In this case, the flag variety is, under the notation in Corollary 2.2.3, $\{(W_i)_{i\in\mathbb{Z}};\ W_i \in \mathrm{Grass}^i(V),\ W_i \subset W_{i+1}\}$.

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LOGARITHMIC STRUCTURES OF FONTAINE-ILLUSIE

By KAZUYA KATO

Introduction. In this note, we present a general formulation of "logarithmic structure" on a scheme found by J. M. Fontaine and L. Illusie. Following their plan, we develop the theory of crystals with logarithmic poles using this logarithmic structure.

The logarithmic structure is "something" which gives rise to differentials with logarithmic poles, crystals and crystalline cohomology with logarithmic poles, . . . etc. For example, a reduced divisor with normal crossings on a regular scheme is such "something," and the logarithmic structure of Fontaine and Illusie is a natural generalization of this example to arbitrary schemes. Their logarithmic structure is defined to be a sheaf of commutative monoids M on the etale site X_{et} of a scheme X, endowed with a homomorphism $M \to \mathcal{O}_X$ satisfying a certain condition. (Cf. Section 1.) For X regular and D a reduced divisor with normal crossings on X, the corresponding M is the sheaf of regular functions on X which are invertible outside D. In general, the homomorphism $M \to \mathcal{O}_X$ is not assumed to be injective.

Algebraic geometry works especially well with smooth morphisms. We can regard the theory of toroidal embeddings as a theory of varieties with smooth logarithmic structures over a field (cf. (3.7)(1)). The logarithmic structure introduces a new range of smoothness, and we expect to have

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good algebraic geometry for smooth morphisms between logarithmic structures. In subsequent papers [HK] [K′], as was the motivation of Fontaine and Illusie, we apply our theory to schemes with semi-stable reduction, which are examples of schemes with smooth logarithmic structures over discrete valuation rings. (Cf. Complement 2 at the end of this note.)

I am very thankful to Fontaine and Illusie for the original definition of the logarithmic structure, their permission for me to develop their theory in this paper, advice and discussions. I was studying originally crystals with log. poles for regular schemes and reduced divisors with normal crossings, and I wished to write a note on log. str.'s of Fontaine-Illusie to know the best formulation of crystals with logarithmic poles.

Discussions between Illusie and M. Raynaud, and between Illusie and P. Deligne gave good influences to the theory.

The theory of logarithmic structures and crystals with logarithmic poles was developed independently by G. Faltings, and some parts of his papers [Fa$_1$] [Fa$_2$] overlap with our study. Our formulation is different from his (cf. Complement 1) and not covered by his theory. The theory of de Rham-Witt complex with logarithmic poles was considered by O. Hyodo [H$_1$] [H$_2$] and by M. Gros (unpublished). The theory of N. Katz [K] on connections with logarithmic poles was the guide for our theory.

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1. Logarithmic structures. In this note, a monoid (resp. a ring) means a commutative monoid (resp. ring) with a unit element. A homomorphism of monoids (resp. rings) is required to preserve the unit elements.

For a monoid M, M^{gp} denotes the associated group $\{ab^{-1}; a, b \in M\}$; $ab^{-1} = cd^{-1} \Leftrightarrow sad = sbc$ for some $s \in M$.

For a scheme X and $x \in X$ and for a sheaf $\mathfrak{F}$ on the etale site X_{et}, $\mathfrak{F}_{\bar{x}}$ denotes the stalk of $\mathfrak{F}$ at the separable closure $\bar{x}$ of x. In particular, $\mathcal{O}_{X,\bar{x}}$ denotes the strict henselization of $\mathcal{O}_{X,x}$.

(1.1). Pre-log. structures. Let X be a scheme. A pre-logarithmic structure on X is a sheaf of monoids M on the etale site X_{et} endowed with a homomorphism $\alpha : M \to \mathcal{O}_X$ with respect to the multiplication on $\mathcal{O}_X$.

A morphism $(X, M) \to (Y, N)$ of schemes with pre-log. str.'s is defined to be a pair (f, h) of a morphism of schemes $f : X \to Y$ and a homomorphism $h : f^{-1}(N) \to M$ such that the diagram

$$\begin{tikzcd} f^{-1}(N) \arrow[r, "h"] \arrow[d] & M \arrow[d] \\ f^{-1}(\mathcal{O}_Y) \arrow[r] & \mathcal{O}_X \end{tikzcd}$$

is commutative. (We use the notation f^{-1}, not f^*, for the inverse image of a sheaf, for we shall make a special use of the notation f^*, cf. (1.4).)

(1.2). Log. structures. A pre-logarithmic structure (M, α) is called a logarithmic structure if

$$\alpha^{-1}(\mathcal{O}_X^*) \cong \mathcal{O}_X^* \quad \text{via} \quad \alpha$$

where $\mathcal{O}_X^*$ denotes the group of invertible elements of $\mathcal{O}_X$. (We shall often identify $\alpha^{-1}(\mathcal{O}_X^*) \subset M$ with $\mathcal{O}_X^*$ via this isomorphism.) A morphism of schemes with log. str.'s is defined as a morphism of schemes with pre-log. str.'s.

(1.3). The log. str. associated to a pre-log. str. For a pre-log. str. (M, α) on X, we define its associated log. str. M^a to be the push out of

$$\begin{tikzcd} \alpha^{-1}(\mathcal{O}_X^*) \arrow[r] \arrow[d] & M \\ \mathcal{O}_X^* & \end{tikzcd}$$

in the category of sheaves of monoids on X_{et}, endowed with

$$M^a \to \mathcal{O}_X; \qquad (a, b) \mapsto \alpha(a)b \quad (a \in M,\ b \in \mathcal{O}_X^*).$$

Then, M^a is universal for homomorphisms of pre-log. str.'s from M to log. str.'s on X.

(*Remark*. If $G \xleftarrow{s} H \xrightarrow{t} M$ is a diagram of monoids with G a group, its push out is described as $(M \oplus G)/\sim$, where $(m, g) \sim (m', g') \Leftrightarrow$ there exist $h_1, h_2 \in H$ such that $mt(h_1) = m't(h_2)$, $gs(h_2) = g's(h_1)$.)

(1.4). The direct image and the inverse image. Let $f: X \to Y$ be a morphism of schemes. For a log. str. M on X, we define the log. str. on Y called the direct image of M, to be the fiber product of sheaves

$$\begin{tikzcd} & f_*(M) \arrow[d] \\ \mathcal{O}_Y \arrow[r] & f_*(\mathcal{O}_X). \end{tikzcd}$$

For a log. str. M on Y, we define the log. str. on X called the inverse image of M and denoted by $f^*(M)$, to be the log. str. associated to the pre-log. str. $f^{-1}(M)$ endowed with the composite map $f^{-1}(M) \to f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$. For a log. str. M on X and for a log. str. N on Y, the following three sets are canonically identified: The set of homomorphisms from N to the direct image of M, the set of homomorphisms from the inverse image of N to M, and the set of extensions of f to a morphism $(X, M) \to (Y, N)$.

The following facts concerning inverse images will be used frequently. Let M be a log. str. on Y.

(1.4.1). $f^{-1}(M/\mathcal{O}_Y^*) \cong (f^*M)/\mathcal{O}_X^*$.

(1.4.2). If M is the log. str. associated to a pre-log. str. M' on Y, $f^*(M)$ coincides with the log. str. associated to the pre-log. str. $f^{-1}(M') \to \mathcal{O}_X$.

(1.5). Examples of log. str.'s. (1) A standard example which we keep in mind is (X, M) where X is a regular scheme with a fixed reduced divisor D with normal crossings, and M is the log. str. on X defined as

$$M = \{g \in \mathcal{O}_X;\ g \text{ is invertible outside } D\} \subset \mathcal{O}_X.$$

The reason why we preferred the etale topology to the Zariski topology in this note is that the definition of "normal crossings" is etale local.

(2) For any scheme X, we call $M = \mathcal{O}_X^* \subset \mathcal{O}_X$ the trivial log. str. on X. This is the initial object in the category of log. str.'s on X. On the other hand, $M = \mathcal{O}_X$ is the final object in this category. The example (1.5)(1) is interpreted to be the direct image of the trivial log. str. on the open subscheme $X - D$.

(3) Let P be a monoid, X a scheme, and assume we are given a homomorphism $P \to \Gamma(X, \mathcal{O}_X)$, or equivalently $P_X \to \mathcal{O}_X$ where P_X denotes the constant sheaf on X corresponding to P. Then, let M be the log. str. associated to the pre-log. str. $P_X \to \mathcal{O}_X$. The log. str. of this type will play important roles in this paper. An interpretation of M is the following. For

a ring R, if $R[P]$ denotes the monoid ring on P over R, $\mathrm{Spec}(R[P])$ has a canonical log. str. associated to the canonical map $P \to R[P]$. The above log. str. M on X is the inverse image of the canonical log. str. on $\mathrm{Spec}(\mathbf{Z}[P])$ under the morphism $X \to \mathrm{Spec}(\mathbf{Z}[P])$. We mention what this M is, under a certain assumption.

Claim. In the above, if P has the property "$ab = ac \Rightarrow b = c$" and if X is a scheme over a ring R such that the induced morphism $X \to \mathrm{Spec}(R[P])$ is flat, then M is identified with the sub-monoid sheaf of $\mathcal{O}_X$ generated by $\mathcal{O}_X^*$ and P.

The proof of this claim will be given at the end of this section.

(1.6). Finite inverse limits. The category of schemes with log. str.'s has finite inverse limits. If (X_λ, M_λ) is a finite inverse system, the inverse limit is (X, M) where X is the inverse limit of the system of schemes X_λ, and M is obtained as follows. Let $p_\lambda : X \to X_\lambda$ be the projection, take the inductive limit M' of the inductive system of sheaves of monoids $p_\lambda^{-1}(M_\lambda)$, and then let M be the log. str. associated to the pre-log. str. $M' \to \mathcal{O}_X$.

(1.7). Logarithmic differentials. Let $\alpha : M \to \mathcal{O}_X$ and $\beta : N \to \mathcal{O}_Y$ be pre-log. str.'s, and let $f : (X, M) \to (Y, N)$ be a morphism. Then, we define the $\mathcal{O}_X$-module $\Omega^1_{X/Y}(\log(M/N))$, which is denoted simply by $\omega^1_{X/Y}$ for simplicity when there is no risk of confusion about the pre-log. str.'s, to be the quotient of

$$\Omega^1_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbf{Z}} M^{gp})$$

($\Omega^1_{X/Y}$ is the usual relative differential module) divided by the $\mathcal{O}_X$-submodule generated locally by local sections of the following forms.

(i) $(d\alpha(a), 0) - (0, \alpha(a) \otimes a)$ with $a \in M$.
(ii) $(0, 1 \otimes a)$ with $a \in \mathrm{Image}(f^{-1}(N) \to M)$.

The class of $(0, 1 \otimes a)$ for $a \in M$ in $\omega^1_{X/Y}$ is denoted by $d\log(a)$.

It is easily seen that if M^a and N^a denote the associated log str.'s, respectively, we have

$$\Omega^1_{X/Y}(\log(M/N)) = \Omega^1_{X/Y}(\log(M^a/N)) = \Omega^1_{X/Y}(\log(M^a/N^a)).$$

If M and N are log. str.'s, we have a surjection

$$\mathcal{O}_X \otimes_{\mathbf{Z}} M^{gp} \to \omega^1_{X/Y}; \qquad a \otimes b \to a.d\log(b),$$

and the kernel is the $\mathcal{O}_X$-submodule generated locally by local sections of the forms

(i) $\alpha(a) \otimes a - \Sigma_i u_i \otimes u_i$ with $a \in M$ and $u_i \in \mathcal{O}_X^*$ such that $\alpha(a) = \Sigma_i u_i$,

(ii) $1 \otimes a$ with $a \in \mathrm{Image}(f^{-1}(N) \to M)$.

