a unit in $\mathcal{O}_{X,x}$, for each n there exists an element a_n of A which maps to a unit in $\mathcal{O}_{X,x}$ and an element ω_n of Ω^1_A such that $a_n d\log u_k$ is the image of ω_n in $\Omega^1_{X/Y,x}$. We can also find elements a_0 and b_0 of A mapping to units in $\mathcal{O}_{X,x}$ such that $(a_0)_x u_0 = (b_0)_x$. Now if a is the product of all the a_k's, we find that

$$\begin{aligned} a\omega &= (a_0\gamma(p_0)u_0)(a_1 d\beta(p_1) + a_1 d\log u_1) \wedge \cdots (a_j d\beta(p_j) + a_j d\log u_j) \\ &= \gamma(p_0)(b_0)_x(a_1 d\beta(p_1) + \omega_{1,x}) \wedge \cdots (a_j d\beta(p_j) + \omega_{j,x}) \end{aligned}$$

Let S denote the set of all elements of P which map to units in $M_{X,x}$ and let P_S be the localization of P by S. Then the map $\beta_x\colon P \to M_{X,x}$ factors through a map $P_S \to M_{X,x}$, and since $P_S \to M_{X,x}$ is still a chart, it follows that the induced map $\overline{P}_S \to \overline{M}_{X,x}$ is an isomorphism and that $P_S \to M_{X,x}$ is exact. Since $\beta_x(kp_0 + p') \geq \beta_x(p_1 + \cdots p_i)$, this relation must also hold in P_S, and hence there exists an $s \in S$ such that $kp_0 + p' + s \geq p_1 + \cdots p_i$ in P. Furthermore, the inverse image of F_x in P_S is the face generated by the image of G, and since p' maps to an element of this face, there exists an element s' of S such that $g =: s' + p' \in G$. Then $k(p_0 + s) + g = kp_0 + ks + s' + p' \geq p_1 + \cdots p_i$, and it follows that $\gamma(p_0+s)d\beta(p_1) \wedge \cdots d\beta(p_j)$ belongs to E^{ij}. By the same token, if $(p'_1, \ldots p'_{j'})$ is any subsequence of $(p_1, \ldots p_j)$, $\gamma(p_0+s)d\beta(p'_1) \wedge \cdots \beta(p'_{j'})$ belongs to $E^{i'j'}$, where $i' =: i-(j-j')$. We have

$$\gamma(s)a\omega = \gamma(p_0 + s)(b_0)_x(a_1 d\beta(p_1) + \omega_{1,x}) \wedge \cdots (a_j d\beta(p_j) + \omega_{j,x}),$$

and since each $\omega_{n,x}$ belongs to E^{11}, $\gamma(s)a\omega \in E^{ij}$. But $a\gamma(s)$ maps to a unit in $\mathcal{O}_{X,x}$, and hence ω is contained in the image of $E \otimes \mathcal{O}_{X,x}$. $\square$

1.6 The Cartier operator

It is not surprising, perhaps, that the logarithmic point of view makes the Cartier operator seem more natural.

Theorem 1.6.1 *Let $f: X \to Y$ be a morphism of fine log schemes in characteristic $p > 0$. Let F_X denote the absolute Frobenius endomorphism of X. Then there is a unique $\mathcal{O}_X$-linear morphism*

$$\sigma: \Omega^1_{X/Y} \to F_{X*}\underline{H}^1(\Omega^{\cdot}_{X/Y})$$

mapping $1 \otimes dlog\, m$ to the class of $dlog\, m$ for every $m \in M_X$. This extends uniquely to a family of morphisms

$$\sigma:^* \Omega^i_{X/Y} \to F_{X*}\underline{H}^i(\Omega^{\cdot}_{X/Y})$$

which is just the pth-power map when $i = 0$ and which is compatible with wedge product.

Proof: The uniqueness of σ on $\Omega^1_{X/Y}$ follows from the fact that $\Omega^1_{X/Y}$ is locally generated by the elements of the form $dlog\, m$ (**??**). Furthermore, one σ is defined on $\Omega^1_{X/Y}$, it evidently extends uniquely in a way compatible with wedge product and Frobenius. Thus we need only prove the existence, in degree 1. The existence depends on the following well-known lemma.

