LOG GEOMETRY

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Contents

Introduction

These notes contain a basic introduction to the *logarithmic geometry* (hereafter *log geometry*) of Fontaine, Illusie, Kato, Olsson, et. al. The slogan is that log geometry is the toric geometry of a ringed topos. This means, roughly, that instead of looking at a monoid and the monoid algebra associated to it, you look at a monoid in a topos mapping to a ring of a topos. Of course, you must impose some limits on these sheaves of monoids so that things don't get too out of hand. And/or, loosely speaking, you localize by identifying monoid morphisms to the ring which "only differ by units of the ring". This notion is much less rigid than toric geometry, so that, for example, there is a rich theory of log structures on positive genus curves and other "irrational" varieties.

A major goal is to treat "limits" of smooth objects in algebraic geometry on the same footing as smooth objects themselves. In order to do this, one has to "remember where the degenerate object came from". For example, if $X \to C$ is a family of smooth varieties degenerating to a singular variety X_0 over a point 0 of C, then one keeps track of the monoid of functions on X which are invertible away from X_0 and the functions on C invertible away from 0. Pulling back these monoids to $X_0 \to 0$ "remembers where X_0 came from".

The formalization of smoothness in log geometry ("log smoothness") is accomplished by abstracting the lifting property defintion of smoothness (or, rather, formal smoothness), then working in a slightly larger category (than schemes, say) where an obvious analog of this lifting property makes sense. The data of the monoid mapping to the structure sheaf helps produce lifts that would not otherwise exist (or, better, prevents maps violating the lifting property from being maps with the additional structure).

A side benefit of the general theory allows one to make sense of adding "log differentials"

$$d \log f \text{ ``} := \text{''}\ f^{-1} df$$

for various functions f on a space. Since $d\log(fg) = d\log f + d\log g$, and one already has $d\log f$ if f is invertible, this also explains how one is led to study monoids mapping to the structure sheaf inducing an isomorphism on invertible elements. The log geometry moniker is derived from this aspect of the theory. As one might expect, there is a well-developed theory of log de Rham complexes and cohomology. We do not particularly touch upon these here.

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1. Monoids

In this section, we introduce the basic theory of monoids as needed in logarithmic geometry.

1.1. **The category of monoids.** Throughout, all monoids are assumed commutative with unit, so a *monoid* is a set P equipped with an associative, commutative binary operation (which we will typically denote $+$), and a distinguished element $0 \in P$ which is an *additive identity element* ($p + 0 = p$ for all $p \in P$). A *morphism* (or *map*) of monoids is a map $f : Q \to P$ between the underlying sets such that $f(q_1 + q_2) = f(q_1) + f(q_2)$ for all $q_1, q_2 \in Q$ and $f(0) = 0$. Morphisms are composed by composing the underlying maps of sets. Monoids form a category denoted **Mon**.

In any monoid, the element 0 is the unique additive identity element, for if p is another additive identity, then we have $0 = 0 + p = p$, using the additive identity element property of p for the first equality, and that of zero for the second. For an element p of a monoid P, an *inverse* of P is an element $p' \in P$ such that $p + p' = 0$. If p'' is another inverse for p, then we compute

$$\begin{aligned} p' &= p' + 0 \\ &= p' + (p + p'') \\ &= (p' + p) + p'' \\ &= 0 + p'' \\ &= p'' \end{aligned}$$

by using associativity, so inverses (if they exist) are unique. An element with an inverse is called *invertible*. We will write $-p$ for the inverse of an invertible element p. The remarks of this paragraph will be used throughout without comment.

Whenever we regard a ring A (commutative with unit as always) as a monoid, it will be under multiplication, and our notation will reflect this.

The category **Mon** is much like the category of abelian groups. It has a zero object (any monoid with a one element underlying set), hence it has a *zero morphism* $0 : Q \to P$ between any two monoids (the unique morphism factoring through the zero object) given by mapping every $q \in Q$ to $0 \in P$. The category **Mon** has all limits, both direct and inverse. The construction of inverse limits is exactly as in the category of sets **Ens**; one simply observes that the inverse limit of the underlying sets has a natural monoid structure making it an inverse limit in the category of monoids (the data of a monoid structure on a set is in terms of inverse limits, and these commute amongst themselves).

The forgetful functor **Mon** $\to$ **Ens** commutes with inverse limits. This is clear from the above remark, but also follows formally from the existence of a left adjoint to this forgetful functor (the free monoid functor of Section 1.9).

Finite products and coproducts (direct sums) in **Mon** coincide. That is, a representative of the inverse limit over a finite, setlike indexing category also represents the direct limit of the same functor. We will use both notations $P \oplus Q$ and $P \times Q$, to emphasize the appropriate categorical operation. For a product of monoids, we always denote the structure maps by $\pi_i : P_1 \times \cdots \times P_n \to P_i$, and we always write

$$f_1 \times \cdots \times f_n : Q \to P_1 \times \cdots \times P_n$$

for the map to a product corresponding to maps $f_i : Q \to P_i$ (so $f = f_1 \times \cdots \times f_n$ is the unique map from Q to the product with $\pi_i f = f_i$).

Every monoid P is a commutative monoid object in the category of sets **Ens**, meaning that the monoid operation $+ : P \times P \to P$ and the inclusion $0 : \{0\} \to P$ of zero are set maps making the diagrams

$$\begin{tikzcd} P \times P \times P \arrow[r, "\pi_1 \times (\pi_2 + \pi_3)"] \arrow[d, "(\pi_1+\pi_2)\times\pi_3"'] & P \times P \arrow[d, "+"] \\ P \times P \arrow[r, "+"] & P \end{tikzcd}$$

$$\begin{tikzcd} P \arrow[r, "\mathrm{Id}_P \times 0"] \arrow[dr, equal] & P \times P \arrow[d, "+"] & P \arrow[l, "0 \times \mathrm{Id}_P"'] \arrow[dl, equal] \\ & P & \end{tikzcd}$$

$$\begin{tikzcd} P \times P \arrow[r, "+"] \arrow[d, "\pi_2 \times \pi_1"'] & P \\ P \times P \arrow[ur, "+"'] & \end{tikzcd}$$

commute. In general, a monoid object of a category $\mathscr{C}$ with terminal object 0 is an object P of $\mathscr{C}$ such that the finite products $P \times \cdots \times P$ exist in $\mathscr{C}$, together with the data of maps $0 \to P$ and $+ : P \times P \to P$ making the above diagrams commute. Notice that, in fact, a monoid is also a monoid object in **Mon** because the maps $0 \to P$ and $+ : P \times P \to P$ are morphisms of monoids.

For a set S and a monoid P, the set $\mathrm{Hom}_{\mathbf{Ens}}(S, P)$ has the structure of a monoid (by the rule $(f+g)(s) := f(s)+g(s)$). In general, if **Ens** is replaced with an arbitrary category $\mathscr{C}$, and P is a monoid object of $\mathscr{C}$, then $\mathrm{Hom}_{\mathscr{C}}(C, P)$ has the structure of a monoid (one adds two morphisms f, g to P by composing their product $f \times g$ with the addition map $+ : P \times P \to P$). In particular, for monoids P, Q, the set

$$P^Q := \mathrm{Hom}_{\mathbf{Mon}}(Q, P)$$

has a natural monoid structure. The monoid P^Q represents the functor

$$\begin{array}{rcl} \mathbf{Mon} & \to & \mathbf{Ens} \\ R & \mapsto & \mathrm{Hom}_{\mathbf{Mon}}(R \times Q, P), \end{array}$$

meaning that there is a natural bijection

$$\mathrm{Hom}_{\mathbf{Mon}}(R, P^Q) \cong \mathrm{Hom}_{\mathbf{Mon}}(R \times Q, P)$$

for any monoid R. We will see in (1.9.12) that P^Q is "nice" when P and Q are "nice" (in various senses).

The *kernel* of a monoid map $f : Q \to P$ (i.e. the inverse limit of $f, 0 : Q \rightrightarrows P$) is given by $f^{-1}(0)$ as usual, but, unlike in an abelian category, having zero kernel does not imply that a morphism of monoids is monic. For example, the morphism $(n_1, n_2) \mapsto n_1+n_2$ from $\mathbb{N}^2 \to \mathbb{N}$ has zero kernel, but is not monic. Monic morphisms in **Mon** are those morphisms which are injective on underlying sets. This is because injectivity can be tested by maps from $\mathbb{N}$, or, what amounts to roughly the same argument, this is a formal consequence of the fact that the forgetful functor **Mon** $\to$ **Ens** is faithful and commutes with inverse limits. We will use the terms "monic" and "injective" interchangably.

1.2. **Direct limits.** The construction of direct limits in **Mon** is sufficiently subtle that we will devote an entire section to it.

Call an equivalence relation $\sim$ on a monoid P a *monoidal equivalence relation* (or *congruence relation*) if

$$p_1 \sim p_1' \text{ and } p_2 \sim p_2' \implies p_1 + p_2 \sim p_1' + p_2'.$$

The intersection of any set of monoidal equivalence relations is again a monoidal equivalence relation. The trivial relation $P \times P$ is also a monoidal equivalence relation, hence any relation $R \subseteq P \times P$ is contained in a smallest monoidal equivalence relation (just intersect all such).

A monoidal equivalence relation is the same thing as an equivalence relation object in the category **Mon**. That is, it is a monomorphism $R \to P \times P$ of monoids such that there exist commutative diagrams

$$\begin{tikzcd} P \arrow[d, "r"'] \arrow[dr, "\Delta = \mathrm{Id}_P \times \mathrm{Id}_P"] & \\ R \arrow[r] & P \times P \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r] \arrow[d, "s"'] & P \times P \arrow[d, "\pi_2 \times \pi_1"] \\ R \arrow[r] & P \times P \end{tikzcd} \qquad \begin{tikzcd} R \times_{P,\pi_2,\pi_1} R \arrow[r] \arrow[d, "t"'] & (P \times P) \times (P \times P) \arrow[d, "\pi_1 \times \pi_4"] \\ R \arrow[r] & P \times P \end{tikzcd}$$

in **Mon**. The diagrams (or rather, the existence of the morphisms r, s, t making them commute) express the reflexive, symmetric, and transitive properties respectively. A *monoidal relation* on a monoid P is a submonoid of $P \times P$. Every relation on P (subset of $P \times P$) is contained in a smallest monoidal relation (the submonoid of $P \times P$ it generates).

Given an arbitrary subset $R \subseteq P \times P$, the smallest monoidal equivalence relation containing R can be constructed as follows:

(1) First form the reflexive, symmetric closure $\overline{R}^{\mathrm{rs}}$ of R:
$$\overline{R}^{\mathrm{rs}} = R \cup \Delta \cup \{(p_2, p_1) : (p_1, p_2) \in R\}$$
(2) Given a reflexive, symmetric relation $R \subseteq P \times P$, form the submonoid $\langle R \rangle$ it generates. This can be done inductively by setting $R_0 := R$,
$$R_{n+1} := \{(p_1 + p_1', p_2 + p_2') : (p_1, p_2), (p_1', p_2') \in R_n\}.$$
Evidently we have $R = R_0 \subseteq R_1 \subseteq \cdots$ and $\langle R \rangle = \cup_n R_n$. The relation $\langle R \rangle$ is again reflexive and symmetric, as is easily proved by showing (by induction) that each R_n is reflexive and symmetric.
(3) Given a reflexive, symmetric, monoidal relation R on P, I claim the transitive closure $\overline{R}^{\mathrm{tr}}$ of R is a monoidal equivalence relation. Since R is symmetric, $\overline{R}$ is equal to the set of pairs (p, p') such that there is a sequence
$$p = p_0, \dots, p_n = p'$$
of elements of P with $(p_i, p_{i+1}) \in R$ for $i = 0, \dots, n-1$. Suppose (r, r') is also in $\overline{R}^{\mathrm{tr}}$ (as witnessed by a sequence $r_0, \dots, r_m$) and we wish to show $(p + r, p' + r') \in \overline{R}^{\mathrm{tr}}$. Since R is symmetric, we may repeat some of the p_i and r_i if necessary to assume $m = n$. The sequence $p_0 + r_0, \dots, p_n + r_n$ then witnesses the desired result because $(p_i + r_i, p_{i+1} + r_{i+1}) \in R$ since R is monoidal.

(It does not work as well to take the transitive closure of a reflexive, symmetric relation before taking its monoidal closure because the resulting relation need not be transitively closed.)

Observe that if $f : Q \to P$ is a morphism of monoids, then

$$Q \times_P Q = \{(q_1, q_2) \in Q \times Q : f(q_1) = f(q_2)\}$$

is a monoidal equivalence relation on Q. Conversely, given a monoidal equivalence relation $\sim$ on Q, the set of equivalence classes $Q/\sim$ has a unique monoid structure making the natural map $Q \to Q/\sim$ a morphism of monoids. Indeed, if we let $[q] \in Q/\sim$ denote the equivalence class of $q \in Q$, then if $Q \to Q/\sim$ is to be a morphism of monoids, we must have $[q] + [q'] = [q + q']$ and conversely, defining an addition law on $Q/\sim$ by this rule makes sense (i.e. is independent of the choices of representatives in each equivalence class) when $\sim$ is monoidal. If we view the monoidal equivalence relation $\sim$ as a monoid homomorphism $s \times t : R \to Q \times Q$, then

$$R \underset{t}{\overset{s}{\rightrightarrows}} Q \longrightarrow Q/\sim$$

is a coequalizer diagram of monoids. In the language of category theory, *every equivalence relation in* **Mon** *is effective.*

It follows from the remarks of the previous paragraphs that the direct limit of a (small) functor $F : \mathscr{C} \to \mathbf{Mon}$ can be constructed as the quotient of $\oplus_{C \in \mathscr{C}} FC$ by the smallest monoidal equivalence relation containing

$$\{(\iota_{FC}(m), \iota_{FD}(n)) : f \in \operatorname{Hom}_{\mathscr{C}}(C, D),\ F(f)(m) = n\}.$$

Here $\iota_{FC} : FC \to \oplus_{C \in \mathscr{C}} FC$ is the structure map to the direct sum, $f : C \to D$ is a morphism in $\mathscr{C}$, and $m \in FC, n \in FD$. Obviously this relation can be rather mysterious, though in principle it can be determined by the procedure given above.

Remark 1.2.1. The direct limit of $F : \mathscr{C} \to \mathbf{Mon}$ and the coequalizer of

$$\bigoplus_{(f:C \to D) \in \mathbf{Mor}_{\mathscr{C}}} \underset{\oplus \iota_{FD} Ff}{\overset{\oplus \iota_{FC}}{\rightrightarrows}} \bigoplus_{C \in \mathscr{C}} FC$$

coincide in any category where these limits exist.

The *cokernel* of $h : Q \to P$ (i.e. a direct limit of $h, 0 : Q \rightrightarrows P$) can be constructed as the quotient of P by the smallest monoidal equivalence relation containing $\{(0, h(q)) : q \in Q\}$. A map with trivial cokernel is not necessarily an epimorphism (right cancellable arrow) in **Mon** (see Example 1.12.1). Even worse, an epimorphism in **Mon** is not necessarily surjective on the underlying sets (we will discuss this in the next section).

We will sometimes write P/Q to denote a ("the") cokernel of h. The diagram

$$\begin{tikzcd} Q \arrow[r] \arrow[d, "h"'] & 0 \arrow[d] \\ P \arrow[r] & P/Q \end{tikzcd}$$

is cocartesian.

For any monoid P, we write P^* for the group of invertible elements (*units*) of P. The units form a submonoid $P^* \hookrightarrow P$ and there is a cartesian diagram

$$\begin{tikzcd} P^* \arrow[r] \arrow[d] & P \times P \arrow[d, "+"] \\ \{0\} \arrow[r, hook] & P \end{tikzcd} \tag{1.2.1.1}$$

in the category of monoids. A monoid is a *group* if $P^* = P$. Evidently a group in this sense is the same thing as an abelian group.

Proposition 1.2.2. *For any morphism of monoids $h : Q \to P$, the relation $\sim$ given by*

$$\{(p_1, p_2) : \exists\, q_1, q_2 \in Q,\ h(q_1) + p_1 = h(q_2) + p_2\}$$

is a monoidal equivalence relation on P and the projection $P \to P/\sim$ is a cokernel of f. If Q is a group, the relation $\sim$ is given by

$$\{(p_1, p_2) : \exists\, q \in Q,\ h(q_1) + p_1 = p_2\}.$$

Proof. First observe that $\sim$ is an equivalence relation. It is reflexive since h preserves zeros and it is clearly symmetric. For transitivity, if $p_1 \sim p_2$ and $p_2 \sim p_3$ then there are elements of Q such that

$$h(q_1) + p_1 = h(q_2) + p_2 \tag{1.2.2.1}$$

$$h(q_2') + p_2 = h(q_3) + p_3. \tag{1.2.2.2}$$

Together, (1.2.2.1) and (1.2.2.2) imply

$$h(q_1 + q_2') + p_1 = h(q_2 + q_3) + p_3,$$

hence $p_1 \sim p_3$. To see that $\sim$ is monoidal, add the equations expressing $p_1 \sim p_2 \wedge p_3 \sim p_4$ to prove $p_1 + p_3 \sim p_2 + p_4$.

To show $P \to P/\sim$ is a cokernel, one either checks the universal property directly, or one checks that $\sim$ is the smallest monoidal equivalence relation $\simeq$ containing

$$\{(0, h(q)) : q \in Q\}.$$

Taking the first approach, suppose $f : P \to M$ has $fh = 0$. Define $P/\sim \to M$ by mapping the class of p to $f(p)$; this is well defined because $fh(q_1) = fh(q_2) = 0$ and certainly the composition $P \to P/\sim \to M$ is f. This is the unique such map simply because $P \to P/\sim$ is surjective.

Taking the second approach, we first observe that $p \simeq h(q) + p$ and $h(q) + p \simeq p$ for all $p \in P, q \in Q$ since $\simeq$ is reflexive, symmetric, and monoidal. Certainly $h(q_1) + p_1 \simeq h(q_2) + p_2$ whenever $h(q_1) + p_1 = h(q_2) + p_2$ because $\simeq$ is reflexive, so we conclude $p_1 \sim p_2$ whenever $h(q_1) + p_1 = h(q_2) + p_2$ for some $q_1, q_2 \in Q$ by transitivity of $\simeq$. This proves $\sim \subseteq \simeq$ and by minimality of $\simeq$ among monoidal equivalence relations, we get the desired equality. □

The above proposition simplifies considerably when Q is a group:

Proposition 1.2.3. *For any morphism of monoids $h : Q \to P$ with Q a group, the cokernel of h is the quotient of P by the equivalence relation $\sim$ where $p_1 \sim p_2$ iff $p_2 = p_1 + h(q)$ for some $q \in Q$.*

Proof. The equivalence relation in question is clearly contained in the one from Proposition 1.2.2. If $h(q_1) + p_1 = h(q_2) + p_2$ for $p_i \in P$, $q_i \in Q$, then adding $h(-q_2)$ to both sides shows that $p_2 = p_1 + h(q_1 - q_2)$. This shows that the equivalence relation in question is the same as the only in Proposition 1.2.2. $\square$

Example 1.2.4. The cokernel of the diagonal map $\Delta : \mathbb{N} \to \mathbb{N}^k$ is isomorphic to $\mathbb{Z}^{k-1}$. One cokernel map $\mathbb{N}^k \to \mathbb{Z}^{k-1}$ is given by

$$(n_1, n_2, \ldots, n_k) \mapsto (n_1 - n_k, n_2 - n_k, \ldots, n_{k-1} - n_k).$$

Example 1.2.5. A monomorphism of monoids is not necessarily the kernel of its cokernel. The inclusion $\mathbb{N} \setminus \{1\} \hookrightarrow \mathbb{N}$ has cokernel $\mathbb{N} \to 0$.

1.3. **Pushouts.** Having discussed the general theory of direct limits and the special case of cokernels in the previous section, we now consider the case of pushouts, which are of great importance in log geometry.

Given a diagram

$$\begin{tikzcd} Q \arrow[r, "h_2"] \arrow[d, "h_1"'] & P_2 \\ P_1 & \end{tikzcd}$$

in **Mon**, we will denote its pushout (direct limit) $P_1 \oplus_Q P_2$. This pushout agrees with the coequalizer of

$$\begin{pmatrix} h_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ h_2 \end{pmatrix} : Q \rightrightarrows P_1 \oplus P_2$$

and can be constructed as the quotient of $P_1 \oplus P_2$ by the smallest monoidal equivalence relation $\sim$ containing

$$\{((h_1 q, 0), (0, h_2 q)) \in (P_1 \oplus P_2) \times (P_1 \oplus P_2) : q \in Q\}.$$

(We drop the parentheses in $h_1(q), h_2(q)$, etc. to avoid a surfeit thereof.) The reflexive, symmetric closure R of this relation is

$$\begin{array}{ll} & \{((h_1 q, 0), (0, h_2 q)) : q \in Q\} \\ \cup & \{((0, h_2 q), (h_1 q, 0)) : q \in Q\} \\ \cup & \{((p_1, p_2), (p_1, p_2)) : (p_1, p_2) \in P_1 \oplus P_2\}. \end{array}$$

Constructing the monoidal closure of R inductively, R_1 is given by

$$\begin{array}{ll} & \{((p_1, h_2 q + p_2), (h_1 q + p_1, p_2)) : q \in Q, p_1 \in P_1, p_2 \in P_2\} \\ \cup & \{((h_1 q + p_1, p_2), (p_1, h_2 q + p_2)) : q \in Q, p_1 \in P_1, p_2 \in P_2\}. \end{array}$$

Each parenthesized subset is closed under addition in $(P_1 \oplus P_2)^2$. We next construct

$$R_2 = \{((h_1 q + p_1, h_2 q' + p_2), (h_1 q' + p_1, h_2 q + p_2)) : q, q' \in Q, p_1 \in P_1, p_2 \in P_2\}.$$

It is straightforward to check that R_2 is reflexive, symmetric, and monoidal, hence $\sim$ is given by its transitive closure $\overline{R}_2^{\mathrm{tr}}$. Explicitly, we have $(p_1, p_2) \sim (p_1', p_2')$ iff there are sequences (for some $n \in \mathbb{N}$)

$$\begin{array}{rcl} q_0, \ldots, q_n & \in & Q \\ q_0', \ldots, q_n' & \in & Q \\ p_{1,0}, \ldots, p_{1,n} & \in & P_1 \\ p_{2,0}, \ldots, p_{2,n} & \in & P_2 \end{array}$$

satisfying the three conditions:

$$\begin{array}{rcll} (h_1q_0 + p_{1,0}, h_2q'_0 + p_{2,0}) & = & (p_1, p_2) & \\ (h_1q'_i + p_{1,i}, h_2q_i + p_{2,i}) & = & (h_1q_{i+1} + p_{1,i+1}, h_2q'_{i+1} + p_{2,i+1}) & i = 1, \dots, n-1 \\ (h_1q'_n + p_{1,n}, h_2q_n + p_{2,n}) & = & (p'_1, p'_2) & \end{array}$$

In particular, if $(p_1, p_2) \sim (p'_1, p'_2)$ is witnessed by such a sequence of length 1, then there are $r_1 \in P_1$, $r_2 \in P_2$ and $q, q' \in Q$ such that

$$\begin{array}{rcl} (p_1, p_2) & = & (r_1 + h_1q, r_2 + h_2q') \\ (p'_1, p'_2) & = & (r_1 + h_1q', r_2 + h_2q). \end{array}$$

Let $[p_1, p_2]$ denote the equivalence class of $(p_1, p_2) \in P_1 \oplus P_2$ under $\sim$. There is a cocartesian diagram

(1.3.0.1)
$$\begin{tikzcd} Q \arrow[r, "h_2"] \arrow[d, "h_1"'] & P_2 \arrow[d] \\ P_1 \arrow[r] & P_1 \oplus_Q P_2 \end{tikzcd}$$

where the maps to the pushout are given by $p_1 \mapsto [p_1, 0]$ and $p_2 \mapsto [0, p_2]$.

It turns out that, under some additional hypotheses, the congruence relation $\overline{R}_2^{\mathrm{tr}}$ has a simple description. We will return to the study of pushouts in Section 1.10. For now, we just note that pushouts under a group are less complicated:

Proposition 1.3.1. *Let Q be a group, $h_i : Q \to P_i$ $(i = 1, 2)$ monoid homomorphisms. The pushout $P_1 \oplus_Q P_2$ is the quotient of $P_1 \oplus P_2$ by the monoidal equivalence relation $\sim$ where $(p_1, p_2) \sim (p'_1, p'_2)$ iff there are $r_1 \in P_1$, $r_2 \in P_2$, and $q, q' \in Q$ such that*

$$\begin{array}{rcl} (p_1, p_2) & = & (r_1 + h_1q, r_2 + h_2q') \\ (p'_1, p'_2) & = & (r_1 + h_1q', r_2 + h_2q). \end{array}$$

Proof. From the discussion above, we see that the monoidal equivalence relation on $P_1 \oplus P_2$ whose quotient is $P_1 \oplus_Q P_2$ is the smallest monoidal equivalence relation containing $\sim$, so it suffices to prove that $\sim$ is a monoidal equivalence relation. It is easy to see that $\sim$ is reflexive, symmetric, and monoidal; the difficulty is to show that it is transitive. Suppose $(p_1, p_2) \sim (p'_1, p'_2)$ is witnessed by r_1, r_2, q, q' as in the statement of the proposition and $(p'_1, p'_2) \sim (p''_1, p''_2)$ is witnessed by s_1, s_2, t, t' so that

$$\begin{array}{rcl} (p'_1, p'_2) & = & (s_1 + h_1t, s_2 + h_2t') \\ (p''_1, p''_2) & = & (s_1 + h_1t', s_2 + h_2t). \end{array}$$

Comparing the two expressions for (p'_1, p'_2) we find

$$\begin{array}{rcl} r_1 + h_1q' & = & s_1 + h_1t \\ r_2 + h_2q & = & s_2 + h_2t. \end{array}$$

Set $w_1 := r_1 + h_1(-t) \in P_1$, $w_2 := r_2 + h_2(-t') \in P_2$. Then we see easily that

$$\begin{array}{rcl} (p_1, p_2) & = & (w_1 + h_1(t + q), w_2 + h_2(t' + q')) \\ (p''_1, p''_2) & = & (w_1 + h_1(t' + q'), w_2 + h_2(t + q)), \end{array}$$

hence $(p_1, p_2) \sim (p''_1, p''_2)$. □

As usual, the description of a *filtered* direct limit is considerably simpler:

Proposition 1.3.2. *If $F : \mathscr{C} \to \mathbf{Mon}$ is a filtered direct limit system of monoids, then*

$$\varinjlim F = \coprod_{C \in \mathscr{C}} FC / \sim,$$

where, for $p \in FC, q \in FD$, we have $p \sim q$ iff there is an object E of $\mathscr{C}$ and $\mathscr{C}$-morphisms $f : C \to E, g : D \to E$ with $F(f)(p) = F(g)(q)$. The sum $[p]+[q]$ is defined using directness to find representatives $p' \in FE$, $q' \in FE$ of $[p], [q]$ and setting $[p] + [q] := [p' + q']$.

In particular, the underlying set of a filtered direct limit of monoids is the direct limit of the underlying sets.

Proof. This is proved in the same manner that it would be proved in the category **Ab** of abelian groups. It is straightforward to check that the addition law in the statement of the theorem is well-defined and that the resulting monoid satisfies the correct universal property. $\square$

1.4. **Surjectivity.** In this section, we investigate the properties of monoid morphisms surjective on underlying sets and we establish various categorical properties of **Mon**.

As mentioned in the previous section, an epimorphism in **Mon** need not be surjective on underlying sets.

Example 1.4.1. The inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$ is an epimorphism in **Mon**. Indeed, if $f_1, f_2 : \mathbb{Z} \to P$ agree on $n \in \mathbb{N}$, then we must have $f_1(-n) = -f_1(n) = -f_2(n) = f_2(-n)$ because inverses are unique.

A map of monoids is *surjective* if it is surjective on underlying sets. Surjective morphisms can be characterized by a natural categorical property without mentioning underlying sets. Recall that in any category, a morphism $C \to D$ is called a *regular epimorphism* iff it is part of some coequalizer diagram $C' \rightrightarrows C \to D$.

Proposition 1.4.2. *For a map of monoids $h : Q \to P$, the following are equivalent:*

(1) The diagram

$$\begin{tikzcd} Q \times_P Q \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & Q \arrow[r, "h"] & P \end{tikzcd}$$

is a coequalizer diagram of monoids.

(2) h is a regular epimorphism in **Mon**.

(3) h is surjective.

Proof. Obviously (1) $\implies$ (2). The converse (2) $\implies$ (1) holds in any category where the fibered product $Q \times_P Q$ exists. Indeed, if $R \rightrightarrows Q \to P$ is a coequalizer diagram, then by the universal property of fibered products, we get a map $R \to Q \times_P Q$ making the diagram

$$\begin{tikzcd} Q \times_P Q \arrow[r, shift left] \arrow[r, shift right] & Q \arrow[r] & P \\ R \arrow[u] \arrow[ur, shift left] \arrow[ur, shift right] & & \end{tikzcd}$$

commute. It now follows easily that the row is a coequalizer diagram, for if $Q \to P'$ equalizes the two projections $Q \times_P Q \rightrightarrows Q$, then by commutativity it also equalizes the parallel arrows $R \rightrightarrows Q$.

The equivalence of (1) and (3) follows from the discussion in Section 1.2 about the construction of direct limits in **Mon**. Indeed, it is clear that any coequalizer map is surjective on underlying sets since we construct the coequalizer as a quotient by an equivalence relation on underlying sets. We also already noted that $Q \times_P Q$ is a monoidal equivalence relation on Q for any map $Q \to P$, and it is clear that if $Q \to P$ is surjective, then P is the quotient of Q by this equivalence relation. □

Recall that a pushout of an epimorphism in any category is again an epimorphism. In any category, an *image factorization* of a morphism $f : X \to Y$ is a factorization of f as a regular epimorphism $X \to \operatorname{Im} f$ followed by a monomorphism $\operatorname{Im} f \hookrightarrow Y$. A formal argument with universal properties shows that any such factorization is unique (in an obvious sense), so the subobject of Y determined by $\operatorname{Im} f$ is well-defined, and called the *image* f. For monoids, we have:

Theorem 1.4.3. *The category of monoids has the following properties.*

(1) Any morphism has an image factorization. Image factorizations are stable under pullback.
(2) A pushout of a surjective map is again surjective.

Proof. For the existence of an image factorization of $f : Q \to P$, take $\operatorname{Im} f$ to be the set theoretic image of f. It is a submonoid of P and clearly $Q \to \operatorname{Im} f$ is surjective, hence it is a regular epimorphism by Proposition 1.4.2. Note that a factorization of f is an image factorization iff the underlying set maps give an image factorization in **Ens**, so the second statement is a formal consequence of the fact that **Mon** $\to$ **Ens** preserves inverse limits and image factorizations are stable under pullback in **Ens**, which is easily checked.

Property (2) is a formal consequence of Proposition 1.4.2, proved as follows. Suppose $Q \to P$ is surjective and

$$\begin{array}{ccc} Q & \longrightarrow & R \\ \downarrow & & \downarrow \\ P & \longrightarrow & M \end{array}$$

is cocartesian. The two projections $Q \times_P Q \to Q$ followed by $Q \to R$ give two maps $Q \times_P Q \to R$ which, by commutativity, agree after composing with $R \to M$, and therefore determine a map $Q \times_P Q \to R \times_M R$ making the diagram

$$\begin{array}{ccc} Q \times_P Q & \longrightarrow & R \times_M R \\ \downarrow\downarrow & & \downarrow\downarrow \\ Q & \longrightarrow & R \\ \downarrow & & \downarrow \\ P & \longrightarrow & M \end{array}$$

commute. By (1.4.2), the left column is a coequalizer diagram and we can prove $R \to M$ is surjective by proving the right column is a coequalizer diagram. To prove this, suppose

$R \to N$ equalizes the two projections $R \times_M R \rightrightarrows R$. Then by commutativity, it also equalizes the two compositions $Q \times_P Q \rightrightarrows R$, so the coequalizer property of the left column yields a map $P \to N$ making the solid diagram

$$\begin{tikzcd}
Q \times_P Q \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & R \times_M R \arrow[d, shift left] \arrow[d, shift right] \\
Q \arrow[r] \arrow[d] & R \arrow[d] \arrow[ddr] \\
P \arrow[r] \arrow[drr] & M \arrow[dr, dotted] \\
& & N
\end{tikzcd}$$

commute. We obtain the dotted arrow making the resulting diagram commute from the cocartesian property of the square. The desired uniqueness property of this arrow follows from the uniqueness property of the cocartesian square and the uniqueness property of the coequalizer column on the left, hence the right column is a coequalizer diagram, as desired. □

Remark 1.4.4. It is much easier to establish (2) by this formal approach than to work directly with the construction of the pushout.

Remark 1.4.5. The fact that **Mon** has finite limits and satisfies (1) of the above proposition means it is a *regular* category. The fact that, furthermore, equivalence relations in **Mon** are effective means it is an *exact* category. These are typical hypotheses placed on categories in non-abelian homological algebra.

Question 1.4.6. Is the pushout of a monomorphism in **Mon** again a monomorphism? This is true under various additional hypotheses (c.f. Corollary 1.10.5).

1.5. **Monoid algebras.** There is a *monoid algebra* functor $\mathbb{Z}[\ \_\ ]$ from **Mon** to the category **An** of (commutative) rings. For a monoid P, $\mathbb{Z}[P]$ is defined to be the ring whose underlying abelian group is free on the elements of P: $\mathbb{Z}[P] := \oplus_{p \in P} \mathbb{Z}[p]$ and whose multiplication law is the unique $\mathbb{Z}$-linear extension of $[p][q] := [p+q]$:

$$\left(\sum_p n_p[p] \right) \left(\sum_p m_p[p] \right) = \sum_p \left(\sum_{q+r=p} n_q m_r \right) [p] \tag{1.5.0.1}$$

(the sums are finite in all cases in the sense that the coefficient of $[p]$ is zero for all but finitely many $p \in P$). Note that some authors use χ^p for the image of $p \in P$ in $\mathbb{Z}[P]$, whereas we use $[p]$. Notice that $1[0]$ is the multiplicative identity in $\mathbb{Z}[P]$. This is one point where the additive notation for the monoid law clashes with the usual notation for multiplication in a ring. Note that $\mathbb{Z}[\ \_\ ]$ applied to the map of monoids $\mathbb{N} \to 0$ gives the ring map

$$\begin{array}{rcl} \mathbb{Z}[x] & \to & \mathbb{Z} \\ x & \mapsto & 1 \end{array}$$

(not the map $x \mapsto 0$).

The functor $\mathbb{Z}[\ \_\ ]$ is left adjoint to the forgetful functor $A \mapsto (A, \cdot)$ from **An** to **Mon**:

$$\mathrm{Hom}_{\mathbf{An}}(\mathbb{Z}[P], A) \cong \mathrm{Hom}_{\mathbf{Mon}}(P, (A, \cdot)).$$

Note that $\mathbb{Z}[P]$ is a free $\mathbb{Z}$-module, so $\mathbb{Z} \to \mathbb{Z}[P]$ is a flat ring map. Obviously $\mathbb{Z}[\ \_\ ]$ takes monomorphisms to monomorphisms.