If we have a cartesian diagram of schemes with log. str.'s

$$\begin{tikzcd} (X', M') \arrow[r, "f"] \arrow[d] & (X, M) \arrow[d] \\ (Y', N') \arrow[r] & (Y, N), \end{tikzcd}$$

we have an isomorphism

$$f^*\omega^1_{X/Y} \cong \omega^1_{X'/Y'}.$$

(1.8). For example, let P and Q be monoids, $Q \to P$ a homomorphism, R a ring, $X = \mathrm{Spec}(R[P])$, $Y = \mathrm{Spec}(R[Q])$, and endow X and Y with the canonical log. str.'s (1.5)(3), respectively. Then,

$$\mathcal{O}_X \otimes_{\mathbf{Z}} (P^{gp}/\mathrm{Image}(Q^{gp})) \cong \omega^1_{X/Y}; \qquad a \otimes b \mapsto a.d\log(b).$$

(1.9). In the situation (1.7), we define $\omega^{\cdot}_{X/Y}$ to be the exterior algebra on the $\mathcal{O}_X$-module $\omega^1_{X/Y}$. It becomes a complex of $f^{-1}(\mathcal{O}_Y)$-modules in the natural way.

(1.10) Proof of the Claim in (1.5)(3). The problem is the injectivity of $M \to \mathcal{O}_X$. By the description of the push out in Remark in (1.3), it suffices to prove the following: If $x \in X$, a, $b \in P$ and $ab^{-1} \in \mathcal{O}^*_{X,\bar{x}}$, then there exists c, $d \in P$ such that c, $d \in \mathcal{O}^*_{X,\bar{x}}$ and $ac = bd$. (Note an element of P is a nonzero-divisor on X by the flatness assumption, and hence the expression $ab^{-1} \in \mathcal{O}^*_{X,\bar{x}}$ makes sense.) Let $\mathfrak{p}$ be the image of x in $\mathrm{Spec}(R[P])$. Then, since $R[P]_{\mathfrak{p}} \to \mathcal{O}_{X,\bar{x}}$ is faithfully flat, we have $ab^{-1} \in R[P]^*_{\mathfrak{p}}$. Hence $\exists f$, $g \in R[P]$ which are not contained in the prime ideal $\mathfrak{p}$ such that $af = bg$ in

$R[P]$. Write $f = \Sigma_c f_c c$, $g = \Sigma_c g_c c$, where $c \in P$, f_c, $g_c \in R$. Take $c \in P$ such that f_c, $c \notin \mathfrak{p}$. Then, the equation $af = bg$ shows that there exists $d \in P$ such that $ac = bd$. Since $c \notin \mathfrak{p}$ and $ab^{-1} \in R[P]_{\mathfrak{p}}^{*}$, we have $d \notin \mathfrak{p}$.

2. Fine log. structures.

(2.1). A log. str. M on a scheme X is called quasi-coherent (resp. coherent) if etale locally on X, there exists a monoid (resp. finitely generated monoid) P and a homomorphism $P_X \to \mathcal{O}_X$ whose associate log. str. is isomorphic to M.

If $(X, M) \to (Y, N)$ is a morphism of schemes with quasi-coherent log. str.'s, $\omega^1_{X/Y}$ (1.7) is a quasi-coherent $\mathcal{O}_X$-module. If furthermore M is coherent and X is noetherian and locally of finite type over Y, it is a coherent $\mathcal{O}_X$-module.

(2.2). A monoid is called integral if "$ab = ac \Rightarrow b = c$" holds. A log. str. M on a scheme X is called integral if M is a sheaf of integral monoids.

(2.3). We call a log. str. "fine" if it is coherent and integral.

In this note, we consider mainly fine logarithmic structures. (To my experience, nonintegral log. str.'s are too much pathological.)

(2.4). The following facts are proved easily.

(2.4.1). If $f : X \to Y$ is a morphism and M is a quasi-coherent (resp. coherent, resp. integral) log. str. on Y, so is $f^*(M)$.

(2.4.2). A quasi-coherent (resp. coherent) log. str. M on a scheme X is integral if and only if etale locally on X, M is isomorphic to the log. str. associated to the pre-log. str. $P_X \to \mathcal{O}_X$ for some integral (resp. finitely generated integral) monoid P.

(2.4.3). If M is coherent (resp. integral), the stalk $M_{\bar{x}}/\mathcal{O}^*_{X,\bar{x}}$ is a finitely generated (resp. integral) monoid for any $x \in X$.

Example (2.5). (1) The log. str. on a regular scheme X corresponding to a reduced divisor with normal crossings (1.5)(1) is fine. Indeed, etale locally on X, write $D = \cup_i$ "$\pi_i = 0$" where "$\pi_i = 0$" are regular closed subschemes of X. Then, M is associated to the pre-log. str.

$$\mathbf{N}^r \to \mathcal{O}_X; \qquad (n_i)_{1 \leq i \leq r} \mapsto \prod_i \pi_i^{n_i}$$

and the monoid $\mathbf{N}^r$ is clearly finitely generated and integral.

(2) If $X = \operatorname{Spec}(k)$ for an algebraically closed field k, there is a bijection between the two sets

{isomorphism classes of integral log. str.'s on X}

{isomorphism classes of integral monoids having no invertible element other than the unit element}

given in the following way. For an integral monoid P as above, the corresponding log. str. is $M = \mathcal{O}_X^* \oplus P$ with

$$M \to \mathcal{O}_X; \qquad (a, b) \to a \quad (\text{resp. } 0) \quad \text{if} \quad b = 1 \quad (\text{resp. } b \neq 1).$$

PROPOSITION (2.6). *The inverse limit of a finite inverse system of schemes with* log. str.'s (1.6) *is coherent if each member of the system is coherent.*

Proof. It is enough to consider finite direct products and equalizers. By the description of finite inverse limits in (1.6), the case of finite direct products is clear, and for equalizers, it is enough to prove the following. Assume we are given two homomorphisms $g, h : M \to N$ of log. str.'s on X, let L' be the co-equalizer of (g, h) in the category of sheaves of monoids, and let L be the log. str. associated to L'. Then, if M, N are coherent, L is coherent. To see this, take finitely generated monoids P, Q and homomorphisms $s : P_X \to M$, $t : Q_X \to N$ which induce $(P_X)^a \cong M$, $(Q_X)^a \cong N$. If we have homomorphisms $g', h' : P \to Q$ compatible with g and h, respectively, then L is associated to the coequalizer of (g', h') in the category of monoids (which is endowed with the induced homomorphism to $\mathcal{O}_X$), and hence is coherent. In general we may not have such (g', h'), but considering the commutative diagram

$$\begin{tikzcd} P \arrow[r, "i_1(\text{resp. } i_2)"] \arrow[d] & Q' = P \oplus P \oplus Q \arrow[d, "t' = (g,h,t)"'] \\ M \arrow[r, "g(\text{resp. } h)"] & N \end{tikzcd}$$

($i_1(x) = (x, 0, 0)$, $i_2(x) = (0, x, 0)$), we see that it is enough to construct etale locally a finitely generated monoid Q'' and a factorization of t' as $Q' \to Q'' \to N$ such that $(Q''_X)^a \cong N$. Fix $x \in X$ and take a system of generators $(a_i)_{1 \leqq i \leqq r}$ of Q'. Then $t'(a_i)_{\bar{x}} = t'(b_i)_{\bar{x}} u_i$ for some $b_i \in Q$ and $u_i \in \mathcal{O}^*_{X,\bar{x}}$. Let Q'' be the monoid $(Q' \oplus \mathbf{N}^r)/\sim$ where $\sim$ is the relation generated by the relations $a_i = b_i e_i (1 \leqq i \leqq r)$ with $(e_i)_i$ the canonical base of $\mathbf{N}^r$. On an etale neighbourhood U of $\bar{x}$, we extend t' to $t'' : Q'' \to M$ by $e_i \mapsto u_i$. Then, $(Q_U)^a \to (Q''_U)^a$ is surjective and the composite $(Q_U)^a \to (Q''_U)^a \to M|_U$ is an isomorphism. Hence $(Q''_U)^a \cong M|_U$.

Proposition (2.7). *The inclusion functor from the category of schemes with fine* log. str.'s *to the category of schemes with coherent* log. str.'s *has a right adjoint.*

Proof. Let (X, M) be a scheme with a coherent log. str. We construct (X', M') over (X, M) with M' fine which is universal for morphisms from schemes with fine log. str.'s. We may work etale locally, and hence assume that we have $X \to \mathrm{Spec}(\mathbf{Z}[P])$ which induces M. Let

$$X' = X \times_{\mathrm{Spec}(\mathbf{Z}[P])} \mathrm{Spec}(\mathbf{Z}[P^{int}])$$

where $P^{int} = \mathrm{Image}(P \to P^{gp})$, and let M' be the log. str. on X' induced by $X' \to \mathrm{Spec}(\mathbf{Z}[P^{int}])$. It is easy to see that this (X', M') is universal.

We shall denote the above universal (X', M') by $(X, M)^{int}$.

(2.8). If (X_λ, M_λ) is a finite inverse system of schemes with fine log. str.'s and (X, M) is its inverse limit (1.6) in the category of schemes with log. str.'s, $(X, M)^{int}$ (2.7) is the inverse limit of (X_λ, M_λ) in the category of schemes with fine log. str.'s. Various properties of morphisms between schemes with fine log. str.'s defined in later sections (smoothness, etaleness, etc.) are preserved by base changes using the fiber products in the category of schemes with fine log. str.'s.

Definition (2.9). (1) For a scheme X with a fine log. str. M, a chart of M is a homomorphism $P_X \to M$ for a finitely generated integral monoid P which induces $(P_X)^a \cong M$.

A chart of M exists etale locally.

(2) For a morphism $f : (X, M) \to (Y, N)$ of schemes with fine log. str.'s, a chart of f is a triple $(P_X \to M, Q_Y \to N, Q \to P)$ where $P_X \to M$,

$Q_Y \to N$ are charts of M and N, respectively, and $Q \to P$ is a homomorphism for which

$$\begin{tikzcd} Q_X \arrow[r] \arrow[d] & P_X \arrow[d] \\ f^{-1}(N) \arrow[r] & M \end{tikzcd}$$

is commutative.

A chart of f also exists etale locally. This fact is deduced easily from

LEMMA (2.10). *Let X be a scheme with a fine* log. str. M, *let $x \in X$, G a finitely generated abelian group, and let $h : G \to M_{\bar{x}}^{gp}$ be a homomorphism such that $G \to M_{\bar{x}}^{gp}/\mathcal{O}_{X,\bar{x}}^*$ is surjective. Let $P = (h^{gp})^{-1}(M_{\bar{x}})$. Then, $P \to M_{\bar{x}}$ is extended to a chart $P_U \to M|_U$ for an etale neighbourhood U of $\bar{x}$.*

Proof. First, P is finitely generated since

$$(*) \qquad P/(\text{a subgroup}) \cong M_{\bar{x}}/\mathcal{O}_{X,\bar{x}}^*$$

and $M_{\bar{x}}/\mathcal{O}_{X,\bar{x}}^*$ is finitely generated. When we extend $P \to M_{\bar{x}}$ to a homomorphism $P_U \to M|_U$ for an etale neighbourhood U of $\bar{x}$, (*) proves $((P_U)^a)_{\bar{x}} \cong M_{\bar{x}}$. This shows that $(P_{U'})^a \cong M|_{U'}$ for an etale neighbourhood U' of $\bar{x} \to U$.