Lemma 1.6.2 *Let X/Y be a scheme, let f and g be sections of $\mathcal{O}_X$, and let p be a prime integer. Then $f^{p-1}df + g^{p-1}dg - (f+g)^{p-1}(df+dg)$ is exact.*

Proof: It suffices to prove this when $X = \operatorname{Spec} \mathbf{Z}[x,y]$, $Y = \operatorname{Spec} \mathbf{Z}$, and $f = x, g = y$. There is a unique $z \in \mathbf{Z}[x,y]$ such that $(x+y)^p - x^p - y^p = pz$. Then $(x+y)^{p-1}(dx+dy) - x^{p-1}dx - y^{p-1}dy = dz$. $\square$

The lemma implies the map $D\colon \mathcal{O}_X \to \underline{H}^1 F_{X*}(\Omega^\cdot_{X/Y})$ sending f to the cohomology class of $F_{X*}(f^{p-1}df)$ is a group homomophism. For $m \in M_X$, let $\delta(m)$ be the class of $F_{X*}(\delta(m))$ in $\underline{H}^1(F_{X*}\Omega^\cdot_{X/Y})$. Then δ defines a homomorphism of sheaves of monoids $M_X \to H^1(F_{X*}\Omega^\cdot_{X/Y})$, which evidently annihilates $f^{-1}M_Y$. We claim that (D,δ) is a log derivation of X/Y with values in $\underline{H}^1(F_{X*}\Omega^\cdot_{X/Y})$. According to (**??**), it suffices to verify that $D\alpha_X(m) = \alpha_X(m)\delta(m)$ for every $m \in M_X$. In fact, writing $[\omega]$ for the cohomology class of ω, we have:

$$\begin{aligned} D\alpha_X(m) &= [F_{X*}(\alpha_X(m)^{p-1}d\alpha_X(m))] \\ &= [F_{X*}(\alpha^p_X(m)dlog\, m] \\ &= \alpha_X(m)[F_{X*}(dlog\, m)] \\ &= \alpha_X(m)\delta(m) \end{aligned}$$

as required.

By the universal property of $\Omega^1_{X/Y}$, there is a unique $\mathcal{O}_X$-linear map

$$\Omega^1_{X/Y} \to H^1(F_{X*}\Omega^\cdot_{X/Y}) \cong F_{X*}H^1(\Omega^\cdot_{X/Y})$$

sending dm to $\delta(m)$ for all $m \in M_X$, and σ is the adjoint to this map. $\square$

In positive characteristic p, the sheaf $T_{X/Y}$ of derivations is not just a LIe algebra, but also a restricted Lie algebra []. We shall see that this is also true for logarithmic derivations. The proof uses the following formula, valid in any characteristic, which was made possible by help from Hendrik Lenstra and the marvelous book [17].

Lemma 1.6.3 *Let $f\colon X \to Y$ be a morphism of coherent log schemes, let (D,δ) be an element of $Der_{X/Y}(\mathcal{O}_X)$, and let m be a section of M_X. Then for each positive integer n,*

$$D^n(\alpha(m)) = \alpha_X(m) \sum_{\pi \in P_n} \prod_{s \in \pi} D^{|s|-1}\delta(m),$$

where P_n is the set of partitions of the set $\{1, \dots n\}$.