Since $\mathbb{Z}[\ \_\ ]$ is a left adjoint, it commutes with direct limits. In particular, for a cartesian diagram (3), the diagram

$$\begin{tikzcd} \mathbb{Z}[P] \arrow[r] \arrow[d] & \mathbb{Z}[Q_1] \arrow[d] \\ \mathbb{Z}[Q_1] \arrow[r] & \mathbb{Z}[Q_1 \oplus_P Q_2] \end{tikzcd}$$

is cartesian in the category **An** of rings:

$$\mathbb{Z}[Q_1 \oplus_P Q_2] \cong \mathbb{Z}[Q_1] \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q_2].$$

In particular, we have $\mathbb{Z}[P] \otimes \mathbb{Z}[Q] \cong \mathbb{Z}[P/Q]$ for a monoid map $Q \to P$. Here we view $\mathbb{Z}$ as the monoid algebra on the zero monoid, so note that $\mathbb{Z}[Q] \to \mathbb{Z}$ maps every $[q] \in \mathbb{Z}[Q]$ to $1 \in \mathbb{Z}$.

1.6. **Groupification.** For any monoid P, let P^{gp} be the associated "Grothendieck" group (or *groupification*) $P^{\mathrm{gp}} := P \oplus P/\sim$, where $(a,b) \sim (c,d)$ whenever there is a $p \in P$ such that $a + d + p = b + c + p$. Note that $\sim$ is a congruence relation on $P \oplus P$ and is, in fact, the smallest congruence relation containing the relation

$$\{((a,b),(c,d)) \in (P \oplus P) \times (P \oplus P) : a + d = b + c\}.$$

The functor $P \mapsto P^{\mathrm{gp}}$ is left adjoint to the inclusion $\mathbf{Ab} \hookrightarrow \mathbf{Mon}$:

$$\mathrm{Hom}_{\mathbf{Mon}}(P, G) \cong \mathrm{Hom}_{\mathbf{Ab}}(P^{\mathrm{gp}}, G).$$

The natural map $P \to P^{\mathrm{gp}}$ is an epimorphism in **Mon**, but not generally a surjection (c.f. Example 1.4.1). This is analogous to the fact that the localization of a ring $A \to S^{-1}A$ at a multiplicative set S is an epimorphism in **An**, but not generally a surjection on underlying sets.

As with the monoid algebra functor, groupification is a left adjoint so it commutes with direct limits. We have, for example

$$(\mathrm{Cok}\, h)^{\mathrm{gp}} = \mathrm{Cok}(h^{\mathrm{gp}}).$$

In light of this, we may unambiguously write $\mathrm{Cok}\, h^{\mathrm{gp}}$ without worrying about the parentheses.

Note that $P \mapsto P^{\mathrm{gp}}$ obviously does not commute with inverse limits: for $h = (1,1) : \mathbb{N}^2 \to \mathbb{N}$, we have $(\mathrm{Ker}\, h)^{\mathrm{gp}} = 0^{\mathrm{gp}} = 0$, but $\mathrm{Ker}(h^{\mathrm{gp}}) \cong \mathbb{Z}$.

1.7. **Integral monoids.** An element p of a monoid P is called *integral* iff

$$p + p_1 = p + p_2 \Longleftrightarrow p_1 = p_2$$

for all $p_1, p_2 \in P$. Evidently every unit is integral. If p and p' are integral elements of P, then

$$\begin{array}{rccl} & (p+p') + p_1 = (p+p') + p_2 & & \\ \Longrightarrow & p + (p' + p_1) = p + (p' + p_2) & & \text{(associativity)} \\ \Longrightarrow & p' + p_1 = p' + p_2 & & \text{(integrality of } p) \\ \Longrightarrow & p_1 = p_2 & & \text{(integrality of } p'), \end{array}$$

so $p+p'$ is integral. Certainly zero is integral, so integral elements of P form a submonoid $i : \operatorname{Int} P \hookrightarrow P$. This submonoid is characterized by the fact that

$$(\operatorname{Int} P) \times P \xrightarrow{i \times \pi_2 \times \pi_2} P \times P \times P \underset{\pi_1+\pi_3}{\overset{\pi_1+\pi_2}{\rightrightarrows}} P$$

is an equalizer diagram of monoids.

A monoid P is called *integral* if $P = \operatorname{Int} P$. An integral monoid is sometimes called *cancellative.*

Proposition 1.7.1. *For a monoid P, the following are equivalent:*

(1) P is integral.
(2) For any $p, p_1, p_2 \in P$ we have $p + p_1 = p + p_2$ iff $p_1 = p_2$.
(3) The diagram

$$P \times P \xrightarrow{\pi_1 \times \pi_2 \times \pi_2} P \times P \times P \underset{\pi_1+\pi_3}{\overset{\pi_1+\pi_2}{\rightrightarrows}} P$$

is an equalizer diagram of monoids.
(4) The adjunction morphism $P \to P^{\mathrm{gp}}$ is monic.
(5) P is a submonoid of a group.

Proof. The equivalence of the first three properties is clear. The equivalence of the last two statements is also clear because if A is a group, any map $P \to A$ factors through $P \to P^{\mathrm{gp}}$, so $P \to A$ can't be monic unless $P \to P^{\mathrm{gp}}$ is monic. To show the equivalence P integral $\iff P \to P^{\mathrm{gp}}$ monic, just recall from the construction of P^{gp} that p_1 and p_2 have the same image in P^{gp} iff there is some $p \in P$ so that $p + p_1 = p + p_2$. ☐

From these criteria, its is clear that a submonoid of an integral monoid is integral. Clearly a product of integral monoids is also integral, so, since an inverse limit of monoids is a submonoid of their product, we conclude that an inverse limit of integral monoids is integral. The case of direct limits is more subtle and will be discussed in Section 1.10.

Let **IMon** denote the full subcategory of **Mon** consisting of integral monoids. The inclusion **IMon** $\hookrightarrow$ **Mon** has a left adjoint $P \mapsto P^{\mathrm{int}}$, where P^{int} is the submonoid of P^{gp} given by the image of $P \to P^{\mathrm{gp}}$.

Example 1.7.2. A ring A is not an integral monoid unless it is the zero ring because

$$0 \cdot a = 0 \cdot 0$$

for all $a \in A$. However, A^* is certainly integral.

Lemma 1.7.3. *Let $Q \to P$ be a morphism of monoids with P integral. Then the cokernel P/Q is an integral monoid.*

Proof. We may use the description of $P/Q \cong P/\sim$ from Proposition 1.2.2. Suppose

$$[p_1] + [p] = [p_2] + [p]$$

in $P/\sim$. Then there are $q_1, q_2 \in Q$ such that

$$h(q_1) + p + p_1 = h(q_2) + p + p_2$$

in P. Since p is integral, this implies $h(q_1) + p_1 = h(q_2) + p_2$ in P and hence $[p_1] = [p_2]$ in $P/\sim$ as desired. ☐

Proposition 1.7.4. *A finite, integral monoid P is a group.*

Proof. Given an element p of P, by finiteness of P we have $np = mp$ for some $0 < m < n$. Writing $np = mp + (n-m)p$, we then have $mp + (n-m)p = mp + 0$, which implies $(n-m)p = 0$ by integrality of mp. This implies $p + (n-m-1)p = 0$, so p is invertible. □

An element s of a monoid P is a *sink* if $s + p = s$ for all $p \in P$. A monoid has at most one sink because if s' is another sink, then $s + s'$ is equal to both s and s'. Obviously a monoid with a sink cannot be integral unless it is the zero monoid. The element 0 is a sink in the multiplicative monoid of any ring.

Example 1.7.5. Let P_N be the monoid with elements $\{0, \dots, N\}$ and composition law

$$a + b := \max(a + b, N)$$

(the second "+" is ordinary addition). Evidently $P_N^{gp} = \{0\}$, so P_N is not integral (unless $N = 0$). The element N is a sink in P_N. Note that

$$\mathbb{Z}[P_N] \cong \mathbb{Z}[x]/\langle x^{N+1} - x^N \rangle \cong \mathbb{Z} \oplus \mathbb{Z}[x]/\langle x^N \rangle.$$

Example 1.7.6. The monoid P_1 from the previous example arises frequently. Up to isomorphism, it is the unique monoid with a two element underlying set which is not a group (i.e. not $\mathbb{Z}/2\mathbb{Z}$). P_1 is isomorphic to the multiplicative monoid $(\mathbb{F}_2, \cdot)$ of the two element field (due to unfortunate notation, the isomorphism exchanges 0 and 1) and, more generally, P_1 is the quotient of the multiplicative monoid of any field by its group of units.

Proposition 1.7.7. *If P is an integral monoid, then the monoid algebra $\mathbb{Z}[P]$ is reduced (has no nilpotent elements). If, furthermore, P^{gp} is torsion free, then $\mathbb{Z}[P]$ is a domain.*

Proof. I do not know if it is possible to prove this directly from the formula (1.5.0.1) for multiplication in $\mathbb{Z}[P]$. We will reduce to the case of $\mathbb{Z}[A]$ for a finitely generated abelian group A, where this can be checked directly using the classification of finitely generated abelian groups. A subring of a reduced ring (resp. an integral domain) is reduced (resp. an integral domain), and $\mathbb{Z}[P] \hookrightarrow \mathbb{Z}[P^{\mathrm{gp}}]$ is monic because $P \hookrightarrow P^{\mathrm{gp}}$ is monic by definition of *integral*, so we may certainly reduce to the case $\mathbb{Z}[A]$ for an abelian group A. Say $\alpha = \sum_p n_p[p]$ is a typical element of $\mathbb{Z}[A]$. Then there are finitely many $p \in A$ for which $n_p \neq 0$, so we see that any finite number of elements of $\mathbb{Z}[A]$ are contained in $\mathbb{Z}[B]$ for some finitely generated subgroup B of A, thus we reduce to the case of $\mathbb{Z}[A]$ for a finitely generated abelian group A. We have $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \cong \mathbb{Z}[t]/\langle t^n - 1\rangle$. We can prove this is reduced by including this ring in, say, $\mathbb{C}[t]/\langle t^n - 1\rangle \cong \oplus_n \mathbb{C}$, which is reduced (the point is that $t^n - 1$ has n distinct roots over $\mathbb{C}$). We also have $\mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[t, t^{-1}]$, which is integral (it is a subring of the field $\mathbb{Q}(t)$), and in particular, reduced. Now, since $\mathbb{Z}[\ \_\ ]$ takes direct sums to tensor products, we have $\mathbb{C}[A] = \oplus_n \mathbb{C}[t_1, t_1^{-1}, \dots, t_k, t_k^{-1}]$ for a general finitely generated abelian group of rank k with torsion subgroup of order n. Since $\mathbb{C}[A]$ contains $\mathbb{Z}[A]$ as a subring, the proof is complete. □

Remark 1.7.8. The converse of Proposition 1.7.7 is false. Consider the unique monoid P_1 on a two element set not isomorphic to $\mathbb{Z}_2$ (see Example 1.7.6). The monoid algebra

$$\mathbb{Z}[P_1] \cong \mathbb{Z}[x]/\langle x^2 - x\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$$

is reduced, but P_1 is not integral.

Remark 1.7.9. Even if P is a sharp, fine monoid, the monoid algebra $\mathbb{Z}[P]$, while reduced, can still have zero divisors. For example, take P equal to the submonoid of $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ generated by $x = (1,0)$ and $y = (1,1)$. This P is clearly sharp and fine and can be presented as the monoid generated by two generators x, y, subject to the relation $2x = 2y$. Note that

$$\begin{aligned} \mathbb{Z}[P] &= \mathbb{Z}[x,y]/(x^2 - y^2) \\ &= \mathbb{Z}[x,y]/((x+y)(x-y)) \end{aligned}$$

is not reduced.

Remark 1.7.10. It is clear from the proof of Proposition 1.7.7 (or other general nonsense) that the result is also valid wth $\mathbb{Z}$ replaced by, say, any field of characteristic zero. This result is no valid in positive characteristic, however. For example, if P is the sharp, fine monoid of the previous remark, then $\mathbb{F}_2[P]$ is not reduced.

Remark 1.7.11. The monoid algebra $\mathbb{Z}[P]$ on an integral monoid P is not generally an integral domain:

$$\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}] \cong \mathbb{Z}[t]/\langle t^2 - 1 \rangle \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Be aware that the terminology is misleading, but unfortunately, completely standard.

1.8. **Extensions.** Our next topic concerns integral extensions of an integral monoid by a group. Such extensions will arise frequently in log geometry with the group given by the group of units of a ring. Let A be a group (abelian, as always), Q an integral monoid. Let $\operatorname{Ext}^1_{\mathbf{Mon}}(Q, A)$ denote the set (of equivalence classes) of diagrams

$$A \xrightarrow{f} P \xrightarrow{g} Q$$

satisfying the conditions:

(1) P is an integral monoid.
(2) f is monic and g is a cokernel of f.

Lemma 1.7.3 implies that Q is also integral in this situation. Two such diagrams are equivalent if there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & P \arrow[r, "g"] \arrow[d, "h", "\cong"'] & Q \arrow[d, equal] \\ A \arrow[r, "f'"] & P' \arrow[r, "g'"] & Q \end{tikzcd}$$

with h an isomorphism of monoids. We call such a diagram an *extension of Q by A*, or just an *extension of integral monoids.* Such an extension is *split* if $P \cong A \oplus Q$ compatibly with the natural maps.

Proposition 1.8.1. *The assignment*

$$(A \to P \to Q) \mapsto (0 \to A \to P^{\mathrm{gp}} \to Q^{\mathrm{gp}} \to 0)$$

defines a natural bijection $\operatorname{Ext}^1_{\mathbf{Mon}}(Q, A) \cong \operatorname{Ext}^1_{\mathbf{Ab}}(Q^{\mathrm{gp}}, A)$. *In particular, an extension of integral monoids is split iff the corresponding extension of groups is split.*

Remark 1.8.2. Also recall that "$\operatorname{Ext}^0_{\mathbf{Mon}}(Q, A)$" $= \operatorname{Hom}_{\mathbf{Mon}}(Q, A) \cong \operatorname{Hom}_{\mathbf{Ab}}(Q^{\mathrm{gp}}, A)$.

Proof. The first issue it to show that $0 \to A \to P^{\mathrm{gp}} \to Q^{\mathrm{gp}} \to 0$ is exact. Surjectivity on the right follows from the surjectivity of $P \to Q$ by right exactness of groupification. Left exactness is clear since the map factors as $A \to P \to P^{\mathrm{gp}}$. For exactness in the middle, suppose $p_1 - p_2$ is zero in Q^{gp}. Integrality of Q implies that p_1 and p_2 have the same image under $P \to Q$, hence we have $p_1 = p_2 + u$ for some $u \in A^*$ by the description of $P \to Q$ in Proposition 1.2.3, so $p_1 - p_2 = u$ in P^{gp}.

Now, to define an inverse to the given map, we take an extension of groups

$$0 \longrightarrow A \xrightarrow{f} G \xrightarrow{g} Q^{\mathrm{gp}} \longrightarrow 0$$

to the extension of monoids obtained by considering the preimage of $Q \subseteq Q^{\mathrm{gp}}$ under g. This is contained in the group G so it is manifestly integral. We have a commutative diagram

$$\begin{tikzcd}
& A \arrow[r] \arrow[d, equal] & g^{-1}[Q] \arrow[r] \arrow[d, hook] & Q \arrow[d, hook] & \\
0 \arrow[r] & A \arrow[r, "f"] & G \arrow[r, "g"] & P^{\mathrm{gp}} \arrow[r] & 0
\end{tikzcd}$$

Surjectivity of $g : g^{-1}[Q] \to Q$ is clear from surjectivity of g. Exactness in the middle is clear from commutativity of the diagram. It is straightforward to check that this construction provides an inverse to the map in the statement of the proposition. □

1.9. **Finiteness.** A monoid P is *sharp* or *unit-free* if $P^* = \{0\}$. More generally, an element p of P is sharp if p is either zero, or not a unit. The sharp elements of P form a submonoid $P^\sharp$ of P because, if p and q are not units, then $p+q$ cannot be a unit, else we could write $(p+q)+r = 0$ and regroup terms using associativity to conclude that p and q are units. A morphism $P \to Q$ of monoids is *sharp* if the induced map of groups $P^* \to Q^*$ is an isomorphism. Evidently P is a sharp monoid iff $P \to 0$ is a sharp morphism iff $0 \to P$ is a sharp morphism. An element $p \in P$ is sharp iff the map $\mathbb{N} \to P$ sending 1 to p is sharp.

The full subcategory of **Mon** consisting of sharp monoids is denoted $\mathbf{Mon}^\sharp$. For any monoid P, the quotient monoid $\overline{P} := P/P^*$ is sharp. The functor $P \mapsto \overline{P}$ from **Mon** to $\mathbf{Mon}^\sharp$ is left adjoint to the inclusion $\mathbf{Mon}^\sharp \hookrightarrow \mathbf{Mon}$.

A monoid is *free* if it is isomorphic to a direct sum of copies of $\mathbb{N} = (\mathbb{N}, +)$. We denote $\oplus_n \mathbb{N}$ by $\mathbb{N}^n$ and write e_1 for the i^{th} standard basis vector. There is a free monoid functor $S \mapsto \oplus_S \mathbb{N}$ from **Ens** to **Mon** which is left adjoint to the forgetful functor $\mathbf{Mon} \to \mathbf{Ens}$:

$$\mathrm{Hom}_{\mathbf{Ens}}(S, P) \cong \mathrm{Hom}_{\mathbf{Mon}}(\oplus_S \mathbb{N}, P).$$

A monoid P is *finitely generated* if there is a surjection $\mathbb{N}^n \to P$ for some finite n. A monoid P is *fine* if it is integral and finitely generated. A monoid P is *finitely presented* if there is a coequalizer diagram

$$\mathbb{N}^m \rightrightarrows \mathbb{N}^n \to P$$

for some finite m, n. More generally, a morphism of monoids $f : Q \to P$ is *finite type* if there is a surjection $f + g : Q \oplus \mathbb{N}^n \to P$ and is *finitely presented* if there is a coequalizer diagram $\mathbb{N}^m \rightrightarrows Q \oplus \mathbb{N}^n \to P$.

An element e of a monoid P is *irreducible* or *primitive* if $a+b = e$ in P implies that a or b is a unit in P. More generally, if $h : Q \to P$ is a morphism of monoids, then an element

p of P is called *Q-primitive* if, whenever $p = p' + h(q)$ for some $q \in Q, p' \in P$, either p' or $h(q)$ must be a unit in P. Evidently the irreducible elements of P are the P-primitive elements of P (quoad the identity map $P \to P$).

In a sharp monoid P, an element e is irreducible iff $a + b = e$ implies $a = 0$ or $b = 0$.

Example 1.9.1. The irreducible elements of $\mathbb{N}^n$ are the standard basis vectors $e_1, \ldots, e_n$. An automorphism induces a bijection on irreducible elements, so it follows that $\mathrm{Aut}(\mathbb{N}^n)$ is the symmetric group S_n. In general, the automorphism group of the free monoid $\oplus_S \mathbb{N}$ is the automorphism group of S in the category of sets.

Lemma 1.9.2. *If P is an integral monoid, then P is fine iff P^* is a finitely generated abelian group and $\overline{P}$ is a finitely generated monoid.*

Proof. (Ogus) If P is fine (integral and finitely generated), then P^{gp} is finitely generated, hence so is $P^* \subseteq P^{\mathrm{gp}}$, and certainly $\overline{P}$ is finitely generated since $P \to \overline{P}$ is finitely generated. Conversely, if $\{p_i\}$ is a finite set of generators for P^* and $\{p'_i\}$ is a finite subset of P whose images generate $\overline{P}$, then it is easy to check that $\{p_i, -p_i, p'_i\}$ generates P. $\square$

Remark 1.9.3. If P is not integral, then $P^* \to P^{\mathrm{gp}}$ need not be injective. For example, if A is any ring regarded as a monoid under multiplication, then $a \cdot 0 = 0 \cdot 0$ for any $a \in A$, so every $a \in A$ maps to zero in A^{gp}. Never-the-less, we will see later in Corollary 1.9.9 that P^* is always a finitely generated abelian group when P is a finitely generated monoid, even without the assumption that P is integral.

Lemma 1.9.4. *If P is a fine, sharp monoid, then*

1. *The set $\mathrm{Irr}(P)$ of (nonzero) irreducible elements of P is finite and generates P.*
2. *The automorphism group of P injects into the group of permutations of $\mathrm{Irr}(P)$, hence is finite.*
3. *If P is nonzero, then it has a nonzero element fixed by any automorphism.*

Proof. (Ogus, Olsson) According to Olsson: To see (1), let $\{p_1, \ldots, p_n\}$ be a finite set of generators for P with n minimal. If $p \in \mathrm{Irr}(P)$, then p must be one of the p_i, for it we write $p \sum_i n_i p_i$, then irreducibility of p implies $\sum_i n_i = 1$. Hence $\mathrm{Irr}(P)$ is finite and there is an inclusion $\mathrm{Irr}(P) \subseteq \{p_1, \ldots, p_n\}$ which we claim is a bijection. Indeed, suppose one of the p_i, say p_n, is not irreducible. Then $p_n = p + q$ for some nonzero $p, q \in P$. The elements p and q must be in the submonoid of P generated by $\{p_1, \ldots, p_{n-1}\}$, otherwise we can write $p = p' + p_n$ for some $p' \in P$ (after possibly interchanging p and q), which implies that q is a unit, a contradiction. But if p and q are in the submonoid generated by $\{p_1, \ldots, p_{n-1}\}$, then so is p_n, contradicting minimality of n.

Statement (2) follows from (1) because any automorphism of P must map $\mathrm{Irr}(P)$ into itself (and it must be monic!), and $\mathrm{Irr}(P)$ generates P. For (3), we may take the sum of the irreducibles. This can't be zero, for then the irreducibles would be units. $\square$

For any monoid P, the relation $p_1 \leq p_2$ iff there is a $p \in P$ such that $p_1 + p = p_2$ defines a *monoidal quasi-ordering* on P, meaning that it is

1. *reflexive*: $p \leq p$ for all $p \in P$,
2. *transitive*: $p_1 \leq p_2$ and $p_2 \leq p_3$ implies $p_1 \leq p_3$, and
3. *monoidal*: $p_1 \leq p_2$ and $p'_1 \leq p'_2$ implies $p_1 + p'_1 \leq p_2 + p'_2$,

but it does *not* necessarily have the property:

$$p_1 \leq p_2 \text{ and } p_2 \leq p_1 \implies p_1 = p_2$$

(this fails miserably if, for example, P is a nontrivial group). The ordering $\leq$ is called the *monoid ordering* of P.

Proposition 1.9.5. *The relation $p_1 \simeq p_2$ iff $p_1 \leq p_2$ and $p_2 \leq p_1$ is a monoidal equivalence relation on P. If P is sharp and integral, then $\simeq$ is the relation of equality, so $P \cong P/\simeq$.*

Proof. Certainly $\simeq$ is an equivalence relation, since this would be true for any reflexive transitive ordering. If $p_1 \simeq p_2$ and $p_1' \simeq p_2'$, then $p_1 \leq p_2$ and $p_1' \leq p_2'$, hence $p_1 + p_1' \leq p_2 + p_2'$ since $\leq$ is monoidal, and we get the reverse equality by interchanging the subscripts $1, 2$. For the second part, if $p_1 \simeq p_2$, then $p_1 + p = p_2$ and $p_2 + p' = p_1$ for some $p, p' \in P$. Substituting the second expression for p_1 in the first expression, and using integrality of the element p_2 implies $p + p' = 0$, hence p and p' are units, so they are zero because P is sharp. $\square$

Theorem 1.9.6. *Let P be a finitely generated monoid.*

(1) Any infinite sequence $p_1, p_2, \ldots$ in P contains an increasing infinite subsequence $p_{i_1} \leq p_{i_2} \leq \ldots$ $(i_1 < i_2 < \ldots)$.
(2) Any decreasing sequence $p_1 \geq p_2 \geq \ldots$ in P eventually has constant image in $P/\simeq$.
(3) Any nonempty subset of P contains a minimal element with respect to $\leq$ and the image of the set of such minimal elements in $P/\simeq$ is finite.
(4) If P is sharp and integral, any decreasing sequence in P is eventually constant, and any nonempty subset of P has a finite nonzero number of minimal elements.

Proof. (Lenstra, Ogus) (1) We first prove this for $P = \mathbb{N}^n$. Here $p_1 \leq p_2$ iff each coordinate $p_{1,i}$ of p_1 is $\leq$ to the corresponding coordinate $p_{2,i}$ of p_2. After passing to an infinite subsequence, we may assume the first coordinates $p_{1,1}, p_{2,1}, p_{3,1} \ldots$ are increasing. Then after passing to an infinite subsequence, we may assume the second coordinates are increasing, and so on, so we get an infinite subsequence where every coordinate is increasing, hence the sequence is increasing in the monoid order on $\mathbb{N}^n$. For an arbitrary finitely generated P, we choose a surjection $f : \mathbb{N}^n \to P$, then lift the sequence $p_1, p_2, \ldots$ to a sequence $\overline{p}_1, \overline{p}_2, \ldots$ in $\mathbb{N}^n$, find an infinite increasing sequence $\overline{p}_{i_1} \leq \overline{p}_{i_2} \leq \ldots$ there, then note that a monoid homomorphism is a nondecreasing function, so $p_{i_1} \leq p_{i_2} \leq \ldots$ in P.

For (2), we may pass to the quotient $P/\simeq$ and prove the sequence is eventually constant. The quotient is again finitely generated, so by (1), there is an infinite increasing sequence in this infinite decreasing sequence. But $\leq$ gives a partial order on $P/\simeq$ so the sequence must be eventually constant.

For (3), we can again work in $P/\simeq$. If there were no minimal element in some nonempty set S, then we could build an infinite strictly decreasing sequence of elements of S, contradicting (2). If there were an infinite number of minimal elements, this would contradict (1).

(4) Follows from the previous results because $\simeq$ is the relation of equality on a fine, sharp monoid (Proposition 1.9.5).

$\square$

Recall that an object C of a category $\mathscr{C}$ is called *small* if, for any filtered direct limit system $(C_i)_{i\in I}$ in $\mathscr{C}$, the natural map

$$\varinjlim \operatorname{Hom}_{\mathbf{Mon}}(C, C_i) \to \operatorname{Hom}_{\mathbf{Mon}}(C, \varinjlim C_i)$$

is a bijection.

Theorem 1.9.7. *For any monoid P, the following are equivalent:*

(1) P is finitely generated.
(2) P is finitely presented.
(3) P is small in the category of monoids.

Every monoid is the direct limit of its finitely presented submonoids.

Proof. (Gabber, Grillet, Ogus) Obviously (2) $\implies$ (1). For the converse, suppose P is finitely generated, and choose a surjection $f : \mathbb{N}^n \to P$. Let $\sim$ be the monoidal equivalence relation

$$R = \{(q, r) \in \mathbb{N}^n \times \mathbb{N}^n : f(q) = f(r)\}$$

on $\mathbb{N}^n$. Certainly we have $P = \mathbb{N}^n/\sim$. It suffices to find a finite subset

$$R' = \{(q_1, r_1), \ldots, (q_m, r_m)\}$$

of R such that R is the smallest monoidal equivalence relation containing R', for then P can be exhibited as the coequalizer of

$$(q_1, \ldots, q_m), (r_1, \ldots, r_m) : \mathbb{N}^m \rightrightarrows \mathbb{N}^n.$$

Let $\leq$ denote the lexicographic well ordering of $\mathbb{N}^n$. For $q \in \mathbb{N}^n$, set

$$\begin{aligned} \mu(q) &:= \min_{\leq}\{r \in \mathbb{N}^n : f(r) = f(q)\} \\ &= \min_{\leq}\{r \in \mathbb{N}^n : (r, q) \in R\}, \end{aligned}$$

so that $\mu(q)$ is the $\leq$-minimum element of $f^{-1}(f(q))$. First I claim that R is the smallest equivalence relation on $\mathbb{N}^n$ containing the relation

$$E' := \{(q, \mu(q)) : q \in \mathbb{N}^n\}.$$

Since $E' \subseteq R$, the equivalence relation E that it generates is certainly contained in R. Suppose E is not all of R and choose $(r_1, r_2) \in E \setminus R$ with $r_1 + r_2$ minimal with respect to $\leq$ among all such elements. Set $r := \mu(r_1) = \mu(r_2)$. By definition of μ, we have $r \leq r_1, r_2$, hence

$$\begin{aligned} r + r_1 &\leq r_1 + r_2 \\ \text{and} \quad r + r_2 &\leq r_1 + r_2 \quad \text{in } \mathbb{N}^n. \end{aligned}$$

If one of these is an equality, it is easy to conclude that $(r_1, r_2) \in E$. If both inequalities are strict, then by minimality of our choice, we have $(r, r_1), (r, r_2) \in E$, and we can conclude $(r_1, r_2) \in E$ using transitivity. This proves the claim.

Next, note that $q \mapsto \mu(q)$ defines a monomorphism (of sets) $\mu : \mathbb{N}^n \to \mathbb{N}^n$ and set

$$\begin{aligned} K &:= \mathbb{N}^n \setminus \mu[\mathbb{N}^n] \\ &= \{k \in \mathbb{N}^n : \mu(k) \neq k\} \\ &= \{k \in \mathbb{N}^n : \mu(k) < k\}. \end{aligned}$$

Since $\mathbb{N}^n$ is a fine, sharp monoid, by Theorem 1.9.6(4), there is a finite subset S of minimal elements of K with respect to the *monoid ordering* of $\mathbb{N}^n$. I claim that the finite subset

$$R' := \{(s, \mu(s)) : s \in S\}$$

of R is as desired. Since $R' \subseteq R$, the congruence relation T generated by R' is also contained in R. By the claim proved in the previous paragraph, to show that $T = R'$ it suffices to show that T containes E'. Suppose not, and choose $(q, \mu(q)) \in E' \setminus T$ with q minimal with respect to $\leq$ among such elements. Certainly $(\mu(q), \mu(q))$ would be in T, so we must have $q \in K$, hence we can write $q = s + r$ for some $s \in S$. Since s is also in K, we have $\mu(s) < s$, hence

$$q' := \mu(s) + r < s + r = q.$$

By minimality of our choice of q, we have $(q', \mu(q')) \in T$. Also, since $(s, \mu(s)), (r, r) \in T$ and T is a congruence relation, we have $(s + r, \mu(s) + r) = (q, q') \in T$. But in particular, this means that $(q, q') \in R$, hence $\mu(q) = \mu(q')$, and we conclude using transitivity that $(q, \mu(q))$ is in T after all. This proves (1) $\Longrightarrow$ (2).

To prove the final statement of the theorem, note that certainly every monoid P can be written as the filtered direct limit $\varinjlim P_i$ of its finitely *generated* submonoids P_i, but these are also finitely presented by what we just proved.

Now, to prove (3) $\Longrightarrow$ (2), we take F to be the filtered direct limit system of finitely presented submonoids of P, so $\varinjlim F = P$. If P is small, then the identity map $P \to P$ factors through the inclusion of a finitely presented submonoids P_i of P, so we must have $P_i = P$.

It remains to prove (2) $\Longrightarrow$ (3). First note that $\mathbb{N}$ is small because a map from $\mathbb{N}$ to a monoid Q is just an element of Q, and the underlying set of a filtered direct limit is the direct limit of the underlying sets (Proposition 1.3.2). Now the desired result follows formally from the fact that, in any category, a finite inverse limit of small objects is small (to prove this, use the universal property of finite inverse limits to write the Hom out of the inverse limit as the inverse limit of the Hom's, and use the fact that finite inverse limits commute with filtered direct limits in the category of sets).

□

Corollary 1.9.8. *Let Q_1, Q_2 be finitely generated monoids, $f_i : Q_i \to P$ monoid homomorphisms. Then the fibered product $Q_1 \times_P Q_2$ is finitely generated. For any finitely generated monoid Q and any two monoid homomorphisms $f_1, f_2 : Q \rightrightarrows P$, the equalizer of the f_i is finitely generated. Any finite direct or inverse limit of finitely generated monoids is finitely generated.*

Proof. We reduce the first statement to the second by noting that $Q_1 \times_P Q_2$ is the equalizer of $f_1\pi_1, f_2\pi_2 : Q_1 \times Q_2 \rightrightarrows P$ and the product (=sum) $Q_1 \times Q_2$ is finitely generated. For the second statement, we choose a surjection $g : \mathbb{N}^n \to Q$, then we note that the equalizer of $f_1 g, f_2 g : \mathbb{N}^n \rightrightarrows Q$ clearly surjects onto the equalizer of the f_i, so we reduce to proving the second statement when $Q = \mathbb{N}^n$; this is proved exactly as we proved (2) $\Longrightarrow$ (1) in the theorem.

The finite generation of a finite direct limit of finitely generated monoids is clear since a finite direct limit is a quotient of a finite direct sum. For finite inverse limits, one reduces formally to finite products and equalizers. □

Corollary 1.9.9. *Suppose P is a finitely generated monoid. Then the group P^* of units in P is a finitely generated abelian group.*

Proof. Let $i : P^* \hookrightarrow P$ be the inclusion. Then the diagram

$$\begin{tikzcd} P^* \arrow[r] \arrow[d, "{(i,-i)}"'] & 0 \arrow[d] \\ P \oplus P \arrow[r, "+"] & P \end{tikzcd}$$

is clearly cartesian, hence P^* is expressed as a finite inverse limit of finitely generated monoids and is hence finitely generated by the previous corollary. $\square$

Remark 1.9.10. If Q is integral, then for any monoid maps $f_1, f_2 : Q \to P$, the equalizer of f_1, f_2 is a submonoid of Q and is hence integral. Clearly a product of integral monoids is integral, so we conclude that any inverse limit of integral monoids is integral (the failure of the analogous statement for direct limits is the subject of the next section). The above corollary implies that any finite inverse limit of fine monoids is fine.

Remark 1.9.11. A submonoid of a finitely generated monoid need not be finitely generated. Consider the submonoid of $\mathbb{N}^2$ generated by $(0, 1), (1, 1), (2, 1), (3, 1), \dots$.

Corollary 1.9.12. *If P and Q are finitely generated monoids, then so is the monoid $P^Q = \mathrm{Hom}_{\mathbf{Mon}}(Q, P)$.*

Proof. By the theorem, Q is finitely presented, so we can write Q as a coequalizer of arrows $f_1, f_2 : \mathbb{N}^n \rightrightarrows \mathbb{N}^m$ for some finite m, n. By the universal property of coequalizers, $\mathrm{Hom}_{\mathbf{Mon}}(Q, P)$ is the equalizer of $f_1^*, f_2^* : \mathrm{Hom}_{\mathbf{Mon}}(\mathbb{N}^m, P) \rightrightarrows \mathrm{Hom}_{\mathbf{Mon}}(\mathbb{N}^n, P)$. Since $\mathrm{Hom}_{\mathbf{Mon}}(\mathbb{N}^n, P) = P^n$ by the adjointness property of the free monoid functor (and similarly with n replaced by m), this equalizer is an equalizer of the maps of finitely generated monoids $P^m \rightrightarrows P^n$ and is hence finitely generated by the previous corollary. $\square$

Remark 1.9.13. If Q is integral, then for any monoid P, the monoid $\mathrm{Hom}_{\mathbf{Mon}}(P, Q) = \mathrm{Hom}_{\mathbf{Mon}}(P^{\mathrm{int}}, Q)$ is easily seen to be integral.

1.10. **Integral morphisms.** Let $h : Q \to P$ be a morphism of monoids. The morphism h is called *integral* iff, for all $q_1, q_2 \in Q$, $p_1, p_2 \in P$, the condition

$$h(q_1) + p_1 = h(q_2) + p_2$$

implies there are $q_3, q_4 \in Q$ and $p \in P$ satisfying the following conditions:

$$\begin{aligned} p_1 &= h(q_3) + p \\ p_2 &= h(q_4) + p \\ q_1 + q_3 &= q_2 + q_4 \end{aligned} \tag{1.10.0.1}$$

(The third condition is automatic when h is monic and P is integral.)