3. Smooth morphisms.

(3.1). We call a morphism of schemes with log. str.'s $i : (X, M) \to (Y, N)$ a closed immersion (resp. an exact closed immersion) if the underlying morphism of schemes $X \to Y$ is a closed immersion and $i^*N \to M$ is surjective (resp. an isomorphism).

(3.2). We shall often consider a commutative diagram of schemes with fine log. str.'s

$$\begin{tikzcd} (T', L') \arrow[r, "s"] \arrow[d, "i"'] & (X, M) \arrow[d, "f"] \\ (T, L) \arrow[r, "t"] & (Y, N) \end{tikzcd}$$

such that i is an exact closed immersion (3.1) and T' is defined in T by an ideal I such that $I^2 = (0)$.

(3.3). Smoothness and etaleness. A morphism $f : (X, M) \to (Y, N)$ of schemes with fine log. str.'s is called smooth (resp. etale) if the underlying morphism $X \to Y$ is locally of finite presentation and if for any commutative diagram as in (3.2), there exists etale locally on T (resp. there exists a unique) $g : (T, L) \to (X, M)$ such that $gi = s$ and $fg = t$.

A standard example of a smooth (resp. etale) morphism is given by the following (3.4). In (3.5), which is the main result of this section, we shall see that all smooth (resp. etale) morphisms are essentially of the type of this standard example.

PROPOSITION (3.4). *Let P, Q be finitely generated integral monoids, $Q \to P$ a homomorphism, R a ring, such that the kernel and the torsion part of the cokernel (resp. the kernel and the cokernel) of $Q^{gp} \to P^{gp}$ are finite groups whose orders are invertible in R. Let*

$$X = \mathrm{Spec}(R[P]), \qquad Y = \mathrm{Spec}(R[Q])$$

and endow them with the canonical log. str.'s *M and N, respectively. Then*, the morphism $(X, M) \to (Y, N)$ *is smooth* (*resp. etale*).

Proof. Consider a commutative diagram as in (3.2). Then, if we embed I in L via the injective homomorphism

$$I \to \mathcal{O}_T^* \subset L; \qquad x \mapsto 1 + x,$$

we have a cartesian diagram

(3.4.1)
$$\begin{tikzcd} L \arrow[r] \arrow[d] & L/I = L' \arrow[d] \\ L^{gp} \arrow[r] & L^{gp}/I = (L')^{gp} \end{tikzcd}$$

By the assumption on $Q^{gp} \to P^{gp}$, we have the following dotted arrow etale locally (resp. uniquely) which makes the diagram commutative;

$$\begin{tikzcd} (L')^{gp} & P^{gp} \arrow[l] \arrow[dl, dashed] \\ L^{gp} \arrow[u] & Q^{gp}. \arrow[l] \arrow[u] \end{tikzcd}$$

By the cartesian diagram (3.4.1), we obtain $P \to L$ which induces the desired morphism $(T, L) \to (X, M)$.

THEOREM (3.5). *Let* $f: (X, M) \to (Y, N)$ *be a morphism of schemes with fine* log. str.'s. *Assume we are given a chart* (2.9) $Q_Y \to N$ *of* N. *Then the following conditions* (3.5.1) *and* (3.5.2) *are equivalent.*

(3.5.1). f is smooth (resp. etale).

(3.5.2). Etale locally on X, there exists a chart $(P_X \to M, Q_Y \to N, Q \to P)$ of f (2.9) extending the given $Q_Y \to N$ satisfying the following conditions (i)(ii).

(i) The kernel and the torsion part of the cokernel (resp. The kernel and the cokernel) of $Q^{gp} \to P^{gp}$ are finite groups of orders invertible on X.

(ii) The induced morphism from $X \to Y \times_{\mathrm{Spec}(\mathbf{Z}[Q])} \mathrm{Spec}(\mathbf{Z}[P])$ is etale (in the classical sense).

Remark (3.6). The proof of (3.5) will show the following facts. We can require in the condition (3.5.2)(i) that $Q^{gp} \to P^{gp}$ is injective, without changing the conclusion of (3.5). In the part concerning the smoothness of f, we can replace the etaleness of the morphism from X to the fiber product in (3.5.2), by the smoothness (also in the classical sense), without changing the conclusion of (3.5).

Examples (3.7). (1) Let k be a field and let X be a scheme over k locally of finite type with a fine log. str. M. Then, by (3.5), (X, M) is smooth over $\mathrm{Spec}(k)$ if and only if etale locally on X, there exists a finitely generated integral monoid P and an etale morphism $X \to \mathrm{Spec}(k[P])$ satisfying the following conditions; $M = P\mathcal{O}_X^* \subset \mathcal{O}_X$, the torsion part of P^{gp} is of order invertible in k. Thus (X, M) corresponds to a toroidal embedding [KKMS] which is locally given by the open immersion

$$X \times_{\mathrm{Spec}(k[P])} \mathrm{Spec}(k[P^{gp}]) \subset X.$$

We assume P^{gp} is torsion free in the usual theory of toroidal embeddings, but essentially, the theory of toroidal embeddings is nothing but the theory of schemes with smooth fine log. str.'s over a field (with respect to the trivial log. str. on the base field).

(2) Let A be a discrete valuation ring, X a regular scheme over A such that etale locally on X, there is a smooth morphism $X \to$

$\mathrm{Spec}(A[T_1, \ldots, T_r]/(T_1 \cdots T_r - \pi))$ for $r \geq 1$ and a prime element π of A. (In this situation, X is called of semi-stable reduction over A.) Then, if M (resp. N) denotes the log. str. on X (resp. $\mathrm{Spec}(A)$) corresponding to the special fiber of X (resp. the closed point of $\mathrm{Spec}(A)$), which is a reduced divisor with normal crossings on a regular scheme, the morphism $(X, M) \to (\mathrm{Spec}(A), N)$ is smooth.

For the proof of (3.5), we use the following facts.

PROPOSITION (3.8). *Let $f : (X, M) \to (Y, N)$ be a morphism of schemes with fine* log. str.'s *such that $f^*N \xrightarrow{\cong} M$. Then f is smooth (resp. etale) if and only if the underlying morphism $X \to Y$ is smooth (resp. etale).*

Proof. Exercise.

PROPOSITION (3.9). *In* (3.2), *assume we are given one morphism $g : (T, L) \to (X, M)$ such that $gi = s$ and $fg = t$. Then there exists a bijection*

$$\{h : (T, L) \to (X, M);\ hi = s,\ fh = t\} \to \mathrm{Hom}_{\mathcal{O}_{T'}}(s^*\omega^1_{X/Y}, I)$$

which sends h to the homomorphism

$$da \to h^*(a) - g^*(a) \quad \text{for} \quad a \in \mathcal{O}_X,$$

$$d\log(a) \to u(a) - 1 \quad \text{for} \quad a \in M,$$

where $u(a)$ is the unique local section of $\mathrm{Ker}(\mathcal{O}_T^* \to \mathcal{O}_{T'}^*) \subset L$ *such that $h^*(a) = g^*(a)u(a)$.*

Proof. Exercise.

The proofs of the following (3.10) (3.12) are reduced to (3.9) in the same way as in the theory of the classical smoothness.

PROPOSITION (3.10). *Let $f : (X, M) \to (Y, N)$ be a smooth morphism of schemes with fine* log. str.'s. *Then the $\mathcal{O}_X$-module $\omega^1_{X/Y}$ is locally free of finite type.*

COROLLARY (3.11). *If $f : (X, M) \to (Y, N)$ is smooth in the diagram* (3.2), *a morphism $g : (T, L) \to (X, M)$ such that $gi = s$ and $fg = t$ exists whenever T is affine.*

Indeed, the obstruction to glueing local g lies in $H^1(T', \mathrm{Hom}_{\mathcal{O}_{T'}}(s^*\omega^1_{X/Y}, I)) = (0)$.

PROPOSITION (3.12). *Let* $(X, M) \xrightarrow{f} (Y, N) \xrightarrow{g} (S, L)$ *be morphisms of schemes with fine* log. str.'s, *and let*

$$f^*\omega^1_{X/S} \xrightarrow{s} \omega^1_{Y/S} \to \omega^1_{X/Y} \to 0$$

be the associated exact sequence. Consider the following conditions.

(i) f *is smooth* (*resp. etale*).

(ii) s *is injective and the image of* s *is locally a direct summand* (*resp.* s *is an isomorphism*).

Then, we have the implication (i) $\Rightarrow$ (ii). If gf is smooth, we have (ii) $\Rightarrow$ (i).

(3.13). *Proof of* (3.5). The implication (3.5.2) $\Rightarrow$ (3.5.1) follows from (3.4) (3.8).

We prove the converse. We construct P etale locally as follows.

Fix $x \in X$. Take elements $t_1, \ldots, t_r$ of $M_{\bar{x}}$ such that $(d\log(t_i))_{1 \leq i \leq r}$ is a basis of $\omega^1_{X/Y,\bar{x}}$ (3.10). Consider

$$\mathbf{N}^r \oplus Q \to M_{\bar{x}}$$

induced by $\mathbf{N}^r \to M_{\bar{x}}$; $(m_i) \to \Pi_i t_i^{m_i}$ and by $Q \to f^{-l}(N)_{\bar{x}} \to M_{\bar{x}}$. Note $M_{\bar{x}}/\mathcal{O}^*_{X,\bar{x}}$ is finitely generated. The map

$$\omega^1_{X/Y,\bar{x}} \to \kappa(\bar{x}) \otimes_{\mathbf{Z}} (M^{gp}_{\bar{x}}/(\mathcal{O}^*_{X,\bar{x}}\ \mathrm{Image}(f^{-1}(N)^{gp}_{\bar{x}}))$$

$$d\log(a) \mapsto 1 \otimes a \quad (a \in M_{\bar{x}})$$

shows that

$$\kappa(\bar{x}) \otimes_{\mathbf{Z}} (\mathbf{Z}^r \oplus Q^{gp}) \to \kappa(\bar{x}) \otimes_{\mathbf{Z}} (M^{gp}_{\bar{x}}/\mathcal{O}^*_{X,\bar{x}})$$

is surjective, and hence the cokernel of

$$\mathbf{Z}^r \oplus Q^{gp} \to M^{gp}_{\bar{x}}/\mathcal{O}^*_{X,\bar{x}}$$

is a finite group annihilated by an integer which is invertible in $\mathcal{O}_{X,\bar{x}}$. By using the fact that $\mathcal{O}^*_{X,\bar{x}}$ is n-divisible, we can easily construct a finitely generated abelian group $G \supset \mathbf{Z}^r \oplus Q^{gp}$ such that $G/(\mathbf{Z}^r \oplus Q^{gp})$ is annihilated by n and such that the map $\mathbf{Z}^r \oplus Q^{gp} \to M_{\bar{x}}^{gp}$ is extended to $h : G \to M_{\bar{x}}^{gp}$ which induces a surjection $G \to M_{\bar{x}}^{gp}/\mathcal{O}_{X,\bar{x}}^*$. Let $P = h^{-1}(M_{\bar{x}})$. Then $Q^{gp} \to P^{gp} = G$ is injective and the torsion part of P^{gp}/Q^{gp} is annihilated by n. We have

$$\mathcal{O}_{X,\bar{x}} \otimes_{\mathbf{Z}} (P^{gp}/Q^{gp}) \cong \omega^1_{X/Y,\bar{x}}.$$

By (3.10), by replacing X by an etale neighbourhood of $\bar{x}$, we have

$$(*) \qquad \mathcal{O}_X \otimes_{\mathbf{Z}} (P^{gp}/Q^{gp}) \cong \omega^1_{X/Y}.$$

Furthermore, by (2.10), $P \to M_{\bar{x}}$ is extended to a chart of $M|_U$ for some etale neighbourhood U of $\bar{x}$. By replacing X with U, we have a morphism g from X to the fiber product $X' = Y \times_{\mathrm{Spec}(\mathbf{Z}[Q])} \mathrm{Spec}(\mathbf{Z}[P])$. It remains to prove g is etale. To see this, endow X' with the inverse image of M' of the canonical log. str. of $\mathrm{Spec}(\mathbf{Z}[P])$. Since the inverse image of M' on X is M, it is sufficient (3.8) to show that $(X, M) \to (X', M')$ is etale. But this follows from (3.12) and $g^*\omega^1_{X'/Y} \cong \omega^1_{X/Y}$ ((1.8) and (*) above).