Proof: Let $\delta_1 := \delta$, and for $n > 1$ define δ_n inductively by

$$\delta_n(m) := \delta(m)\delta_{n-1}(m) + D\delta_{n-1}(m).$$

Then when $n = 1$, it follows from the definition of a log derivation that

$$D^n\alpha_X(m) = \alpha_X(m)\delta_n(m)$$

for any $m \in M_X$. If the above equation holds for n, then

$$\begin{aligned} D^{n+1}\alpha_X(m) &= D\alpha_X(m)\delta_n(m) + \alpha_X(m)D\delta_n(m) \\ &= \alpha_X(m)\delta(m)\delta_n(m) + \alpha_X(m)D\delta_n(m) \\ &= \alpha_X(m)\delta_{n+1}(m). \end{aligned}$$

Thus $D^n\alpha_X(m) = \alpha_X(m)\delta_n(m)$ for all n, and it remains to prove that

$$\delta_n = \sum_{\pi \in P_n} \prod_{s \in \pi} D^{|s|-1} \circ \delta$$

for every n. This is trivial for $n = 1$ and we proceed by induction on n.

For each $\pi \in P_n$, let π^* be the partition of $\{1, \ldots n+1\}$ obtained by adjoining $\{n+1\}$ to π, and for each pair (s, π) with $\pi \in P_n$ and $s \in \pi$, let π_s be the partition of $\{1, \ldots n+1\}$ obtained by adding $n+1$ to s. Let $P_n^* := \{\pi^* : \pi \in P_n\}$ and $P_\pi^* := \{\pi_s : s \in \pi\}$. In this way we obtain all the partitions of $\{1, \ldots n+1\}$, and so P_{n+1} can be written as a disjoint union of sets

$$P_{n+1} = P_n^* \bigsqcup \{P_\pi^* : \pi \in P_n\}.$$

By the definition of δ_n and the product rule,

$$\begin{aligned} \delta_{n+1} &= \delta \cdot \delta_n + D \circ \delta_n \\ &= \delta \cdot \sum_{\pi \in P_n} \prod_{s \in \pi} D^{|s|-1} \circ \delta + D \circ \sum_{\pi \in P_n} \prod_{s \in \pi} D^{|s|-1} \circ \delta \\ &= \sum_{\pi \in P_n} \prod_{s \in \pi} \delta D^{|s|-1} \circ \delta + \sum_{\pi \in P_n} \sum_{s' \in \pi \setminus \{s\}} \prod_{s \in \pi} (D^{|s|} \circ \delta)(D^{|s'|-1} \circ \delta) \\ &= \sum_{\pi \in P_n} \prod_{t \in \pi^*} D^{|t|-1} \circ \delta + \sum_{t \in \pi \in P_n} \prod_{t \in \pi_s} D^{|t|-1} \circ \delta \\ &= \sum_{\pi \in P_{n+1}} \prod_{t \in \pi} D^{|s|-1} \circ \delta \end{aligned}$$

$\square$

Proposition 1.6.4 *Let $f: X \to Y$ be a morphism of coherent log schemes in characteristic p. Then $T_{X/Y}$ has the structure of a restricted Lie algebra, with pth power operator defined by*

$$(D, \delta)^{(p)} = (D^p, F_X^* \circ \delta + D^{p-1} \circ \delta).$$

Proof: If π is any element of P_n and $|\pi| = r$, choose an ordering $(s_1, s_2, \ldots s_r)$ of π with $|s_1| \geq |s_2| \ldots |s_r|$, and let $I(\pi) =: (s_1|, |s_2|, \ldots |s_r|)$. Then $I(\pi)$ is independent of the chosen ordering, and $\pi \mapsto I(\pi)$ is a function from P_n to the set of finite sequences I of positive integers. Its (nonempty) fibers are exactly the orbits of P_n under the natural action of the symmetric group S_n. For each sequence I, let $c(I) =: |\{\pi \in P_n : I(\pi) = I\}|$. Then the formula of (1.6.3) be rewritten

$$D^n \alpha_X(m) = \alpha_X(m) \sum_I c(I) \prod_j D^{I_j - 1} \delta(m).$$

The cyclic group $\mathbf{Z}/n\mathbf{Z}$ acts on P_n through its inclusion in S_n; it is clear that the only elements of P_n fixed under this action are the two trivial partitions, with n elements and with 1 element, respectively. In particular, if $n = p$ is prime, all the other orbits have cardinality divisible by p. Thus modulo p the formula reduces to

$$D^p \alpha_X(m) = \alpha_X(m) \delta(m)^p + \alpha_X(m) D^{p-1} \delta(m).$$

Let

$$\delta^{(p)}(m) := \delta(m)^p + D^{p-1}\delta(m) = (F_X^* \circ \delta + D^{p-1} \circ \delta)(m)$$

Then $\delta^{(p)}: M_X \to \mathcal{O}_X$ is a homomorphism of monoids and $(D^p, \delta^{(p)})$ is a logarithmic derivation.