Despite having the most mysterious definition, integral morphisms between integral monoids are an extremally important class of morphisms. Given the delicacy of direct limits in **Mon** (Section 1.2 it is perhaps not surprising that the pushout $P_1 \oplus_Q P_2$ of maps $h_i :: Q \to P_i$ of integral monoids need not be integral. Recall from Section 1.2 that the pushout $P_1 \oplus_Q P_2$ can be constructed as the quotient of $P_1 \oplus P_2$ by the transitive closure $\overline{R}_2^{\mathrm{tr}}$ of the relation

$$R_2 = \{((h_1q + p_1, h_2q' + p_2), (h_1q' + p_1, h_2q + p_2)) : q, q' \in Q, p_1 \in P_1, p_2 \in P_2\}.$$

Consider the relation S on $P_1 \oplus P_2$ where $((p_1, p_2), (p_1', p_2'))$ is in S iff there are $q, q' \in Q$ such that

$$(h_1q + p_1, h_2q' + p_2) = (h_1q' + p_1', h_2q + p_2')$$

in $P_1 \oplus P_2$. We say q, q' *witnesses* $((p_1, p_2), (p_1', p_2')) \in S$.

Lemma 1.10.1. *The relation S is a monoidal equivalence relation (congruence relation) containing R_2, hence containing the transitive closure of R_2. Suppose at least one of the following holds:*

(1) P_1 is a group.
(2) P_1 and P_2 are integral monoids and h_1 is integral.

Then $\overline{R}_2^{\mathrm{tr}} = S$, hence $P_1 \oplus_Q P_2$ is given by the quotient of $P_1 \oplus P_2$ by the monoidal equivalence $\sim$ where $(p_1, p_2) \sim (p_1', p_2')$ iff there are $q, q' \in Q$ such that

$$(h_1q + p_1, h_2q' + p_2) = (h_1q' + p_1', h_2q + p_2')$$

in $P_1 \oplus P_2$.

Proof. Obviously S is reflexive and symmetric (reverse the roles of q, q' for symmetry). For transitivity, suppose $((p_1, p_2), (p_1', p_2')) \in S$, witnessed by q, q' and $((p_1', p_2'), (p_1'', p_2'')) \in S$, witnessed by r, r'. Then we easily find that

$$(h_1(q + r) + p_1, h_2(q' + r') + p_2) = (h_1(q' + r') + p_1'', h_1(q + r) + p_2''),$$

so $q + r, q' + r'$ witnesses $((p_1, p_2), (p_1'', p_2'')) \in S$. The proof that S is monoidal is nearly identical.

To prove that $R_2 \subseteq S$, just note that a typical element $((h_1q + p_1, h_2q' + p_2), (h_1q' + p_1, h_2q + p_2))$ of R_2 is in S because q', q witnesses it:

$$((h_1q' + h_1q + p_1, h_2q + h_2q' + p_2), (h_1q + h_1q' + p_1, h_2q' + h_2q + p_2)).$$

To prove S is contained in the transitive closure of R_2 when P_1 is a group, suppose $((p_1, p_2), (p_1', p_2')) \in S$, so there are $q, q' \in Q$ such that

$$\begin{aligned} h_1q + p_1 &= h_1q' + p_1' \\ h_2q' + p_2 &= h_2q + p_2'. \end{aligned}$$

Set $p = p_1 - h_1q' = p_1' - h_1q$ in P_1. By definition of R we have

$$\begin{array}{rcl} & ((h_1q' + p, p_2), (p, h_2q' + p_2)) & \in R \\ = & ((p_1, p_2), (p, h_2q' + p_2)) & \in R \\ = & ((p_1, p_2), (p, h_2q + p_2')) & \in R \end{array}$$

and

$$\begin{array}{rcl} & ((h_1q + p, p_2'), (p, h_2q + p_2')) & \in R \\ = & ((p_1', p_2'), (p, h_2q + p_2')) & \in R \end{array}$$

so we conclude $((p_1, p_2), (p_1', p_2')) \in \overline{R}_2^{\mathrm{tr}}$ by transitivity.

To prove S is contained in the transitive closure of R_2 under hypothesis (2), suppose $((p_1,p_2),(p_1',p_2')) \in S$, so there are $q,q' \in Q$ such that

$$h_1 q + p_1 = h_1 q' + p_1' \tag{1.10.1.1}$$
$$h_2 q' + p_2 = h_2 q + p_2'. \tag{1.10.1.2}$$

Then since h_1 satisfies the equational criterion, there are $p \in P, q_3, q_4 \in Q$ such that

$$p_1 = h_1 q_3 + p \tag{1.10.1.3}$$
$$p_1' = h_1 q_4 + p \tag{1.10.1.4}$$
$$q + q_3 = q' + q_4. \tag{1.10.1.5}$$

Adding (1.10.1.2) and h_2 applied to (1.10.1.5) and using integrality of $h_2 q, h_2 q'$ in P_2, we find

$$h_2 q_3 + p_2 = h_2 q_4 + p_2'.$$

By definition of R_2, we have

$$\begin{aligned} & ((p_1,p_2),(p,h_2q_4+p_2')) \\ = \; & ((h_1q_3+p,p_2),(p,h_2q_4+p_2')) \in R_2 \end{aligned}$$

and

$$\begin{aligned} & ((p_1',p_2'),(p,h_2q_4+p_2')) \\ = \; & ((h_1q_4+p,p_2'),(p,h_2q_4+p_2')) \in R_2, \end{aligned}$$

so $((p_1,p_2),(p_1',p_2')) \in \overline{R}_2^{\mathrm{tr}}$ by transitivity.

□

Proposition 1.10.2. *For a morphism $h : Q \to P$ between integral monoids, the following are equivalent:*

(1) For any morphism $h' : Q \to P'$ with P' integral, the pushout $P \oplus_Q P'$ is integral.
(2) h satisfies the equational criterion for integrality.

Proof. (K. Kato) For (1) $\Longrightarrow$ (2), suppose $q_1, q_2 \in Q$, $p_1, p_2 \in P$ satisfy:

$$h(q_1) + p_1 = h(q_2) + p_2. \tag{1.10.2.1}$$

Let $P' := (Q \oplus \mathbb{N}^2)/\sim$, where $\sim$ is the monoidal equivalence relation defined by

$$(q,m,n) \sim (q',m',n')$$

iff the following hold:

$$\begin{aligned} m + n &= m' + n' \\ q + mq_1 + nq_2 &= q' + m'q_1 + n'q_2. \end{aligned}$$

One sees easily that this P' is integral, and that

$$[q_1,0,1] = [q_2,1,0] \tag{1.10.2.2}$$

in P'. Regard P' as a monoid under Q via $q \mapsto [q,0,0]$. By assumption, $P \oplus_Q P'$ is integral. I claim that

$$[p_1,[0,1,0]] = [p_2,[0,0,1]] \tag{1.10.2.3}$$

in P'. Since P' is integral and we have the equality (1.2.2.1), it suffices to prove that

$$[p_1,[q_1,1,1]] = [p_2,[q_2,1,1]],$$

and this is obvious from (1.10.2.2). Recall from the beginning of the section that the pushout $P \oplus_Q P'$ is given as the quotient of $P \oplus P'$ by the transitive closure of the relation $R_2 \subseteq (P \oplus P')^2$ consisting of pairs

$$((h(q) + p, [r + q', m, n]), (h(q') + p, [r + q, m, n]))$$

where $p \in P, [r, m, n] \in P', q, q' \in Q$. The equality (1.10.2.3) means that

$$((p_1, [0, 1, 0]), (p_2, [0, 0, 1])) \in (P \oplus P')^2$$

is in the transitive closure of R_2. For simplicity, let us just treat the case where it is actually in R_2 (the general case is similar, but the notation is cumbersome). Then we have

$$\begin{aligned} (p_1, [0, 1, 0]) &= (h(q) + p, [r + q', m, n]) \\ (p_2, [0, 0, 1]) &= (h(q') + p, [r + q, m, n]) \end{aligned}$$

for some $p \in P, [r, m, n] \in P', q, q' \in Q$. This means:

$$\begin{aligned} p_1 &= h(q) + p \\ p_2 &= h(q') + p \\ m + n &= 1 \\ q_1 &= r + q' + mq_1 + nq_2 \\ q_2 &= r + q + mq_1 + nq_2. \end{aligned}$$

The third equality above is irrelevant for our purposes, but the first two equalities, together with the obvious consequence

$$q_1 + q = q_2 + q'$$

of the third equality imply that h satisfies the equational criterion for integrality as desired.

For (2) $\implies$ (1), we use the description of $P \oplus_Q P'$ as the quotient of $P \oplus P'$ by the relation $\sim$ of Lemma 1.10.1. Suppose

$$[p, p'] + [p_1, p_1'] = [p, p'] + [p_2, p_2']$$

in P. Then there are $q, q' \in Q$ satisfying the conditions:

$$\begin{aligned} h(q) + p + p_1 &= h(q') + p + p_2 && \in P \\ h'(q') + p' + p_1' &= h'(q) + p' + p_2' && \in P'. \end{aligned}$$

By integrality of $p \in P$ and $p' \in P'$ we can drop the p and p' from both sides of these equalities to get equalities proving $[p_1, p_1'] = [p_2, p_2']$. ☐

A morphism of integral monoids satisfying the equivalent conditions in the lemma is called *integral*. We will refer to the first condition as the *pushout criterion* and the second condition as the *equational criterion (for integrality)*. If we say simply that $h : Q \to P$ is an *integral morphism*, then it is understood that P and Q are integral monoids.

The basic facts about integral morphisms are summed up in the following

Proposition 1.10.3. *Let P and Q be integral monoids.*

1. *If $h : Q \to P$ is a morphism with $h[Q] \subseteq P^*$, then h is an integral morphism.*
2. *The zero morphism $0 : Q \to P$ is an integral morphism.*
3. *If Q or P is a group, then any morphism $Q \to P$ is an integral morphism.*
4. *A pushout of an integral morphism is an integral morphism.*

(5) *The composition of two integral morphims is an integral morphism.*
(6) *For any morphism $h : Q \to P$, the cokernel map $P \to P/Q$ is an integral morphism.*
(7) *A morphism $h : Q \to P$ is integral iff the induced map of sharp monoids $\overline{h} : \overline{Q} \to \overline{P}$ is an integral morphism.*
(8) *If Q is generated by one element, then any morphism $Q \to P$ is integral.*

Proof. (1) Take $q_3 = q_2, q_4 = q_1$, $p = p_1 - h(q_2) = p_2 - h(q_1)$ in the equational criterion.

(2) is a consequence of (1), or just take $p = p_1 = p_2, q_3 = q_2, q_4 = q_1$ in the equational criterion.

(3) is an immediate consequence of (1).

(4) This is immediate from the pushout criterion and the fact that a pushout of a pushout is a pushout of the original.

(5) This follows from the pushout criterion and the fact that a pushout of gf is a pushout of f followed by a pushout of g.

(6) is a special case of (4).

(7) We will use the equational criterion for both implications. Note that $\overline{P}$ and $\overline{Q}$ are integral monoids and the maps $P \to \overline{P}, Q \to \overline{Q}$ are integral morphisms by (3) and (6).

($\Longrightarrow$) Suppose $\overline{h}[q_1] + [p_1] = \overline{h}[q_2] + [p_2]$ in $\overline{P}$. Then $h(q_1) + p_1 = h(q_2) + p_2 + u$ in P for some $u \in P^*$. Since h is integral, there are $q_3, q_4 \in Q, p \in P$ such that

$$\begin{aligned} p_1 &= h(q_3) + p \\ p_2 + u &= h(q_4) + p \\ q_1 + q_3 &= q_2 + q_4 \end{aligned}$$

and hence $[q_3], [q_4] \in \overline{Q}, [p] \in \overline{P}$ satisfy

$$\begin{aligned} [p_1] &= \overline{h}[q_3] + [p] \\ [p_2] &= \overline{h}[q_4] + [p] \\ [q_1] + [q_3] &= [q_2] + [q_4]. \end{aligned}$$

($\Longleftarrow$) Suppose $h(q_1) + p_1 = h(q_2) + p_2$. Then $\overline{h}[q_1] + [p_1] = \overline{h}[q_2] + [p_2]$, so by integrality of $\overline{h}$ there are $q_3, q_4 \in Q, p \in P$, and units $u_1, u_2 \in P^*, v \in Q^*$ such that

$$\begin{aligned} p_1 &= h(q_3) + p + u_1 \\ p_2 &= h(q_4) + p + u_2 \\ q_1 + q_3 &= q_2 + q_4 + v. \end{aligned}$$

After replacing q_4 by $q_4 + v$ and u_2 by $u_2 + h(-v)$ we may assume $v = 0$. Substituting for p_1, p_2 in the first supposition, we have

$$h(q_1 + q_3) + p + u_1 = h(q_2 + q_4) + p + u_2.$$

By integrality of the element $h(q_1 + q_3) + p = h(q_2 + q_4) + p$ of P, we conclude $u_1 = u_2$. The elements $q_3, q_4 \in Q$, $p + u_1 = p + u_2 \in P$ therefore satisfy the desired relations.

(8) Use the equational criterion. Let q be a generator of Q. Suppose

$$h(q_1) + p_1 = h(q_2) + p_2.$$

Write $q_1 = mq, q_2 = nq$. Without loss of generality, we may assume $m \geq n$. Rewriting the equation, we have

$$h((m-n)q) + h(nq) + p_1 = h(nq) + p_2.$$

Set $p = p_1, q_3 = 0, q_4 = (m-n)q$. Using integrality of the element $h(nq)$ in the above equation, we see that $p_2 = h(q_4) + p$. The other necessary equalities are obvious.

□

Proposition 1.10.4. *For a morphism $h : Q \to P$ of integral monoids, the following are equivalent:*

(1) h is monic and integral.
(2) The induced map of monoid algebras $\mathbb{Z}[Q] \to \mathbb{Z}[P]$ is flat.
(3) The map of monoid algebras $k[Q] \to k[P]$ is flat for any field k.

Proof. Proposition 4.1 of [K. Kato]. □

Corollary 1.10.5. *The pushout of an integral monomorphism $P \hookrightarrow Q_1$ along a map $P \to Q_2$ with Q_2 integral is again an integral monomorphism.*

Proof. This is a consequence of the proposition, together with the fact that $\mathbb{Z}[\,\_\,]$ preserves pushouts and the pushout of a flat ring map is flat. □

The following theorem is an important characterization of integral monomorphisms of fine, sharp monoids. In the case of a monomorphism $h : Q \hookrightarrow P$ of sharp monoids, recall that $p \in P$ is called *Q-primitive* iff, whenever $p = p' + h(q)$ for $p' \in P, q \in Q$, either p' or q is zero.

Theorem 1.10.6. *Let $h : Q \hookrightarrow P$ be a monomorphism of fine, sharp monoids. Then the following are equivalent:*

(1) h is integral.
(2) Every equivalence class $[p] \in P/Q$ contains a unique Q-primitive element and every element $p \in P$ has a unique expression $p = p' + q$ with p' Q-primitive and $q \in Q$.

Proof. (1) $\implies$ (2) By Theorem 1.9.6 each such equivalence class has finitely many minimal elements with respect to the monoid ordering. Suppose there is more than one such minimal element and let p_1, p_2 be two such elements. By the cokernel description of Proposition 1.2.2, the fact that $[p_1] = [p_2] \in P/Q$ implies there are $q_1, q_2 \in Q$ with $p_1 + q_1 = p_2 + q_2$. By the equational criterion, we can write $p_1 = q_3 + p, p_2 = q_4 + p$ for some $p \in P$. But then $[p] = [p_1] = [p_2]$ and $p \leq p_1, p_2$, so $p = p_1 = p_2$ by minimality and the fact that the monoid order equivalence relation is equality (1.9.5). The second statement follows from the fact that if $p' + q = p + q'$, then $[p] = [p']$, so if p, p' are minimal elements in $[p]$, then they must agree.

(2) $\implies$ (1) We will check the equational criterion. If $q_1 + p_1 = q_2 + p_2$, then $[p_1] = [p_2] \in P/Q$, so if p is the unique Q-primitive in $[p_1] = [p_2]$, then we can write $p_1 = p + q_3, p_2 = p + q_4$ for some $q_3, q_4 \in Q$. Substituting in the original equality and using integrality of the element p, we find $q_1 + q_3 = q_2 + q_4$. □

The property of being Q-primitive enjoys the following "stability under pushout":

Lemma 1.10.7. *Let $h : Q \to P$ be an integral monomorphism of sharp, integral monoids, satisfying Condition 2 of Theorem 1.10.6 and let $f : Q \to R$ be any morphism to a sharp, integral monoid. Then the pushout $P \oplus_Q R$ is a sharp, integral monoid, and the pushout map $R \to P \oplus_Q R$ is an integral monomorphism (with the same cokernel as h) satisfying Condition 2 of Theorem 1.10.6. Furthermore, for any Q-primitive $p \in P$, $[p, 0] \in P \oplus_Q Q$ is R-primitive.*

Proof. The map $p \mapsto [p, 0]$ induces an isomorphism on cokernels for formal reasons (direct limits commute amongst themselves). Due to the integrality assumption on h, $P \oplus_Q R$ is integral, $R \to P \oplus_Q R$ is an integral morphism, and (by Lemma 1.10.1) $P \oplus_Q R$ is the quotient of $P \oplus R$ by the monoidal equivalence relation $\sim$ where $(p, r) \sim (p', r')$ iff there are $q, q' \in Q$ with

$$(p + q, r + f(q')) = (p' + q', r' + f(q)).$$

All the statements follow easily from this. □

The following simple variant of the above lemma, which makes no mention of primitivity, is simple and useful for describing a pushout of monoids *as a set*; this is particularly useful when one has to describe a pushout of sheaves of monoids.

Lemma 1.10.8. *Let $Q \hookrightarrow P$ be an integral monomorphism of integral monoids. Suppose that there is a subset $S \subseteq P$ such that the addition map $(s, q) \mapsto s + q$ defines a bijection of sets $S \times Q \to P$. Let $f : Q \to R$ be an arbitrary morphism of monoids with R integral. Then the map $(s, r) \mapsto [s, r]$ defines a bijection of sets $S \times R \to P \oplus_Q R$.*

Proof. The surjectivity requires no assumptions at all about Q, P, R or the maps between them, and only uses surjectivity of the original addition map. Given $[p, r] \in P \oplus_Q R$, just find $s \in S$ and $q \in Q$ so $p = s + q$. Then the computation

$$[p, r] = [s, r] + [q, 0] = [s, r] + [0, f(q)] = [s, r + f(q)]$$

shows $[p, r]$ is in the image of the map in question. For injectivity, suppose $[s, r] = [s', r']$ in $P \oplus_Q R$ for some $s, s' \in S$, $r, r' \in R$. We want to show that $s = s'$ and $r = r'$. Now, using all our hypotheses on Q, P, R and $Q \hookrightarrow P$, we can appeal to Lemma 1.10.1 so conclude that $[s, r] = [s', r']$ implies the existence of $q, q' \in Q$ such that $s + q = s' + q'$ in P and $r + f(q') = r' + f(q)$ in R. By the injectivity of the original addition map, the first of these equalities implies $s = s'$ and $q = q'$, so the second of the equalities says $r + f(q) = r' + f(q)$. But this implies $r = r'$ because R is integral. □

1.11. **Types of morphisms.** In this section we define several important types of morphisms of monoids and study their basic properties.

A monoid homomorphism $h : Q \to P$ is of *Kummer type* iff h is monic and for all $p \in P$ there is a positive integer n such that np is in the image of h. The morphism h is *exact* iff the diagram

$$\begin{tikzcd} Q \arrow[r, "h"] \arrow[d] & P \arrow[d] \\ Q^{\mathrm{gp}} \arrow[r, "h^{\mathrm{gp}}"] & P^{\mathrm{gp}} \end{tikzcd}$$

is cartesian. The morphism h is called *vertical* iff $\operatorname{Cok} h$ is a group. These notions will be most pertinent when Q, P are integral monoids, but we can make the definitions without that assumption.

Lemma 1.11.1. *Suppose $h : Q \to P$ is a morphism of Kummer type with P integral and Q saturated. Then h is exact.*

Proof. Since h is monic, we suppress it from the notation. Suppose $q_1 - q_2 = p$ in P^{gp} for some $q_1, q_2 \in Q$ and $p \in P$. We want to prove that $q_1 - q_2 \in Q$. We have $q_1 = p + q_2$ in P. Since h is of Kummer type, there is an $n \in \mathbb{Z}_{>0}$ and a $q \in Q$ with $np = q$, hence $nq_1 = q + nq_2$, hence $n(q_1 - q_2) \in Q$, hence $q_1 - q_2 \in Q$ by saturation. □

Proposition 1.11.2. *Suppose $h : Q \to P$ is a morphism of integral monoids so that the induced map $h^* : Q^* \to P^*$ on units is an isomorphism of abelian groups. Let $\overline{h} : \overline{Q} \to \overline{P}$ be the induced map of sharp monoids. Then:*

1. *(1) h is integral iff $\overline{h}$ is integral.*
2. *(2) h is of Kummer type iff $\overline{h}$ is of Kummer type.*
3. *(3) h is exact iff $\overline{h}$ is exact.*
4. *(4) $\operatorname{Cok} h = \operatorname{Cok} \overline{h}$. In particular, h is vertical iff $\overline{h}$ is vertical.*

Proof. Exercise. □

1.12. **Saturation.** An integral monoid is *saturated* if, for any $p \in P^{\mathrm{gp}}$ with $np \in P$, we have $p \in P$. The inclusion $\mathbf{Mon}^{\mathrm{int}}$ into the category $\mathbf{Mon}^{\mathrm{sat}}$ also has a left adjoint $P \mapsto P^{\mathrm{sat}}$, where P^{sat} is the submonoid

$$\{p \in P^{\mathrm{gp}} : \exists\, n > 0,\ np \in P\}$$

of P^{gp}. For an arbitrary monoid P, one can set $P^{\mathrm{sat}} := (P^{\mathrm{int}})^{\mathrm{sat}}$ and obtain a similar adjoint to $\mathbf{Mon} \to \mathbf{Mon}^{\mathrm{sat}}$.

More generally, a morphism of monoids $h : P \to Q$ is a *saturated morphism* if, whenever nq is in the image of h for some $q \in Q$, $n > 0$, then q is in the image of h. The subset

$$\{q \in Q : \exists\, n > 0,\ nq \in h(P)\}$$

of Q is a submonoid of Q called the *saturation* of P in Q. In case P is integral and we take $Q = P^{\mathrm{gp}}$, we recover the previous notion.

Example 1.12.1. The morphism

$$h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \mathbb{N}^2 \to \mathbb{N}^2$$

1. (1) is monic.
2. (2) has trivial cokernel.
3. (3) has image $\{(m, n) : m \geq n\} \subset \mathbb{N}^2$.
4. (4) is saturated.
5. (5) is an epimorphism in $\mathbf{Mon}^{\mathrm{int}}$ but not in $\mathbf{Mon}$.
6. (6) is not integral.

To see that h is not integral, we could use the equational criterion: $h(e_1) + e_2 = h(e_2) + 0$ but there are no $q_3, q_4, p \in \mathbb{N}^2$ satisfying

$$\begin{cases} e_2 & = & h(q_3) + p \\ 0 & = & h(q_4) + p. \end{cases}$$

Alternatively, it is enough to note that the induced map $\mathbb{Z}[\mathbb{N}^2] \to \mathbb{Z}[\mathbb{N}^2]$ is not flat. Indeed, this map is

$$\begin{array}{rcl} \mathbb{Z}[x,y] & \to & \mathbb{Z}[x,z] \\ y & \mapsto & xz, \end{array}$$

which is not flat (it is one of the standard charts for the blowup of $\mathbb{A}^2$ at the origin).

To see that h is not an epimorphism, consider the two maps $f_1, f_2 : \mathbb{N}^2 \to (\mathbb{F}_2, \cdot)$ given by:

$$\begin{array}{rclcrcl} f_1(e_1) & = & 0 & \quad & f_2(e_1) & = & 0 \\ f_1(e_2) & = & 0 & & f_2(e_2) & = & 1 \end{array}$$

We have $f_1 h = f_2 h = f_1$, but $f_1 \neq f_2$. However, if P is integral, and $f_1, f_2 : \mathbb{N}^2 \rightrightarrows P$ are two parallel arrows with the same composition g, then applying the f_i to

$$e_2 + h(e_1) = h(e_2)$$

we get $f_1(e_2) + g(e_1) = f_2(e_2) + g(e_1)$ hence $f_1(e_2) = f_2(e_2)$ by integrality and certainly $f_1(e_1) = f_2(e_1) = g(e_1)$ so we can conclude $f_1 = f_2$.

Lemma 1.12.2. *Let P be an integral monoid. Then there is a natural isomorphism $P^{\text{sat}}/P^* = (P/P^*)^{\text{sat}}$.*

Proof. This is an easy but tedious exercise with the definitions. □

A monoid is *fine* if it is integral and finitely generated. The category $\mathbf{Mon}^f$ of fine monoids is a full subcategory of the category $\mathbf{Mon}^{fg}$ of finitely generated monoids. The inclusion $\mathbf{Mon}^f \hookrightarrow \mathbf{Mon}^{fg}$ has a left adjoint $P \mapsto P^{\text{int}}$ (if P is finitely generated, then so is P^{int} because $P \to P^{\text{int}}$ is surjective).

A monoid is *fs* if it is fine and saturated. The category $\mathbf{Mon}^{fs}$ is a full subcategory of the category $\mathbf{Mon}^f$ of fine monoids.

Theorem 1.12.3. *Let P be a fine monoid. Then P^{sat} is fine.*

Proof. Of course P^{sat} is integral since $P^{\text{sat}} \subseteq P^{\text{gp}}$. The issue is to prove that it is finitely generated. We first reduce to the case where P is sharp: Suppose the result is known for sharp, fine monoids, and consider an arbitrary fine monoid P. Since the result is known for the sharp fine monoid P/P^*, we know by Lemma 1.12.2 that P^{sat}/P^* is finitely generated, hence $P^{\text{sat}}/(P^{\text{sat}})^*$ is also finitely generated as it is a quotient of the latter. We also know $(P^{\text{sat}})^*$ is finitely generated because it is a subgroup of P^{gp}, so we conclude that P^{sat} is finitely generated by Lemma 1.9.2.

We next reduce to the case where P is sharp *and* P^{gp} is torsion free: Suppose P is sharp and fine. Then we can write $P^{\text{gp}} = A \oplus T$ where $A \cong \mathbb{Z}^n$ for some n and T is a finite abelian group. Set $P' := P \cap A$, viewing A and P as submonoids of P^{gp}. Since P is sharp and integral, so is the submonoid P', and P' is finitely generated by Corollary 1.9.8, so P' is sharp and fine. Clearly $(P')^{\text{gp}} \subseteq A$ because $P' \subseteq A$, hence $(P')^{\text{gp}}$ is torsion free. On the other hand, one easily sees that $P^{\mathrm{sat}} = (P')^{\mathrm{sat}} \oplus T$, so it is enough to know the result for P'.

Now it is enough to prove the result for a sharp fine monoid P where $M := P^{\mathrm{gp}} \cong \mathbb{Z}^n$ is torsion free. View P as a submonoid of $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $C(P) \subseteq M_{\mathbb{Q}}$ be the "cone spanned by P," i.e., the submonoid of $M_{\mathbb{Q}}$ consisting of finite $\mathbb{Q}_{\geq 0}$-linear combinations of elements of P. Then I claim $P^{\mathrm{sat}} = C(P) \cap M$. Indeed, given $m \in C(P) \cap M$, one sees by "clearing denominators" that $dm \in P$ for some $d \in \mathbb{Z}_{>0}$, so we have $C(P) \cap M \subseteq P^{\mathrm{sat}}$. On the other hand, given any $m \in P^{\mathrm{gp}} = M$ with $m \in P^{\mathrm{sat}}$, we can find $d \in \mathbb{Z}_{>0}$ so that $dm =: p$ is in P, then the expression $m = (1/d)p$ in $M_{\mathbb{Q}}$ shows that $m \in C(P) \cap M$, so we have $P^{\mathrm{sat}} \subseteq C(P) \cap M$.

The fact that

$$P^{\mathrm{sat}} = C(P) \cap M$$

is finitely generated is now a standard result called *Gordan's Lemma*. The subset $C(P) \subseteq M_{\mathbb{Q}}$ is clearly a *cone* in the sense that it is closed under scalar multiplication by $\mathbb{Q}_{\geq 0}$. It is *rational* because P is finitely generated, and it is *strongly convex* in the sense that $\{0\}$ is the only $\mathbb{Q}$ linear subspace of $M_{\mathbb{Q}}$ contained in $C(P)$. This is because P is sharp: if there were some nontrivial $\mathbb{Q}$ linear subspace of $M_{\mathbb{Q}}$ contained in $C(P)$, then we could find a nonzero $c \in C(P)$ such that $-c$ is also in $C(P)$. After clearing denominators, we see that this would contradict $P^* = \{0\}$.

Gordan's Lemma is easily proved as follows: Take $p_1, \dots, p_k \in P$ generating P. Consider the set

$$K := \left\{ \sum_{i=1}^{k} t_i p_i \in M_{\mathbb{Q}} : t_i \in [0,1] \cap \mathbb{Q} \right\}.$$

Then on compactness grounds (replace $\mathbb{Q}$ by $\mathbb{R}$ everywhere if you want to take this literally) the set $K \cap M$ is finite, and one easily proves that it generates $C(P) \cap M$. $\square$

Theorem 1.12.4. *Let $h : Q \to P$ be a morphism of integral monoids, and let $L \subseteq P$ be the saturation of Q in P. Then $\mathbb{Z}[L] \subseteq \mathbb{Z}[P]$ is the integral closure of $\mathbb{Z}[Q]$ in $\mathbb{Z}[P]$.*

Proof. Replacing Q by $h(Q) \subseteq P$ if necessary, we can assume h is a monomorphism and suppress it from the notation. For $l \in L$, we have $nl = q$ for some $n \in \mathbb{Z}_{>0}$, $q \in Q$, so $[l]^n = [q]$ in $\mathbb{Z}[P]$, hence $[l] \in \mathbb{Z}[P]$ is a root of the monic polynomial $x^n - [q] \in \mathbb{Z}[Q][x]$, so $[l]$ is integral over $\mathbb{Z}[Q]$, hence $\mathbb{Z}[L]$ is integral over $\mathbb{Z}[Q]$ because the set of integral elements always forms a subring. We have proved that the integral closure of $\mathbb{Z}[Q]$ in $\mathbb{Z}[P]$ at least *contains* $\mathbb{Z}[L]$.

Showing that the integral closure of $\mathbb{Z}[Q]$ in $\mathbb{Z}[P]$ is contained in $\mathbb{Z}[L]$ is more difficult. Consider a typical element $f = \sum_{i=1}^m a_i[p_i]$ of $\mathbb{Z}[P]$. We can assume each integer a_i is non-zero. Suppose f is integral over $\mathbb{Z}[Q]$, so f satisfies

$$g_0 + g_1 f + \cdots + g_{n-1} f^{n-1} + f^n \quad = \quad 0 \tag{1.12.4.1}$$

in $\mathbb{Z}[P]$ for some $g_i \in \mathbb{Z}[Q]$. We want to prove $f \in \mathbb{Z}[L]$. I.e., we want to show that each p_i is in L. Think about the coefficient of $[np_i]$ in (1.12.4.1). There is an obvious (nonzero!) contribution of a_i^n to this coefficient coming from the f^n term. The other contributions to this coefficient occur only when we can write

$$np_i \quad = \quad A_{i1}p_1 + A_{i2}p_2 + \cdots + A_{im}p_m + q_i \tag{1.12.4.2}$$

for some $A_{ij} \in \mathbb{N}$ with $\sum_{j=1}^m A_{ij} < n$ and some $q_i \in Q$. In particular, in order for (1.12.4.1) to hold, we must have at least one expression of the form (1.12.4.2) for each $i \in \{1, \dots, m\}$. I claim that this alone will be enough to conclude that each p_i is in L. For $i \neq j$, set $a_{ij} := -A_{ij}$, and set $a_{ii} := n - A_{ii}$. Then the equations (1.12.4.2) imply that we have an "augmented matrix"

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} & | & q_1 \\ a_{21} & a_{22} & \cdots & a_{2m} & | & q_2 \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} & | & q_m \end{pmatrix}$$

with $a_{ij} \in \mathbb{Z}$, $q_i \in Q$ satisfying the following conditions:

(1) For $i \neq j$, $a_{ij} \in \mathbb{Z}_{\leq 0}$ is a non-positive integer.
(2) For each i, $\sum_{j=1}^m a_{ij}$ is a positive integer (in particular, each a_{ii} must be positive).
(3) For each i, $\sum_{j=1}^m a_{ij} p_j = q_i$ in P^{gp}.

Now we will perform "Positive Integral Gaussian Elimination" to this augmented matrix.[1] The point is that matrices satisfying the conditions above are invariant under the following operation: For any $i \neq j$, we can replace row i with

$$a_{jj} \operatorname{row} i - a_{ij} \operatorname{row} j.$$

Indeed, $a_{jj} > 0$ and $-a_{ij} \geq 0$, so this clearly preserves the last two conditions. For $k \neq i, j$, the new entry in row i column k will be

$$a_{jj} a_{ij} - a_{ij} a_{jk},$$

which is ≤ 0 because

$$a_{jj} > 0, \ a_{ij} \leq 0, \ a_{jk} \leq 0.$$

The key point is that the new entry in row i column j is zero. Now we perform the following simple algorithm:

(1) Set $c := 1$ and proceed to the next step.
(2) Assume at this step that we are given a matrix satisfying the three properties above, together with the property: For any $j < c$, the only nonzero entry in column j is a_{jj}. Then, for each $i \neq c$, replace row i with

$$a_{cc} \operatorname{row} i - a_{ic} \operatorname{row} c.$$

The resulting matrix still satisfies the three properties above, and now, for any $j < c + 1$, the only nonzero entry in column j is a_{jj}.
(3) If $c = m$, stop, otherwise increase c by one and return to Step 2.

This algorithm clearly terminates after finitely-many steps to yield a new matrix of the form

$$\begin{pmatrix} a_1 & 0 & \cdots & 0 & | & q_1' \\ 0 & a_2 & \cdots & 0 & | & q_2' \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & a_m & | & q_m' \end{pmatrix}$$

also satisfying the above properties. That is, we have $a_i p_i = q_i'$ for some $a_i \in \mathbb{Z}_{>0}$, $q_i' \in Q$, hence $p_i \in L$ as desired. $\square$

[1] See Jacobson's *Basic Algebra* for discussion of Gaussian Elimination over $\mathbb{Z}$.

Example 1.12.5. Let $P = \langle 2, 3\rangle$ be the submonoid of $\mathbb{N}$ generated by $\{2, 3\}$. The monoid P is called the *cusp monoid* (or the monoid of possible football scores). It is the coequalizer of

$$2, 3 : \mathbb{N} \rightrightarrows \mathbb{N}.$$

Certainly P is integral because $\mathbb{N}$ is integral. However, $P^{\mathrm{gp}} \to \mathbb{N}^{\mathrm{gp}} \cong \mathbb{Z}$ is an isomorphism because $1 = 3 - 2$, but 1 is not in P, so P is not saturated.

The inclusion $P \hookrightarrow \mathbb{N}$ is not integral because $\mathbb{Z}[P] \to \mathbb{Z}[\mathbb{N}]$ is not flat (Spec of this is the resolution of the cuspital curve by $\mathbb{A}^1$). This can also be checked using the equational criterion: Take $q_1 = 3, q_2 = 2, p_1 = 1, p_2 = 2$. Then $q_1 + p_1 = q_2 + p_2$, but

$$\begin{cases} 1 &= q_3 + p \\ 2 &= q_4 + p \end{cases}$$

has no solution.