The theory of infinitesimal liftings for smooth morphisms hold in the logarithmic situation as follows. This (3.14) and the related theorem (4.12) were obtained following faithfully suggestions of L. Illusie.

PROPOSITION (3.14). *Let $f : (X, M) \to (Y, N)$ be a smooth morphism between schemes with fine* log. str.'s, *and let $i : (Y, N) \to (\tilde{Y}, \tilde{N})$ with N fine be an exact closed immersion* (3.2) *such that Y is defined in $\tilde{Y}$ by a nilpotent ideal I of $\mathcal{O}_{\tilde{Y}}$. Then we have:*

(1) *If X is affine, a smooth lifting of (X, M, f) exists and is unique up to isomorphism. Here, by a smooth lifting of (X, M, f), we mean a scheme $\tilde{X}$ with a fine* log. str. *$\tilde{M}$ endowed with a smooth morphism $\tilde{f} : (\tilde{X}, \tilde{M}) \to (\tilde{Y}, \tilde{N})$ and with an isomorphism*

$$g : (X, M) \cong (\tilde{X}, \tilde{M}) \times_{(\tilde{Y}, \tilde{N})} (Y, N) \quad \textit{over} \quad (Y, N).$$

(2) *Assume $I^2 = (0)$. Then, for a smooth lifting $(\tilde{X}, \tilde{M}, \tilde{f}, g)$ of (X, M, f), there exists a canonical isomorphism*

$$\mathrm{Aut}(\tilde{X}, \tilde{M}, \tilde{f}, g) = \mathrm{Hom}_{\mathcal{O}_X}(\omega^1_{X/Y}, I\mathcal{O}_{\tilde{X}}).$$

(3) *Assume $I^2 = (0)$. If we are given one fixed smooth lifting of (X, M, f), we have a bijection from the set of all isomorphism classes of smooth liftings of (X, M, f) to*

$$H^1(X, \mathcal{H}om_{\mathcal{O}_X}(\omega^1_{X/Y}, I\mathcal{O}_{\tilde{X}})).$$

(4) *A smooth lifting of (X, M, f) exists if $I^2 = (0)$ and if*

$$H^2(X, \mathcal{H}om_{\mathcal{O}_X}(\omega^1_{X/Y}, I\mathcal{O}_{\tilde{X}})) = (0).$$

Proof. Once we prove that a smooth lifting exists etale locally on X, the statements in (3.14) are deduced from it by the classical arguments as in SGA I Exposé 3. Etale locally we have a chart ($P_X \to M$, $Q_Y \to N$, $Q \to P$) of f satisfying the condition (3.5.2) such that $Q_Y \to N$ factors through a chart $Q_Y \to \tilde{N}$ of $\tilde{N}$. Let

$$X' = Y \times_{\mathrm{Spec}(\mathbf{Z}[Q])} \mathrm{Spec}(\mathbf{Z}[P]), \qquad \tilde{X}' = \tilde{Y} \times_{\mathrm{Spec}(\mathbf{Z}[Q])} \mathrm{Spec}(\mathbf{Z}[P]).$$

Lift the etale scheme X over X' to an etale scheme $\tilde{X}$ over $\tilde{X}'$ (this is classical; SGA I Exposé 1), and endow $\tilde{X}$ with the inverse image $\tilde{M}$ of the canonical log. str. on $\mathrm{Spec}(\mathbf{Z}[P])$. Then, $(\tilde{X}, \tilde{M}, \tilde{f}, g)$ with the evident definitions of $\tilde{f}$, g is a smooth lifting.

4. Several types of morphisms. We define integral morphisms (4.3), exact morphisms (4.6), and morphisms of Cartier type (4.8), describe their properties, and prove a Cartier isomorphism (4.12).

Proposition (4.1). (1) *Let $h : Q \to P$ be a homomorphism of integral monoids. Then, the following conditions* (i) *and* (iv) *are equivalent* (*resp.* (ii), (iii) *and* (v) *are equivalent*).

(i) *For any integral monoid Q' and for any homomorphism $g : Q \to Q'$, the push out of $P \leftarrow Q \to Q'$ in the category of monoids is integral.*

(ii) *The homomorphism $\mathbf{Z}[Q] \to \mathbf{Z}[P]$ induced by h is flat.*

(iii) *For any field k, the homomorphism $k[Q] \to k[P]$ induced by h is flat.*

(iv) *If $a_1, a_2 \in Q$, $b_1, b_2 \in P$ and $h(a_1)b_1 = h(a_2)b_2$, there exist a_3, $a_4 \in Q$ and $b \in P$ such that $b_1 = h(a_3)b$, $b_2 = h(a_4)b$, and $a_1a_3 = a_2a_4$.*

(v) *The condition* (iv) *is satisfied and h is injective.*

(2) *Let* $f:(X, M) \to (Y, N)$ *be a morphism of schemes with integral* log. str.'s. *Then, for* $x \in X$, *the conditions* (i)–(v) *in* (1) *for* $Q = (f^*N)_{\bar{x}}$ *and* $P = M_{\bar{x}}$ *are equivalent, and they are equivalent to each of* (i)–(v) *for* $Q = f^{-1}(N/\mathcal{O}_Y^*)_{\bar{x}}$ *and* $P = (M/\mathcal{O}_X^*)_{\bar{x}}$.

(4.2). *Proof of* (4.1). (1) We omit the proof of (4.1)(2) since it is easy by considering the conditions (4.1)(1)(iv) and (v).

(i) $\Rightarrow$ (iv). Let $a_1, a_2 \in Q$, $b_1, b_2 \in P$ and $h(a_1)b_1 = h(a_2)b_2$. Define $Q' = (Q \oplus \mathbf{N}^2)/\sim$, where $\sim$ is the equivalence relation

$$(c, m, n) \sim (c', m', n') \Leftrightarrow$$

$$m + n = m' + n' \quad \text{and} \quad c a_1^m a_2^n = c' a_1^{m'} a_2^{n'},$$

and let P' be the push out of $P \leftarrow Q \to Q'$. Since Q' is integral and (i) is satisfied, P' is also integral and we see from this that $(b_1, 1, 0)$ and $(b_2, 0, 1)$ coincide in P'. It follows that there exists a sequence $v_0, \ldots, v_r$ of elements of $P \oplus \mathbf{N}^2$ such that $v_0 = (b_1, 1, 0)$, $v_r = (b_2, 0, 1)$ and such that for each $i = 1, \ldots, r$, there exist $c, c' \in Q$, $m, n, m', n' \in \mathbf{N}$ and $w \in P \oplus \mathbf{N}^2$ satisfying $v_{i-1} = (h(c), m, n)w$, $v_i = (h(c'), m', n')w$, $m + n = m' + n'$, $c a_1^m a_2^n = c' a_1^{m'} a_2^{n'}$. As is easily seen, this implies that there exist $a_3, a_4 \in Q$ and $b \in P$ such that $b_1 = h(a_3)b$, $b_2 = h(a_4)b$ and $a_1 a_3 = a_2 a_4$.

(iv) $\Rightarrow$ (i). Let P' be the push out. We prove the surjection $P' \to (P')^{int}$ is bijective. Let $b_1, b_2 \in P$, $c_1, c_2 \in Q'$ and assume $b_1 c_1$ and $b_2 c_2$ coincide in $(P')^{int}$. Then, an easy observation on the push out of $P^{gp} \leftarrow Q^{gp} \to (Q')^{gp}$ shows that there exist $a_1, a_2 \in Q$ such that $h(a_1)b_1 = h(a_2)b_2$ in P and $g(a_2)c_1 = g(a_1)c_2$ in Q'. By the condition (iv), we have $b_1 = h(a_3)b$, $b_2 = h(a_4)b$, $a_1 a_3 = a_2 a_4$ for some $a_3, a_4 \in Q$, $b \in P$. We have $g(a_3)c_1 = g(a_4)c_2$ in Q', and hence we have in P' (not only in $(P')^{int}$),

$$b_1 c_1 = (h(a_3)b)c_1 = b(g(a_3)c_1) = b(g(a_4)c_2) = (h(a_4)b)c_2 = b_2 c_2.$$

(iii) $\Rightarrow$ (v). We show first h is injective. Let $a_1, a_2 \in Q$ and let k be any field. As is easily seen, the kernel of the multiplication by $a_1 - a_2$ on $k[Q]$ is generated, as an ideal, by elements of the form $\Sigma_{1 \leq i \leq n} c_i$ $(n \geqq 1, c_i \in Q)$ such that $a_1 c_i = a_2 c_{i+1}$ for $i = 1, \ldots, n-1$ and $a_1^n = a_2^n$. By the flatness of $k[Q] \to k[P]$, the images of these elements in $k[P]$ generates as an ideal, the kernel of the multiplication by $h(a_1) - h(a_2)$ on $k[P]$. If $h(a_1) = h(a_2)$, the above elements satisfy $h(c_1) = \cdots = h(c_n)$ and hence $k[P]$ is generated as an ideal by the elements $h(\Sigma_{1 \leq i \leq n} c_i) = nh(c_1)$. Hence n is invertible in any field k and hence $n = 1$, that is, $a_1 = a_2$.

Next assume $h(a_1)b_1 = h(a_2)b_2$, $a_1, a_2 \in Q$ and $b_1, b_2 \in P$. Let S be the kernel of

$$k[Q] \oplus k[Q] \to k[Q]; \qquad (f, g) \mapsto a_1 g - a_2 f.$$

By the flatness, the kernel T of

$$k[P] \oplus k[P] \to k[P]; \qquad (f, g) \mapsto h(a_1)g - h(a_2)f$$

is generated as a $k[P]$-module, by the image of S. Since $(b_1, b_2) \in T$, we can write

$$(*) \qquad b_1 = \sum_{1 \leq i \leq r} h(c_i)f_i, \qquad b_2 = \sum_{1 \leq i \leq r} h(d_i)f_i, \qquad a_1 c_i = a_2 d_i$$

for some $c_i, d_i \in k[Q]$, $f_i \in k[P]$ $(1 \leq i \leq r)$. The expression of b_1 in (*) shows that there are $a_3 \in Q$ and $b \in P$ and i such that a_3 appears in c_i, b appears in f_i, and $b_1 = h(a_3)b$. By $a_1 c_i = a_2 d_i$, there exist $a_4 \in Q$ which appear in d_i such that $a_1 a_3 = a_2 a_4$. We have $b_2 = h(a_4)b$ by

$$h(a_2)b_2 = h(a_1)b_1 = h(a_1 a_3)b = h(a_2)h(a_4)b.$$

(v) $\Rightarrow$ (ii). The $\mathbf{Z}[Q]$-module $\mathbf{Z}[P]$ becomes a filtered inductive limit of free $\mathbf{Z}[Q]$-modules which are direct sums of $\mathbf{Z}[Q]$-modules of the form $\mathbf{Z}[Q]b$ with $b \in P$.

Definition (4.3). Let $f : (X, M) \to (Y, N)$ be a morphism of schemes with integral log. str.'s. We say f is integral if for any $x \in X$, the equivalent conditions in (4.1)(2) are satisfied.