$\partial^{(p)} =: (D^p, \delta^{(p)})$ is again a logarithmic derivation.

???? Furthermore, the axioms for a restricted lie algebra, as well as **??**, will hold, by the general formula of Hochschild [11, Lemma 1].

□

Proposition 1.6.5 *Let X/Y be a morphism of fine log schemes in characteristic $p > 0$. Then there is a unique $\mathcal{O}_X$-bilinear pairing:*

$$\mathcal{C}: T_{X/Y} \times \underline{H}^1(F_{X*}(\Omega^{\cdot}_{X/Y})) \to F_{X*}(\mathcal{O}_X)$$

sending a pair $(\partial, [F_{X*}\omega])$ to $\langle \partial^{(p)}, \omega\rangle - \partial^{p-1}\langle \partial, \omega\rangle$. If $\omega \in \Omega^1_{X/Y}$ and $\sigma(\omega)$ is the corresponding element of $\underline{H}^1(F_{X*}(\Omega^{\cdot}_{X/Y}))$, then

$$\mathcal{C}(\partial, \sigma(\omega)) = F_X^*\langle \partial, \omega\rangle.$$

Proof: Fix $\partial \in T_{X/Y}$ and define, for $\omega \in \Omega^1_{X/Y}$, $\mathcal{C}_\partial(\omega) := \langle \partial^{(p)}, \omega\rangle - D^{p-1}\langle \partial, \omega\rangle$. C_∂ is evidently additive in ω and linear over pth powers of sections of $\mathcal{O}_X$. Furthermore, if $f \in \mathcal{O}_X$, $C_\partial(df) = \partial^p(f) - \partial^{p-1}(\partial f) = 0$. This proves that $\mathcal{C}_\partial(\omega)$ depends only the cohomology class of ω and that the function $\mathcal{C}$ in the proposition is well-defined. To prove that $\mathcal{C}(\partial, \sigma(\omega)) = F_X^*\langle \partial, \omega\rangle$, note that both sides are $\mathcal{O}_X$-linear in ω, and so it suffices to prove the formula if $\omega = d\log m$. If $\partial = (D, \delta)$, then

$$\begin{aligned}
\mathcal{C}(\partial, \sigma(d\log m)) &= \mathcal{C}(\partial, F_{X*}(d\log m) \\
&= \langle \partial^{(p)}, d\log m\rangle - D^{p-1}\langle \partial, d\log m\rangle \\
&= \delta(p)(m) - D^{p-1}\delta(m) \\
&= F_X^*(\delta(m) + D^{p-1}\delta(m) - D^{p-1}(\delta(m) \\
&= F_X^*(\delta(m))
\end{aligned}$$

This formula also proves that the pairing $\mathcal{C}$ is additive in ∂, at least on the image of σ. For the proof in the general case..... □

Remark 1.6.6 The pairing defined in (**??**) induces an $\mathcal{O}_X$-linear map

$$F_{X*}\underline{H}^1(\Omega^{\cdot}_{X/Y}) \to \mathrm{Hom}_{\mathcal{O}_X}(T_{X/Y}, F_{X*}(\mathcal{O}_X).$$

If $\Omega^1_{X/Y}$ is locally free, the target of the above arrow can be canonically identified with $F_{X*}(\Omega^1_{X/Y}$, and so $\mathcal{C}$ can be identified with a map

$$\mathcal{C}\colon F_{X*}\underline{H}^1(\Omega^{\cdot}_{X/Y}) \to F_{X*}(\Omega^1_{X/Y}).$$

This is the log version of the classical , and the formula of (**??**) shows that it is inverse to σ.

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