Example 1.12.6. Any affine toric variety is the monoid algebra on a finitely generated, saturated submonoid of $\mathbb{Z}^n$. We do not intend to give any significant coverage of the theory of toric varieties. Here is one small example: the coequalizer P of $(1, 1, 0, 0), (0, 0, 1, 1) : \mathbb{N} \rightrightarrows \mathbb{N}^4$ is called the *conifold monoid*. We have the obvious presentation

$$P = \{x, y, z, w : x + y = z + w\}$$

so $\mathbb{Z}[P] \cong \mathbb{Z}[x, y, z, w]/\langle xy - zw\rangle$ is the coordinate ring of the affine cone on a smooth quadric $\operatorname{Proj} \mathbb{Z}[x, y, z, w]/\langle xy - zw\rangle \cong \mathbb{P}^1 \times \mathbb{P}^1$.

1.13. **Nilpotents.** Let $h : Q \hookrightarrow P$ be a monomorphism of sharp monoids (suppressed in ensuing notation). Consider the ideal

$$\begin{aligned} I_Q &:= (Q \setminus \{0\}) + P \\ &= \{q + p : q \in Q \setminus \{0\}, p \in P\} \end{aligned}$$

generated by $Q \setminus \{0\} \subseteq P$. We say that an element $p \in P$ is *nilpotent for* h iff $p \notin I_Q$ but $np \in I_Q$ for some $n > 0$. We say that h *has nilpotents* iff there is some nilpotent element $p \in P$.

For example, if $h : \mathbb{N} \hookrightarrow \mathbb{N}$ is multiplication by $n \in \mathbb{Z}_{>0}$, then $I_Q = \{n, n + 1, \dots\}$, and $1, \dots, n - 1$ are the nilpotents for h.

The ideal $I_Q \subseteq P$ gives rise to an ideal

$$I_Q^{\mathbb{Z}} := \{\sum_i a_i[p_i] : a_i \in \mathbb{Z}, p_i \in I_Q\}$$

of the monoid algebra $\mathbb{Z}[P]$ as in Section 2.1, and hence (by "flatness") an ideal

$$\begin{aligned} I_Q^A &:= I_Q^{\mathbb{Z}} \otimes_{\mathbb{Z}} A \\ &= \{\sum_i a_i[p_i] : a_i \in A, p_i \in I_Q\} \end{aligned}$$

of $A[P] = A \otimes_{\mathbb{Z}} \mathbb{Z}[P]$ for any ring A.

Lemma 1.13.1. *Let $h : Q \hookrightarrow P$ be a monomorphism of sharp monoids. If p is nilpotent for h, then for any nonzero ring A, $[p] \in A[P]/I_Q^A$ is a nonzero nilpotent. The following partial converse holds: If P is sharp* and integral, *and A is a nonzero reduced ring such that $A[P]/I_Q^A$ has a nonzero nilpotent, then h has a nonzero nilpotent.*

Proof. The first statement is obvious. For the second statement, suppose $f = \sum_{i=1}^m a_i[p_i] \in A[P]$ becomes a nontrivial nilpotent in $A[P]/I_Q^A$. After possibly replacing f with a different element of $A[P]$ having the same image in $A[P]/I_Q^A$, we can assume that every a_i is nonzero, and $p_1, \dots, p_m \notin I_Q$. Since f maps to a nilpotent in $A[P]/I_Q^A$ we can find $n > 1$ so that $f^n = \sum_{j=1}^s b_j[r_j]$ for some $r_1, \dots, r_s \in I_Q$. If some np_i appears in the list $r_1, \dots, r_s$, then p_i is manifestly nilpotent for h and we're done, so we can suppose this is not the case. Then it must be that, for every i, the coefficient of $[np_i]$ in f^n is zero. On the other hand, since A is reduced and $a_i \neq 0$, $a_i^n \neq 0$, so there is an obvious nonzero contribution of a_i^n to this coefficient. The only other contributions to this coefficient in f^n occur when we can write

$$np_i = \sum_{j \neq i} n_{ij} p_j \tag{1.13.1.1}$$

for some $n_{ij} \in \mathbb{N}$ with $\sum_{j \neq i} n_{ij} < n$. Choose one such expression (1.13.1.1) for each i, and set $a_{ij} = n$ when $i = j$, $a_{ij} := -n_{ij}$ when $i \neq j$. Then (a_{ij}) is an $m \times m$ integer matrix with the properties:

(1) The off-diagonal entries are nonpositive.
(2) The sum of the entries in any row is positive.
(3) For any row i, we have $\sum_{j=1}^m a_{ij} p_j = 0$ in P^{gp}.

As in the proof of Theorem 1.12.4, we can perform Positive Integral Gaussian Elimination to this matrix to obtain a new $m \times m$ integer matrix (b_{ij}) satisfying all of the properties above where the off-diagonal entries are all zero. But then $b_{ii} p_i = 0$ violates sharpness of P. $\square$

The following Lemma is used in the study of log curves.

Lemma 1.13.2. *Let $h : Q \hookrightarrow P$ be an integral monomorphism of fine, sharp monoids, without nilpotents. Then:*

(1) If P saturated, then the quotient $P^{\mathrm{gp}}/Q^{\mathrm{gp}}$ is torsion free, and, at least when $P^{\mathrm{gp}}/Q^{\mathrm{gp}} \cong \mathbb{Z}$, the quotient P/Q is saturated (hence is isomorphic to $0, \mathbb{N}$, or $\mathbb{Z}$).
(2) If $P/Q \cong \mathbb{N}$, then there is a unique $p \in P$ such that $(h, p) : Q \oplus \mathbb{N} \to P$ is an isomorphism.
(3) If $P/Q \cong \mathbb{Z}$, then there is a unique $q_0 \in Q$ and $p_1, p_{-1} \in P$ (unique up to $p_1 \leftrightarrow p_{-1}$) such that the diagram below is cocartesian.

$$\begin{tikzcd} \mathbb{N} \arrow[r, "q_0"] \arrow[d, "\Delta"'] & Q \arrow[d, "h"] \\ \mathbb{N} \oplus \mathbb{N} \arrow[r, "p_1, p_2"] & P \end{tikzcd}$$

Proof. (1) Suppose $P^{\mathrm{gp}}/Q^{\mathrm{gp}}$ has nontrivial torsion, so there are $p_1, p_2 \in P$ such that $p_1 - p_2 \notin Q^{\mathrm{gp}}$ but $np_1 - np_2 \in Q^{\mathrm{gp}}$ for some $n \in \mathbb{Z}_{>0}$, i.e.

$$np_1 + q_1 = np_2 + q_2$$

for some $q_1, q_2 \in Q$. By Theorem 1.10.6, we can find Q-primitive elements p_1', p_2' with the same image in P/Q as p_1, p_2, so after possibly replacing p_i with p_i', we can assume p_1, p_2

are Q-primitive. Again by Theorem 1.10.6, we can write

$$\begin{aligned} np_1 &= a_1 + b \\ np_2 &= a_2 + b \end{aligned}$$

for some $a_1, a_2 \in Q$, and some Q-primitive $b \in P$ (i.e. b is the unique Q-primitive representative of the common image of np_1, np_2 in P/Q). If $a_1, a_2 = 0$, then we have $n(p_1 - p_2) = 0$ and $n(p_2 - p_1) = 0$ in P^{gp}, hence $p_1 - p_2, p_2 - p_1 \in P$ because P is saturated. But then $p_1 - p_2 = 0$ because P is sharp, and this certainly contradicts $p_1 - p_2 \notin Q^{\text{gp}}$. So it must be that one of a_1, a_2, say a_1, is nonzero. Then $np_1 = a_1 + b$ is manifestly in I_Q. Now, if p_1 were not in I_Q, then it would be nilpotent, so we can now assume that p_1 is in I_Q. Then, since p_1 is Q-primitive, it must actually be that $p_1 \in Q$. But then np_1 is certainly zero in P/Q, so we must have $b = 0$, hence $np_2 = a_2$ is in Q, and, in particular, $np_2 \in I_Q$. I claim that p_2 is not in I_Q, hence is nilpotent (a contradiction). Indeed, if p_2 were in I_Q, then in fact p_2 would be in Q because it is Q-primitive, and this would violate $p_1 - p_2 \notin Q^{\text{gp}}$.

It remains to prove that P/Q is saturated when $P^{\text{gp}}/Q^{\text{gp}} \cong \mathbb{Z}$. Choose an identification $P^{\text{gp}}/Q^{\text{gp}} \cong \mathbb{Z}$, so we can view P/Q as a submonoid of $\mathbb{Z}$. Since its groupification is $\mathbb{Z}$, we must be able to find $m, n \in P/Q \subseteq \mathbb{Z}$ relatively prime. Then we can find $a, b \in \mathbb{N}$ such that $am - bn = \pm 1$. Now, for $m \in P/Q \subseteq \mathbb{Z}$, let $p_m \in P$ denote the unique Q-primitive lift of m. Then for any $t \in \mathbb{N}$ we must have $tp_m = p_{mt}$, otherwise p_m would be nilpotent. Consider the element $ap_m - bp_n \in P^{\text{gp}}$. If we can show that this element is in P, then we are done because this element maps to $\pm 1 \in \mathbb{Z}$, so we would have $\pm 1 \in P/Q$ and any submonoid of $\mathbb{Z}$ containing ± 1 is clearly saturated. Now, since P is saturated, it is enough to show that $m(ap_m - bp_n) \in P$. For this, it suffices to show that $amp_m = bmp_n + p_m$, or, equivalently, $p_{amm} = p_{bmn} + p_m$. By Theorem 1.10.6 we can write

$$p_{bmn} + p_m = p_{amm} + q$$

for some $q \in Q$, so it is enough to prove $q = 0$. But if q were not zero, then p_m would be nilpotent since

$$(bn + 1)p_m = bnp_m + p_m = p_{bmn} + p_m = p_{amm} + q.$$

(2) By Theorem 1.10.6, for each $n \in \mathbb{N}$, there is a unique Q-primitive $p_n \in P$ mapping to n in $P/Q \cong \mathbb{N}$. Using Theorem 1.10.6, it is clear that the set map $n \mapsto p_n$ provides a splitting iff it is a monoid homomorphism (iff $p_n = np_1$ for all $n \in \mathbb{N}$, in which case $p = p_1$ is as desired). Again by Theorem 1.10.6, we can write

$$p_1 = p_1 + 0, \; 2p_1 = p_2 + q_2, \; 3p_1 = p_3 + q_3, \; \dots$$

for unique $q_i \in Q$, so our map is a monoid homomorphism iff all these q_i are zero. If one of them isn't, then for some $n > 1$ we have $np_1 \in I_Q$, even though p_1 itself if not in I_Q because p_1 is Q-primitive but not in Q. The uniqueness of the p yielding such a splitting is clear from Theorem 1.10.6 because such a p must be Q-primitive and map to $1 \in P/Q = \mathbb{N}$ since $(0, 1)$ certainly has these properties in $Q \oplus \mathbb{N}$.

(3) By Theorem 1.10.6, for each $n \in \mathbb{Z}$, there is a unique Q-primitive $p_n \in P$ mapping to n in $P/Q \cong \mathbb{Z}$ and we can write $p_1 + p_{-1} = q_0$ for a unique $q_0 \in Q$ since $0 \in P$ is certainly the unique Q-primitive mapping to $0 \in \mathbb{Z}$ (i.e. $p_0 = 0$). Define set maps $\mathbb{N} \to Q$ by $n \mapsto nq_0$ and $\mathbb{N}^2 \to P$ by $(m, n) \mapsto p_m + p_{-n}$. By the same argument as in the previous proof, we must have $p_n = np_1$ and $p_{-n} = np_{-1}$, otherwise p_1 (or p_{-1}) will be nilpotent. The uniqueness of q_0, p_1, p_{-1} follows from Lemma 1.10.7 and the fact that $(1, 0), (0, 1) \in \mathbb{N}^2$ are the unique Q-primitive lifts of $1, -1 \in Q/P$ (for an approproate indentification $Q/P \cong \mathbb{Z}$;

the ambiguity $p_1 \leftrightarrow p_{-1}$ results only from this choice of identification). Now at least the diagram commutes: $nq_0 = n(p_1 + p_{-1}) = p_n + p_{-n}$, so we just need to check that the natural map

$$\begin{array}{rcl} Q \oplus_{\mathbb{N}} (\mathbb{N} \oplus \mathbb{N}) & \to & P \\ [q, (m, n)] & \mapsto & q + p_m + p_{-n} \end{array}$$

is an isomorphism. Surjectivity is clear since, according to the theorem, any p can be written $p = p_n + q$ for some $n \in \mathbb{Z}$. For injectivity, suppose $q + p_m + p_{-n} = q' + p_{m'} + p_{-n'}$ in P. Then in particular, they map to the same element of the cokernel so we must have $m - n = m' - n'$. After possibly exchanging the primed and unprimed terms, we may assume $m' \leq m$, so that $m - m' \in \mathbb{N}$. Since the two ways $\mathbb{N}$ maps into the pushout have to agree, we have

$$\begin{array}{rcl} [q', (m', n')] + (m - m') & = & [q', (m, n)] \\ [q, (m, n)] + (m - m') & = & [(m - m')q_0 + q, (m, n)]. \end{array}$$

Now we have

$$q' + p_m + p_{-n} = (m - m')q_0 + q + p_m + p_{-n}$$

in P so it follows from integrality of P and the uniqueness part of Theorem 1.10.6 that $q' = (m - m')q_0 + q$ in Q. Since we know that the pushout monoid is integral (because h is integral), we conclude the equality $[q', (p_{m'}, p_{n'})] = [q, (p_m, p_n)]$ in the pushout from the integrality of the element $m - m'$. $\square$

Condition (1) can hold even if h has nilpotents: For example, for the morphism $(2, 3) : \mathbb{N} \to \mathbb{N}^2$, we have $\mathbb{N}^2/\mathbb{N} \cong \mathbb{Z}$ (via $(1, 0) \mapsto 3, (0, 1) \mapsto -2$), but $(1, 1) \in \mathbb{N}^2$ is nilpotent since $(5, 5) = (3, 2) + (2, 3)$.

2. Monoidal algebraic geometry

In this section, we import the basic ideas of algebraic geometry into the theory of monoids. We discuss the *prime spectrum* $\operatorname{Spec} P$ of a monoid P, the notion of a *fan* (the analog of a scheme), then we discuss modules over monoids and (quasi-)coherent sheaves on fans.

2.1. Ideals and faces.

2.1. Ideals and faces. An *ideal* of a monoid P is a subset $I \subsetneq P$ such that, for any $i \in I, p \in P$, $p + i \in I$. An ideal $\mathfrak{p} \subsetneq P$ is *prime* iff $\mathfrak{p} \neq P$, and, whenever $x + y \in I$, one of x, y is in I. A *face* of P is a submonoid $F \subseteq P$ such that $P \setminus F$ is an ideal (necessarily prime) of P. The *codimension* of a face $F \subseteq P$ is the rank of the abelian group $(P/F)^{\mathrm{gp}}$ (defined when this abelian group is finitely generated).

Every monoid has a smallest prime ideal, $\emptyset$, and a largest prime ideal $P \setminus P^*$, so every monoid is "local". The notation $\mathfrak{m}_P := P \setminus P^*$ is often convenient. For a prime ideal $\mathfrak{p} \subseteq P$, we call $P_{\mathfrak{p}} := (P \setminus \mathfrak{p})^{-1} P$ the *localization* of P at $\mathfrak{p}$.

The *dimension* of a monoid P is the largest integer n such that there is a strictly increasing chain $\emptyset \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ of prime ideals $\mathfrak{p}_i$ in P (or ∞ if there are arbitrarily long such chains).

2.2. **Prime spectrum.** As in the case of rings, the set $\operatorname{Spec} P$ of prime ideals of a monoid P (its *prime spectrum*) has a topology where a basic open set is a set of the form

$$U_p = \{\mathfrak{p} \in \operatorname{Spec} P : p \notin \mathfrak{p}\}.$$

Note $U_p \cap U_q = U_{p+q}$ and $U_0 = \operatorname{Spec} P$. The corresponding basic closed sets are denoted

$$\begin{aligned} Z_p &:= \operatorname{Spec} P \setminus U_p \\ &= \{\mathfrak{p} \in \operatorname{Spec} P : p \in \mathfrak{p}\}. \end{aligned}$$

A standard argument with Zorn's Lemma shows that Z_p is empty iff $p \in P^*$. We have

$$\cap_{p \in P \setminus P^*} Z_p = \{\mathfrak{m}_P\},$$

so $\mathfrak{m}_P$ is a closed point of $\operatorname{Spec} P$ and $\mathfrak{m}_P$ is in any non-empty closed subset of $\operatorname{Spec} P$. We have $\mathfrak{p} \subseteq \mathfrak{q}$ iff $\mathfrak{q} \in \{\mathfrak{p}\}^-$ in $\operatorname{Spec} P$. In particular, $\{\emptyset\}^- = \operatorname{Spec} P$ and $\operatorname{Spec} P$ is an irreducible topological space with a unique generic point. Similarly, a closed subspace Z of $\operatorname{Spec} P$ is irreducible iff it has a unique minimal prime $\mathfrak{p}$, in which case $\mathfrak{p}$ is the unique generic point of Z (i.e. $\operatorname{Spec} P$ is *sober*). A monoid homomorphism $h : P \to Q$ induces a continuous map

$$\begin{aligned} \operatorname{Spec} h : \operatorname{Spec} Q &\to \operatorname{Spec} P \\ \mathfrak{p} &\mapsto h^{-1}(\mathfrak{p}) \end{aligned}$$

because $(\operatorname{Spec} h)^{-1}(U_p) = U_{h(p)}$. For any such h, we clearly have $(\operatorname{Spec} h)(\emptyset) = \emptyset$, so if $\operatorname{Spec} h$ is a closed map of topological spaces, then it is necessarily a surjective map of topological spaces. Consequently, it is relatively rare for h to induce a closed embedding on spaces without being an isomorphism, even in situations where one might expect this to happen. For example:

Lemma 2.2.1. *If $h : Q \to P$ is surjective, then* $\operatorname{Spec} h$ *is an embedding (not necessarily closed!) of spaces.*

Proof. Since h is surjective, $h^{-1}(\mathfrak{p}) = h^{-1}(\mathfrak{q})$ clearly implies $\mathfrak{p} = \mathfrak{q}$, so $\operatorname{Spec} h$ is monic. For $p \in P$, if we choose a lift $q \in Q$ with $h(q) = p$, then $(\operatorname{Spec} h)^{-1}(U_q) = U_p$, so every basic open subset of $\operatorname{Spec} P$ is obtained by intersecting an open subset of $\operatorname{Spec} Q$ with $\operatorname{Spec} P$, hence $\operatorname{Spec} h$ is an embedding. □

Similarly, the whole space $\operatorname{Spec} P$ is the only neighborhood of $\mathfrak{m}_P$, so any $h : P \to Q$ where $\operatorname{Spec} h$ is open and $h^{-1}(\mathfrak{m}_Q) = \mathfrak{m}_Q$ has $\operatorname{Spec} h$ surjective.

Lemma 2.2.2. *The sharpening map $f : P \to \overline{P}$ induces a homeomorphism* $\operatorname{Spec} \overline{P} \to \operatorname{Spec} P$.

Proof. Certainly f is surjective, so $\operatorname{Spec} f$ is an embedding by the previous lemma, so it is enough to show that $\operatorname{Spec} f$ is surjective. Given $\mathfrak{p} \in \operatorname{Spec} P$, its image $f(\mathfrak{p}) \in \operatorname{Spec} \overline{P}$ under the surjection f is certainly an ideal of $\overline{P}$—we claim it is prime. Indeed, if $\overline{p}_1 + \overline{p}_2 \in f(\mathfrak{p})$, then $p_1 + p_2 + u = p$ for some $p \in \mathfrak{p}$ and some lifts $p_1, p_2 \in P$ of the $\overline{p}_i$ and some unit $u \in P^*$. But then $p_1 + p_2 + u \in \mathfrak{p}$, so, since $\mathfrak{p}$ is prime one of the p_i is in $\mathfrak{p}$, hence one of the $\overline{p}_i$ is in $f(\mathfrak{p})$. By similar reasoning we see that $f^{-1}(f(\mathfrak{p})) = \mathfrak{p}$. □

If A is a ring, then a prime ideal $p \in \operatorname{Spec} A$ may also be viewed as a prime ideal of the multiplicative monoid $(A, \cdot)$. In particular, if P is a monoid and $h : P \to A$ is a monoid homomorphism, then $h^{-1}(p)$ is a prime ideal of P.

An ideal $I \subseteq P$ gives rise to an ideal

$$\mathbb{Z}[I] \quad := \quad \{\sum_i a_i[p_i] : a_i \in \mathbb{Z}, p_i \in I\}$$

of the monoid algebra $\mathbb{Z}[P]$.

Lemma 2.2.3. *If $\mathfrak{p} \subseteq P$ is a prime ideal, then $\mathbb{Z}[\mathfrak{p}] \subseteq \mathbb{Z}[P]$ is a prime ideal.*

Proof. □

Similarly, for any ring A,

$$\begin{aligned} I^A &:= I^{\mathbb{Z}} \otimes_{\mathbb{Z}} A \\ &= \{\sum_i a_i[p_i] : a_i \in A, p_i \in I\} \end{aligned}$$

is an ideal of $A[P] = A \otimes_{\mathbb{Z}} \mathbb{Z}[P]$. This reflects the fact that the quotient ring $\mathbb{Z}[P]/I^{\mathbb{Z}}$ is flat over $\mathbb{Z}$ (it is isomorphic as an abelian group to $\oplus_{P \setminus I}\mathbb{Z}$).

3. Modules

4. Log Rings

In this section, we develop the basic theory of log rings, which play the role of "affine log schemes" in log geometry.

4.1. **Log structures.** A *prelog structure* on a ring A is a morphism of monoids $\alpha : P \to A$. A *log structure* is a prelog structure $\alpha : P \to A$ such that

$$\alpha|\alpha^{-1}(A^*) : \alpha^{-1}(A^*) \quad \to \quad A^* \tag{4.1.0.1}$$

is an isomorphism (of monoids). For simplicity, we will refer to the prelog structure $\alpha : P \to A$ as P, leaving the map α and the ring A implicit. A (pre)log ring is a ring with (pre)log structure. If $P \to A$ is a log ring, we will view A^* as a submonoid of P by identifying it with $\alpha^{-1}(A^*)$ via the isomorphism (4.1.0.1). A prelog structure P is *integral* if P is an integral monoid (1.7).

A *morphism* of prelog rings $(P \to A) \to (Q \to B)$ is a commutative diagram

$$\begin{array}{ccc} P & \longrightarrow & A \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ Q & \longrightarrow & B \end{array}$$

where f is a ring homomorphism and h is a morphism of monoids. Prelog rings form a category **PLogAn** where morphisms are composed in the obvious way. Log rings form a full subcategory **LogAn** of **PLogAn**.

A morphism of prelog structures on a ring A is defined by considering the case $A = B$, $f = \mathrm{Id}_A$ in the definition of a morphism of prelog rings. We usually just write $P \to Q$ for a morphism of prelog structures. Note that a morphism of prelog structures is nothing more than a morphism in the slice category $\mathbf{Mon}/A$. The categories of log structures and prelog structures on a ring A are denoted $\mathbf{Log}_A$, $\mathbf{PLog}_A$ respectively. Log structures are a full subcategory of prelog structures.

Given a prelog structure $\alpha : P \to A$, we can form an *associated log structure* $P^a \to A$. Take $P^a := A^* \oplus_{\alpha^{-1}A^*} P$ to be the pushout of

$$\begin{tikzcd} \alpha^{-1}A^* \arrow[r, hook] \arrow[d, "\alpha"] & P \\ A^* & \end{tikzcd}$$

in the category of monoids. The maps $\alpha : P \to A$ and the inclusion $A^* \hookrightarrow A$ define, by the universal property of pushouts, a map of monoids $\alpha^a : P^a \to A$. It is easy to check that this is a log structure and that any morphism of prelog structures $P \to Q$ to a log structure Q factors uniquely through the natural map $P \to P^a$. The functor $P \mapsto P^a$ defines a left adjoint to the inclusion functor $\mathbf{Log}_A \hookrightarrow \mathbf{PLog}_A$:

$$\mathrm{Hom}_{\mathbf{Log}_A}(P^a, Q) \cong \mathrm{Hom}_{\mathbf{PLog}_A}(P, Q).$$

By Lemma 1.10.1, the pushout $P^a := A^* \oplus_{\alpha^{-1}A^*} P$ can be given concretely as the quotient of $A^* \oplus P$ by the monoidal equivalence relation

$$\{((u,p),(u',p')) : \exists\, r, r' \in \alpha^{-1}A^*,\ u\alpha(r) = u'\alpha(r'),\ p + r' = p' + r\}.$$

Using this description, α^a is given by $[u,p] \mapsto u\alpha(p)$.

A morphism

$$\begin{tikzcd} P \arrow[r] \arrow[d, "h"'] & A \arrow[d, "f"] \\ Q \arrow[r] & B \end{tikzcd}$$

of prelog rings induces a morphism of log rings

$$\begin{tikzcd} P^a \arrow[r] \arrow[d, "h"'] & A \arrow[d, "f"] \\ Q^a \arrow[r] & B \end{tikzcd}$$

given by $h^a[p,u] = [hp, fu]$ in the above description of P^a, Q^a.

Example 4.1.1. Suppose $f : A \to B$ is a ring homomorphism, S is a multiplicative subset of A, and T is a multiplicative subset of B containing $f[S]$ (e.g. $T = f[S]$). Assume the multiplicative sets S, T do not contain zero. Set

$$\begin{aligned} P &:= \{a \in A : a \in (S^{-1}A)^*\} \\ Q &:= \{b \in B : b \in (T^{-1}B)^*\}. \end{aligned}$$

Here $A \to S^{-1}A$ is the localization of A with respect to the multiplicative set S, and we write a for the image of $a \in A$ under this map, by abuse of notation. Then the inclusions $P \hookrightarrow A$, $Q \hookrightarrow B$ are log structures, and

$$\begin{tikzcd} P \arrow[r] \arrow[d, "f"'] & A \arrow[d, "f"] \\ Q \arrow[r] & B \end{tikzcd}$$

is a morphism of log rings. These log structures are integral as long as A and B are integral domains.

Given a log ring $P \to A$, the quotient P/A^* is called the *characteristic monoid* of the log ring and is denoted $\overline{P}$. Note that, if P is a log structure on A, then $P^* = A^*$, so this terminology does not conflict with the previous usage of $\overline{P}$ to denote the sharp monoid obtained from P by quotienting P by its group of units.

Lemma 4.1.2. *If $\alpha : P \to A$ is an integral prelog structure on A, then the associated log structure $P^a \to A$ is integral. (The pushout $P \oplus_{\alpha^{-1}A^*} A^*$ is an integral monoid.) If P is an integral log structure, then $P \to \overline{P}$ is an integral morphism. In particular, the characteristic monoid of an integral log structure is integral.*

Proof. The monoid $\alpha^{-1}A^*$ is a submonoid of the integral monoid P and is hence integral. The monoid A^* is a group and is, in particular, integral. The map $\alpha : \alpha^{-1}A^* \to A^*$ is hence an integral morphism of integral monoids because its codomain is a group (Proposition 1.10.3 (3)). Since P is integral, the pushout P^a of the integral morphism is integral since this is one of the criteria of Proposition 1.10.2.

The second statement follows from Proposition 1.10.3 (6) applied to $P^* \to P$.

□

4.2. **Pullback and pushforward.** Given a log structure $P \to A$ and a ring map $f : A \to B$, the prelog structure on B associated to the composition $P \to A \to B$ will be denoted $f^{-1}P$. It is a log structure iff f induces an isomorphism $A^* \cong B^*$. The log structure $(f^{-1}P)^a$ on B associated to $f^{-1}P$ is called the *inverse image* (or *pullback*) log structure and is denoted f^*P. There is a natural morphism of log rings

$$\begin{tikzcd} P \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ f^*P \arrow[r] & B \end{tikzcd}$$

which is clearly the map of log rings associated to the map of prelog rings

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d, equal] & A \arrow[d, "f"] \\ P \arrow[r, "f\alpha"'] & B \end{tikzcd}$$

(the bottom horizontal arrow is $f^{-1}P$).

Given a (pre)log ring $\alpha : P \to B$ and a map of rings $f : A \to B$, the fibered product $P \times_B A$ defines a (pre)log structure on A called the *direct image log structure* (or *pushforward*) and denoted $f_*^{\log}P$. When P is a log structure, the map $P \times_B A \to A$ is a log structure because, given any $u \in A^*$, $f(u)$ is in B^* so, since P is a log structure, there is a unique $p \in P$ with $\alpha(p) = f(u)$, so $(p, u) \in P \times_B A$ is the unique preimage of u in $P \times_B A$. There is a morphism of (pre)log rings:

$$\begin{tikzcd} f_*^{\log}P \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ P \arrow[r] & B \end{tikzcd}$$

Remark 4.2.1. The naming conventions $f_*^{\log}, f^*$ are contradictory to the variance for reasons that will become clear later. Work in $\mathbf{LogAn}^{\mathrm{op}}$ if you wish. The superscript log is added to avoid confusion later with the direct image of sheaves.

For a ring morphism $f : A \to B$, the functor $f_*^{\log} : \mathbf{Log}_B \to \mathbf{Log}_A$ is right adjoint to $f^* : \mathbf{Log}_A \to \mathbf{Log}_B$. In fact, we have

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Log}_A}(P, f_*^{\log} Q) &\cong \mathrm{Hom}_{\mathbf{PLog}_B}(f^{-1}P, Q) \\ &\cong \mathrm{Hom}_{\mathbf{Log}_B}(f^*P, Q) \end{aligned}$$

for any log structure Q on B. (The first bijection is trivial, and the second follows from the universal property of associated log structures.)

Example 4.2.2. The log structure discussed in Example 4.1.1 is nothing but the direct image of the trivial log structure on $S^{-1}A$ under the localization map $A \to S^{-1}A$.

Formation of associated log structures commutes with inverse images:

Theorem 4.2.3. *Let $f : A \to B$ be a ring homomorphism. Then the diagram of functors*

$$\begin{tikzcd} \mathbf{PLog}_A \arrow[r, "P \mapsto P^a"] \arrow[d, "f^{-1}"'] & \mathbf{Log}_A \arrow[d, "f^*"] \\ \mathbf{PLog}_B \arrow[r, "Q \mapsto Q^a"] & \mathbf{Log}_B \end{tikzcd}$$

commutes up to a natural natural equivalence of functors. In particular, if a morphism $P \to Q$ of prelog structures on A induces an isomorphism on associated log structures, then the same is true of the morphism $f^{-1}P \to f^{-1}Q$ of prelog structures on B.

Proof. Let P be a prelog structure on A, so $P^a = P \oplus_{\alpha^{-1}A^*} A^*$, and

$$(f^{-1}P)^a = P \oplus_{(f\alpha)^{-1}B^*} B^*.$$

By definition of the inverse image log structure, we have

$$f^*(P^a) := (f^{-1}(P^a))^a = ((P \oplus_{\alpha^{-1}A^*} A) \oplus B^*) \,/\, \sim,$$

where $([p, u], v) \sim ([p', u'], v')$ iff there are $[r, w], [r', w'] \in f^{-1}(P^a)$ satisfying the conditions:

(1) $[p + u, rw] = [p' + r', u'w']$
(2) $(f\alpha)(r), (f\alpha)(r') \in B^*$ or, equivalently, $(f\alpha)(r) \cdot f(w), (f\alpha)(r') \cdot f(w') \in B^*$.
(3) $(f\alpha)(r) \cdot f(w) \cdot v' = (f\alpha)(r') \cdot f(w') \cdot v$ in B^*.

Define a map

$$\begin{aligned} f^*(P^a) &\to (f^{-1}P)^a \\ [[p, u], v] &\mapsto [p, f(u)v] \end{aligned}$$

and a map

$$\begin{aligned} (f^{-1}P)^a &\to f^*(P^a) \\ [p, v] &\mapsto [[p, 1], v]. \end{aligned}$$

It is straightforward to check, using the above description of $f^*(P^a)$, that these maps are well defined. One composition is clearly the identity. The equality

$$[[p, u], v] = [[p, 1], f(u)v]$$

in $f^* P^a$ is witnessed by the pair $[0,1], [0,u] \in f^{-1}(P^a)$. These maps are clearly natural in P, hence they define the desired natural equivalence.

The first "natural" in the "natural natural equivalence" of the theorem means that this natural equivalence makes the obvious diagram commute for composable ring maps $A \to B \to C$, which is clear from the formulas.

□

Remark 4.2.4. For a prelog structure $Q \to B$ on B, the adjunction morphisms yield a natural map

$$\begin{array}{rcl} (f_*Q)^a & \to & f_*(Q^a) \\ [(p,a),a'] & \mapsto & ([p,f(a')],aa') \end{array} \tag{4.2.4.1}$$

which is not generally an isomorphism.

Proposition 4.2.5. *The inverse image of an integral (pre)log structure is integral. If $f : A \to B$ is a monomorphism of rings, and $\alpha : P \to B$ is an integral (pre)log structure, then the direct image f_*P is an integral (pre)log structure on A.*

Proof. Clearly the inverse image of an integral prelog structure is integral, since the prelog inverse image has the same monoid, so the first statement follows from the previous proposition. For the second statement, if

$$(p_1, a_1) + (p, a) = (p_2, a_2) + (p, a)$$

in $f_*P = P \times_B A$, then integrality of P implies $p_1 = p_2$, so $\alpha(p_1) = f(a_1) = \alpha(p_2) = f(a_2)$, hence $a_1 = a_2$ because f is monic. □

The direct image of log structures is evidently not as well behaved as the inverse image. Luckily, it is also used less frequently.

Example 4.2.6. Suppose $\alpha : P \to A$ is a prelog ring and $\mathfrak{p}$ is a prime ideal of A. Let $l : A \to A_{\mathfrak{p}}$ denote the localization at $\mathfrak{p}$. Let $F \subseteq P$ denote the set of those $p \in P$ where $\alpha(p) \in A \setminus \mathfrak{p}$. That is, $F = (l\alpha)^{-1}(A_{\mathfrak{p}}^*)$. Since $\mathfrak{p}$ is a prime ideal, F is a submonoid of P. The complement $P \setminus F$ is equal to $\alpha^{-1}(\mathfrak{p})$, so it is a prime ideal, hence F is a face, as discussed in Section 2.1. Note that F contains $\alpha^{-1}(A^*)$ and $F/\alpha^{-1}(A^*)$ is a face of the characteristic monoid $\overline{P} = P/\alpha^{-1}(A^*)$. The quotient of $\overline{P}$ by the face $F/\alpha^{-1}(A^*)$ of coincides with P/F, and P/F is nothing but the characteristic monoid of the inverse image prelog structure $l\alpha : P \to A_{\mathfrak{p}}$.

4.3. **Charts and finiteness.** If P is a log structure on a ring A, then a *chart* for P is a prelog structure Q on A and a morphism of prelog structures $Q \to P$ inducing an isomorphism $Q^a \cong P$ on associated log structures. Charts for P form a full subcategory of $\mathbf{PLog}_A/P$. A chart $Q \to P$ is *finitely generated* (resp. *fine*, *saturated*, etc.) if the monoid Q is finitely generated (resp. fine, saturated, etc.). A log structure P is *coherent* if it has a finitely generated chart. The full subcategory of $\mathbf{Log}_A$ consisting of coherent log structures is denoted $\mathbf{CohLog}_A$.