That f is integral (Resp. In the case M and N are fine, that f is integral) is equivalent to the following

(4.3.1). For any scheme Y' with an integral (resp. a fine) log. str. N' and for any $(Y', N') \to (Y, N)$, the log. str. of the fiber product $(X, M) \times_{(Y,N)} (Y', N')$ is integral.

(The implication "(4.1)(i) for $Q = (f^*N)_{\bar{x}}$ and $P = M_{\bar{x}}$ holds for any $x \in X \Rightarrow$ (4.3.1)" is proved easily. The implication "(4.3.1) $\Rightarrow$ (4.1)(iv) for $Q = (f^*N)_{\bar{x}}$ and $P = M_{\bar{x}}$ holds for any $x \in X$" is proved by the method of the proof of (iv) $\Rightarrow$ (i) given in (4.2).)

Corollary (4.4). *A morphism $f : (X, M) \to (Y, N)$ of schemes with integral* log. str.'s *is integral in each of the following cases*:

(i) *M is isomorphic to the inverse image of N.*

(ii) *For any $y \in Y$, the monoid $(N/\mathcal{O}_Y^*)_{\bar{y}}$ is generated by one element.*

Proof. For the case (ii), consider the condition (iv) in (4.1)(1).

For example, the morphism $(X, M) \to (\mathrm{Spec}(A), N)$ in (3.7)(2) (the semi-stable reduction situation) is integral.

Corollary (4.5). *If a morphism f of schemes with fine* log. str.'s *is smooth and integral, the underlying morphism $X \to Y$ is flat.*

Proof. By using (ii) of (4.1)(1), we can find etale locally a chart $(P_X \to M,\ Q_Y \to N,\ Q \to P)$ satisfying (3.5.2) such that $\mathbf{Z}[Q] \to \mathbf{Z}[P]$ is flat.

Definition (4.6). (1) We say a homomorphism of integral monoids $h : Q \to P$ is exact if $Q = (h^{gp})^{-1}(P)$ in Q^{gp} where $h^{gp} : Q^{gp} \to P^{gp}$.

(2) We say a morphism of schemes with integral log. str.'s $f : (X, M) \to (Y, N)$ is exact if the homomorphism $(f^*N)_{\bar{x}} \to M_{\bar{x}}$ is exact for any $x \in X$.

Following facts are proved easily. Exact morphisms are stable under composition. For fine log. str.'s, exact morphisms are stable under base changes in the category of schemes with fine log. str.'s in the sense of (2.8). An integral morphism is exact. If f is exact, the homomorphism $f^*N \to M$ is injective. (This last fact shows that a closed immersion (3.1) between schemes with integral log. str.'s is exact if and only if it is an exact closed immersion in the sense of (3.1).)

Now we consider characteristic p.

Definition (4.7). Let p be a prime number. For a scheme X over $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$ and a log. str. M on X, we define the absolute frobenius $F_{(X,M)} : (X, M) \to (X, M)$ as follows. The morphism of schemes underlying $F_{(X,M)}$ is the usual absolute frobenius $F_X : X \to X$, and the homomorphism $F_X^{-1}(M) \to M$ is the multiplication by p on M under the canonical identification of $F_X^{-1}(M)$ with M.

Definition (4.8). Let $f:(X, M) \to (Y, N)$ be a morphism of schemes with integral log. str.'s. Assume Y is a scheme over $\mathbf{F}_p$ with p a prime number. We say that f is of Cartier type if f is integral and the morphism $(f, F_{(X,M)})$ from (X, M) to the fiber product of

$$\begin{tikzcd} & (X, M) \arrow[d, "f"] \\ (Y, N) \arrow[r, "F_{(Y,N)}"] & (Y, N) \end{tikzcd}$$

is exact.

For example, f is of Cartier type if f has locally a chart of the form $Q = \mathbf{N}$, $P = \mathbf{N}^r (r \geqq 1)$, and $Q \to P$ is the diagonal map. (This happens in the semi-stable reduction situation (3.7)(2).)

Morphisms of Cartier type are stable under compositions and base changes.

(4.9). Let p be a prime number and let $f:(X, M) \to (Y, N)$ be a morphism of schemes with integral log. str.'s over $\mathbf{F}_p$. We say f is weakly purely inseparable if the following (i)–(iii) are satisfied.

(i) The map $X \to Y$ of underlying topological spaces is a homeomorphism.

(ii) For $x \in X$ and $a \in M_{\bar{x}}$, there exists $n \geqq 0$ such that $a^{p^n} \in \mathrm{Image}(f^{-1}(N)_{\bar{x}})$.

(iii) If $x \in X$ and if $a, b \in f^{-1}(N)_{\bar{x}}$ have the same image in $M_{\bar{x}}$, $a^{p^n} = b^{p^n}$ for some $n \geqq 0$.

We say f is purely inseparable if it is exact and weakly purely inseparable.

PROPOSITION (4.10). *Let $f:(X, M) \to (Y, N)$ be a morphism of schemes with fine* log. str.'s.

(1) *Assume that for any $x \in X$ and $a \in M_{\bar{x}}$, there exists $n \geqq 1$ such that $a^n \in \mathrm{Image}((f^*N)_{\bar{x}} \to M_{\bar{x}})$. Then, etale locally on X, f has a factorization $f = f'f''$ such that f' is an etale morphism of schemes with fine* log. str.'s *and f'' is exact.*

(2) *Assume Y is a scheme over $\mathbf{F}_p$ for a prime number p and f is weakly purely inseparable* (4.9). *Then f has a unique factorization $f = f'f''$ such that f' is an etale morphism between schemes with fine* log. str.'s *and f'' is purely inseparable* (4.9).

Proof. In the situation of (1) (resp. (2)), it is easily seen that etale locally on X, there exists a chart $(P_X \to M, Q_Y \to N, h : Q \to P)$ of f satisfying the following condition (i) (resp. (i) and (ii)).

(i) For any $a \in P$, there exists $n \geqq 1$ (resp. $n \geqq 0$) such that $a^n \in h(Q)$. (resp. $a^{p^n} \in h(Q)$.

(ii) For any $a, b \in Q$ such that $h(a) = h(b)$, there exists $n \geqq 0$ such that $a^{p^n} = b^{p^n}$.)

Let $Q' = (h^{gp})^{-1}(P)$ where $h^{gp} : Q^{gp} \to P^{gp}$, let

$$Y' = Y \times_{\mathrm{Spec}(\mathbf{Z}[Q])} \mathrm{Spec}(\mathbf{Z}[Q']),$$

and endow Y' with the inverse image N' of the canonical log. str. of $\mathrm{Spec}(\mathbf{Z}[Q'])$. Then, $f' : (Y', N') \to (Y, N)$ is etale by (3.5). We prove that $f'' : (X, M) \to (Y', N')$ is exact. Note that $Q' \to P$ is exact. Let $x \in X$, and write the homomorphisms $P \to M_{\bar{x}}$, $Q' \to (f''^{*}N')_{\bar{x}}$, $Q' \to P$, $(f''^{*}N')_{\bar{x}} \to M_{\bar{x}}$ by s, t, h', g, respectively. Our task is to prove that if a, $b \in (f''^{*}N')_{\bar{x}}$ and $g(a) \in g(b)M_{\bar{x}}$, then $a \in b(f''^{*}N')_{\bar{x}}$. We may assume $a = t(a_0)$, $b = t(b_0)$ for some $a_0, b_0 \in Q'$. We have $h'(a_0)c = h'(b_0)d$ for some $c, d \in P$ such that the image of c in $M_{\bar{x}}$ belongs to $\mathcal{O}^{*}_{X,\bar{x}}$. Take $n \geqq 1$ such that $c^n = h'(e)$, $e \in Q'$. Then, $h'(a_0 e) = h'(b_0)c^{n-1}d$. Since h' is exact, we have $a_0 e \in b_0 Q'$. Since $s(e) \in \mathcal{O}^{*}_{X,\bar{x}}$, we have $a \in b(f''^{*}N')_{\bar{x}}$. This completes the proof of (1). Furthermore, in the situation of (2), f'' is weakly purely inseparable as is easily seen, and we obtain the local existence of the factorization in (2). It remains to prove the uniqueness of the factorization in (2), from which the global existence follows from the local existence. Assume we have two factorizations $(X, M) \to (Y'_1, N'_1) \to (Y, N)$ and $(X, M) \to (Y'_2, N'_2) \to (Y, N)$ of f satisfying the condition stated in (2). Let $q_i : ((Y'_1, N'_1) \times_{(Y,N)} (Y'_2, N'_2))^{int} \to (Y'_i, N'_i)$ be the projections ($i = 1, 2$). Then, q_i is etale and purely inseparable. Hence q_i is an isomorphism by the following (4.11) (which we apply to the case where s and t are isomorphisms). This proves the uniqueness.

Lemma (4.11). *Let p be a prime number and let*

$$\begin{tikzcd} (T', L') \arrow[r, "s"] \arrow[d, "i"'] & (X, M) \arrow[d, "f"] \\ (T, L) \arrow[r, "t"] & (Y, N) \end{tikzcd}$$

be a commutative diagram of schemes with fine log. str.'s *over* $\mathbf{F}_p$ *such that* i *is purely inseparable, and such that* f *is etale. Then, there exists a unique morphism* $h : (T, L) \to (X, M)$ *such that* $hi = s$ *and* $fh = t$.

Proof. By taking a chart of f satisfying the condition (3.5.2), the proof proceeds just as the proof of (3.4).

THEOREM (4.12) (Cf. [DI], [Il$_2$].). *Let* p *be a prime number and let* $f : (X, M) \to (Y, N)$ *be a smooth morphism of schemes with fine* log. str.'s *over* $\mathbf{F}_p$. *Let* $f' : (X', M') \to (Y, N)$ *be the base change of* f *by the absolute frobenius* $F_{(Y,N)} : (Y, N) \to (Y, N)$, *let*

$$(X, M) \xrightarrow{F} (X', M') \to (X, M)$$

be the factorization of $F_{(X,M)}$ *characterized by the property* $f = f'F$, *and consider the factorization*

$$(X, M) \xrightarrow{g} (X'', M'') \xrightarrow{h} (X', M')^{int}$$

of $F^{int} : (X, M) \to (X', M')^{int}$ *given by* (4.10)(2).

(1) *Assume* f *is smooth. Let* s *be the composite morphism* $(X'', M'') \to (X', M') \to (X, M)$. *Then we have a canonical isomorphism of* $\mathcal{O}_{X''}$-modules

$$C^{-1} : \omega^q_{X''/Y} \to \mathcal{H}^q(\omega_{X/Y})$$

for any $q \in \mathbf{Z}$ characterized by

$$C^{-1}(a d\log(s^*(b_1)) \wedge \cdots \wedge d\log(s^*(b_q)))$$
$$= g^*(a) d\log(b_1) \wedge \cdots \wedge d\log(b_q)$$

$(a \in \mathcal{O}_{X''},\ b_1, \ldots, b_q \in M)$.

(2) *Assume* f *is smooth and integral. Assume we are given a scheme with a fine* log. str. $(\tilde{Y}, \tilde{N})$ *such that* $\tilde{Y}$ *is flat over* $\mathbf{Z}/p^2\mathbf{Z}$, *and an isomorphism*

$$(Y, N) \cong (\tilde{Y}, \tilde{N}) \times_{\mathrm{Spec}(\mathbf{Z}/p^2\mathbf{Z})} \mathrm{Spec}(\mathbf{F}_p)$$

where Spec($\mathbf{Z}/p^2\mathbf{Z}$) *and* Spec($\mathbf{F}_p$) *are endowed with trivial* log. str.'s. *Then, there exists a canonical bijection between the set of all isomorphism classes of smooth liftings of* (X'', M'') *over* $(\tilde{Y}, \tilde{N})$ (3.14) *and the set of all splittings of* $\tau \leqq_1 \omega^{\cdot}_{X/Y}$ *in the derived category of the category of* $\mathcal{O}_{X''}$*-modules* (*cf*. [DI] *Section* 3). *If a smooth lifting of* (X, M) *over* $(\tilde{Y}, \tilde{N})$ *exists, there is an isomorphism*

$$\tau_{<p}\, \omega^{\cdot}_{X/Y} \cong \bigoplus_{0\leqq i<p} \omega^{i}_{X''/Y}[-i]$$

in the derived category of the category of $\mathcal{O}''_X$*-modules.*

(3) *Assume the following* (i)–(iii).

(i) *f is smooth and of Cartier type.* (*Note that* $(X'', M'') = (X', M')$ *in this case.*)

(ii) *The underlying morphism* $X \to Y$ *is proper.*

(iii) *Etale locally on* Y, *there exist* $(\tilde{Y}, \tilde{N})$ *as in* (2) *and a smooth lifting of* (X', M') *over* $(\tilde{Y}, \tilde{N})$.

Then, the Hedge spectral sequence

$$E_1^{s,t} = R^t f_* \omega^s_{X/Y} \Rightarrow R^{s+t} f_* \omega^{\cdot}_{X/Y}$$

satisfies $E_1^{s,t} = E_\infty^{s,t}$ *for* s, t *such that* $s + t < p$. *Furthermore, the* $\mathcal{O}_Y$*-modules* $R^q f_* \omega^{\cdot}_{X/Y}$ *for* $q < p$ *are locally free and commute with any base changes.*

Proof. Since the proof is a simple modification of those given in [DI] (classically smooth case) and in [Il$_2$] (the case of morphisms of "semi-stable reduction type" between smooth schemes), we give here only the proof of the Cartier isomorphism (4.12)(1), and left the other part of the proof to the reader. (As in [Il$_2$], the other part is deduced from (1) and (3.14) by the arguments in [DI].) I just note that the assumption f is integral in (2) is used to have the flatness over $\mathbf{Z}/p^2\mathbf{Z}$ (4.5) of smooth liftings of (X'', M''). Now we prove (1). By a standard argument, we may assume that there is a cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Spec}(\mathbf{F}_p[P]) \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathrm{Spec}(\mathbf{F}_p[Q]) \end{array}$$

where P, Q are finitely generated integral monoids with a homomorphism $Q \to P$ such that $Q^{gp} \to P^{gp}$ is injective and the torsion part of P^{gp}/Q^{gp} is of order invertible on Y, and M and N are the inverse images of the canonical log. str.'s, respectively. Let H be the submonoid of P containing Q defined by

$$H = \{a \in P;\ a = b^p c \text{ in } P^{gp} \text{ for some } b \in P^{gp} \text{ and } c \in Q^{gp}\}.$$

Then we have

$$X'' = Y \times_{\mathrm{Spec}(\mathbf{F}_p[Q])} \mathrm{Spec}(\mathbf{F}_p[H]).$$

In this identification, $X'' \to X$ (resp. $X'' \to Y$, resp. $X \to X''$) is given by $F_{(Y,N)} \times (a \mapsto a^p;\ P \to H)$; $Y \times_{\mathrm{Spec}(\mathbf{F}_p[Q])} \mathrm{Spec}(\mathbf{F}_p[H]) \to Y \times_{\mathrm{Spec}(\mathbf{F}_p[Q])} \mathrm{Spec}(\mathbf{F}_p[P])$ (resp. $pr_1 : Y \times_{\mathrm{Spec}(\mathbf{F}_p[Q])} \mathrm{Spec}(\mathbf{F}_p[H] \to Y$, resp. id. $\times$ $(H \hookrightarrow P))$; $Y \times_{\mathrm{Spec}(\mathbf{F}_p[Q])} \mathrm{Spec}(\mathbf{F}_p[P]) \to Y \times_{\mathrm{Spec}(\mathbf{F}_p[Q])} \mathrm{Spec}(\mathbf{F}_p[H]))$. For $\nu \in P^{gp}/Q^{gp} \otimes_{\mathbf{Z}} \mathbf{F}_p$, let E_ν be the $\mathbf{F}_p[Q]$-submodule of $\mathbf{F}_p[P]$ generated by elements of P which belong to ν, and define the complex $C_\nu^{\cdot}$ by

$$C_\nu^q = \mathcal{O}_Y \otimes_{\mathbf{F}_p[Q]} E_\nu \otimes_{\mathbf{F}_p} \wedge^q_{\mathbf{F}_p}((P^{gp}/Q^{gp}) \otimes \mathbf{F}_p)$$

with the differential $C_\nu^q \to C_\nu^{q+1}$ induced by

$$\wedge^q_{\mathbf{F}_p}((P^{gp}/Q^{gp}) \otimes \mathbf{F}_p) \to \wedge^{q+1}_{\mathbf{F}_p}((P^{gp}/Q^{gp}) \otimes \mathbf{F}_p); \qquad a \mapsto \nu \wedge a.$$

We have

$$\omega^{\cdot}_{X/Y} = \oplus_\nu C_\nu^{\cdot}.$$

The complex $C_\nu^{\cdot}$ is acyclic if $\nu \neq 0$, and so $\omega^{\cdot}_{X/Y}$ is quasi-isomorphic to $C_0^{\cdot}$. On the other hand, the differential of $C_0^{\cdot}$ is zero, $E_0 = \mathbf{F}_p[H]$, and

$$C_0^q = \mathcal{O}_Y \otimes_{\mathbf{F}_p[Q]} \mathbf{F}_p[H] \otimes_{\mathbf{F}_p} \wedge^q_{\mathbf{F}_p}((P^{gp}/Q^{gp}) \otimes \mathbf{F}_p) \cong \omega^q_{X''/Y}$$

(cf. (1.8)).

Remark (4.13). In (4.12)(1), if $Y = \mathrm{Spec}(k)$ for a field k and N is the trivial log. str., and if X is normal, X'' coincides with the normalization of X'.

5. Crystalline sites. A fact for log. str.'s which is different from the classical facts is that the crystalline cohomology theory is easier than the ℓ ($\neq$ char.)-adic etale cohomology theory. (I have not yet a good definition of the etale site for a log. str. A related problem is to define the K-group of a scheme with a log. str.)

(5.1) As a base, we take a 4-ple (S, L, I, γ) where S is a scheme such that $\mathcal{O}_S$ is killed by a nonzero integer, L is a fine log. str. on S, I is a quasi-coherent ideal on S, and γ is a PD (=divided power) structure on I.

(5.2) Let (S, L, I, γ) be as above, let (X, M) be a scheme with a fine log. str. over (S, L) such that γ extends to X. Then, we define the crystalline site $((X, M)/(S, L, I, \gamma))_{crys}$ (denoted also simply by $(X/S)^{\log}_{crys}$ if there is no risk of confusion) as follows. An object is a 5-ple (U, T, M_T, i, δ) where U is an etale scheme over X, (T, M_T) is a scheme with a fine log. str. over (S, L), i is an exact closed immersion (3.1) $(U, M) \to (T, M_T)$ over (S, L), and δ is a PD-structure on the ideal of $\mathcal{O}_T$ defining U which is compatible with γ. A morphism is defined in the evident way. A covering is a covering for the usual etale topology forgetting the log. str.'s.

The structure sheaf $\mathcal{O}_{X/S}$ on $(X/S)^{\log}_{crys}$ is defined by

$$\mathcal{O}_{X/S}(U, T, M_T, i, \delta) = \Gamma(T, \mathcal{O}_T).$$

We sometimes abbreviate (U, T, M_T, i, δ) simply as T. We sometimes denote $\gamma_n(a)$ and $\delta_n(a)$ as $a^{[n]}$.

We have the following fact by applying (1.4.1) to $U \to T$; If $g : T' \to T$ is a morphism in $(X/S)^{\log}_{crys}$, $g^*(M_T) \to M_{T'}$ is an isomorphism.

We have a logarithmic version of the PD-envelope:

PROPOSITION (5.3). *Let (S, I, γ) be as in (5.1). (We forget L here). Let $\mathcal{C}$ be the category of closed immersions (3.1) $i : (X, M) \to (Y, N)$ of schemes with* log. str.'s *over S such that M is fine and N is coherent. (By definition, a morphism $i' \to i$ is a commutative diagram*

$$\begin{array}{ccc} (X', M') & \xrightarrow{i'} & (Y', N') \\ \downarrow & & \downarrow \\ (X, M) & \xrightarrow{i} & (Y, N) \end{array}$$

over S.) Let $\mathcal{C}'$ be the category of pairs (i, δ) where i is an exact closed immersion (3.1) $(X, M) \to (Y, N)$ *of schemes with fine* log. str.'s *over S and δ is a PD-structure on the ideal of Y defining X which is compatible with γ. Then, the canonical functor $\mathcal{C}' \to \mathcal{C}$ has a right adjoint.*

Definition (5.4). In (5.3), let $i : (X, M) \to (Y, N)$ be an object of $\mathcal{C}$ and let $(\tilde{i} : (\tilde{X}, \tilde{M}) \to (\tilde{Y}, \tilde{N}), \delta)$ be the result of applying the right adjoint functor to i. We call $(\tilde{i}, \delta)$ (or sometimes $(\tilde{Y}, \tilde{N})$) the PD-envelope of (X, M) in (Y, N) with respect to γ, and denote it by $D_{(X,M)}((Y, N)/(S, I, \gamma))$ (or simply by $D_X^{\log}(Y)$).

(5.5). The construction of the PD-envelope given below shows the following facts:

(5.5.1). If i is an exact closed immersion, $D_X^{\log}(Y)$ coincides with the usual PD-envelope $D_X(Y)$ endowed with the inverse image of N.

(5.5.2). If γ extends to Y, $(\tilde{X}, \tilde{M}) \to (X, M)$ is an isomorphism.

(5.5.3). $\tilde{M}$ always coincides with the inverse image of M.

(5.6). *Proof of* (5.3). We construct $(\tilde{i}, \delta)$ of (5.4). We may assume N is fine, since $(\tilde{i}, \delta)$ for $(X, M) \to (Y, N)$ is the same thing with $(\tilde{i}, \delta)$ for $(X, M) \to (Y, N)^{int}$ (2.7). We may work etale locally, and hence we have a factorization $i = gi'$ with $i' : (X, M) \to (Z, M_Z)$ an exact closed immersion and g etale (4.10)(1). Let $(\tilde{i} : \tilde{X} \to D, \delta)$ be the PD-envelope of i' with respect to γ in the usual sense, and endow $\tilde{X}$ (resp. D) with the inverse image $\tilde{M}$ (resp. M_D) of M (resp. M_Z). It is not hard to see that $(\tilde{i} : (\tilde{X}, \tilde{M}) \to (D, M_D), \delta)$ has the desired universal property.

Example (5.7). Let k be a field of characteristic $p > 0$, let $X = \mathrm{Spec}(k[T])$, $Y = \mathrm{Spec}(k[T_1, T_2])$, $X \to Y$ by $T_i \mapsto T\ (i = 1, 2)$. Endow X (resp. Y) with log. str. M (resp. N) corresponding to the divisor "$T = 0$" (resp. "$T_1 = 0$" $\cup$ "$T_2 = 0$"). Take the base $S = \mathrm{Spec}(k)$, $I = (0)$. Then the PD-envelope $D_X^{\log}(Y)$ is the usual PD-envelope of X in $Z = \mathrm{Spec}(k[T_1, T_2, T_1T_2^{-1}, T_2T_1^{-1}]) = \mathrm{Spec}(k[T_1, V, V^{-1}])$ ($V = T_1T_2^{-1}$, $V \mapsto 1$ on X) endowed with the inverse image of the log. str. M_Z on Z corresponding to the divisor "$T_1 = 0$" (= "$T_2 = 0$"). Indeed, the closed immersion $(X, M) \to (Y, N)$ is not exact, but $(X, M) \to (Z, M_Z)$ is an exact closed immersion and $(Z, M_Z) \to (Y, N)$ is etale.