A *characteristic chart* is a chart $Q \to P$ such that the composition $Q \to P \to \overline{P}$ is an isomorphism. Characteristic charts play a fundamental role in log geometry and it is highly desirable to establish their existence when possible. For integral log rings, we will give a simple, complete answer to this question of existence in Theorem 4.3.12.

A chart $Q \to P$ is *monic* if the map $h : Q \to P$ is monic (this has little to do with whether $Q \to A$ is monic). Any chart gives rise to a monic chart by considering the inclusion of its image $h[Q] \hookrightarrow P$. This is again a chart because we have a factorization $Q^a \to h[Q]^a \to P$ where the composition is an isomorphism and the left map is surjective.

If

$$\begin{tikzcd} P \arrow[r] \arrow[d, "h"'] & A \arrow[d, "f"] \\ Q \arrow[r] & B \end{tikzcd}$$

is a morphism of log rings, then a *chart* for this morphism is a commutative diagram

$$\begin{tikzcd} P' \arrow[r] \arrow[d, "h'"'] & P \arrow[r] \arrow[d, "h"'] & A \arrow[d, "f"] \\ Q' \arrow[r] & Q \arrow[r] & B \end{tikzcd}$$

where $P' \to P$ is a chart for and $Q' \to Q$ is a chart for Q.

Example 4.3.1. Clearly $(A^* \hookrightarrow A)$ is a log structure. It is initial in the category of log structures on A and is the log structure associated to the prelog structure $0 \to A$. Similarly, $(\mathrm{Id}_A : A \to A)$ is a log structure on A, terminal in the category of log structures.

Example 4.3.2. If P is any monoid and A is any ring, then the natural map of monoids $P \to A[P]$ defines a prelog structure on the monoid ring $A[P]$.

Remark 4.3.3. It would be too restrictive to demand that a log structure P be given by a finitely generated monoid, because then even the trivial log structure would not generally be finitely generated because A^* is not generally a finitely generated abelian group. Note that the trivial log structure is certainly coherent since it is the log structure associated to $0 \to A$.

The basic facts about log rings are proved below.

Lemma 4.3.4. *Given a diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & E \arrow[d, dotted] \\ C \arrow[r] & D \arrow[r] & F \end{tikzcd}$$

of solid arrows as indicated in a category with zero object, suppose

(1) The square is cocartesian and
(2) The right two horizontal arrows are cokernels of the horizontal arrows just left of them.

Then there is a unique isomorphism $E \cong F$ as indicated by the dotted arrow making the diagram commute.

Proof. This is an elementary exercise using the universal properties. □

Corollary 4.3.5. *For a prelog structure $\alpha : P \to A$ on a ring A, the characteristic monoid $\overline{P^a}$ of the associated log structure is naturally isomorphic to the quotient $P/\alpha^{-1}A^*$. In particular, the characteristic monoid of a coherent log structure is finitely generated.*

Proof. Apply Lemma 4.3.4 to the diagram

$$\begin{tikzcd} \alpha^{-1}A^* \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & P/\alpha^{-1}A^* \\ A^* \arrow[r] & P^a \arrow[r] & \overline{P} \end{tikzcd}$$

used to construct P^a. The second statement follows because a quotient of a finitely generated monoid is finitely generated. $\square$

For an integral log structure P on a ring A, the diagram $A^* \to P \to \overline{P}$ is called the *characteristic extension* of P. By the above Proposition, it is an element of $\operatorname{Ext}^1_{\mathbf{Mon}}(\overline{P}, A^*)$ (see Section 1.8).

Example 4.3.6. In fact, for any ring A, any sharp, integral monoid $\overline{P}$, and any integral extension

$$A^* \to P \to \overline{P} \in \operatorname{Ext}^1_{\mathbf{Mon}}(\overline{P}, A^*),$$

there is a (non-unique!) log structure on A with the given extension as its characteristic sequence. Indeed, the map

$$\begin{array}{rcl} P & \to & A \\ p & \mapsto & \begin{cases} p, & p \in A^* \\ 0, & p \in P \setminus A^* \end{cases} \end{array}$$

defines a log structure because if one of p_1, p_2, say p_1, is not in A^*, but $p_1 + p_2 \in A^*$, then p_1 would be a nontrivial invertible element in the sharp monoid $\overline{P}$. In particular, for the split extension $\overline{P} \oplus A^*$, we have a log structure $\overline{P} \oplus A^* \to A$ which is sometimes written $(p, u) \mapsto u0^p$. We will refer to a log structure of this form as *split*.

Lemma 4.3.7. *A morphism $h : Q \to P$ of integral log structures on a ring A is an isomorphism iff the induced map $\overline{h} : \overline{Q} \to \overline{P}$ on characteristics is an isomorphism.*

Proof. The implication ($\Longrightarrow$) is trivial. For ($\Longleftarrow$), we will prove directly that h is bijective on the underlying sets. If $h(q_1) = h(q_2)$, then $\overline{h}[q_1] = \overline{h}[q_2]$, so $[q_1] = [q_2]$, hence $q_1 = q_2 + u$ for some $u \in Q^* = A^* \subseteq Q$. Since $0 + h(q_1) = h(q_2) + u$, we must have $u = 0$ by integrality of $h(q_1) = h(q_2)$. This proves h is monic.

Given $p \in P$, since $\overline{h}$ is an isomorphism, there is $q \in Q$ such that $h(p) = q + u$ for some $u \in Q^* = A^*$, but u is also in $P^* = A^*$, so $h(p - u) = q$ and hence h is an epimorphism on underlying sets. $\square$

Remark 4.3.8. The integrality assumption cannot be dropped. Let $\mathbb{R}_{\geq 0}$ be the monoid of nonnegative real numbers under multiplication and consider the "Kato-Nakayama log structure"

$$\begin{array}{rcl} \alpha : S^1 \times \mathbb{R}_{\geq 0} & \to & \mathbb{C} \\ (u, r) & \mapsto & ur \end{array}$$

on the field $\mathbb{C}$ (which we will use later in Remark 9.1.2). The action of $\mathbb{C}^* = S^1 \times \mathbb{R}_{>0}$ on $S^1 \times \mathbb{R}_{\geq 0}$ has two orbits: the orbit of the identity $(1, 1)$ and the orbit of $(1, 0)$, and the orbit of their product is the orbit of $(1, 0)$, so the characteristic monoid is the two-element "sink monoid" $P_1 = \{0, 1\}$ (with $1 + 1 = 1$) of Example 1.7.5. Consider the monoid

$$P = \{(u, r) \in S^1 \times \mathbb{R}_{\geq 0} : u = 1 \text{ if } r = 0\}$$

where the multiplication law is given as usual on the submonoid $\mathbb{C}^* = S^1 \times \mathbb{R}_{>0}$, and where multiplication by the distinguished element $(1,0)$ (the unique element of $P \setminus \mathbb{C}^*$) is defined by

$$(r,u) \cdot (1,0) = (1,0)$$

(for any $(r,u) \in P$). The product map $(u,r) \mapsto ru$ defines a log structure $P \to \mathbb{C}$ and there is a map of log structures on $\mathbb{C}$ from the Kato-Nakayama log structure to P given, necessarily, by the identity on $\mathbb{C}^*$, and by mapping the entire submonoid $S^1 \times \{0\}$ of the Kato-Nakayama log structure to the distinguished element $(1,0) \in P$. This map is clearly not an isomorphism, though it induces an isomorphism on characteristics.

Proposition 4.3.9. *Let $P \to A$ be a log structure on a ring A. The following are equivalent:*

(1) P has a fine, monic chart.
(2) P is integral and coherent.
(3) P is integral and its characteristic monoid $\overline{P}$ is finitely generated.

Remark 4.3.10. Without the integrality assumption, finite generation of the characteristic monoid does not imply that a log structure is coherent. For example, the Kato-Nakayama log structure has trivial characteristic monoid but is not coherent.

Proof. (1) $\implies$ (2) follows from the definitions and Proposition 4.3.9.

(2) $\implies$ (3) follows from Corollary 4.3.5.

(3) $\implies$ (1) Since $\overline{P}$ is finitely generated, there is a surjection $\mathbb{N}^n \to \overline{P}$. Since $\mathbb{N}^n$ is free, this lifts to a map $f : \mathbb{N}^n \to P$. Consider the image Q of f. It is manifestly finitely generated and is a submonoid of the integral monoid P, hence it is integral. Furthermore, we have $Q/(Q \cap A^*) \cong \overline{P}$. By Corollary 4.3.5, $Q/(Q \cap A^*)$ is the characteristic monoid of the log structure associated to Q and this isomorphism is clearly the map on characteristic monoids induced by the map of log structures $Q^a \to P$, so we conclude $Q^a \cong P$ by Lemma 4.3.7. □

A slight variant on the last part of the above proof will be used frequently. The various lemmas concerning construction of charts will be called *chart lemmas*.

Lemma 4.3.11. First Chart Lemma. *Let P be an integral log structure, $h : G \to P^{\mathrm{gp}}$ a morphism of abelian groups such that the composition $G \to \overline{P}^{\mathrm{gp}}$ is surjective. Then the inclusion of the image of $h : h^{-1}P \to P$ in P is an integral, monic chart for P. If P is coherent (i.e. fine) and G is finitely generated, it is a fine, monic chart.*

Proof. This follows from the same argument in the proof of (3) $\implies$ (1) in the previous theorem. In the second statement, the finite generation of $h[h^{-1}P]$ is an issue. Luckily, the finitely generated group $h^{-1}P^*$ surjects onto $(h[h^{-1}P])^*$ and the monoid in question has the same characteristic as P, finite generation follows from Lemma 1.9.2 because $\overline{P}$ is finitely generated (4.3.9). □

Theorem 4.3.12. *An integral log structure P on a ring A has a characteristic chart iff its characteristic extension*

$$(0 \to A^* \to P^{\mathrm{gp}} \to \overline{P}^{\mathrm{gp}} \to 0) \in \mathrm{Ext}^1_{\mathbf{Ab}}(A^*, \overline{P}^{\mathrm{gp}})$$

is trivial. If P is a fine log structure, this is a fine, monic chart.

Proof. ($\Longrightarrow$) is clear. For the other implication, apply the First Chart Lemma (4.3.11) with $G = \overline{P}^{\text{gp}}$ and h a section of $P^{\text{gp}} \to \overline{P}^{\text{gp}}$. $\square$

Corollary 4.3.13. *Any integral log structure $P \to k$ on an algebraically closed field is split (i.e. is of the form $P = (P/k^*) \oplus k^*$.)*

Proof. If k is algebraically closed, then k^* is a divisible abelian group (to divide $u \in k^*$ by n, find a solution to $X^n - u$ in the algebraically closed field k), hence an injective object in **Ab**, so any extension by k^* splits, hence the result follows from the proposition and the fact that log structures on a field are uniquely determined by the characteristic sequence. $\square$

Corollary 4.3.14. *Any integral log structure $P \to \mathscr{O}_{X,x}$ on the fppf local ring of a scheme X at an fppf point $x \in X$ admits a characteristic chart.*

Proof. The group $\mathscr{O}^*_{X,x}$ is divisible since adjoining an n^{th} root of any element a of a ring A yields an fppf map $A \to A[x]/(x^n - a)$, so $\mathscr{O}^*_{X,x}$ contains n^{th} roots of any of its elements for any $n \in \mathbb{N}$. $\square$

Corollary 4.3.15. *Let P be a fine log structure on a strictly Henselian local ring (A, m_A). Suppose the torsion part of $\overline{P}^{\text{gp}}$ has order invertible in A. Then there is a fine, monic chart $\overline{P} \to P$ for P using the characteristic monoid.*

Proof. Using Theorem 4.3.12, it suffices to show that $0 \to A^* \to P^{\text{gp}} \to \overline{P}^{\text{gp}} \to 0$ is split. Since $\overline{P}^{\text{gp}}$ is finitely generated, it splits as the direct sum of a free abelian group and its torsion part. Since $\operatorname{Ext}^1_{\mathbf{Ab}}(\overline{P}^{\text{gp}}, A^*)$ also splits accordingly, we may assume $\overline{P}^{\text{gp}}$ is a torsion group. Let n be its order. I claim multiplication by n (rather: the n^{th} power map) $A^* \to A^*$ is surjective. Indeed, to find an n^{th} root of $u \in A^*$, we consider the equation $X^n - u \in A[X]$. Since n is invertible in A, the reduction $X^n - \overline{u}$ mod m_A is a separable polynomial in the separably closed residue field, so it has a root. Since A is Henselian, this root lifts to a root of $X^n - u$ in A, proving the claim.

Next, note that Ext^2 always vanishes in **Ab**, so Ext^1 is right exact, hence our surjection $\cdot n : A^* \to A^*$ induces a surjection

$$\operatorname{Ext}^1_{\mathbf{Ab}}(A^*, \overline{P}^{\text{gp}}) \to \operatorname{Ext}^1_{\mathbf{Ab}}(A^*, \overline{P}^{\text{gp}}).$$

On the other hand, this map is the same as the map on Ext^1 induced by multiplication by n in $\overline{P}^{\text{gp}}$. But the latter map is zero, hence so is the map it induces on Ext^1. The desired vanishing follows because the zero map is surjective.

$\square$

Lemma 4.3.16. Second Chart Lemma. *Let*

$$\begin{tikzcd} P \arrow[r] \arrow[d, "h"'] & A \arrow[d, "f"] \\ Q \arrow[r] & B \end{tikzcd}$$

be a morphism of integral log rings, $Q' \to Q$ a chart for Q, $P' \to P$ a chart for P. Then there is a chart for the morphism

$$\begin{tikzcd} P' \arrow[r] \arrow[d, "h'"'] & P \arrow[r] \arrow[d, "h"'] & A \arrow[d, "f"] \\ Q'' \arrow[r] & Q \arrow[r] & B \end{tikzcd}$$

where $Q'' \to Q$ is monic, and $Q' \to Q$ factors through $Q'' \to Q$. If $P' \to P$ and $Q' \to Q$ are fine charts, we may arrange that this is a fine, monic chart.

In particular, any morphism of fine log rings has a fine chart.

Proof. Consider the commutative diagram:

$$\begin{tikzcd} P' \arrow[r] \arrow[d] & P \arrow[d, "h"'] \\ P' \oplus Q' \arrow[r] \arrow[d] & Q \arrow[d] \\ (P')^{\mathrm{gp}} \oplus (Q')^{\mathrm{gp}} \arrow[r] & Q^{\mathrm{gp}} \end{tikzcd}$$

Let g be the bottom horizontal arrow, and let $\overline{g}$ be the composition g and the projection $Q^{\mathrm{gp}} \to \overline{Q}^{\mathrm{gp}}$. Since $Q' \to Q$ is a chart, $Q' \to \overline{Q}$ is surjective, hence so is its groupification and so is $\overline{g}$. By the First Chart Lemma (4.3.11), the inclusion of $Q'' := g^{-1}Q$ into Q is a chart with the desired properties. $\square$

Example 4.3.17. We will conclude this section by constructing an example of a non-split fine log structure on a field. It suffices to find a field k and a sharp, fine monoid P such that

$$\mathrm{Ext}^1_{\mathbf{Ab}}(P^{\mathrm{gp}}, k^*) \neq 0,$$

for then we can build a non-split (integral) log structure on k as in Example 4.3.6. This integral log structure will be fine by Proposition 4.3.9. Let us try to arrange that the finitely generated abelian group P^{gp} be as small as possible. It cannot be zero or even torsion free, for then the Ext group will vanish. It cannot be finite, for then P itself would be a group (1.7.4), hence not sharp, so the smallest possibility is $P^{\mathrm{gp}} = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Indeed, the submonoid P of $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ generated by $(1,1)$ and $(1,0)$ is a fine, sharp monoid generating $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ as a group. Now we just need a field k so that

$$\mathrm{Ext}^1_{\mathbf{Ab}}(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, k^*) \cong \mathrm{Ext}^1_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, k^*) \neq 0.$$

We may take $k = \mathbb{F}_3$, for then $k^* = \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$ and we have the nontrivial extension

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Taking the direct sum of this sequence and the exact sequence

$$0 \to 0 \to \mathbb{Z} \to \mathbb{Z} \to 0$$

and applying Proposition 1.8.1, we get a non-split integral extension of monoids

$$(k^* \to Q \to P) \in \mathrm{Ext}^1_{\mathbf{Mon}}(P, k^*)$$

which is the characteristic sequence of a non-split fine log structure $Q \to \mathbb{F}_3$.

We can even arrange that the field k be separably closed. For example, let k be the separable closure of the purely transcendental field extension $\mathbb{F}_p(t)$, for some prime p. Then

$$\operatorname{Ext}^1_{\mathbf{Ab}}(\mathbb{Z}/p\mathbb{Z}, k^*) \quad \cong \quad k^*/(k^*)^p$$

is nonzero because t does not have a p^{th} root in k since the minimal polynomial of t is not separable (p^{th} roots are unique in characteristic p). Just as above, we can consider the submonoid P of $\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ generated by $(1,1)$ and $(1,0)$. This P is sharp (and manifestly finitely generated) and clearly $P^{\text{gp}} = \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. As above, a nontrivial extension of $\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ by k^* gives rise to a non-split integral extension of monoids

$$(k^* \to Q \to P) \in \operatorname{Ext}^1_{\mathbf{Mon}}(P, k^*)$$

which is the characteristic sequence of a non-split fine log structure $Q \to k$.

4.4. **Integral morphisms.** A morphism

$$\begin{tikzcd} Q \arrow[r] \arrow[d, "h"'] & A \arrow[d, "f"] \\ P \arrow[r] & B \end{tikzcd}$$

of prelog rings is an *integral morphism* if h is an integral morphism of monoids (in particular, P, Q must be integral prelog structures).

Proposition 4.4.1. *If*

$$\begin{tikzcd} Q \arrow[r] \arrow[d, "h"'] & A \arrow[d, "f"] \\ P \arrow[r] & B \end{tikzcd}$$

is an integral morphism of prelog rings, then the induced morphism

$$\begin{tikzcd} Q^a \arrow[r] \arrow[d, "h^a"'] & A \arrow[d, "f"] \\ P^a \arrow[r] & B \end{tikzcd}$$

of associated log rings is an integral morphism.

Proof. It is probably possible to prove this formally from the pushout criterion, but the most direct proof uses the equational criterion. □

Remark 4.4.2. The converse is certainly not true. The monoids P, Q and the map h could be arbitrarily bad, but the associated log structures could both be trivial.

4.5. **Limits and Smallness.** Recall that a category I is called *filtered* iff it satisfies the following two conditions:

(1) It is *directed* in the sense that for any two objects i, i' of I, there is a third object i'' of I and morphisms $i \to i''$ and $i' \to i''$.
(2) It has *weak coequalizers* in the sense that for any parallel arrows $f, g : i \rightrightarrows i'$ there is a morphism $h : i' \to i''$ so that $hf = hg$.

Let I be a filtered category. A subcategory J of I is *dense* if there is an object i of I such that J is the full subcategory of I consisting of objects j with $\operatorname{Hom}_I(i,j) \neq \emptyset$. Evidently any two dense subcategories contain a third. A dense subcategory J is a cofinal (filtered) subcategory of I, so the direct limit of any functor $A : I \to \mathscr{C}$ to a category $\mathscr{C}$ depends only on its restriction $A|J : J \to \mathscr{C}$ to any dense subcategory J. We sometimes say "for sufficiently large i" to mean "on a dense subcategory $J \subseteq I$".

In this section, we consider a fixed I-indexed direct limit system $A : I \to \mathbf{An}$ in the category $\mathbf{An}$ of rings. We denote it by (A_i). A *prelog structure* on (A_i) is a factorization $P : J \to \mathbf{PLogAn}$ of the restriction of A to some dense subcategory $J \subseteq I$ through the forgetful functor $\mathbf{PLogAn} \to \mathbf{An}$. We denote a prelog structure on (A_i) by $(P_i \to A_i)$.

We are only interested in prelog structures up to restricting to a dense subcategory $J \subseteq I$, so a *morphism* of prelog structures from $P : J \to \mathbf{PLogAn}$ to $Q : K \to \mathbf{PLogAn}$ will be represented by a natural transformation $h : P|L \to Q|L$ between the restrictions of P, Q to some dense subcategory $L \subseteq J, K$ inducing the identity natural transformation $(A_i) \to (A_i)$ upon composition with $\mathbf{PLogAn} \to \mathbf{An}$. Two such natural transformations $h : P|L \to Q|L$ and $k : P|M \to Q|M$ are regarded as the same morphism in the category $\mathbf{PLog}(A_i)$ of prelog structures on (A_i) iff they agree after restricting to some dense subcategory $N \subseteq L, M$. We denote a morphism in $\mathbf{PLog}(A_i)$ by

$$(h_i) : (P_i \to A_i) \to (Q_i \to A_i).$$

A prelog structure $(P_i \to A_i)$ is called a *log structure* iff $P_i \to A_i$ is a log structure for sufficiently large i. Log structures on (A_i) form a full subcategory $\mathbf{Log}(A_i)$ of $\mathbf{PLog}(A_i)$ and there is an associated log structure functor

$$\begin{array}{rcl} a : \mathbf{PLog}(A_i) & \to & \mathbf{Log}(A_i) \\ (P_i \to A_i) & \mapsto & (P_i^a \to A_i) \end{array}$$

which is left adjoint to the forgetful functor $\mathbf{Log}(A_i) \to \mathbf{PLog}(A_i)$.

Remark 4.5.1. Our category $\mathbf{PLog}(A_i)$ is related to the category of ind objects in $\mathbf{PLogAn}$ with underlying ind ring (A_i), but our category has fewer morphisms. Two functors $P, Q : I \to \mathbf{PLogAn}$ (lying over (A_i)) can be isomorphic in the category of ind objects in $\mathbf{PLogAn}$ without being isomorphic in $\mathbf{PLog}(A_i)$. Our more restrictive notion of morphisms is important in applications.

In the applications we have in mind, the indexing category I will be the opposite category of the category $\mathbf{Vois}_x$ of neighborhoods of a point x of a topos X (see Section 5.1 and the references there for definitions). Given a "(pre)log ring object" $\mathcal{M}_X \to \mathscr{O}_X$ of X, we will consider the ind (pre)log ring

$$\begin{array}{rcl} \mathbf{Vois}_x^{\mathrm{op}} & \to & \mathbf{PLogAn} \\ U & \mapsto & \mathcal{M}_X(U) \to \mathscr{O}_X(U). \end{array}$$

Specifically, given two log structures $\mathcal{M}_X, \mathcal{N}_X$ on the same ring object $\mathscr{O}_X$, we will often want to compare three types of morphisms:

(1) Maps $\mathcal{M}_X|U \to \mathcal{N}_X|U$ defined on some neighborhood U of x
(2) Maps between the corresponding ind (pre)log rings (which are the identity on the level of ind rings)
(3) Maps $\mathcal{M}_{X,x} \to \mathcal{N}_{X,x}$ of (pre)log structures on the stalk ring $\mathscr{O}_{X,x}$

A prelog structure $(P_i \to A_i)$ is called *essentially constant* iff the functor $i \mapsto P_i$ is isomorphic to the constant functor $i \mapsto P$ for some monoid P for sufficiently large i. An essentially constant prelog structure is called *finitely generated* if the monoid P can be taken to be finitely generated. A log structure $(P_i \to A_i)$ is called *quasi-coherent* (resp. *coherent*) iff it is isomorphic to the associated log structure $(P_i^a \to A)$ of an essentially constant prelog structure (resp. finitely generated essentially constant prelog structure). Let $\mathbf{CohLog}(A_i)$ denote the full subcategory of $\mathbf{Log}(A_i)$ consisting of coherent log structures.

Let A be the direct limit of (A_i) in $\mathbf{An}$. Then we have a functor

$$\begin{array}{rcl} \varinjlim : \mathbf{PLog}(A_i) & \to & \mathbf{PLog}(A) \\ (P_i \to A_i) & \mapsto & (\varinjlim P_i \to A). \end{array}$$

Proposition 4.5.2. *The functor* $\varinjlim$ *takes log structures on* (A_i) *to log structures on* A *and the diagram of functors*

$$\begin{tikzcd} \mathbf{PLog}_{(A_i)} \arrow[r, "\varinjlim"] \arrow[d, "(\_ ^a)"'] & \mathbf{PLog}_A \arrow[d, "\_ ^a"] \\ \mathbf{Log}_{(A_i)} \arrow[r, "\varinjlim"'] & \mathbf{Log}_A \end{tikzcd}$$

commutes up to natural natural isomorphism.

Proof. The point is that the natural map $\varinjlim A_i^* \to (\varinjlim A_i)^*$ is an isomorphism. For a prelog structure (P_i) on (A_i), it is straightforward to see that both $\varinjlim (P_i^a)$ and $(\varinjlim P_i)^a$ can be described as the set of pairs (p_i, u_i), where p_i is in some P_i, $u_i \in A_i^*$, modulo the relation $(p_i, u_i) \sim (p_j, u_j)$ iff for some $k \in I$, there are I-morphisms $i, j \to k$ and r_k, r_k' in P_k and $u_k, u_k' \in \alpha^{-1} A_k^*$ such that $p_i + r_k = p_j + r_k$ in P_k (writing p_i, p_j as abuse of notation for their images under $P_i \to P_k, P_j \to P_k$) and $\alpha_k(r_k)u_j = \alpha_k(r_k')u_i$ (under similar abuse of notation). $\square$

The main result of this section is the following:

Theorem 4.5.3. *The restriction of* $\varinjlim$ *to the category of coherent log structures defines a fully faithful functor*

$$\varinjlim : \mathbf{CohLog}(A_i) \to \mathbf{Log}(A)$$

whose essential image is the category $\mathbf{CohLog}(A)$ *of coherent log structures on* A*, hence*

$$\varinjlim : \mathbf{CohLog}(A_i) \to \mathbf{CohLog}(A)$$

defines an equivalence of categories.

Proof. The fact that $\varinjlim$ takes coherent log structures on (A_i) to coherent log structures on A is an immediate consequence of the definitions and the previous proposition.

Full faithfulness. Let (Q_i) be a coherent log structure on (A_i). Choose a finitely generated chart $P \to (Q_i)$, so we have isomorphisms $P_i^a \cong Q_i$ for all sufficiently large i. Note that the induced map $P \to \varinjlim Q_i$ on direct limits induces an isomorphism

$P^a \cong \varinjlim Q_i$ of log structures on A by Proposition 4.5.2. Let (Q_i') be an arbitrary log structure on (A_i) and let

$$h : \varinjlim Q_i \to \varinjlim Q_i'$$

be a morphism of log structures on A. We want to show that there is a unique morphism $(h_i) : (Q_i) \to (Q_i')$ in $\mathbf{Log}_{(A_i)}$ with $h = \varinjlim h_i$.

By Theorem 1.9.6, P is small, so the composition $P \to \varinjlim Q_i \to \varinjlim Q_i'$ factors as $k_i : P \to Q_i'$ followed by the structure map $Q_i' \to \varinjlim Q_i'$. Composing k_i with the restriction $Q_i' \to Q_j'$, we get a map $(k_i) : P \to (Q_i')$ inducing the above composition on direct limits. Since Q_i' is a log structure and $P_i^a \cong Q_i$, the maps k_i factor through maps $h_i = k_i^a : Q_i \to Q_i'$ (at least for sufficiently large i) so (k_i) factors through $(h_i) : Q_i \to Q_i'$. By Proposition 4.5.2, we have

$$\varinjlim h_i = \varinjlim k_i^a = (\varinjlim k_i)^a = h.$$

For uniqueness, suppose $(h_i') : (Q_i) \to (Q_i')$ is another morphism with $\varinjlim h_i' = h$. We want to show $h_i = h_i'$ for all sufficiently large i. Let $(k_i') : P \to Q_i'$ be the composition of $P \to (Q_i)$ and (h_i'), so that $(h_i') = (k_i'^a)$. Now, we do not necessarily have $\varinjlim k_i = \varinjlim k_i'$, so we cannot use Theorem 1.9.6 to conclude that $k_i = k_i'$ for all sufficiently large i. In fact, this may not even be true; all we want to show is that $h_i = k_i^a = k_i'^a = h_i'$ for sufficiently large i, which is a weaker statement that we can prove with only a little more work.

Choose generators $p_1, \dots, p_n$ of P. Let us use the usual description of $P_i^a \cong Q_i$ (for sufficiently large i) as the quotient of $P \oplus A_i^*$. Then any map of log structures $f : P_i^a \to Q_i'$ is determined by the $f[p_j, 1]$ because any $[p, u] \in P_i^a$ can be written as

$$[0, u] + \sum_j n_j [p_j, 1]$$

for some $n_j \in \mathbb{N}, u_i \in A_i^*$. So, to arrange that $h_i = k_i^a$ and $h_i' = k_i'^a$ agree, all we need to do is arrange that $h_i[p_j, 1] = h_i'[p_j, 1]$ for $j = 1, \dots, n$. But we know the images of $h_i[p_j, 1]$ and $h_i'[p_j, 1]$ in $\varinjlim Q_i'$ are equal (they are both given by $h[p_j, 1]$), so, since there are only finitely many j, these equalities hold for all sufficiently large i.

Essential surjectivity. Let Q be a coherent log structure on A and let $P \to Q$ be a chart with P finitely generated. By Theorem 1.9.6, P is a small object in **Mon**, so the structure map $P \to A$ factors through some $A_i \to A$. Passing to the dense subcategory determined by i and using the compositions $P \to A_i \to A_j$, we see that P determines a constant prelog structure on (A_i). Obviously the direct limit prelog structure on A is P, so $(\varinjlim P)^a = P^a \cong Q$. On the other hand, the log structure (P_i^a) on (A_i) associated to the constant prelog structure P is manifestly coherent, and we have $\varinjlim P_i^a \cong (\varinjlim P)^a$ by Proposition 4.5.2.

□

Here is a typical application:

Corollary 4.5.4. *Let P, Q be coherent log structures on a ring A. Let $l : A \to A_\wp$ be the localization of A at a prime ideal $\wp$ and suppose the inverse image log structures l^*P*

and l^*Q on $A_\wp$ are isomorphic. Then there is some $f \in A \setminus \wp$ such that the log structures P_f^a, Q_f^a on A_f obtained as inverse images of P, Q under $A \to A_f$ are isomorphic.

Proof. The localization $A_\wp$ is a filtered direct limit of the rings A_f over elements $f \in A \setminus \wp$. We view P, Q as constant prelog structures on the filtered direct limit system of rings (A_f) via the compositions

$$P \to A \to A_f, \qquad Q \to A \to A_f.$$

Since $l * P$ and l^*Q are the log structures associated to the prelog structures given by the compositions

$$P \to A \to A_\wp = \varinjlim A_f, \qquad Q \to A \to A_\wp = \varinjlim A_f$$

we certainly have

$$l^*P = (\varinjlim P)^a, \qquad l^*Q = (\varinjlim Q)^a.$$

On the other hand, coherence of P, Q clearly implies that the associated log structures $(P_f^a), (Q_f^a)$ on (A_f) are coherent and we have $l^*P \cong \varinjlim P_f^a$ and $l^*Q \cong \varinjlim Q_f^a$ by Proposition 4.5.2 so the result follows from the theorem. $\square$

5. Log Ringed Topoi

The goal of this section is to generalize the discussion in Section 4 on log structures on a ring to log structures on a sheaf of rings (a ring object of a topos). In the long run, we will focus on the topoi commonly associated with schemes, but there is no real reason to do so at this point. In my opinion, the extra generality we adopt here serves only to clarify the situation, since we may isolate the problems of log geometry from those of any particular setting (schemes, topological spaces, etc.).

We would like to assume the reader is familiar with Tome 1 (Exposés I-IV) of [SGA4], but actually the exposition below should be fairly self-contained if the reader takes some standard categorical nonsense for granted, or restricts attention to some familiar examples (étale site of a scheme, topological spaces, etc.). We will return to more explicit examples in the next section; the purpose of this section is mainly to lay down a general theory that we can refer back to as needed.

5.1. **Sites and topoi.** The purpose of this section is to set up notation and to recall the basic definitions and results about sites and topoi (c.f. Tome 1 of [SGA4]).

A *site* is a category X equipped with a *topology.* A topology assigns to each object U of X a set $\mathbf{Cov}_U$ of X-morphisms to U called *covers.*[2] This assignment is required to satisfy the following properties:

(1) An isomorphism is a cover.
(2) A cover of a cover is a cover.
(3) The pullback $V' \to V$ of any cover $U' \to U$ along any morphism $V \to U$ exists and is a cover of V.

We will denote a site and its underlying category by the same symbol when the topology is clear from context.

We will assume for simplicity that every site X has a terminal object, which we will also call X by abuse of notation. This abuse of notation is eminently compatible with the examples we have in mind.[3]

An extreme example is the *coarse* topology where $\mathbf{Cov}_U$ is the set of isomorphisms $U' \to U$. Another important example is the *canonical topology*, where $\mathbf{Cov}_U$ is the set of maps $U' \to U$ such that, for any $V \to U$, the fibered product $V' := V \times_U U'$ exists, and for any object W, the diagram

$$\operatorname{Hom}_X(V, W) \to \operatorname{Hom}_X(V', W) \rightrightarrows \operatorname{Hom}_X(V' \times_V V', W)$$

[2]Technically speaking, this is a *pretopology* in the sense of [SGA4.II.1.3]. For each object U of X and each cover $U' \to U$, we can consider the full subcategory of X/U consisting of those objects $\pi : V \to U$ where π factors through $U' \to U$. The set $J(U)$ of all such full subcategories of X/U is called the set of *covering seives* (French: *cribles couvrant*) of U and the data of all the $J(U)$ defines a topology on $\mathscr{C}$ in the sense of [SGA4.II.1.1]. One can make a more general definition of "site," but it is rarely useful in practice, and the resulting notion of "topos" would not be any different: Any category equivalent to the category of sheaves on a site (in the [SGA4.II.1.1] sense of site) is also equivalent to the category of sheaves on a site (in our sense of site).

[3]Again, even with this "restrictive" definition of "site," the notion of a topos will be unchanged.

is an equalizer diagram of sets. A cover in the canonical topology is also called a *canonical cover*. A topology $\mathbf{Cov}$ is called *subcanonical* iff every cover $(U' \to U) \in \mathbf{Cov}_U$ is a canonical cover.

The basic example is the category $\mathbf{Ouv}_X$ of topological spaces $f : Y \to X$ over X where f is locally a homeomorphism onto its image. A cover in $\mathbf{Ouv}_X$ is a surjective morphism $Y' \to Y$ of such topological spaces over X.

Following the conventions of [SGA4], we write $\mathscr{C}^\wedge := \operatorname{Hom}_{\mathbf{Cat}}(\mathscr{C}^{\text{op}}, \mathbf{Ens})$ for the category of *presheaves* on a category $\mathscr{C}$. If $\mathscr{F}$ is a presheaf on $\mathscr{C}$, $f : C \to C'$ is a $\mathscr{C}$-morphism, and $s \in \mathscr{F}(C')$, then we write $s|_C$ for the image of s under $\mathscr{F}(f) : \mathscr{F}(C') \to \mathscr{F}(C)$. According to Yoneda's Lemma, the functor

$$\begin{array}{rcl} h : \mathscr{C} & \to & \mathscr{C}^\wedge \\ U & \mapsto & h_U \end{array}$$

assigning the presheaf

$$h_U : V \mapsto \operatorname{Hom}_{\mathscr{C}}(U, V)$$

to an object U of $\mathscr{C}$ is fully faithful and we have

$$\mathscr{F}(U) = \operatorname{Hom}_{\mathscr{C}^\wedge}(h_U, \mathscr{F})$$

for every presheaf $\mathscr{F}$ on $\mathscr{C}$. A presheaf isomorphic to one of the presheaves h_U is called *representable*. Notice that a topology $\mathbf{Cov}$ is subcanonical iff every presheaf h_U is a sheaf. We usually abuse notation and simply write U for h_U.