Remark (5.8). Let (S, L) be as in (5.1) (we forget here I and γ). We can define the n-th infinitesimal neighbourhood in the logarithmic sense as follows, similarly to the PD-envelopes. For $n \geqq 0$, let $\mathcal{C}_n$ be the category of exact closed immersions $(X, M) \to (Y, N)$ of schemes over (S, L) such that X is defined in Y by an ideal J with the property $J^{n+1} = 0$. Then the canonical functor $\mathcal{C}_n \to \mathcal{C}$ ($\mathcal{C}$ is as in (5.3)) has a right adjoint. Indeed, let (Z, M_Z) be as in (5.6), and let D be the n-th infinitesimal neighbourhood of X in Z in the usual sense endowed with the inverse image M_D of M_Z. Then, $(X, M) \to (D, M_D)$ is the desired universal object. In the case of the diagonal embedding $(X, M) \to (Y, N) = (X, M) \times_{(S,L)} (X, M)$ with $n = 1$, if we denote by J the ideal of X in D, we have

$$\omega^1_{X/S} \cong J/J^2. \tag{5.8.1}$$

(5.9). We have the functoriality of the crystalline topoi. Let

$$\begin{array}{ccc} (X', M') & \xrightarrow{f} & (X, M) \\ \downarrow & & \downarrow \\ (S', L', I', \gamma') & \longrightarrow & (S, L, I, \gamma) \end{array}$$

be a commutative diagram where the assumptions of (5.1) (5.2) are satisfied by both $(X, M)/(S, L, I, \gamma)$ and $(X', M')/(S', L', I', \gamma')$. Then we have the morphism of topoi

$$f_{crys} : ((X'/S')^{\log}_{crys})^{\sim} \to ((X/S)^{\log}_{crys})^{\sim}$$

(the $\sim$ denote the topoi associated to sites) characterized by

$$f_{crys*}(\mathfrak{F})(U, T, M_T, i, \delta) = \mathrm{Mor}((U, T, M_T, i, \delta)^{\sim}, \mathfrak{F})$$

where $(U, T, M_T, i, \delta)^{\sim}$ is the sheaf on $(X'/S')^{\log}_{crys}$ whose value in $(U', T', M'_{T'}, i', \delta')$ is the set of all pairs (g, h) of morphisms $g : (U', M') \to (U, M)$, $h : (T', M'_{T'}) \to (M, T)$ for which the diagram

$$\begin{array}{ccccccc} (X', M') & \longrightarrow & (U', M') & \longrightarrow & (T', M'_{T'}) & \longrightarrow & (S', L') \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow \\ (X, M) & \longrightarrow & (U, M) & \longrightarrow & (T, M_T) & \longrightarrow & (S, L) \end{array}$$

commutes and such that h commutes with δ and δ'.

The proof of the fact f_{crys*} determines a morphism of topoi (i.e. f_{crys*} has a left adjoint which commutes with finite inverse limits) is proved by the same way as in the classical theory of crystalline topoi ([B] Chapter 3, Section 2), by using the notion (5.4) of PD-envelopes.

6. Crystals and crystalline cohomology. In this section, let (S, L, I, γ) be as in (5.1) and let $f : (X, M) \to (S, L)$ be a morphism of schemes such that M is fine and γ extends to X.

Definition (6.1). A crystal on $(X/S)^{\log}_{crys}$ is a sheaf of $\mathcal{O}_{X/S}$-modules $\mathfrak{F}$ on $(X/S)^{\log}_{crys}$ satisfying the following condition: For any morphism $g : T' \to T$ in $(X/S)^{\log}_{crys}$, if we denote by $\mathfrak{F}_T$ and $\mathfrak{F}_{T'}$ the sheaves on T_{et} and T'_{et} induced by $\mathfrak{F}$ respectively, $g^*(\mathfrak{F}_T) \to \mathfrak{F}_{T'}$ is an isomorphism.

THEOREM (6.2). *Let (Y, N) be a scheme with a fine* log. str. *which is smooth over (S, L), and let $(X, M) \to (Y, N)$ be a closed immersion* (3.1). *Denote the PD-envelope of (X, M) in (Y, N) as (D, M_D). Then, the following two categories* (a) (b) *are equivalent.*

(a) *The category of crystals on $(X/S)^{\log}_{crys}$.*

(b) *The category of $\mathcal{O}_D$-modules $\mathfrak{M}$ on D_{et} endowed with an additive map*

$$\nabla : \mathfrak{M} \to \mathfrak{M} \otimes_{\mathcal{O}_Y} \omega^1_{Y/S}$$

having the following properties (i)–(iii).

(i) $\nabla(ax) = a\nabla(x) + x \otimes da$ *for* $a \in \mathcal{O}_D$ *and* $x \in \mathfrak{M}$.

(ii) *The composite*

$$\mathfrak{M} \xrightarrow{\nabla} \mathfrak{M} \otimes_{\mathcal{O}_Y} \omega^1_{Y/S} \xrightarrow{\nabla} \mathfrak{M} \otimes_{\mathcal{O}_Y} \omega^2_{Y/S}$$

is zero, where we extend ∇ to

$$\mathfrak{M} \otimes_{\mathcal{O}_Y} \omega^q_{Y/S} \to \mathfrak{M} \otimes_{\mathcal{O}_Y} \omega^{q+1}_{Y/S};$$

$$x \otimes \omega \mapsto \nabla(x) \wedge \omega + x \otimes d\omega.$$

(iii) *Let $x \in X$ and let t_i $(1 \leqq i \leqq r)$ be elements of $M_{\bar{x}}$ such that $(d\log(t_i))_{1\leqq i\leqq r}$ is a basis of $\omega^1_{Y/S,\bar{x}}$. Then, for any i and for any $a \in \mathfrak{M}_{\bar{x}}$, there exist $m_1, \ldots, m_k, n_1, \ldots, n_k \in \mathbf{N}$ such that*

$$\Big(\prod_{1\leqq i\leqq r, 1\leqq j\leqq k} (\nabla^{\log}_{t_i} - m_j)^{n_j}\Big)(a) = 0.$$

Here $\nabla^{\log}_{t_i}$ is defined by: if $\nabla(a) = \Sigma_{1\leqq i\leqq r}\, a_i \otimes d\log(t_i)$, then $\nabla^{\log}_{t_i}(a) = a_i$.

(It is proved as in the classical case, that if the condition (iii) holds for one choice of $(t_i)_{1\leqq i\leqq r}$, then it holds for any choice of $(t_i)_{1\leqq i\leqq r}$.)

Remark (6.3). *If $t_i \in \mathcal{O}^*_{X,\bar{x}}$* and ∇_{t_i} denotes $t_i^{-1}\nabla^{\log}_{t_i}$, then

$$(\nabla_{t_i})^n = t_i^{-n} \prod_{0\leqq j\leqq n-1} (\nabla^{\log}_{t_i} - j).$$

Thus, the condition (b) (iii) in (6.2) is the natural logarithmic version of the classical notion of the "quasi-nilpotence" ([B] Chapter 2, Section 4.3).

THEOREM (6.4). *Let (X, M), (Y, N) and D be as in* (6.2), *let $\mathfrak{F}$ be a crystal on $(X/S)^{\log}_{crys}$, and let $\mathfrak{M}$ be the corresponding $\mathcal{O}_D$-module with ∇. Then,*

$$Ru^{\log}_{X/S*}(\mathfrak{F}) \cong \mathfrak{M} \otimes_{\mathcal{O}_Y} \omega^{\cdot}_{Y/S}.$$

Here $u^{\log}_{X/S}$ is the canonical morphism $((X/S)^{\log}_{crys})^{\sim} \to (X_{et})^{\sim}$ characterized by

$$(u^{\log}_{X/S})_*(\mathfrak{F})(U) = \textit{the global section of } \mathfrak{F} \textit{ on } (U/S)^{\log}_{crys}$$

for a sheaf $\mathfrak{F}$ on $(X/S)^{\log}_{crys}$.

For the proofs of (6.2) and (6.4), the following (6.5) is essential.

PROPOSITION (6.5). *Under the assumption of* (6.2), *let $(D', M_{D'})$ be the PD-envelope* (5.4) *of the diagonal morphism $(X, M) \to (Y, N) \times_{(S,L)} (Y, N)$, and let $p_1, p_2 : (D', M_{D'}) \to (D, M_D)$ be the first and the second projections, respectively. Let $x \in X$, take $t_1, \ldots, t_r \in N_{\bar{x}}$ such that $(d\log(t_i))_{1\leqq i\leqq r}$ is a basis of $\omega^1_{Y/S,\bar{x}}$, and let u_i $(1 \leqq i \leqq r)$ be the elements of* $\mathrm{Ker}(\mathcal{O}^*_{D',\bar{x}} \to \mathcal{O}^*_{X,\bar{x}}) \subset (M_{D'})_{\bar{x}}$ *defined by $p_2^*(t_i) = p_1^*(t_i)u_i$ (the existence* *of u_i follows from $(M_{D'})_{\bar{x}}/\mathcal{O}^*_{D',\bar{x}} \cong M_{\bar{x}}/\mathcal{O}^*_{X,\bar{x}}$). Then we have the description of D'*

$$\mathcal{O}_{D,\bar{x}}\langle T_1, \ldots, T_r\rangle \cong \mathcal{O}_{D',\bar{x}}; \qquad T_i^{[n]} \mapsto (u_i - 1)^{[n]}$$

where $\mathcal{O}_{T,\bar{x}}\langle T_1, \ldots, T_r\rangle$ denotes the PD-polynomial ring.

(6.6). *Proof of* (6.5). By the construction of the PD-envelopes in (5.6), we may assume that $(X, M) \to (Y, N)$ is an exact closed immersion. Etale locally at x, take an exact closed immersion $(X, M) \to (Z, M_Z)$ where M_Z is fine and (Z, M_Z) is etale over $((Y, N) \times_{(S,L)} (Y, N))^{int}$ (4.10)(1). Let $q_i : Z \to Y\,(i = 1, 2)$ be the two projections. Then, the stalks at $\bar{x}$ of the sheaves M_Z and $q_i^*(N)\,(i = 1, 2)$ coincide. So, by replacing Z by an etale neighbourhood of $\bar{x} \to Z$, we may assume that $M_Z = q_i^*N\,(i = 1, 2)$. Then, q_i are smooth in the usual sense by (3.8). Since D (resp. D') is the usual PD-envelope of X in Y (resp. Z), and since $q_1^*(t_i)^{-1}q_2^*(t_i) - 1$ $(1 \leq i \leq r)$ form a smooth coordinate of Z over Y with respect to (say) q_1 and their restrictions to X are zero, the statement of (6.5) follows.