A presheaf $\mathscr{F}$ on a site X is called a *sheaf* iff, for any cover $U' \to U$, the diagram

$$\mathscr{F}(U) \to \mathscr{F}(U') \rightrightarrows \mathscr{F}(U' \times_U U')$$

is an equalizer diagram of sets. The presheaf $\mathscr{F}$ is said to be *separated* iff the induced map from $\mathscr{F}(U)$ to the equalizer of $\mathscr{F}(U') \rightrightarrows \mathscr{F}(U' \times_U U')$ is monic. The full subcategory of $X^\wedge$ consisting of sheaves is denoted $X^\sim$.

For example, for a topological space X, the category of sheaves on the site $\mathbf{Ouv}_X$ defined above is equivalent to the usual category of sheaves on X.[4]

The inclusion $X^\sim \to X^\wedge$ admits a left adjoint

$$\begin{array}{rcl} X^\wedge & \to & X^\sim \\ \mathscr{F} & \mapsto & \mathscr{F}^+ \end{array}$$

called the *sheafification functor*, constructed as follows (ignoring set theoretic issues). For any presheaf $\mathscr{F}$ on X, and any object U of X, define a functor

$$\begin{array}{rcl} \operatorname{Eq}_U \mathscr{F} : \mathbf{Cov}_U^{\text{op}} & \to & \mathbf{Ens} \\ (U' \to U) & \mapsto & \varprojlim \mathscr{F}(U') \rightrightarrows \mathscr{F}(U' \times_U U') \end{array}$$

and define a presheaf $\operatorname{Eq} \mathscr{F}$ by

$$U \mapsto \varinjlim_{\mathbf{Cov}_U^{\text{op}}} \operatorname{Eq}_U \mathscr{F}.$$

The fact that $\operatorname{Id}_U : U \to U$ is a cover means there is a natural map $\mathscr{F}(U) \to (\operatorname{Eq} \mathscr{F})(U)$ and hence a map of presheaves $\mathscr{F} \to \operatorname{Eq} \mathscr{F}$. One can check that $\operatorname{Eq} \mathscr{F}$ is a separated presheaf for any presheaf $\mathscr{F}$ and that $\operatorname{Eq} \mathscr{F}$ is a sheaf for any separated presheaf $\mathscr{F}$. Then we have $\mathscr{F}^+ = \operatorname{Eq} \operatorname{Eq} \mathscr{F}$.

[4] In [SGA4], this category of sheaves on X is denoted $\mathbf{Top}(X)$.

Using that covers form a cofiltered category, it can be shown that the sheafification functor $\mathscr{F} \mapsto \mathscr{F}^+$ commutes with finite inverse limits (i.e. is *left exact*) and, being a left adjoint, it commutes with all direct limits. (c.f. [SGA4.II.3])

For a functor $F : \mathscr{C} \to \mathscr{D}$ and an object D of $\mathscr{D}$, write $\mathbf{Vois}_D$ for the *category of neighborhoods*[5] of D, whose objects are pairs consisting of an object C of $\mathscr{C}$ and a $\mathscr{D}$-morphism $D \to FC$. A morphism from $D \to FC$ to $D \to FC'$ is a $\mathscr{C}$-morphism $C \to C'$ whose image under F makes the obvious diagram commute. An object of $\mathbf{Vois}_D$ will be called a *neighborhood* of D and a morphism in $\mathbf{Vois}_D$ will be called a *shrinking* (of its codomain).

The category of presheaves $\mathscr{C}^\wedge$ is contravariantly functorial in $\mathscr{C}$. That is, a functor $F : \mathscr{C} \to \mathscr{D}$ induces a functor

$$\begin{array}{rcl} F_* : \mathscr{D}^\wedge & \to & \mathscr{C}^\wedge \\ \mathscr{G} & \mapsto & \mathscr{G} \circ F \end{array}$$

called the *direct image functor*. Ignoring set-theoretic issues, the functor F_* always admits a left adjoint

$$\begin{array}{rcl} F^{-1}_{\mathrm{pre}} : \mathscr{C}^\wedge & \to & \mathscr{D}^\wedge \\ \mathscr{F} & \mapsto & F^{-1}_{\mathrm{pre}}\mathscr{F} \end{array}$$

constructed by setting

$$(F^{-1}_{\mathrm{pre}}\mathscr{F})(D) := \varinjlim_{D \to FC} \mathscr{F}(C).$$

More precisely, the direct limit is the direct limit, over the category $\mathbf{Vois}_D^{\mathrm{op}}$, of the functor

$$\begin{array}{rcl} \mathbf{Vois}_D^{\mathrm{op}} & \to & \mathbf{Ens} \\ (D \to FC) & \mapsto & \mathscr{F}(C) \end{array}$$

It is straightforward to check [SGA4.I.5.1] that this is a left adjoint to F_* (it is a left Kan extension), so we have a natural bijection

$$\mathrm{Hom}_{\mathscr{D}^\wedge}(F^{-1}_{\mathrm{pre}}\mathscr{F}, \mathscr{G}) \cong \mathrm{Hom}_{\mathscr{C}^\wedge}(\mathscr{F}, F_*\mathscr{G})$$

for every presheaf $\mathscr{F}$ on $\mathscr{C}$ and every presheaf $\mathscr{G}$ on $\mathscr{D}$.

Lemma 5.1.1. *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. The diagram of functors*

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "h"'] & \mathscr{D} \arrow[d, "h"] \\ \mathscr{C}^\wedge \arrow[r, "F^{-1}_{\mathrm{pre}}"] & \mathscr{D}^\wedge \end{tikzcd}$$

commutes up to natural isomorphism.

[5] In the proof of [SGA4.I.5.1], Grothendieck and Verdier use the notation I^D_f for our $\mathbf{Vois}_D$. This category is nothing but the so-called *comma category* $D \downarrow \mathscr{C}$ of [MacLane].

Proof. We must construct a natural isomorphism $h_{FC} \cong F_{\text{pre}}^{-1} h_C$ in $\mathscr{D}^\wedge$ for every object C of $\mathscr{C}$. Let $\mathscr{F}$ be a presheaf on $\mathscr{D}$. Then we have natural isomorphisms

$$\begin{array}{rcll} \operatorname{Hom}_{\mathscr{D}^\wedge}(h_{FC}, \mathscr{F}) & \cong & \mathscr{F}(FC) & \text{(Yoneda)} \\ & =: & (F_*\mathscr{F})(C) & \\ & \cong & \operatorname{Hom}_{\mathscr{C}^\wedge}(h_C, F_*\mathscr{F}) & \text{(Yoneda)} \\ & \cong & \operatorname{Hom}_{\mathscr{D}^\wedge}(F_{\text{pre}}^{-1} h_C, \mathscr{F}) & \text{(Adjointness).} \end{array}$$

Since this holds for any $\mathscr{F}$, we get the desired natural isomorphism from Yoneda's Lemma. □

If $\mathscr{C}$ and $\mathscr{D}$ are sites, then a functor $F : \mathscr{C} \to \mathscr{D}$ between their underlying categories is called *continuous* if $F_* : \mathscr{D}^\wedge \to \mathscr{C}^\wedge$ takes sheaves to sheaves (i.e. F_* takes $\mathscr{D}^\sim \subseteq \mathscr{D}^\wedge$ into $\mathscr{C}^\sim \subseteq \mathscr{D}^\wedge$). If F is continuous, then the composition of F_{pre}^{-1} (restricted to the category of sheaves on $\mathscr{C}$) and the sheafification functor define a functor

$$\begin{array}{rcl} F^{-1} : \mathscr{C}^\sim & \to & \mathscr{D}^\sim \\ \mathscr{F} & \mapsto & (F_{\text{pre}}^{-1})^+ \mathscr{F} \end{array}$$

which is left adjoint to

$$F_* : \mathscr{D}^\sim \to \mathscr{C}^\sim.$$

Indeed, for a sheaf $\mathscr{F}$ on $\mathscr{C}$ and a sheaf $\mathscr{G}$ on $\mathscr{D}$, we have natural isomorphisms:

$$\begin{array}{rcl} \operatorname{Hom}_{\mathscr{D}^\sim}(F^{-1}\mathscr{F}, \mathscr{G}) & := & \operatorname{Hom}_{\mathscr{D}^\sim}((F_{\text{pre}}^{-1}\mathscr{F})^+, \mathscr{G}) \\ & \cong & \operatorname{Hom}_{\mathscr{D}^\wedge}(F_{\text{pre}}^{-1}\mathscr{F}, \mathscr{G}) \\ & \cong & \operatorname{Hom}_{\mathscr{C}^\wedge}(\mathscr{F}, F_*\mathscr{G}) \\ & = & \operatorname{Hom}_{\mathscr{C}^\sim}(\mathscr{F}, F_*\mathscr{G}) \end{array}$$

The first isomorphism is by the adjointness property of sheafification, the second is the adjointness of F_{pre}^{-1} and F_* and the final equality is a matter of definitions (sheaves are a full subcategory of presheaves, and $F_*\mathscr{G}$ is a sheaf by definition of a continuous functor). Even if F is not continuous, we still have the functor $F^{-1} : \mathscr{D}^\sim \to \mathscr{C}^\sim$ obtained by composing F_{pre}^{-1} and sheafification, though we do not necessarily have any use for it since it does not have the adjointness property we want.

Proposition 5.1.2. *Suppose $\mathscr{C}$ and $\mathscr{D}$ are sites and $F : \mathscr{C} \to \mathscr{D}$ is a functor between their underlying categories. Each condition below implies the subsequent condition:*

(1) $\mathscr{C}$ has finite inverse limits and F is left exact.
(2) The category of neighborhoods $\mathbf{Vois}_D$ *is cofiltered for every D in $\mathscr{D}$.*
(3) $F_{\text{pre}}^{-1} : \mathscr{D}^\wedge \to \mathscr{C}^\wedge$ is left exact.
(4) $F^{-1} : \mathscr{D}^\sim \to \mathscr{C}^\sim$ is left exact.

Remark 5.1.3. We assume our sites have terminal objects, so (1) is equivalent to

(1)′ $\mathscr{C}$ has cartesian products and F preserves them and takes the terminal object of $\mathscr{C}$ to the terminal object of $\mathscr{D}$.

Proof. (1) $\Longrightarrow$ (2) To see that $\mathbf{Vois}_D$ is codirected, suppose $D \to FC$ and $D \to FC'$ are two objects of $\mathbf{Vois}_D$, and consider the induced map $D \to FC \times FC' \cong F(C \times C')$. To see that $\mathbf{Vois}_D$ has weak equalizers, suppose $FC \rightrightarrows FC'$ are parallel arrows in $\mathbf{Vois}_D$. By definition of $\mathbf{Vois}_D$, we have parallel arrows $C \rightrightarrows C'$ in $\mathscr{C}$ and a $\mathscr{D}$-morphism $D \to FC$ so that the two compositions $D \to FC \rightrightarrows FC'$ agree. If we let E be the equalizer of $C \rightrightarrows C'$, then by exactness of F,

$$FE \to FC \rightrightarrows FC'$$

is an equalizer diagram in $\mathscr{D}$, so $D \to FC$ factors through a map $D \to FE$, hence we may regard so $D \to FE$ as an object of $\mathbf{Vois}_D$ with a map to FC equalizing the parallel arrows.

(2) $\implies$ (3) Let $\mathscr{F}_i$ be a finite inverse system of presheaves on $\mathscr{D}$. The inverse limit of presheaves (as well as the direct limit) is formed objectwise, so we have

$$(\varprojlim \mathscr{F}_i)(D) = \varprojlim \mathscr{F}_i(D).$$

(You can easily check that this has the correct universal property.) We wish to prove that the natural map

$$\varprojlim (F_{\mathrm{pre}}^{-1}\mathscr{F}_i) \to F_{\mathrm{pre}}^{-1}(\varprojlim \mathscr{F}_i)$$

is an isomorphism. That is, for each object D, we want to prove that the natural map

$$\varprojlim \left((F_{\mathrm{pre}}^{-1}\mathscr{F}_i)(D)\right) \to \left(F_{\mathrm{pre}}^{-1}(\varprojlim \mathscr{F}_i)\right)(D)$$

is an isomorphism. By definition of F_{pre}^{-1} this map is the natural map

$$\varprojlim \varinjlim_{\mathbf{Vois}_D^{\mathrm{op}}} \mathscr{F}_i(D) \to \varinjlim_{\mathbf{Vois}_D^{\mathrm{op}}} \varprojlim \mathscr{F}_i(D),$$

which is an isomorphism under hypothesis (2) because $\mathbf{Vois}_D^{\mathrm{op}}$ is filtered and filtered direct limits of sets commute with finite inverse limits (c.f. [SGA4.I.2.8] or [MacLane]).

(3) $\implies$ (4) because the sheafication functor is left exact and a composition of left exact functors is left exact. $\square$

A *morphism of sites* [SGA4.IV.4.9] $f : X \to Y$ is a continuous functor $f : Y \to X$ between the underlying categories (in the opposite direction!) such that $f^{-1} : Y^\sim \to X^\sim$ is left exact. With this definition of morphisms, sites form a 2-category denoted **Sit**. We will assume, for simplicity, that a morphism of sites preserves terminal objects (the functor $f : Y \to X$ takes Y to X).

A *topos* is a category equivalent to the category of sheaves on a site.

Proposition 5.1.4. *Let X be a site, let $X^\sim$ be the topos of sheaves on X and let $f : \mathscr{F} \to \mathscr{G}$ be a morphism in $X^\sim$. The following are equivalent:*

(1) f is an epimorphism.
(2) f is a canonical cover.
(3) For any object U of X and any $s \in \mathscr{G}(U)$ there is a cover $U' \to U$ of U in X and a section $t \in \mathscr{F}(U')$ with $f(U')(t) = \mathscr{F}(U' \to U)(s)$.

Proof. This is an exercise with definitions that we leave to the reader. $\square$

For simplicity, we call a morphism f satisfying the equivalent conditions of Proposition 5.1.4 a *cover*. We sometimes say that an object of $X^\sim$ is a *cover* to mean that the map to the terminal object of $X^\sim$ is a cover. Similarly, if X is a topos, we say that an object Y of X is a *cover* iff the map from Y to the terminal object of X is a cover. The following corollary is straightforward to prove from the proposition:

Corollary 5.1.5. *Let X be a site, $X^\sim$ the topos of sheaves on X, U an object of X. Then the sheafified Yoneda functor $V \mapsto h_V^+$ takes covers of U to covers of h_U^+, and the covers of h_U^+ arising in this manner are cofinal in the category of all covers of h_U^+.*

For example, for any category $\mathscr{C}$, the category $\mathscr{C}^\wedge$ is a topos (presheaves on $\mathscr{C}$ are sheaves on $\mathscr{C}$ in the coarse topology). Every topos has a terminal object, which we also denote X by abuse of notation. A *morphism of topoi* $f : X \to Y$ is a functor $f_* : X \to Y$, together with an exact left adjoint $f^{-1} : Y \to X$ of f_*. Since f^{-1} is a left adjoint it automatically commutes with all direct limits, so the nontrivial requirement is that f^{-1} should commute with finite inverse limits. We refer to f_* as the *direct image* and f^{-1} as the *inverse image*. Since f_* and f^{-1} both commute with finite inverse limits, they take objects defined in terms of such limits (group objects, ring objects, monoid objects and so forth) to the corresponding objects. For any topos X and any object Y of X, the category X/Y of objects of X over Y is also a topos and the obvious forgetful functor $X/Y \to X$ is (the direct image functor for) a morphism of topoi (with inverse image functor $X \to X/Y$ given by $U \mapsto (\pi_1 : Y \times U \to Y)$).

For example, for any site X, the forgetful functor, and its left adjoint (sheafification) define a morphism of topoi

$$(+, \text{forget}) : X^\sim \to X^\wedge.$$

Topoi form a 2-category **Top** whose 2-morphisms are pairs of natural transformations (θ^{-1}, θ_*) between the direct and inverse image functors which satisfy an obvious compatibility with the adjunction isomorphisms. There is a functor

$$\begin{array}{rcl} \mathbf{Top} : \mathbf{Sit} & \to & \mathbf{Top} \\ X & \mapsto & X^\sim \end{array}$$

which takes a morphism of sites $f : X \to Y$ to the pair of adjoint functors

$$\mathbf{Top} f = (f^{-1}, f_*) : X^\sim \to Y^\sim.$$

Note that f_* takes sheaves on X to sheaves on Y and f^{-1} is left exact by definition of a morphism of sites.

The category of sets **Ens** is a topos (it is equivalent to the category of presheaves on the punctual category). Every topos admits a morphism to **Ens** by considering the functor

$$\begin{array}{rcl} X & \to & \mathbf{Ens} \\ \mathscr{F} & \mapsto & \mathscr{F}(X) := \operatorname{Hom}_X(X, \mathscr{F}) \end{array}$$

and its left adjoint

$$\begin{array}{rcl} \mathbf{Ens} & \mapsto & X \\ S & \mapsto & \underline{S}, \end{array}$$

where $\underline{S}$ is the sheafification of the constant presheaf $U \mapsto S$ in the canonical topology of X (or, equivalently, $\underline{S}$ is the categorical coproduct of "S copies of the terminal object"). The left adjoint is exact because coproducts commute with finite inverse limits in a topos (c.f. [SGA4.II.4.3] and [SGA4.II.4.8] for general categorical properties of topoi) . Furthermore, it is straightforward to check that every morphism of topoi to **Ens** is uniquely 2-isomorphic to such a morphism (i.e. the category of 1-morphisms to **Ens** is a punctual category) so **Ens** is a terminal object in **Top**. A topos is called *punctual* if the essentially unique morphism to the terminal object is an equivalence.

A *point* [SGA4.IV.6] of a topos X is a morphism of topoi $x = (x^{-1}, x_*) : \mathbf{Ens} \to X$. A point of a site X is, by definition, a point of its topos of sheaves $X^\sim$. If x is a point of X and $\mathscr{F}$ is an object of X, the inverse image $x^{-1}\mathscr{F}$ is called the *stalk* of $\mathscr{F}$ at x and is denoted $\mathscr{F}_x$. The functor x^{-1} is called the *stalk functor* (French: *foncteur fibre*) of x. The stalk functor $\mathscr{F} \mapsto \mathscr{F}_x$ preserves direct limits, finite inverse limits, and takes epimorphisms in X to surjections of sets. Points of X form a category

$$\mathbf{Pt}_X := \mathbf{HomTop}(\mathbf{Ens}, X).$$

If x is a point of a topos X, then a *neighborhood* [SGA4.IV.6.8] of x is a pair (U, u) where U is an object of X and u is an element of the set $x^{-1}U = U_x$ (i.e. a neighborhood of the punctual set under the functor $x^{-1} : X \to \mathbf{Ens}$ according our previous definition of "neighborhood"). Neighborhoods of x form a category $\mathbf{Vois}_{x/X}$ (or just $\mathbf{Vois}_x$ if X is clear from context) where a morphism $(U, u) \to (V, v)$ is an X morphism $f : U \to V$ such that $f_x(u) = v \in V_x$. Using the fact that X has finite inverse limits preserved by x^{-1}, one easily sees that $\mathbf{Vois}_x$ is cofiltered. If x is a point of a site X, then a *neighborhood* of x is a pair (U, u) where U is an object of X and $u \in (h_U^+)_x = x^{-1}h_U^+$. Neighborhoods of x form a category $\mathbf{Vois}_{x/X}$. Evidently a neighborhood of x in the site X is, in particular, a neighborhood of x in the topos $X^\sim$, and it is a standard fact that $\mathbf{Vois}_{x/X} \to \mathbf{Vois}_{x/X^\sim}$ is cofinal. One can compute the stalk $\mathscr{F}_x$ as

$$\mathscr{F}_x = \varinjlim_{(U,u)\in\mathbf{Vois}_x^{\mathrm{op}}} \mathscr{F}(U),$$

and formation of the stalk commutes with sheafification:

$$(\mathscr{F}^+)_x = \varinjlim_{(U,u)\in\mathbf{Vois}_x^{\mathrm{op}}} \mathscr{F}(U)$$

for any presheaf $\mathscr{F}$ on a site X [SGA4.IV.6.8.4].

Example 5.1.6. For a topological space X, the category of points $\mathbf{Pt}(X^\sim)$ in the topos of sheaves on X is equivalent to the category associated to the ordered set of points of X_{sob}, ordered by specialization. Here X_{sob} is obtained from X by first taking the quotient $X/\sim$ by the equivalence relation $x \sim y$ if x is in the closure of $\{y\}$ and y is in the closure of $\{x\}$, then adjoining a generic point for each irreducible closed subset of the resulting space. If X is Hausdorff or is the space underlying a scheme, then $X = X_{\mathrm{sob}}$. See [SGA4.IV.7.1.6].

For later use, let us note the following:

Lemma 5.1.7. *Let X be a topos, Y a cover of (the terminal object of) X, x a point of X. Then the neighborhoods (U, u) of x where U ranges over X/Y are cofinal in $\mathbf{Vois}_x$.*

Proof. The basic point is that $\pi_1 : U \times Y \to U$ is an epimorphism for any object U because epimorphisms in a topos are preserved under fiber products, and the stalk functor takes epimorphisms to surjections. □

5.2. **Ringed sites.** A *ringed topos* (resp. *ringed site*) $(X, \mathscr{O}_X)$ is a topos (resp. site) X together with a ring object $\mathscr{O}_X$ of X (resp. of $X^\sim$).

A morphism of ringed topoi (resp. ringed sites) $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ consists of a morphism of topoi (resp. sites) $f : X \to Y$ and a morphism $f^\sharp : f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ of ring objects of X (resp. $X^\sim$). By adjointness, the morphism $f^\sharp$ is the thing as a morphism

$f^\flat : \mathscr{O}_Y \to f_*\mathscr{O}_X$ of ring objects of Y (resp. $Y^\sim$). Ringed topoi (resp. ringed sites) form a 2-category denoted **TopAn** (resp. **SitAn**).

If no confusion seems likely, we will denote a ringed topos or site $(X, \mathscr{O}_X)$ simply by the letter X, reserving the notation $\mathscr{O}_X$ for the ring object. In particular, for a scheme X, we write X_T for the ringed site $(X_T, \mathscr{O}_X)$, and $X_T^\sim$ for the ringed topos $(X_T^\sim, \mathscr{O}_X)$.

A morphism of ringed topoi (resp. ringed sites) $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is called *local* if

(5.2.0.1)
$$\begin{tikzcd} f^{-1}\mathscr{O}_Y^* \arrow[r] \arrow[d, hook] & \mathscr{O}_X^* \arrow[d, hook] \\ f^{-1}\mathscr{O}_Y \arrow[r, "f^\sharp"] & \mathscr{O}_X \end{tikzcd}$$

is cartesian in X (resp. $X^\sim$). In other words: anything mapping to a unit of $\mathscr{O}_X$ under $f^\sharp$ is a unit in $f^{-1}\mathscr{O}_Y$.

Let us investigate this a little. If x is a point of X and f is a local morphism of ringed topoi as above, then applying the exact functor x^{-1} to the cartesian square (5.2.0.1) gives a cartesian diagram

(5.2.0.2)
$$\begin{tikzcd} \mathscr{O}_{Y,f(x)}^* \arrow[r] \arrow[d, hook] & \mathscr{O}_{X,x}^* \arrow[d, hook] \\ \mathscr{O}_{Y,f(x)} \arrow[r, "f_x^\sharp"] & \mathscr{O}_{X,x} \end{tikzcd}$$

of sets. If $\mathscr{O}_{X,x}$ and $\mathscr{O}_{Y,f(x)}$ happen to local rings, this says exactly that $f_x^\sharp$ is a local homomorphism of local rings. Conversely, if the above diagram is cartesian for every point x of X, and X has enough points, then we can conclude that (5.2.0.1) is cartesian as well, as follows. Certainly the right square in (5.2.0.1) commutes (a ring map takes units to units!), so we get a map $f^{-1}\mathscr{O}_Y^* \to f^{-1}\mathscr{O}_Y \times_{\mathscr{O}_X} \mathscr{O}_X^*$ which, according to (5.2.0.2), becomes an isomorphism after applying x^{-1} for any point x of X. But if X has enough points, the x^{-1} form a conservative system.

This discussion proves

Proposition 5.2.1. *A morphism of locally ringed topological spaces $f : X \to Y$ induces a local morphism of ringed sites*

$$\mathbf{Ouv}\, f : (\mathbf{Ouv}_X, \mathscr{O}_X) \to (\mathbf{Ouv}_Y, \mathscr{O}_Y)$$

and ringed topoi

$$\mathbf{Top}\, f : (X^\sim, \mathscr{O}_X) \to (Y^\sim, \mathscr{O}_Y)$$

iff f is a local morphism of locally ringed spaces in the usual sense ($f_x : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is a local morphism of local rings for every point $x \in X$).

The subcategory of **TopAn** (resp. **SitAn**) whose morphisms are local morphisms is a 2-category denoted **TopLocAn** (resp. **SitLocAn**).

We will see in Section **??** that a morphism of schemes induces a local morphism of ringed in all the usual topologies (Zariski, étale, fppf, etc.).

5.3. **Monoid objects.** The purpose of this section is to codify some properties of sheaves of monoids that we intend to use with little explicit mention in the sequel.

For any category $\mathscr{C}$, we write $\mathbf{Mon}_{\mathscr{C}}, \mathbf{Ab}_{\mathscr{C}}, \mathbf{An}_{\mathscr{C}}$ etc. for the category of monoid objects (commutative with zero), abelian group objects, ring objects, etc. in $\mathscr{C}$. As in Section 4, a ring object is regarded as a monoid object via multiplication.

If P is a monoid object of a topos X, then it is easy to see that the presheaf

$$U \mapsto \operatorname{Hom}_X(U, P)^*$$

is representable. The object representing it is denoted P^* and called the group of units of P. It is an object of $\mathbf{Ab}_X$ and fits into a cartesian diagram

$$\begin{tikzcd} P^* \arrow[r] \arrow[d] & P \times P \arrow[d, "+"] \\ X \arrow[r, "0"] & P \end{tikzcd}$$

where $+$ is the binary operation for the monoid P, X is the terminal object of X, and 0 is the identity of the monoid P. The group of units is regarded as a submonoid of P by composing the map $P^* \to P \times P$ with either projection.

Given a morphism of topoi or sites $f : X \to Y$, notice that, because P^* is defined in terms of a cartesian diagram (finite inverse limit) and the exact functor f^{-1} commutes with such limits, it follows that

$$(f^{-1}P)^* = f^{-1}(P^*),$$

so we may write $f^{-1}P^*$ unambiguously for this group object of X.

Groupification of a sheaf of monoids is slightly subtle. For a sheaf of monoids $\mathcal{P}$ on a site X, the presheaf

$$\mathcal{P}^{\mathrm{gp}}_{\mathrm{pre}} : U \mapsto \mathcal{P}(U)^{\mathrm{gp}}$$

is not generally a sheaf because groupification involves a coequalizer and taking sections won't commute with taking this coequalizer. However, sheafifying this presheaf gives a sheaf $\mathcal{P}^{\mathrm{gp}}$, and the functor $\mathcal{P} \mapsto \mathcal{P}^{\mathrm{gp}}$ is left adjoint to the inclusion $\mathbf{Ab}_{X^\sim} \hookrightarrow \mathbf{Mon}_{X^\sim}$ as usual.

The groupification P^{gp} of a monoid object P can be constructed formally in any category of monoid objects (having the appropriate equalizers and coequalizers) as follows. Define a monoid (object) R as the equalizer

$$\begin{tikzcd} R \arrow[r, "\iota"] & P^4 \arrow[r, shift left, "\pi_1+\pi_4"] \arrow[r, shift right, "\pi_2+\pi_3"'] & P \end{tikzcd}$$

and consider the coequalizer Q in the diagram

$$\begin{tikzcd} R \arrow[r, shift left, "\iota_1 \times \iota_2"] \arrow[r, shift right, "\iota_3 \times \iota_4"'] & P^2 \arrow[r, "p"] & Q \end{tikzcd} .$$

Evidently R is a relation on P^2, but it need not be an equivalence relation since it may not be transitive. I claim $p(\mathrm{Id} \times 0) : P \to Q$ is a groupification of P.

First of all, to see that Q is a group, observe that the composition of p and the twist map $\pi_2 \times \pi_1 : P^2 \to P^2$ defines a map $P^2 \to Q$ which "equalizes" the parallel arrows, hence defines a map $- : Q \to Q$ making the diagram

$$\begin{tikzcd} P^2 \arrow[r, "p"] \arrow[d, "\pi_2 \times \pi_1"'] & Q \arrow[d, "-"] \\ P^2 \arrow[r, "p"] & Q \end{tikzcd}$$

commute. It is easy to see that this is in fact an inverse map for Q. Now we need to check that $P \to Q$ has the appropriate universal property. Given a map to a group object $f : P \to A$, we get a map $Q \to A$ by considering

$$f\pi_1 - f\pi_2 : P^2 \to A.$$

To see that this equalizes the parallel arrows, we need to check that

$$(f\pi_1 - f\pi_2)(\iota_1 \times \iota_2) = (f\pi_1 - f\pi_2)(\iota_3 \times \iota_4).$$

But since Q is a group, it is enough to check that

$$f\pi_1\iota + f\pi_4\iota = f\pi_2\iota + f\pi_3\iota,$$

but after factoring out the f this is clear from the definition of R. The composition $P \to Q \to A$ is $(f\pi_1 - f\pi_2)(\mathrm{Id} \times 0) = f$ and it is straightforward to see that this is the unique such map by using the fact that $\mathrm{Id} \times 0$, and $0 \times \mathrm{Id} : P \to P^2$ are jointly surjective together with the formula for inversion in Q.

Lemma 5.3.1. *Let X be a topos, x a point of X, $\mathcal{P}$ a monoid object of X. Then $(\mathcal{P}_x)^{\mathrm{gp}} \cong (\mathcal{P}^{\mathrm{gp}})_x$.*

Proof. The stalk functor x^{-1} is exact. The discusssion above shows that the groupification of a monoid object is obtained by taking various equalizers and coequalizers. Since x^{-1} commutes with these, the result is clear. $\square$

In light of the Lemma, we will write $\mathcal{P}_x^{\mathrm{gp}}$ without ambiguity as to whether the groupification or the stalk was taken first.

Proposition 5.3.2. *For a sheaf of monoids $\mathcal{P}$ on a site X, the following are equivalent:*

(1) $\mathrm{Hom}_{X^\wedge}(\mathscr{F}, \mathcal{P})$ *is an integral monoid for every presheaf* $\mathscr{F}$ *on* X.
(2) $\mathrm{Hom}_{X^\sim}(\mathscr{F}, \mathcal{P})$ *is an integral monoid for every sheaf* $\mathscr{F}$ *on* X.
(3) $\mathcal{P}(U)$ *is an integral monoid for every object* U *of* X.
(4) $\mathcal{P} \to \mathcal{P}^{\mathrm{gp}}$ *is monic.*

Proof. Obviously (1) $\Longrightarrow$ (2). For (2) $\Longrightarrow$ (3), consider the presheaf h_U represented by U. Then we have

$$\begin{aligned} \mathcal{P}(U) &\cong \mathrm{Hom}_{X^\wedge}(h_U, \mathcal{P}) \\ &\cong \mathrm{Hom}_{X^\sim}(h_U^+, \mathcal{P}) \end{aligned}$$

by a variant of Yoneda's Lemma and the universal property of sheafification.

For (3) $\Longrightarrow$ (4), first note that (3) easily implies that $\mathcal{P}^{\mathrm{gp}}_{\mathrm{pre}}$ is separated, hence the map to its sheafification $\mathcal{P}^{\mathrm{gp}}_{\mathrm{pre}} \to \mathcal{P}^{\mathrm{gp}}$ is monic. Since (3) certainly implies $\mathcal{P} \to \mathcal{P}^{\mathrm{gp}}_{\mathrm{pre}}$ is monic, the implication follows.

Finally, for (4) $\implies$ (1), note that a map of sheaves is monic iff this is true in the category of presheaves, and the monomorphism of presheaves $\mathcal{P} \to \mathcal{P}^{\mathrm{gp}}$ factors through $\mathcal{P} \to \mathcal{P}^{\mathrm{gp}}_{\mathrm{pre}}$, hence this latter map is also monic. By definition of "monic" this meanes that for any presheaf $\mathscr{F}$,

$$\mathrm{Hom}_{X^\wedge}(\mathscr{F}, \mathcal{P}) \to \mathrm{Hom}_{X^\wedge}(\mathscr{F}, \mathcal{P}^{\mathrm{gp}}_{\mathrm{pre}})$$

is monic. But the latter monoid is easily seen to be isomorphic to $\mathrm{Hom}_{X^\wedge}(\mathscr{F}, \mathcal{P})^{\mathrm{gp}}$, so the implication follows. $\square$

A sheaf of monoids $\mathcal{P}$ satisfying these equivalent properties will be called *integral*. In an arbitrary category $\mathscr{C}$, we can use (2) as the definition of an integral monoid object. In a topos, where the groupification P^{gp} of a monoid P can surely be formed, we may use the equivalent properties (2), (4) to define an integral monoid object. We define an *integral morphism* of monoid objects in a category $\mathscr{C}$ to be a morphism $Q \to P$ of integral monoid objects such that

$$\mathrm{Hom}_{\mathscr{C}}(C, Q) \to \mathrm{Hom}_{\mathscr{C}}(C, P)$$

is an integral morphism of monoids (Section 1.10) for any object C of $\mathscr{C}$. In a site or topos, one can work out various equivalent definitions...

Lemma 5.3.3. *Let X be a site, $\mathcal{P}$ an integral monoid object of $X^\wedge$ (a presheaf of integral monoids on X). The associated sheaf $\mathcal{P}^+$ is an integral monoid object of $X^\sim$.*

Proof. Groupification commutes with sheafification by exactness of sheafification and the description of groupification via limits. Since $\mathcal{P}$ is integral, certainly it injects into its presheaf groupification: $\mathcal{P} \hookrightarrow \mathcal{P}^{\mathrm{gp}}$. Using the previous observation and exactness of sheafification again, we see that $\mathcal{P}^+ \to (\mathcal{P}^+)^{\mathrm{gp}} \cong (\mathcal{P}^{\mathrm{gp}})^+$ is monic, so we get the desired result by (4). $\square$

Lemma 5.3.4. *Suppose $P \to Q_1$, $P \to Q_2$ are morphisms of integral monoid objects in a topos so that at least one of the morphisms is integral. Then the pushout $Q_1 \otimes_P Q_2$ is an integral monoid.*

Proof. We may assume $X = \mathscr{C}^\sim$ for some site $\mathscr{C}$. The presheaf

$$(Q_1 \otimes_P Q_2)_{\mathrm{pre}} : U \mapsto Q_1(U) \otimes_{P(U)} Q_2(U)$$

(the pushout is in the category of monoids) is an integral monoid object of $\mathscr{C}^\wedge$ by Proposition 1.10.2. We have

$$(Q_1 \otimes_P Q_2) = (Q_1 \otimes_P Q_2)^+_{\mathrm{pre}},$$

so the result follows from Lemma 5.3.3. $\square$

Proposition 5.3.5. *Let X be a site. If $\mathcal{P}$ is an integral sheaf of monoids on X, then $\mathcal{P}_x$ is an integral monoid for every point x of X. If X has enough points, then the converse holds.*

Proof. Using the criterion (4) in the previous proposition, this is clear from the fact that groupification commutes with taking stalks (Lemma 5.3.1). $\square$

Lemma 5.3.6. *Let $f : X \to Y$ be a morphism of topoi. For any monoid P of Y, there is a natural isomorphism $(f^{-1}P)^{\mathrm{gp}} \cong f^{-1}(P^{\mathrm{gp}})$. For any monoid P of X, there is a natural morphism $(f_*P)^{\mathrm{gp}} \to f_*(P^{\mathrm{gp}})$ which is monic if P is integral.*

Proof. The first statement follows from exactness of f^{-1} and the construction of the groupification via limits. For the second statement, we return to the construction of P^{gp}, which starts with the equalizer diagram

$$R \xrightarrow{\iota} P^4 \overset{\pi_1+\pi_4}{\underset{\pi_2+\pi_3}{\rightrightarrows}} P .$$

Recall from the beginning of this section that P^{gp} is constructed as the coequalizer

$$R \overset{\iota_1\times\iota_2}{\underset{\iota_3\times\iota_4}{\rightrightarrows}} P^2 \xrightarrow{\pi_1-\pi_2} P^{\mathrm{gp}} . \tag{5.3.6.1}$$

By left exactness of f_*, the diagram

$$f_*R \xrightarrow{\iota} (f_*P)^4 \overset{\pi_1+\pi_4}{\underset{\pi_2+\pi_3}{\rightrightarrows}} f_*P$$

is also an equalizer diagram, so $(f_*P)^{\mathrm{gp}}$ is given by the coequalizer

$$f_*R \overset{\iota_1\times\iota_2}{\underset{\iota_3\times\iota_4}{\rightrightarrows}} (f_*P)^2 \xrightarrow{\pi_1-\pi_2} (f_*P)^{\mathrm{gp}} .$$

Pushing forward the coequalizer map from (5.3.6.1) gives the natural map.