(6.7). *Proof of* (6.2). We follow the classical theory ([B] Chapter 4, Section 1). Let $(\mathcal{M}, \nabla)$ be an object of (b). We show how to define the corresponding object of (a). Let $x \in X$, let $(t_i)_i$ be as in (b)(iii), and let D' and $(u_i)_i$ be as in (6.5). Then we have an isomorphism at $\bar{x}$

$$\eta : p_2^*\mathcal{M} \cong p_1^*\mathcal{M}; \tag{6.7.1}$$

$$1 \otimes a \mapsto \sum_{n \in \mathbf{N}^r} \Big(\prod_{1 \leq i \leq r} (u_i - 1)^{[n_i]}\Big) \otimes \Big(\prod_{1 \leq i \leq r, 0 \leq j \leq n_i - 1} (\nabla_{t_i}^{\log} - j)\Big)(a)$$

$(a \in \mathcal{M})$, which satisfies the "transitivity condition"

$$p_{13}^*(\eta) = p_{23}^*(\eta)p_{12}^*(\eta)$$

([B] Chapter 2, 1.3.1) on the PD-envelope D'' of (X, M) in $(Y, N) \times_{(S,L)} (Y, N) \times_{(S,L)} (Y, N)$ where p_{12}, p_{13}, p_{23} are the projections $D'' \to D'$. Now we obtain an object $\mathcal{F}$ of (a) from $(\mathcal{M}, \nabla)$ as follows. Let (U, T, M_T, i, δ) be an object of $(X/S)^{\log}_{crys}$. Then, etale locally on T, $(X, M) \to (D, M_D)$ is extended to a morphism $h : (T, M_T) \to (D, M_D)$ over (S, L) by the smoothness of $(Y, N) \to (S, L)$. We define $\mathcal{F}_T$ to be $h^*\mathcal{M}$ etale locally on

T. If we have two such $h_i : (T, M_T) \to (D, M_D)$ $(i = 1, 2)$, they define $h' : (T, M_T) \to (D', M_{D'})$ such that $h_i = p_i h'$, and the isomorphism (6.7.1) induces $h_1^* \mathfrak{M} \cong h_2^* \mathfrak{M}$. Thus $\mathfrak{F}$ is independent of h and defined globally on T.

Conversely if we have an object $\mathfrak{F}$ of (a), let $\mathfrak{M} = \mathfrak{F}_D$. Then, the defining condition of the crystal gives an isomorphism $\eta : p_2^* \mathfrak{M} = p_1^* \mathfrak{M}$ satisfying the "transitivity condition". By writing η in the form

$$1 \otimes a \mapsto \sum_{n \in \mathbf{N}^r} \prod_{1 \leq i \leq r} (u_i - 1)^{[n_i]} \otimes \eta_n(a),$$

we define ∇ by

$$\nabla(a) = \sum_{1 \leq i \leq r} \eta_{e_i}(a) \otimes d \log(t_i)$$

where $(e_i)_{1 \leq i \leq r}$ is the canonical base of $\mathbf{N}$.

(6.9). *Proof of* (6.4). This is also a repetition of the classical argument. For an $\mathcal{O}_D$-module $\mathfrak{N}$ on D_{et}, let $L(\mathfrak{N})$ be the crystal on $(X/S)^{\log}_{crys}$ corresponding to the $\mathcal{O}_D$-module $p_1^* \mathfrak{N}$ with

$$\nabla : p_1^* \mathfrak{N} \to p_1^* \mathfrak{N} \otimes_{\mathcal{O}_Y} \omega^1_{Y/S}; \qquad a \otimes v \mapsto v \otimes da$$

$(a \in \mathcal{O}_{D'}, v \in \mathfrak{N}$, with D' as in (6.5)). Here d is the composite map

$$\mathcal{O}_{D'} \to \mathcal{O}_{D'} \otimes_{\mathcal{O}_Z} \omega^1_{Z/S} \cong \mathcal{O}_{D'} \otimes_{\mathcal{O}_Y} \omega^1_{Y/S}$$

with Z as in (6.6). The same argument as in the classical theory shows

$$R(u^{\log}_{X/S})_* L(\mathfrak{N}) = \mathfrak{N},$$

where we identified X_{et} with D_{et}. If $\mathfrak{F}$ is a crystal corresponding to $(\mathfrak{M}, \nabla)$, we have a resolution

$$\mathfrak{F} \to L(\mathfrak{M} \otimes_{\mathcal{O}_Y} \omega^{\cdot}_{Y/S}).$$

We obtain from this

$$R(u^{\log}_{X/S})_* \mathfrak{F} \cong (u^{\log}_{X/S})_* L(\mathfrak{M} \otimes_{\mathcal{O}_Y} \omega_{Y/S}) \cong \mathfrak{M} \otimes_{\mathcal{O}_Y} \omega^{\cdot}_{Y/S}.$$

The following (6.10) (the base change theorem) and (6.11) (the Künneth formula) are proved by the same method in the classical theory ([B] Chapter 5, Sections 3 and 4).

THEOREM (6.10). *Assume we are given a commutative diagram*

$$\begin{array}{ccc} (X', M') & \xrightarrow{g'} & (X, M) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ (Y', N') & \xrightarrow{g} & (Y, N) \\ \downarrow & & \downarrow \\ (S', L', I', \delta) & \xrightarrow{h} & (S, L, I, \delta) \end{array}$$

where all the log. str.'s *are fine, Y is quasi-compact, f is smooth and integral, the underlying morphism $X \to Y$ of f is quasi-separated, and the upper square is cartesian. Let $\mathfrak{F}$ be a quasi-coherent crystal on $(X/S)^{\log}_{crys}$ which is flat over $\mathcal{O}_{X/S}$. Then,* we have a canonical isomorphism

$$Lg^*_{crys}(Rf_{crys*}(\mathfrak{F})) \cong Rf'_{crys*}(g'_{crys}{}^*(\mathfrak{F})).$$

THEOREM (6.12). *Let $f : (Y, M_Y) \to (X, M_X)$ and $g : (Z, M_Z) \to (X, M_X)$ be smooth integral morphisms between schemes with fine* log. str.'s *over (S, L), and assume that X is quasi-compact and the underlying morphisms $Y \to X$, $Z \to X$ are quasi-compact and quasi-separated. Let (V, M_V) be the fiber product of (Y, M_Y) and (Z, M_Z) over (X, M_X) with $p : (V, M_V) \to (Y, M_Y)$, $q : (V, M_V) \to (Z, M_Z)$, $h : (V, M_V) \to (X, M_X)$, and let $\mathcal{E}$ (resp. $\mathfrak{F}$) be a quasi-coherent crystal on $(Y/S)^{\log}_{crys}$ (resp. $(Z/S)^{\log}_{crys}$) which is flat over $\mathcal{O}_{Y/S}$ (resp. $\mathcal{O}_{Z/S}$). Then, we have a canonical isomorphism*

$$Rf_{crys*}(\mathcal{E}) \otimes^L_{\mathcal{O}_{X/S}} Rg_{crys*}(\mathfrak{F}) \cong Rh_{crys*}(p^*_{crys}(\mathcal{E}) \otimes_{\mathcal{O}_{V/S}} q^*_{crys}(\mathfrak{F})).$$

Complement 1. We explain the relation between the log. str. of this note and that of Faltings in [Fa$_2$]. A definition of log. str. equivalent to that of Faltings was found by Deligne ([D]) independently.

The log. str. of Faltings in [Fa$_2$] on a scheme X is a family $(\mathcal{L}_i, x_i)_{1 \le i \le r}$ of invertible sheaves $\mathcal{L}_i$ on X and global sections x_i of $\mathcal{L}_i$. An equivalent definition (take the dual) given in [D] is that a log. str. on X is a family $(\mathcal{L}_i, s_i)_{1 \le i \le r}$ of invertible sheaves $\mathcal{L}_i$ on X and homomorphisms s_i :

$\mathcal{L}_i \to \mathcal{O}_X$ of $\mathcal{O}_X$-modules. To compare with our log. str., we adopt the latter definition of Deligne for it is nearer to our definition, and we call the log. str. $(\mathcal{L}_i, s_i)$ in the sense of Deligne the DF. log. str.

We claim that a DF. log. str. on X is equivalent to a pair (M, t) of a fine log. str. (in our sense) M on X and a homomorphism $t : \mathbf{N}^r_X \to M/\mathcal{O}^*_X$ which lifts etale locally on X to a chart $\mathbf{N}^r_X \to M$ of M. Indeed, for such a pair (M, t), we define $(\mathcal{L}_i, s_i)$ as follows. Let $(e_i)_{1 \leqq i \leqq r}$ be the cannical base of $\mathbf{N}^r$. Then, the inverse image of $t(e_i)$ under $M \to M/\mathcal{O}^*_X$ is a principal homogeneous space over $\mathcal{O}^*_X$ and corresponds to an invertible sheaf $\mathcal{L}_i$, and the homomorphism $M \to \mathcal{O}_X$ defines s_i. Conversely, we can reconstruct (M, t) from $(\mathcal{L}_i, s_i)$ as follows. Define first the pre-log. str. M' to be the sheaf of pairs (n, a) where $n \in \mathbf{N}^r$ and a is a local generator of $\otimes_i \mathcal{L}_i^{\otimes n_i}$, endowed with the homomorphism $M' \to \mathcal{O}_X$ induced by $\otimes_i s_i^{\otimes n_i} : \otimes_i \mathcal{L}_i^{\otimes n_i} \to \mathcal{O}_X$. Then, define M to be the log. str. associated to M', and t to be the composite of the inverse of the isomorphism

$$M'/\mathcal{O}^*_X \to \mathbf{N}^r_X; \qquad (n, a) \mapsto n$$

with the canonical homomorphism $M'/\mathcal{O}^*_X \to M/\mathcal{O}^*_X$.

The log. str. of Fontaine-Illusie is more general than the DF. log. str.: A fine log. str. M on a scheme X has a chart of the form $\mathbf{N}^r_X \to M$ with $r \geqq 0$ on an etale neighbourhood of $x \in X$ if and only if

$$M_{\bar{x}}/\mathcal{O}^*_{X,\bar{x}} \cong \mathbf{N}^s$$

for some $s \geqq 0$. One has also that for a finitely generated integral monoid P in which the unit element is the only invertible element, and for a non-zero ring R, the canonical log. str. on $\mathrm{Spec}(R[P])$ comes from a DF. log. str. if and only if $P \cong \mathbf{N}^r$ for some r. Log. str.'s which do not come from DF. log. str.'s appear, for example, by taking a product of schemes with semi-stable reduction over a dvr., or by a ramified extension of the base dvr. of a scheme with semi-stable reduction.

Complement 2. The crystalline cohomology theory in this note is applied to the semi-stable reduction situation (3.7)(2) as follows. Let $X \to \mathrm{Spec}(A)$ be as in (3.7)(2), let k be the residue field of A, and let Y be the special fiber $X \otimes_A k$ of X. Endow Y (resp. $\mathrm{Spec}(k)$) with the inverse image $\bar{M}$ (resp. $\bar{N}$) of the log. str. M on X (resp. N on $\mathrm{Spec}(A)$) in (3.7)(2). Assume k is perfect and $\mathrm{char}(k) = p > 0$, and let $W_n(k)$ be the ring of Witt vectors of length n, and endow $\mathrm{Spec}(W_n(k))$ with the log. str. N_n associated to $\mathbf{N} \to W_n(k)$; $1 \mapsto 0$. By fixing a prime element π of A, we have morphisms

$$(Y, \bar{M}) \to (\mathrm{Spec}(k), \bar{N}) \to (\mathrm{Spec}(W_n(k)), N_n)$$

where the second arrow is induced from $\mathbf{N} \to A$; $1 \mapsto \pi$. (Then, $N_1 \to \bar{N}$ is an isomorphism.) Consider the crystalline cohomology of (Y, M) over the base $(\mathrm{Spec}(W_n(k)), N_n)$ with the usual PD str. on the ideal $pW_n(k)$. Then this crystalline cohomology is very important, and serves as the mixed characteristic analogue of the limit Hodge str. [S]. For the details and the relation with the de Rham-Witt complex in [H₂], cf. [HK] and [K′].

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