For the second statement, we first claim that

$$\begin{tikzcd} R \arrow[r, "\iota_3\times\iota_4"] \arrow[d, "\iota_1\times\iota_2"'] & P^2 \arrow[d, "\pi_1-\pi_2"] \\ P^2 \arrow[r, "\pi_1-\pi_2"] & P^{\mathrm{gp}} \end{tikzcd}$$

is cartesian in X. Adopting the usual notation $A(B) = \operatorname{Hom}_X(B, A)$, we must prove that

$$\begin{tikzcd} R(U) \arrow[r, "\iota_3\times\iota_4"] \arrow[d, "\iota_1\times\iota_2"'] & P(U)^2 \arrow[d, "\pi_1-\pi_2"] \\ P(U)^2 \arrow[r, "\pi_1-\pi_2"] & P^{\mathrm{gp}}(U) \end{tikzcd}$$

is cartesian for any object U of X. If $(p_1, p_2), (p_1', p_2')$ is in the cartesian product, then $p_1 - p_2 = p_1' - p_2'$ in $P^{\mathrm{gp}}(U)$, so by integrality, $p_1 + p_2' = p_2 + p_1'$ in $P(U)$, hence (p_1, p_2, p_1', p_2') is in $R(U) \subseteq P(U)^4$, which proves the claim.

Since f_* preserves this cartesian diagram, in the integral case, we have a coequalizer diagram

$$(f_*P)^2 \times_{f_*(P^{\mathrm{gp}})} (f_*P)^2 \rightrightarrows (f_*P)^2 \xrightarrow{\pi_1-\pi_2} (f_*P)^{\mathrm{gp}} .$$

Now, suppose $s, t \in (f_*P)^{\mathrm{gp}}(U)$ agree in $f_*(P^{\mathrm{gp}})(U)$ and we want to show $s = t$. By the construction of coequalizers in a topos, we can find a cover $U' \to U$ of U (in the canonical topology) so that the pullbacks of s, t to U lift to $s', t' \in (f^*P)(U')^2$. Since s, t agree in $f_*(P^{\mathrm{gp}})$, we have

$$(s', t') \in (f_*P)(U)^2 \times_{f_*(P^{\mathrm{gp}})(U)} (f_*P)(U)^2,$$

so the restrictions of s and t to $(f_*P)^{\mathrm{gp}}(U')$ agree, hence $s = t$ because U' is a cover.

$\square$

In light of the lemma, we will write $f^{-1}P^{\text{gp}}$ without ambiguity.

Lemma 5.3.7. *If $f : X \to Y$ is a morphism of topoi, then for any integral monoid P of X, f_*P is integral, and for any integral monoid P of Y, $f^{-1}P$ is integral.*

Proof. For the direct image this follows from adjointness:

$$\mathrm{Hom}_Y(U, f_*P) \cong \mathrm{Hom}_X(f^{-1}U, P)$$

is integral if P is integral. For the inverse image, P integral implies $P \to P^{\text{gp}}$ is monic, hence $f^{-1}P \to f^{-1}P^{\text{gp}}$ is monic since f^{-1} is left exact, so $f^{-1}P$ is integral (we implicitly use the previous proposition). □

5.4. **Log structures.** Let X be a ringed topos or site. A *prelog structure* on X is a morphism $\alpha_X : \mathcal{P} \to \mathscr{O}_X$ from a monoid object $\mathcal{P}$ of X (or $X^\sim$) to $\mathscr{O}_X$. A morphism of prelog structures is a morphism of monoid objects commuting with the maps to $\mathscr{O}_X$. Prelog structures on X form a category $\mathbf{PLog}_X$. A prelog structure $\mathcal{P}$ is called *integral* if $\mathcal{P}$ is integral (i.e. satisfies the equivalent properties in Proposition 5.3.2).

If $\alpha : \mathcal{P} \to \mathscr{O}_X$ is a prelog structure on X, then for each object U of X, we have a prelog ring $\alpha(U) : \mathcal{P}(U) \to \mathscr{O}_X(U)$. Thus we may think of a prelog structure as a prelog ring object in X where the underlying ring object is $\mathscr{O}_X$.

A prelog structure $\alpha_X : \mathcal{M}_X \to \mathscr{O}_X$ is called a *log structure* iff

$$\alpha_X|\alpha_X^{-1}\mathscr{O}_X^* : \alpha_X^{-1}\mathscr{O}_X^* \to \mathscr{O}_X^* \tag{5.4.0.1}$$

is an isomorphism of monoids. Equivalently,

$$\alpha_X(U)|\alpha_X(U)^{-1}\mathscr{O}_X(U)^* : \alpha_X(U)^{-1}\mathscr{O}_X(U)^* \to \mathscr{O}_X(U)^* \tag{5.4.0.2}$$

should be an isomorphism for every object U of X, so we may think of a log structure on X as a log ring object of X.

We usually denote a log structure simply by $\mathcal{M}_X$, leaving α_X implicit. We also suppress notation for the inverse of (5.4.0.2) and thus view $\mathscr{O}_X^*$ as a submonoid of $\mathcal{M}_X$. By definition, a morphism of log structures is a morphism of prelog structures, so log structures on $\mathscr{O}_X$ form a full subcategory of the category of prelog structures on $\mathscr{O}_X$. As in Section 4, the inclusion functor has a left adjoint

$$\begin{array}{rcl} \_^{\,a} : \mathbf{PLog}_X & \to & \mathbf{Log}_X \\ \mathcal{P} & \mapsto & \mathcal{P}^a \end{array}$$

given by the pushout

$$\mathcal{P}^a := \mathcal{P} \oplus_{\alpha^{-1}\mathscr{O}_X^*} \mathscr{O}_X^* \tag{5.4.0.3}$$

in the category $\mathbf{Mon}_X$ (or $\mathbf{Mon}_{X^\sim}$ for a site).

Example 5.4.1. The inclusion $\mathscr{O}_X^* \hookrightarrow \mathscr{O}_X$ defines a log structure called the *trivial log structure* on $\mathscr{O}_X$. It is initial in the category of log structures on X. Similarly, the identity map $\mathscr{O}_X \to \mathscr{O}_X$ gives the *terminal log structure*, terminal in the category of log structures on $\mathscr{O}_X$.

Observe that $\alpha : \mathcal{M}_X \to \mathscr{O}_X$ is a log structure iff there is a cartesian diagram

$$\begin{tikzcd} \mathscr{O}_X^* \arrow[r, equal] \arrow[d] & \mathscr{O}_X^* \arrow[d] \\ \mathcal{P} \arrow[r, "\alpha"] & \mathscr{O}_X . \end{tikzcd}$$

If this is the case, then since, for any object U of X, the functor $\operatorname{Hom}_X(U, \_)$ tautologically commutes with inverse limits, the diagram

$$\begin{tikzcd} \mathscr{O}_X(U)^* \arrow[r, equal] \arrow[d] & \mathscr{O}_X(U)^* \arrow[d] \\ \mathcal{M}_X(U) \arrow[r, "\alpha"] & \mathscr{O}_X(U) \end{tikzcd}$$

is also cartesian and we see that $\alpha(U) : \mathcal{M}_X(U) \to \mathscr{O}_X(U)$ is a log structure on $\mathscr{O}_X(U)$. In other words, a log structure is a log ring object in the appropriate category.

As in Section 4.1, for a log structure $\mathcal{M}_X$, the quotient $\overline{\mathcal{M}}_X := \mathcal{M}_X/\mathscr{O}_X^*$ is called the *characteristic monoid* of $\mathcal{M}_X$. The characteristic monoid of any log structure is sharp.

Lemma 5.4.2. *If $\mathcal{P} \to \mathscr{O}_X$ is an integral prelog structure, then the associated log structure $\mathcal{P}^a$ is integral.*

Proof. Use the proof of Lemma 4.1.2, but substitute Lemma 5.3.4 for Proposition 1.10.2. □

Proposition 5.4.3. *If $\alpha : \mathcal{P} \to \mathscr{O}_X$ is a prelog structure on $\mathscr{O}_X$, then the characteristic monoid $\overline{\mathcal{P}^a}$ of the associated log structure is the quotient of $\mathcal{P}$ by $\alpha^{-1}\mathscr{O}_X^*$.*

Proof. This follows formally from Lemma 4.3.4 just as in the proof of Corollary 4.3.5. □

Proposition 5.4.4. *If $(\alpha : \mathcal{P} \to \mathscr{O}_X) \to (\beta : \mathcal{Q} \to \mathscr{O}_X)$ is a morphism of integral prelog structures on $\mathscr{O}_X$, then the induced morphism of log structures on $\mathscr{O}_X$ is an isomorphism iff the induced map*

$$\mathcal{P}/\alpha^{-1}\mathscr{O}_X^* \to \mathcal{Q}/\beta^{-1}\mathscr{O}_X^*$$

is an isomorphism. In particular, a morphism of integral log structures on $\mathscr{O}_X$ is an isomorphism iff the induced map on characteristics is an isomorphism.

Proof. In light of the previous proposition, it suffices to prove the second statement. The proof of this is similar to the proof of Lemma 4.3.7. □

Our next task is to show that formation of associated log structures on a site commutes with sheafification. This will be useful for making connections with the theory we built up in Section 4. For a ringed site X, let $\mathbf{PLog}_{X^\wedge}$ be the category of presheaf prelog structures on $\mathscr{O}_X$, so an object of $\mathbf{PLog}_{X^\wedge}$ is a monoid object $\mathcal{P}$ of $X^\wedge$ and a map of monoid objects $\alpha : \mathcal{P} \to \mathscr{O}_X$. As usual, such a presheaf prelog structure is called a presheaf log structure if $\alpha : \alpha^{-1}\mathscr{O}_X^* \to \mathscr{O}_X^*$ is an isomorphism.

The inclusion $\mathbf{Log}_{X^\wedge} \hookrightarrow \mathbf{PLog}_{X^\wedge}$ of the full subcategory of log structures admits a left adjoint

$$\begin{array}{rcl} -^a_{\mathrm{pre}} : \mathbf{PLog}_{X^\wedge} & \to & \mathbf{Log}_{X^\wedge} \\ \mathcal{P} & \mapsto & (\mathcal{P} \oplus_{\alpha^{-1}\mathscr{O}_X^*} \mathscr{O}_X^*)_{\mathrm{pre}}. \end{array}$$

Here $(\mathcal{P} \oplus_{\alpha^{-1}\mathscr{O}_X^*} \mathscr{O}_X^*)_{\mathrm{pre}}$ is the pushout in $\mathbf{Mon}_{X^\wedge}$. We retain the subscript "pre" because, when $\mathcal{P}$ happens to be a sheaf, this pushout can differ from $\mathcal{P} \oplus_{\alpha^{-1}\mathscr{O}_X^*} \mathscr{O}_X^*$. That is, the sheaf $\mathcal{P}^a$ may not be isomorphic to the presheaf $\mathcal{P}^a_{\mathrm{pre}}$.

The presheaf $\mathcal{P}^a_{\mathrm{pre}}$ associates to each object U of X the (monoid of the) log ring associated to the prelog ring $\mathcal{P}(U) \to \mathscr{O}_X(U)$, as discussed in Section 4. We again emphasize that this need not be a sheaf, even if $\mathcal{P}$ is a sheaf.

Theorem 5.4.5. *Let X be a ringed site. The diagram of functors*

$$\begin{tikzcd} \mathbf{PLog}_{X^\wedge} \arrow[r, "+"] \arrow[d, "-^a_{\mathrm{pre}}"'] & \mathbf{PLog}_{X^\sim} \arrow[d, "-^a"] \\ \mathbf{Log}_{X^\wedge} \arrow[r, "+"] & \mathbf{Log}_{X^\sim} \end{tikzcd}$$

commutes. That is, for any presheaf prelog structure $\mathcal{P}$ on X, there is a natural isomorphism $(\mathcal{P}^a_{\mathrm{pre}})^+ \cong (\mathcal{P}^+)^a$ of log structures on X.

Proof. Let $\alpha : \mathcal{P} \to \mathscr{O}_X$ be a presheaf prelog structure on $\mathscr{O}_X$. Consider the cartesian diagram

(5.4.5.1)
$$\begin{tikzcd} \alpha^{-1}\mathscr{O}_X^* \arrow[r] \arrow[d] & \mathscr{O}_X^* \arrow[d] \\ \mathcal{P} \arrow[r, "\alpha"] & \mathscr{O}_X \end{tikzcd}$$

in the category of presheaves $X^\wedge$. Left exactness of the sheafification functor implies that

$$\begin{tikzcd} (\alpha^{-1}\mathscr{O}_X^*)^+ \arrow[r] \arrow[d] & \mathscr{O}_X^* \arrow[d] \\ \mathcal{P}^+ \arrow[r, "\alpha^+"] & \mathscr{O}_X \end{tikzcd}$$

is also cartesian, so we get an isomorphism

(5.4.5.2)
$$(\alpha^{-1}\mathscr{O}_X^*)^+ \cong (\alpha^+)^{-1}\mathscr{O}_X^*.$$

The associated log structure $\mathcal{P}^a_{\mathrm{pre}}$ is obtained as the pushout of the upper left part of the diagram (5.4.5.1) in the category $\mathbf{Mon}_{X^\wedge}$:

$$\begin{tikzcd} \alpha^{-1}\mathscr{O}_X^* \arrow[r] \arrow[d] & \mathscr{O}_X^* \arrow[d] \\ \mathcal{P} \arrow[r] & \mathcal{P}^a_{\mathrm{pre}} \end{tikzcd}$$

Right exactness of sheafification implies

$$\begin{array}{ccc} (\alpha^{-1}\mathscr{O}_X^*)^+ & \longrightarrow & \mathscr{O}_X^* \\ \downarrow & & \downarrow \\ \mathcal{P} & \longrightarrow & (\mathcal{P}^a_{\mathrm{pre}})^+ \end{array}$$

is cocartesian in $\mathbf{Mon}_{X^\sim}$. Using the isomorphism (5.4.5.2) we see that

$$\begin{array}{ccc} (\alpha^+)^{-1}\mathscr{O}_X^* & \longrightarrow & \mathscr{O}_X^* \\ \downarrow & & \downarrow \\ \mathcal{P}^+ & \longrightarrow & (\mathcal{P}^a_{\mathrm{pre}})^+ \end{array}$$

is cocartesian in $\mathbf{Mon}_{X^\sim}$. But this latter cocartesian square defines $(\mathcal{P}^+)^a$, so we get the desired isomorphism $(\mathcal{P}^a_{\mathrm{pre}})^+ \cong (\mathcal{P}^+)^a$. □

Corollary 5.4.6. *Suppose $\mathcal{Q} \to \mathcal{P}$ is a morphism of presheaf prelog structures on a ringed site X so that*

$$\mathcal{Q}(U) \to \mathcal{P}(U)$$

induces an isomorphism

$$\mathcal{Q}(U)^a \to \mathcal{P}(U)^a$$

of log structures on $\mathscr{O}_X(U)$ for every object U of X. Then the map of prelog structures $\mathcal{Q}^+ \to \mathcal{P}^+$ induces an isomorphism $(\mathcal{Q}^+)^a \to (\mathcal{P}^+)^a$.

Proof. The hypothesis says that the map of presheaf associated log structures $\mathcal{Q}^a \to \mathcal{P}^a$ is an isomorphism, so this remains an isomorphism after sheafification: $(\mathcal{Q}^a)^+ \cong (\mathcal{P}^a)^+$. The result then follows from the theorem. □

Corollary 5.4.7. *Let X be a ringed site. Suppose $Q \to P$ is a morphism of prelog structures on $\Gamma(X, \mathscr{O}_X)$ inducing an isomorphism on associated log structures. Then the induced map of prelog structures $\underline{Q} \to \underline{P}$ on $\mathscr{O}_X$ induces an isomorphism on associated log structures.*

Proof. The map $\underline{Q} \to \underline{P}$ is obtained by sheafifying $\underline{Q}_{\mathrm{pre}} \to \underline{P}_{\mathrm{pre}}$, so by the previous corollary, it suffices to show that $\underline{Q}_{\mathrm{pre}}(U) \to \underline{P}_{\mathrm{pre}}(U)$ induces an isomorphism on associated log structures for every object U of X. But this map is just $Q \to P$ and is interpreted as the map between the inverse image prelog structures (in the sense of Section 4.2) under the restriction map $\mathscr{O}_X(X) \to \mathscr{O}_X(U)$, so the result follows from Proposition 4.2.3. □

5.5. **Pullback and pushforward.** If $f : X \to Y$ is a morphism of ringed topoi and $\alpha_Y : \mathcal{M}_Y \to \mathscr{O}_Y$ is a log structure on Y, then the *inverse image log structure* $f^*\mathcal{M}_Y$ on $\mathscr{O}_X$ is defined to be the log structure on $\mathscr{O}_X$ associated to the prelog structure given by the composition

$$f^{-1}\mathcal{M}_Y \xrightarrow{f^{-1}\alpha_Y} f^{-1}\mathscr{O}_Y \xrightarrow{f^\sharp} \mathscr{O}_X .$$

If $\mathcal{M}_X$ is a log structure on $\mathscr{O}_X$, then one can check that the prelog structure $f_*^{\log}\mathcal{M}_X$ defined by the cartesian diagram

$$\begin{tikzcd} f_*^{\log}\mathcal{M}_X \arrow[r] \arrow[d] & f_*\mathcal{M}_X \arrow[d, "f_*\alpha_X"] \\ \mathscr{O}_Y \arrow[r, "f^\flat"] & f_*\mathscr{O}_X \end{tikzcd}$$

defines a log structure on $\mathscr{O}_Y$ called the *direct image log structure*.

For a log structure $\mathcal{M}_Y$ on Y, there is a natural morphism $f^{-1}\mathcal{M}_Y \to f^*\mathcal{M}_Y$ in $\mathbf{Mon}_X$ and for any log structure $\mathcal{M}_X$ on X, there is a natural map $f_*^{\log}\mathcal{M}_X \to f_*\mathcal{M}_X$ in $\mathbf{Mon}_Y$.

A *log ringed topos* is a ringed topos X together with a log structure on X. We will typically denote a log ringed topos by $X^\dagger$, denoted the underlying topos X, the ring object $\mathscr{O}_X$, and the log structure $\alpha_X : \mathcal{M}_X \to \mathscr{O}_X$. A morphism of log ringed topoi $f : X^\dagger \to Y^\dagger$ is a morphism of ringed topoi, together with a morphism $f^\dagger : f^*\mathcal{M}_Y \to \mathcal{M}_X$ of log structures on X. This is the same as a morphism $f^\dagger : f^{-1}\mathcal{M}_Y \to \mathcal{M}_X$ of prelog structures on X. Log ringed topoi form a 2-category $\mathbf{TopAn}^\dagger$. It has a sub-2-category $\mathbf{TopLocAn}^\dagger$ with the same objects, but whose morphisms are required to be local morphisms on the underlying ringed topoi.

If $f : X^\dagger \to Y^\dagger$ is a morphism of log ringed topoi, then the cokernel of $f^\dagger : f^*\mathcal{M}_Y \to \mathcal{M}_X$ is called the *relative characteristic monoid* and is denoted $\mathcal{M}_{X/Y}$. In case $Y^\dagger$ is just $(X, \mathscr{O}_X)$ with the trivial log structure (and the underlying morphism of ringed topoi is the identity), $f^\dagger$ is just the inclusion of $\mathscr{O}_X^*$ in $\mathcal{M}_X$, we recover the definition of the characteristic monoid $\overline{\mathcal{M}}_X$.

The following propositions contain some basic facts about inverse image log structures. Notice that pullback of log structures behaves best when the morphism of ringed topoi is local.

Proposition 5.5.1. *If $f : X \to Y$ is a* local *morphism of ringed topoi and $\mathcal{M}_Y$ is a log structure on Y, then there is a natural isomorphism $\overline{f^*\mathcal{M}_Y} \cong f^{-1}\overline{\mathcal{M}}_Y$.*

Proof. Consider the diagram

$$\begin{tikzcd} f^{-1}\mathscr{O}_Y^* \arrow[r, equal] \arrow[d] & f^{-1}\mathscr{O}_Y^* \arrow[r] \arrow[d, hook] & \mathscr{O}_X^* \arrow[d, hook] \\ f^{-1}\mathcal{M}_Y \arrow[r, "f^{-1}\alpha_Y"] & f^{-1}\mathscr{O}_Y \arrow[r, "f^\sharp"] & \mathscr{O}_X. \end{tikzcd}$$

The right square is cartesian by definition of a local morphism and the left square is cartesian because it is obtained by applying the exact functor f^{-1} to a square which is cartesian by definition of a log structure, so we conclude that the big square is cartesian. In other words, the preimage of $\mathscr{O}_X^*$ under the composition $f^\sharp \circ f^{-1}\alpha_Y$ is $f^{-1}\mathscr{O}_Y^*$. The inverse image log structure $f^*\mathcal{M}_Y$ is, by definition, the log structure associated to the prelog structure given by this composition, so we get

$$\overline{f^*\mathcal{M}_Y} \cong (f^{-1}\mathcal{M}_Y)/(f^{-1}\mathscr{O}_Y^*) \cong f^{-1}\overline{\mathcal{M}}_Y,$$

using Proposition 5.4.3 for the first isomorphism and right exactness of f^{-1} for the second isomorphism. $\square$

Proposition 5.5.2. *If $f : X \to Y$ is a morphism of ringed topoi and $\mathcal{M}_Y$ is an integral log structure on Y, then $f^*\mathcal{M}_Y$ is an integral log structure on X.*

Proof. $f^{-1}\mathcal{M}_Y$ is integral by Lemma 5.3.7, so $f^*\mathcal{M}_Y := (f^{-1}\mathcal{M}_Y)^a$ is integral by Lemma 5.4.2. □

Formation of associated log structures commutes with inverse images:

Proposition 5.5.3. *If $f : X \to Y$ is a morphism of ringed topoi and $\alpha : \mathcal{P} \to \mathscr{O}_Y$ is a prelog structure on Y, then there is a natural isomorphism $(f^{-1}\mathcal{P})^a \cong f^*(\mathcal{P}^a)$ of log structures on X.*

Proof. The morphism f tautologically factors as

$$(X, \mathscr{O}_X) \to (X, f^{-1}\mathscr{O}_Y) \to (Y, \mathscr{O}_Y),$$

so we can prove the proposition separately for each of the two morphisms.

For the morphism on the right, the exactness of f^{-1} implies that f^{-1} already takes log structures to log structures, so we have $f^{-1} = f^*$. We have a cocartesian diagram

$$\begin{tikzcd} \alpha^{-1}\mathscr{O}_Y^* \arrow[r] \arrow[d] & \mathscr{O}_Y^* \arrow[d] \\ \mathcal{P} \arrow[r] & \mathcal{P}^a \end{tikzcd}$$

in Y by definition of $\mathcal{P}^a$. By exactness of f^{-1}, the diagram

$$\begin{tikzcd} f^{-1}(\alpha^{-1}\mathscr{O}_Y^*) \arrow[r] \arrow[d] & f^{-1}\mathscr{O}_Y \arrow[d] \\ f^{-1}\mathcal{P} \arrow[r] & f^{-1}(\mathcal{P}^a) \end{tikzcd}$$

is cocartesian in X. Exactness of f^{-1} gives $f^{-1}(\alpha^{-1}\mathscr{O}_X^*) \cong (f^{-1}\alpha)^{-1}(f^{-1}\mathscr{O}_Y^*)$, so

$$\begin{tikzcd} (f^{-1}\alpha)^{-1}(f^{-1}\mathscr{O}_Y^*) \arrow[r] \arrow[d] & f^{-1}\mathscr{O}_Y \arrow[d] \\ f^{-1}\mathcal{P} \arrow[r] & f^{-1}(\mathcal{P}^a) \end{tikzcd}$$

is cocartesian in X. But $f^{-1}\mathscr{O}_Y$ is the structure sheaf of X for this morphism, so this cocartesian diagram defines $(f^{-1}\mathcal{P})^a$, hence we have the desired isomorphism.

For the morphism on the left, this is a question about two ring objects in the same topos X. We may write $\mathscr{O}_Y$ for $f^{-1}\mathscr{O}_Y$ and forget about the underlying morphism of topoi (retaining only the ring morphism $f^\sharp : \mathscr{O}_Y \to \mathscr{O}_X$), and we can assume $X = \mathscr{C}^\sim$ for a site $\mathscr{C}$.

According to the definitions, $\mathcal{P}^a$ is the log structure associated to the prelog structure $\mathcal{P} \to \mathscr{O}_Y$ and then $f^*(\mathcal{P}^a)$ is the log structure associated to the prelog structure

$$\mathcal{P}^a \to \mathscr{O}_Y \to \mathscr{O}_X.$$

Similarly, $(f^{-1}\mathcal{P})^a$ is the log structure associated to the prelog structure

$$\mathcal{P} \to \mathscr{O}_Y \to \mathscr{O}_X.$$

According to Theorem 5.4.5, we have an isomorphism $\mathcal{P}^a \cong (\mathcal{P}^a_{\mathrm{pre}})^+$, where $\mathcal{P}^a_{\mathrm{pre}}$ is the presheaf $\mathcal{P}^a_{\mathrm{pre}}$ associating to any object U of $\mathscr{C}$, the (monoid part of the) log structure associated to $\mathcal{P}(U) \to \mathscr{O}_Y(U)$. By the same theorem, $(f^{-1}\mathcal{P})^a$ is the sheaf associated to the presheaf $(f^{-1}\mathcal{P})^a_{\mathrm{pre}}$ associating to any object U of $\mathscr{C}$ the monoid part of the log structure associated to the prelog structure

$$\mathcal{P}(U) \to \mathscr{O}_Y(U) \to \mathscr{O}_X(U).$$

Applying the same theorem to the presheaf prelog structure

$$\mathcal{P}^a_{\mathrm{pre}} \to \mathscr{O}_Y \to \mathscr{O}_X$$

on $\mathscr{O}_X$, we get an isomorphisms

$$((\mathcal{P}^a_{\mathrm{pre}})^a_{\mathrm{pre}})^+ \cong ((\mathcal{P}^a_{\mathrm{pre}})^+)^a \cong (\mathcal{P}^a)^a = f^*(\mathcal{P}^a).$$

(The notation is bad: the first "a" is formation of associated log structures on $\mathscr{O}_Y$ and the second "a" is formation of associated log structures on $\mathscr{O}_X$.) Now we see that it suffices to prove that there is a natural isomorphism of presheaf prelog structures $(f^{-1}\mathcal{P})^a_{\mathrm{pre}} \cong (\mathcal{P}^a_{\mathrm{pre}})^a_{\mathrm{pre}}$. This is exactly Theorem 4.2.3.

$\square$

Proposition 5.5.4. *Let $f : X^\dagger \to Y^\dagger$ be a morphism of log ringed topoi, $\mathcal{M}_Y \to \mathcal{N}_Y$ a morphism of log structures on $\mathscr{O}_Y$. Define another log structure $\mathcal{N}_X$ on $\mathscr{O}_X$ by taking the log structure associated to the prelog structure*

$$\mathcal{N}_X^{\mathrm{pre}} := \mathcal{M}_X \oplus_{f^{-1}\mathcal{M}_Y} f^{-1}\mathcal{N}.$$

Then $f = (f, f^\dagger)$ naturally induces a morphism of log ringed topoi

$$(f, g^\dagger) : (X, \mathcal{N}_X) \to (Y, \mathcal{N}_Y)$$

with the same underlying morphism of ringed topoi and the same relative characteristic: $\mathcal{M}_{X/Y} = \mathcal{N}_{X/Y}$.

Proof. By definition we have a cocartesian diagram

$$\begin{tikzcd} f^{-1}\mathcal{M}_Y \arrow[r] \arrow[d, "f^\dagger"'] & f^{-1}\mathcal{N}_Y \arrow[d, "g^\dagger_{\mathrm{pre}}"] \\ \mathcal{M}_X \arrow[r] & \mathcal{N}_X^{\mathrm{pre}} \end{tikzcd}$$

in $\mathbf{PLog}_X$, so we get the natural morphism by taking

$$g^\dagger := (g^\dagger_{\mathrm{pre}})^a : f^*\mathcal{N}_Y \to \mathcal{N}_X.$$

Since formation of associated log structures is a left adjoint, it preserves cocartesian squares, so

$$\begin{tikzcd} f^*\mathcal{M}_Y \arrow[r] \arrow[d, "f^\dagger"'] & f^*\mathcal{N}_Y \arrow[d, "g^\dagger"] \\ \mathcal{M}_X \arrow[r] & \mathcal{N}_X \end{tikzcd}$$

is a cocartesian diagram of monoids, hence the second statement follows from Lemma 4.3.4.

$\square$

5.6. **Types of morphisms.** A morphism of log ringed topoi $f : X^\dagger \to Y^\dagger$ is called *strict* if $f^\dagger : f^*\mathcal{M}_Y \to \mathcal{M}_X$ is an isomorphism. f is called *integral* if $f^\dagger$ is an integral morphism (c.f. Sections 1.10 and 5.3). In particular, for f to be integral, we require $\mathcal{M}_Y$ (or rather, $f^*\mathcal{M}_Y$) and $\mathcal{M}_X$ to be integral monoids. f is called *vertical* if $\mathcal{M}_{X/Y}$ is a group.

5.7. **Coherence and charts.** A log structure $\mathcal{M}_X$ on a ringed topos X is called *quasi-coherent* iff there are a cover Y (of the terminal object of X), a monoid P, and a morphism homomorphism $P \to \mathcal{M}_X(Y)$ such that the corresponding map $\underline{P}_Y \to \mathcal{M}_X|Y$ induces an isomorphism $\underline{P}^a_Y \cong \mathcal{M}_X|Y$ on associated log structures. Such a morphism $P \to \mathcal{M}_X(Y)$ will be called a *chart* (for $\mathcal{M}_X$). A chart is called *finitely generated* (resp. integral, fine, ...) iff P is finitely generated (resp. integral, fine, ...). A log structure $\mathcal{M}_X$ is called *coherent* iff it admits a finitely generated chart, and *fine* iff it is coherent and integral (we will soon see that this implies that it has a fine chart).

Many of the finiteness properties of coherent/fine log rings (Section 4.3) carry over to coherent/fine log structures on a topos by using the smallness results of Section 4.5.

Proposition 5.7.1. *Let $h : \mathcal{M} \to \mathcal{N}$ be a morphism of coherent log structures on a ringed topos X and let x be a point of X. If h induces an isomorphism*

$$h_x : \mathcal{M}_x \to \mathcal{N}_x$$

on stalks at x, then it induces an isomorphism of log structures on a neighborhood of x.

Proof. By definition of coherent there is some cover Y on which $\mathcal{M}, \mathcal{N}$ have charts. Since the neighborhoods of x that factor through Y are cofinal in the neighborhoods of x (Lemma 5.1.7), we can assume, after possibly replacing X with X/Y, that there are finitely generated monoids Q, P, and morphisms $Q \to \mathcal{M}(X)$, $P \to \mathcal{N}(X)$ inducing isomorphisms $\underline{Q}^a \cong \mathcal{M}$ and $\underline{P}^a \cong \mathcal{N}$. By Theorem 5.4.5, we have $\underline{Q}^a = (\underline{Q}^a_{\mathrm{pre}})^+$, where $\underline{Q}^a_{\mathrm{pre}}$ is the presheaf

$$U \mapsto (Q \to \mathscr{O}_X(U))^a$$

(and similarly with Q replaced by P), so it is enough to prove that these presheaves are isomorphic on a neighborhood of x. We may view $\underline{P}^a_{\mathrm{pre}}$ and $\underline{Q}^a_{\mathrm{pre}}$ as coherent log structures on the filtered direct limit system of rings

$$\{\mathscr{O}_X(U) : (U, u) \in \mathbf{Vois}_x^{\mathrm{op}}\}$$

as in Section 4.5. The stalk functor commutes with sheafification, so we may view h_x as an isomorphism between the limit log structures on $\mathscr{O}_{X,x} = \varinjlim_U \mathscr{O}_X(U)$. The result now follows from Theorem 4.5.3. $\square$

The following is the global analog of the First Chart Lemma (4.3.11):

Lemma 5.7.2. *Let $\mathcal{M}_X$ be a fine log structure on a ringed topos X, G a finitely generated abelian group, x a point of X, and $h : G \to \mathcal{M}^{\mathrm{gp}}_{X,x}$ a map of groups inducing a surjection $G \to \overline{\mathcal{M}}^{\mathrm{gp}}_{X,x}$ onto the stalk of the groupified characteristic monoid at x. Then $P := h[h^{-1}\mathcal{M}_{X,x}]$ is a finitely generated integral submonoid of $\mathcal{M}_{X,x}$ and its inclusion $i : P \hookrightarrow \mathcal{M}_{X,x}$ lifts to a chart on a neighborhood of x.*

Proof. Note that $\mathcal{M}_{X,x} \to \mathscr{O}_{X,x}$ is a fine log structure on the ring $\mathscr{O}_{X,x}$, so by the First Chart Lemma (4.3.11), P is a fine monoid and $i : P \to \mathcal{M}_{X,x}$ induces an isomorphism $P^a \cong \mathcal{M}_{X,x}$ of log structures on $\mathscr{O}_{X,x}$. Since P is small (Theorem 1.9.7), there is no trouble in lifting $i : P \to \mathcal{M}_{X,x}$ to a map $j : P \to \mathcal{M}_X(U)$ on some neighborhood of x. But then, this lifted map j induces an isomorphism $\underline{P}^a_V \cong \mathcal{M}_X|V$ of log structures on some (possibly smaller) neighborhood V of x by Proposition 5.7.1. $\square$

6. Log Schemes

In this section, we will specialize the general results of Section 5 to the sites and topoi of algebraic geometry.

6.1. **Algebrogeometric sites.** Let X be a scheme. We will be primarily interested in the following sites (and their topoi of sheaves). In all cases, the category will be a full subcategory of the category of schemes locally of finite presentation over X and the covers will be families $\{f_i : U_i \to U\}$ such that $U = \cup_i f[U_i]$ (on underlying topological spaces), so it will suffice to indicate the objects.

(1) The objects of the *small Zariski site* X_{Zar} of X are Zariski open immersions (maps that are isomorphisms Zariski locally on domain and codomain) $Y \to X$.
(2) The objects of the *small étale* site $X_{\text{ét}}$ are étale maps $U \to X$.
(3) The objects of the *small Nisnevich site* X_{Nis} of X are étale maps $f : U \to X$ with the property that, for any $u \in X$, the natural map $\operatorname{Spec} k(f(u)) \to X$ factors through f.
(4) The *small root site* X_r of X is the smallest subcategory of $\mathbf{Sch}/X$ containing the Zariski open immersions and satisfying: Given $(Y \to X)$ in X_r, $u \in \Gamma(Y, \mathscr{O}_Y^*)$, and $n \in \mathbb{Z}_{>0}$, the natural map from $\operatorname{Spec}_Y \mathscr{O}_Y[x]/(x^n - u)$ to Y (note that this map is affine and manifestly of finite presentation) is a morphism in X_r.

We will use X_t to refer to one of these "small" sites.

There are several other possibilities. For example, the objects of the *small qff site* X_{qff} are flat, quasi-finite morphisms $U \to X$ of locally finite presentation. Of course, there are many other possibilities. We also have the corresponding big sites (denoted using the corresponding capital subscripts $X_{\mathrm{CC}}, X_{\mathrm{ZAR}}$, etc.), which are topologies on $\mathbf{Sch}/X$ whose covers are families $\{U_i \to U\}$ where each $U_i \to U$ is an object of the corresponding small site of U, and $U = \cup_i U_i$ as usual. Finally, there is the *fppf site* X_{fppf} of X, which is the topology on the category of schemes locally of finite presentation over X where a cover is a family $\{U_i \to U\}$ with $U = \cup_i U_i$ and where each $U_i \to U$ is flat and locally of finite presentation.

We will use the symbol X_T generically to refer to any of the sites mentioned in this section. Notice that X (rather $\mathrm{Id}_X : X \to X$) is the terminal object of X_T.

Remark 6.1.1. It is often possible to make sense of the site X_T when X is something other than a scheme. For example, X could be an algebraic space or Deligne-Mumford stack, or even an Artin stack (though here the small sites are typically too small to be useful). Note that a Zariski open immersion is determined by the underlying map of topological spaces. This notion makes sense in other categories of ringed spaces, for example, in the category of analytic spaces.

Remark 6.1.2. The "Zariski site" of course makes sense for any ringed topological space X. The corresponding topos $X^{\sim}_{\mathrm{Zar}}$ is equivalent to the topos of sheaves on X in the usual sense (sheaves on the site $\mathbf{Ouv}_X$ of open subsets of X with the usual topology).

Remark 6.1.3. People will disagree about the definitions of the "Zariski site," "étale site," etc., but they will not disagree about the corresponding topos of sheaves. For example, some people would say that the Zariski site of X is the category of open subsets of the space underlying X, whose morphisms are inclusions, and whose covers are families

$\{U_i \to U\}$ with $U = \cup_i U_i$. This site differs from our Zariski site, but they certainly have the same sheaves.

Each of the sites X_T is covariantly functorial[6] in X. If $f : X \to Y$ is a morphism of schemes, there is an induced functor

$$\begin{array}{rcl} f_T : Y_T & \to & X_T \\ (U \to Y) & \mapsto & (U \times_Y X \to X) \end{array}$$

(the underlying category X_T is contravariantly functorial in X) which defines a morphism of sites $X_T \to Y_T$ by the criteria of Proposition 5.1.2 and the fact that f_T clearly takes covers to covers. Note that $f_T(Y) = Y \times_Y X$ so this morphism of sites preserves the terminal objects. Given an object $U \to X$ of X_T, a neighborhood (or a T-neighborhood if we want to emphasize the topology) of U is an object $V \to Y$ of Y_T and a commutative diagram:

$$\begin{tikzcd} & & V \arrow[d] \\ U \arrow[r] \arrow[urr] & X \arrow[r, "f"] & Y \end{tikzcd}$$

This is the same thing as a morphism $U \to V \times_Y X = f_T(V \to Y)$. The left adjoint $f^{-1}_{T,\,\mathrm{pre}}$ is given by

$$f^{-1}_{T,\,\mathrm{pre}}\mathscr{F}(U) := \varinjlim \mathscr{F}(V),$$

where the direct limit is over the (opposite) category of neighborhoods $U \to V$ of U. We will drop the subscripts T when they are clear from context.

If the map $f : X \to Y$ is already an object of Y_T (this is always true if T is one of the big topologies), then for any $V \to X$ in X_T, the composition $V \to X \to Y$ is in Y_T and is cofinal among T-neighborhoods of $V \to X$, so f^{-1}_{pre} is given simply by restriction: $f^{-1}_{\mathrm{pre}}\mathscr{F}(V \to X) = \mathscr{F}(V \to X \to Y)$. Furthermore, if $\mathscr{F}$ is a sheaf on Y_T, then clearly $f^{-1}_{\mathrm{pre}}\mathscr{F}$ is clearly already a sheaf on X_T, so we have $f^{-1}\mathscr{F}(V \to X) = \mathscr{F}(V \to X \to Y)$. In this case, we just write $\mathscr{F}|_X$ for $f^{-1}\mathscr{F}$.

For any scheme X, there are natural morphisms of topoi

$$X^{\sim}_{\mathrm{fppf}} \to X^{\sim}_{\mathrm{QFF}} \to X^{\sim}_{\mathrm{\acute{E}T}} \to X^{\sim}_{\mathrm{NIS}} \to X^{\sim}_{\mathrm{ZAR}}$$

and

$$X^{\sim}_{\mathrm{\acute{e}t}} \to X^{\sim}_{\mathrm{Nis}} \to X^{\sim}_{\mathrm{Zar}}.$$

The direct image part of each of these maps is given simply by "restriction". For example, a sheaf $\mathscr{F}$ on $X_{\mathrm{\acute{e}t}}$ is, in particular, a sheaf on X_{Zar} because every object $U \to X$ of X_{Zar} is also an object of $X_{\mathrm{\acute{e}t}}$ (Zariski open immersions are étale) so it makes sense to evaluate $\mathscr{F}$ on U (similarly, a Zariski cover is, in particular, an étale cover, so the resulting presheaf is a sheaf). We will write, for example $\mathscr{F}|_{X_{\mathrm{\acute{e}t}}}$ for the restriction of an fppf sheaf on X to the étale site of X. We will write $\mathscr{F}^{\mathrm{fppf}}$ for the inverse image of a Zariski (Nisnevich, étale, etc.) sheaf under the morphism of topoi $X_{\mathrm{fppf}} \to X_{\mathrm{Zar}}$.

For any scheme X, the category of points of the topos $X^{\sim}_{\mathrm{Zar}}$ is equivalent to the category of points of the underlying topological space of X, with morphisms given by specialization

[6]pseudo-functorial actually, since one makes choices of representations of limits

(see Example 5.1.6). Neighborhoods of x in the usual sense are cofinal in the category of neighborhoods $\mathbf{Vois}_x$ in the sense of neighborhoods of points of a topos.

For any separably closed field k, observe that the site $(\operatorname{Spec} k)_{\text{ét}}$ is punctual, meaning that the topos $(\operatorname{Spec} k)_{\text{ét}}^{\sim}$ is equivalent to $\mathbf{Ens}$ with the equivalence given by

$$\mathscr{F} \mapsto \mathscr{F}(\operatorname{Spec} k) = \Gamma(\operatorname{Spec} k, \mathscr{F}).$$

A morphism of schemes $x : \operatorname{Spec} k \to X$ therefore determines a point

$$\begin{array}{rcl} X_{\text{ét}}^{\sim} & \to & \mathbf{Ens} \\ \mathscr{F} & \mapsto & \mathscr{F}_x := \Gamma(\operatorname{Spec} k, x_{\text{ét}}^{-1}\mathscr{F}) \end{array}$$

of the site $X_{\text{ét}}$ (which is, by definition, a point of the topos $X_{\text{ét}}^{\sim}$). If one considers all such maps where $\operatorname{Spec} k$ is a separable closure of the residue field $k(x)$ of the image point $x \in X$ ("geometric points" in the sense of [SGA4.VIII.7]), then these points form a conservative family [SGA4.VIII.3.5(b)], and in fact, the category of points of the étale topos is equivalent to the category of such maps, with morphisms given by "specialization" (see the precise statement in [SGA4.VIII.7.9] for the definition—it is fairly involved). In particular, a morphism of sheaves $f : \mathscr{F} \to \mathscr{G}$ on $X_{\text{ét}}$ is an isomorphism iff $f_x : \mathscr{F}_x \to \mathscr{G}_x$ is an isomorphism for every geometric point $x : \operatorname{Spec} k \to X$ of X.

If we view $x : \operatorname{Spec} k \to X$ as a point of $X_{\text{ét}}$ in the sense of topos theory, then recall that a neighborhood of x is a pair (U, u) where $U = (f : U \to X) \in X_{\text{ét}}$ and $u \in (h^+U)_x$. For the sake of clarity, let us prove that this u is the same thing as a factorization

$$\begin{tikzcd} & U \arrow[d, "f"] \\ \operatorname{Spec} k \arrow[ur, "u"] \arrow[r, "x"'] & X \end{tikzcd}$$

(One sometimes sees this as the definition of an étale neighborhood of a point.) To prove this, note that we in fact have equalities

$$\begin{array}{rcl} (h^+U)_x & = & \Gamma(\operatorname{Spec} k, x^{-1}(h^+U)) \\ & = & \Gamma(\operatorname{Spec} k, h^+(x_{\text{ét}}(U))) \\ & = & \Gamma(\operatorname{Spec} k, h^+(U \times_X \operatorname{Spec} k)) \\ & = & \operatorname{Hom}_{(\operatorname{Spec} k)_{\text{ét}}}(\operatorname{Spec} k, X \times_U \operatorname{Spec} k) \\ & = & \operatorname{Hom}_{\mathbf{Sch}/X}(\operatorname{Spec} k, U), \end{array}$$

the first equality is the definition of the stalk, the second is Lemma 5.1.1, the third is the definition of the functor

$$x_{\text{ét}} : X_{\text{ét}} \to (\operatorname{Spec} k)_{\text{ét}},$$

the forth is either because the étale topology is subcanonical, or because sheafification commutes with evaluation on $\operatorname{Spec} k$ in $(\operatorname{Spec} k)_{\text{ét}}$, and the last is the universal property of fibered products.

We would like to endow each X_T with a sheaf of rings and thereby regard it as a ringed site. A basic theorem of descent theory asserts that the presheaf

$$X \mapsto \Gamma(X, \mathscr{O}_X)$$

on the category of schemes is an fppf sheaf (an object of $(\operatorname{Spec} \mathbb{Z})_{\text{fppf}}^{\sim}$). Or rather, a basic theorem of descent theory asserts that the fppf topology is subcanonical so, in particular, the canonical sheaf Γ represented by $\mathbb{A}^1$ is an fppf sheaf. Hence the restriction of this sheaf to any X_T determines a sheaf of rings on X_T denoted $\mathscr{O}_X$.

The rest of this section is devoted to proving that, for a morphism of schemes $f : X \to Y$, the induced morphism of ringed sites $f_T : X_T \to Y_T$ is a local morphism. This will follow rather tautologically from the fact that a morphism of schemes is, by definition, a *local* morphism of locally ringed spaces.

Lemma 6.1.4. *Let $h : U \to V$ be a local morphism of locally ringed spaces, $s \in \Gamma(V, \mathscr{O}_V)$ a section with $h^\sharp s \in \Gamma(U, \mathscr{O}_U)^*$. Then there is an open subset $W \subseteq V$, and a factorization $U \to W \hookrightarrow V$ of h with $s|_W \in \mathscr{O}_V(W)^*$.*

Proof. For each point u of U, $h^\sharp_u(s) = (h^\sharp s)_u \in \mathscr{O}^*_{U,u}$, so by locality $s \in \mathscr{O}^*_{V,f(u)}$, hence s is invertible on a neighborhood of $f(u)$. Covering $f[U] \subseteq V$ by such neighborhoods, we produce the desired W. $\square$

Proposition 6.1.5. *For any morphism of schemes $f : X \to Y$, the induced morphism of ringed sites $f_T : X_T \to Y_T$ is a local morphism.*

Proof. By exactness of sheafification, it suffices to prove that

$$\begin{tikzcd} (f^{-1}_{T,\mathrm{pre}}\mathscr{O}^*_Y)(U) \arrow[r, "f^\sharp_T"] \arrow[d] & \mathscr{O}^*_X(U) = \Gamma(U, \mathscr{O}_U)^* \arrow[d] \\ (f^{-1}_{T,\mathrm{pre}}\mathscr{O}_Y)(U) \arrow[r, "f^\sharp"] & \mathscr{O}_X(U) = \Gamma(U, \mathscr{O}_U) \end{tikzcd}$$

is cartesian for any $U \to X$ in X_T. A section of $(f^{-1}_{T,\mathrm{pre}}\mathscr{O}_Y)(U)$ is represented by a $V \to Y$ in Y_T and a factorization

$$\begin{tikzcd} & & V \arrow[d] \\ U \arrow[urr, "h"] \arrow[r] & X \arrow[r, "f"] & Y \end{tikzcd}$$

(a neighborhood of U) and a section $s \in \mathscr{O}_Y(V) = \Gamma(V, \mathscr{O}_V)$. Assuming that $f^\sharp_T(s) = h^\sharp s$ is in $\Gamma(U, \mathscr{O}_U)^*$, we wish to show that $s \in (f^{-1}_{T,\mathrm{pre}}\mathscr{O}^*_Y)(U)$. That is, we want to produce a shrinking W of V so that $s|_W \in \mathscr{O}^*_Y(W) = \Gamma(W, \mathscr{O}_W)^*$. This follows from the lemma above and the fact that $W \to Y$ is in Y_T because the property of being a Zariski open immersion, an étale map, etc. is preserved under passing to an open subset of the domain. $\square$

6.2. **Log structures.** For a scheme X and one of our topologies T, a T *log structure* on X is a log structure on the ringed site $X^\sim_T$, so we have Zariski log structures, Nisnevich log structures, fppf log structures, etc. A T *log scheme* (or just a *log scheme* if T is clear from context) is a scheme X eqipped with a T log structure. We will write $X^\dagger$ for a log scheme and write $\alpha_X : \mathcal{M}_X \to \mathscr{O}_X$ for its log structure, as usual. A *morphism* of log schemes $X^\dagger \to Y^\dagger$ is a morphism of log ringed topoi whose underlying morphism of ringed topoi is induced by a morphism of schemes (and is hence a local morphism by Proposition 6.1.5).

If we just require the underlying morphism of ringed topoi to be local, then the second statement is probably somewhat redundant because the functor

$$\begin{array}{rcl} \mathbf{Sch} & \to & \mathbf{TopLocAn} \\ X & \mapsto & (X_T, \mathscr{O}_X) \end{array}$$

can be shown to be fully faithful for various T.[7] This is not technically relevant, but we mention it for cultural reasons.

Let $\mathbf{LSch}_T$ denote the category of T log schemes. We will simply write $\mathbf{LSch}$ if the topology T is clear from context (it will most often be the étale topology).

6.3. **Log rings to log schemes.** For any of our topologies T, there is a functor

$$\begin{array}{rcl} \operatorname{Spec} : \mathbf{PLogAn}^{\mathrm{op}} & \to & \mathbf{LSch}_T \\ P \to A & \mapsto & \operatorname{Spec}(P \to A) \end{array}$$

from the category of prelog rings to the category of T log schemes defined as follows. To a prelog ring $P \to A$, we associated the scheme $\operatorname{Spec} A$ together with the log structure associated to the prelog structure $\underline{P} \to \mathscr{O}_{\operatorname{Spec} A}$ corresponding to the map

$$P \to \Gamma(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A}) = A.$$

This log structure is manifestly quasi-coherent.

If $X = \operatorname{Spec} A$ is affine, then α is just a map $P \to A$, which is a prelog structure on A. Unfortunately, taking global sections of $\underline{P}^a \to \mathscr{O}_X$ does not necessarily recover the log structure $P^a \to A$ associated to α, even in the Zariski topology.

Example 6.3.1. For example, if α is the zero map $\mathbb{N} \to \mathbb{C} \times \mathbb{C}$, then $P^a = \mathbb{N} \oplus \mathbb{C}^* \times \mathbb{C}^*$, but global sections of $\underline{P}^a \to \mathscr{O}_X$ are given by

$$(\mathbb{N} \oplus \mathbb{C}^*)^2 \to \mathbb{C} \times \mathbb{C}.$$

The topology on $\mathbb{C} \times \mathbb{C}$ doesn't make much difference.

In general it seems to me that the determination of $\underline{P}^a(\operatorname{Spec} A)$ can be quite difficult for a prelog structure $P \to A$. The difficulties do not seem to arise solely from disconnected X as the above example might suggest.

6.4. **Coherence.**

6.5. **Cartesian products.** The category of T log schemes has cartesian products. To construct the cartesian product $X_1^\dagger \times_{Y^\dagger} X_2^\dagger$ in a diagram

$$\begin{tikzcd} X_1^\dagger \times_{Y^\dagger} X_2^\dagger \arrow[r] \arrow[d] & X_2^\dagger \arrow[d, "f_2"] \\ X_1^\dagger \arrow[r, "f_1"] & Y^\dagger \end{tikzcd}$$

[7] For $T = \mathrm{Zar}$, this follows from the results of [SGA4.IV.4.2]. For $T = \text{ét}$, recall that, for a scheme X, one can recover the underlying topological space of X as the set of sieves ("opens") in $X_{\text{ét}}^\sim$, ordered by inclusion [SGA4.VIII.6.1].

set $g = f_1\pi_1 = f_2\pi_2$ and endow the scheme $X_1 \times_Y X_2$ with the pushout log structure defined by the cocartesian diagram

$$\begin{tikzcd} g^*\mathcal{M}_Y \arrow[r, "\pi_2^* f_2^\dagger"] \arrow[d, "\pi_1^* f_1^\dagger"'] & \mathcal{M}_{X_2} \arrow[d] \\ \mathcal{M}_{X_1} \arrow[r] & \mathcal{M}_{X_1 \times_Y X_2} \end{tikzcd}$$

6.6. **Constructibility of $\overline{\mathcal{M}}_X^{\mathrm{gp}}$.** Recall that a sheaf of a abelian groups $\mathscr{F}$ on a topological space X is called *constructible* if there are a finite partition

$$X = \coprod_{i=1}^{n} X_i$$

of X into locally closed (i.e. constructible) subspaces and finitely generated abelian groups A_i such that, locally on X_i, $\mathscr{F}|_{X_i}$ is isomorphic to the constant sheaf $\underline{A}_i$. Here $\mathscr{F}|_{X_i}$ is abuse of notation for $\iota_i^{-1}\mathscr{F}$, where $\iota_i : X_i \to X$ is the inclusion.

Similarly, a sheaf of abelian groups $\mathscr{F}$ on $X_{\text{ét}}$ is called *constructible* if the same condition holds with "locally on X_i" replaced by "étale locally on X_i".

Proposition 6.6.1. *Let X be a quasi-compact scheme, $\mathcal{M}_X$ a coherent log structure on $X_{\text{ét}}$. Then $\overline{\mathcal{M}}_X^{\mathrm{gp}}$ is constructible.*

6.7. **Automorphisms of log structures.** Let $X^\dagger$ be a log scheme. Let $\mathrm{Aut}(\mathcal{M}_X)$ (or just Aut if $\mathcal{M}_X$ is clear from context) be the presheaf of automorphisms (of log structures, equivalently sheaves of monoids over $\mathscr{O}_X$) of $\mathcal{M}_X$:

$$\begin{aligned} \mathrm{Aut}(\mathcal{M}_X) : \mathbf{Sch}/X &\to \mathbf{Ens} \\ (f : Y \to X) &\mapsto \mathrm{Aut}(f^*\mathcal{M}_X). \end{aligned}$$

Of course $\mathrm{Aut}(f^*\mathcal{M}_X)$ means $\mathrm{Aut}_{\mathbf{Mon}_Y/\mathscr{O}_Y}(f^*\mathcal{M}_X)$, but we will always use the former notation.

In [Olsson4], Olsson proves

Theorem 6.7.1. *If $\mathcal{M}_X$ is a fine log structure, the presheaf* Aut *is a locally separated algebraic space of finite type over X.*

We will prove this theorem under some additional hypotheses on $\mathcal{M}_X$ which simplify the proof and make it more self contained. There are several steps to the proof:

(1) Show that Aut is a sheaf in the étale topology.
(2) Show that the diagonal
$$\Delta : \mathrm{Aut} \to \mathrm{Aut} \times_X \mathrm{Aut}$$
is representable by quasi-compact locally closed immersions.
(3) Produce an étale cover of Aut by a scheme finitely presented over X.

Step 1 is not particularly difficult, but is not totally obvious because $f^*\mathcal{M}_X \neq f^{-1}\mathcal{M}_X$, so we cannot simply appeal to the theory of étale descent of étale sheaves (gluing). In fact, we will show in Section 7 that Aut is a sheaf in the fppf topology, which is not much harder than showing it is a sheaf in the étale topology. (This stronger statement, together with a general existence theorem for algebraic spaces, is needed to complete the proof without the additional assumptions we will impose.) For now, we will simply assume without proof that Aut is an étale sheaf. Step 2 is straightforward.

The idea of Step 3 is to produce a finite set $P_i \to \mathcal{M}_X(U_i)$ of (étale local) fine, sharp charts for $\mathcal{M}_X$ so that, locally, any automorphism of $\mathcal{M}_X$ agrees with one commuting with an automorphism of P_i. Finally, we construct, with our bare hands, a scheme representing automorphisms commuting with a given automorphism of P_i. Since this last step is the main idea, we prove it first:

Proposition 6.7.2. *Suppose $h : P \to \mathcal{M}_X(X)$ is a chart for $\mathcal{M}_X$ with P finitely generated, and let σ be an automorphism of P. Let* $\operatorname{Aut}_\sigma$ *be the subpresheaf of* Aut *taking $f : Y \to X$ to the set of $a \in \operatorname{Aut}(f^*\mathcal{M}_X)$ making the diagram*

$$\begin{tikzcd} P \arrow[r, "\sigma"] \arrow[d] & P \arrow[d] \\ f^*\mathcal{M}_X(Y) \arrow[r] & f^*\mathcal{M}_X(Y) \end{tikzcd}$$

commute. Then:

(1) $\operatorname{Aut}_\sigma$ *is open in* Aut *and*
(2) $\operatorname{Aut}_\sigma$ *is (representable by) a scheme finitely presented and affine over X.*

Proof. For the second part, let $C_\sigma := \operatorname{Spec} \mathscr{O}_X[P^{\mathrm{gp}}]/J$ where J is the ideal generated by the set of sections

$$\{\alpha(\sigma(p))[p] - \alpha(p) : p \in P\} \subseteq \Gamma(X, \mathscr{O}_X[P^{\mathrm{gp}}]).$$

Here α is shorthand for the composition of h and $\alpha_X : \mathcal{M}_X(X) \to \mathscr{O}_X(X)$, and $[p]$ is the image of p in the global sections of the group algebra $\mathscr{O}_X[P^{\mathrm{gp}}]$. This is finitely presented (and clearly affine) over X because P^{gp} is finitely generated (so $\mathscr{O}_X[P]$ is a finitely presented $\mathscr{O}_X$-algebra) and because one can check easily that to generate J we only need the above expressions where p runs over a set of generators for P.

Let $g : C_\sigma \to X$ be the projection. There is an automorphism $b \in \operatorname{Aut}(\mathcal{M}_X)(C_\sigma)$ given by

$$p \mapsto [p] + \sigma(p) \in g^*\mathcal{M}_X.$$

(Note here that $[p] \in \Gamma(C_\sigma, \mathscr{O}_{C_\sigma})^* \subseteq g^*\mathcal{M}_X$, so this expression makes sense.) Evidently

$$(f : Y \to C_\sigma) \mapsto f^*b \in \operatorname{Aut}_Y(f^*g^*\mathcal{M}_X)$$

defines a map of presheaves $C_\sigma \to \operatorname{Aut}_\sigma$, which we claim is an isomorphism. Given a map $f : Y \to X$ and an automorphism $a \in \operatorname{Aut}_\sigma(f^*\mathcal{M}_X)$, the universal property of the relatively affine scheme C_σ says that to factor f through g is to lift $f^\sharp : f^{-1}\mathscr{O}_X \to \mathscr{O}_Y$ to a map $f^{-1}\mathscr{O}_X[P^{\mathrm{gp}}]/J \to \mathscr{O}_Y$, which we can do by mapping $[p]$ to the unique unit $u_p \in \mathscr{O}_Y^*$ for which $a(p) = u_p + \sigma(p)$. Such a unit exists and this map kills the ideal J exactly because $a \in \operatorname{Aut}_\sigma(f^*\mathcal{M}_X) \subseteq \operatorname{Aut}(f^*\mathcal{M}_X)$. Chasing through the definitions, it is clear that a is the pullback of b under the lifting morphism $Y \to C_\sigma$. Evidently this construction is compatible with pullback, so we have defined a morphism of presheaves $\operatorname{Aut}_\sigma \to C_\sigma$ so that the composition $\operatorname{Aut}_\sigma \to C_\sigma \to \operatorname{Aut}_\sigma$ is the identity. Checking that the other composition is the identity amounts to checking that applying this construction to Id : $C_\sigma \to C_\sigma$ and the automorphism $b \in \mathrm{Aut}(g^*\mathcal{M}_X)$ yields the identity map of C_σ, which is obvious. $\square$

7. Comparison of Topologies

In this section we will be concerned with the differences (or lack thereof) between log structures on schemes using different topologies. We will phrase propositions in terms of diagrams of ringed topoi because we feel this generality clarifies the proofs.

The results in this section are due to M. Olsson with minor differences in phrasing, though any errors herein are my own. I've made an effort to provide more detail in some of the proofs, since we have the luxury of using all our previous results.

7.1. Generalities.

In this section, we will consider a pseudo-commutative diagram

(7.1.0.1)
$$\begin{tikzcd} X'' \arrow[r, "p_2"] \arrow[d, "p_1"'] & X' \arrow[d, "f"] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

in **TopAn**. Set $g := fp_1 \cong fp_2$.

Proposition 7.1.1. *Let $\mathcal{M}_X$ be an integral log structure on X. If the adjunction sequences*

(7.1.1.1)
$$\mathscr{O}_X \longrightarrow f_*f^*\mathscr{O}_X \rightrightarrows g_*g^*\mathscr{O}_X$$

(7.1.1.2)
$$\overline{\mathcal{M}}_X \longrightarrow f_*\overline{f^*\mathcal{M}_X} \rightrightarrows g_*\overline{g^*\mathcal{M}_X}$$

are exact, then so is the adjunction sequence

(7.1.1.3)
$$\mathcal{M}_X \longrightarrow f_*f^*\mathcal{M}_X \rightrightarrows g_*g^*\mathcal{M}_X.$$

Remark 7.1.2. We systematically suppress notation for the 2-isomorphism $fp_1 \cong fp_2$ and proceed as if there is an equality $fp_1 = fp_2$. For example, this 2-isomophism is used to identify $g_*g^* = (fp_1)_*(fp_1)^*$ and $(fp_2)_*(fp_2)^*$ to form the parallel arrows.

Remark 7.1.3. The sequence (7.1.1.1) is the same as the sequence

(7.1.3.1)
$$\mathscr{O}_X \longrightarrow f_*\mathscr{O}_{X'} \rightrightarrows g_*\mathscr{O}_{X''}\ ,$$

by definition of the functors $f^* := f^{-1}\ \_ \otimes_{f^{-1}\mathscr{O}_X} \mathscr{O}_{X'}$ and g^*.

Remark 7.1.4. If the morphisms are local (i.e. the diagram is in **TopLocAn**), then the adjunction sequence (7.1.1.2) is naturally isomorphic to

(7.1.4.1)
$$\overline{\mathcal{M}}_X \longrightarrow f_*f^{-1}\overline{\mathcal{M}}_X \longrightarrow g_*g^{-1}\overline{\mathcal{M}}_X$$

by Proposition 5.5.1. It is generally easier in practice to conclude existence of this sequence since it only involves the usual inverse image functors f^{-1} and g^{-1}, and not the pullback of log structures f^*, g^*.

Remark 7.1.5. We always have $(f_*\mathscr{O}_{X'})^* \cong f_*(\mathscr{O}_{X'}^*)$ because the right adjoint f_* commutes with all inverse limits, so we will write $f_*\mathscr{O}_{X'}^*$ unambiguously.

Remark 7.1.6. In the conclusion of the theorem, f_* and g_* denote the usual direct image functors. However, the exactness of the sequence (7.1.1.3) implies the exactness of the adjunction sequence

(7.1.6.1)
$$\mathcal{M}_X \longrightarrow f_*^{\log}f^*\mathcal{M}_X \rightrightarrows g_*^{\log}g^*\mathcal{M}_X$$

as follows. By definition of $f_*^{\log}$, we have a cartesian diagram:

$$\begin{tikzcd} f_*^{\log} f^* \mathcal{M}_X \arrow[r] \arrow[d] & f_* f^* \mathcal{M}_X \arrow[d] \\ \mathscr{O}_X \arrow[r] & f_* \mathscr{O}_{X'} \end{tikzcd}$$

The map $\alpha_X : \mathcal{M}_X \to \mathscr{O}_X$ and the adjunction morphism determine a map $\mathcal{M}_X \to f_*^{\log} f^* \mathcal{M}_X$. Replacing f by g we get a similar map. These maps fit into a commutative diagram

$$\begin{tikzcd} \mathcal{M}_X \arrow[d] \arrow[dr] & & \\ f_*^{\log} f^* \mathcal{M}_X \arrow[r] & f_* f^* \mathcal{M}_X \arrow[r, shift left] \arrow[r, shift right] \arrow[dr, shift left] \arrow[dr, shift right] & g_* g^* \mathcal{M}_X \\ & & g_*^{\log} g^* \mathcal{M}_X \arrow[u] \end{tikzcd}$$

from which the exactness of (7.1.6.1) follows from the exactness of (7.1.1.3) by a familiar diagram chase.

Proof. We have to analyze the commutative diagram below.

$$\begin{tikzcd} \mathscr{O}_X^* \arrow[r] \arrow[d] & f_* \mathscr{O}_{X'}^* \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & g_* \mathscr{O}_{X''}^* \arrow[d] \\ \mathcal{M}_X \arrow[r] \arrow[d] & f_* f^* \mathcal{M}_X \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & g_* g^* \mathcal{M}_X \arrow[d] \\ \overline{\mathcal{M}}_X \arrow[r] & f_* \overline{f^* \mathcal{M}_X} \arrow[r, shift left] \arrow[r, shift right] & g_* \overline{g^* \mathcal{M}_X} \end{tikzcd}$$

The bottom row is exact by hypothesis and the exactness of the top row follows from the hypothesized exactness of (7.1.1.1). The bottom left vertical arrow is surjective, but the other two vertical arrows on the bottom may not be surjective because f_* and g_* are not right exact. However, the top vertical arrows are still kernels of the vertical arrows beneath them because f_* and g_* are left exact.

This will be enough to conclude the exactness of the middle row by a diagram chase, as follows. Proving that $\mathcal{M}_X \to f_* f^* \mathcal{M}_X$ is monic is straightforward. Once this is known, surjectivity onto the equalizer of $f_* f^* \mathcal{M}_X \rightrightarrows g_* g^* \mathcal{M}_X$ can be proved locally. Say s is a local section in this equalizer. Commutativity, exactness of the bottom row, and surjectivity of the bottom left vertical map imply that, at least after passing to a cover, there is some $t \in \mathcal{M}_X$ whose image in $f_* f^* \mathcal{M}_X$ has the same image as s in $f_* \overline{f^* \mathcal{M}_X}$. Using the fact that f_* preserves equalizers (together with Proposition 1.2.3 to be very careful), we can write $t = s + u$ for some $u \in f_* \mathscr{O}_{X'}^*$. Let $u_1, u_2 \in g_* \mathscr{O}_{X''}^*$ be the images of u under the top parallel arrows. Since s and t have the same image under the middle parallel arrows, we conclude $t = r + u_1 = r + u_2$ in $g_* g^* \mathcal{M}_X$, and hence $u_1 = u_2$ in $g_* g^* \mathcal{M}_X$ by integrality (integrality of $\mathcal{M}_X$ implies integrality of $g_* g^* \mathcal{M}_X$ by Lemma 5.3.7). Since the top right vertical arrow is monic, we have $u_1 = u_2$, hence exactness of the top row implies $u \in \mathscr{O}_X^* \subseteq \mathcal{M}_X$, so $s = t - u$ is in $\mathcal{M}_X$. $\square$

Let X be a ringed site. Then an $\mathscr{O}_X^*$-sheaf is a sheaf $\mathscr{F}$ on X together with an action of the group $\mathscr{O}_X^*$ on $\mathscr{F}$ (a morphism $a : \mathscr{O}_X^* \times \mathscr{F} \to \mathscr{F}$ respecting the multiplication and identity of $\mathscr{O}_X^*$). Evidently $\mathscr{O}_X^*$ is itself an $\mathscr{O}_X^*$-sheaf with the action given by multiplication. $\mathscr{O}_X^*$-sheaves form a category where a morphism from $\mathscr{F}$ to an $\mathscr{O}_X^*$-sheaf $\mathscr{G}$ (with action $b : \mathscr{O}_X^* \times \mathscr{G} \to \mathscr{G}$) is a morphism $f : \mathscr{F} \to \mathscr{G}$ so that

$$fa = b(\mathrm{Id} \times f) : \mathscr{O}_X^* \times \mathscr{F} \to \mathscr{G}.$$

An $\mathscr{O}_X^*$-sheaf $\mathscr{F}$ is called an $\mathscr{O}_X^*$-torsor if there is a cover $\{U_i \to X\}$ and isomorphisms of $\mathscr{O}_X^*|_{U_i}$-sheaves $\mathscr{F}|_{U_i} \cong \mathscr{O}_X^*|_{U_i}$ on X/U_i.

If $f : X' \to X$ is a morphism of ringed sites, then f_* takes $\mathscr{O}_{X'}^*$-sheaves to $\mathscr{O}_X^*$-sheaves by defining an action of $\mathscr{O}_X^*$ on $f_*\mathscr{F}$ by

$$(f_*a)(f^\flat \times \mathrm{Id}) : \mathscr{O}_X^* \times f_*\mathscr{F} \to f_*\mathscr{F}.$$

Proposition 7.1.7. *Let $f : X' \to X$ be a local morphism of ringed sites, $\mathcal{M}_{X'}$ an integral log structure on X'. Assume the following:*

(1) $f^\flat : \mathscr{O}_X \to f_*\mathscr{O}_{X'}$ is an isomorphism.
(2) The adjunction morphism $f^{-1}f_*\overline{\mathcal{M}}_{X'} \to \overline{\mathcal{M}}_{X'}$ is an isomorphism.
(3) For any object U of X and any $\mathscr{O}_{X'}^*|_{f(U)}$-torsor $\mathscr{F}$, the $\mathscr{O}_X^*|_U$-sheaf $f_*\mathscr{F}$ is an $\mathscr{O}_X^*|_U$-torsor.

Then the adjunction morphism $f^*f_*\mathcal{M}_{X'} \to \mathcal{M}_{X'}$ is an isomorphism.

Remark 7.1.8. Assumption (1) implies $f_*^{\log}\mathcal{M}_{X'} \to f_*\mathcal{M}_{X'}$ is an isomorphism.

Proof. $f^*f_*\mathcal{M}_{X'}$ is integral by Lemma 5.3.7 and 5.5.2, so by 5.4.4, it suffices to prove that $\overline{f^*f_*\mathcal{M}}_{X'} \to \overline{\mathcal{M}}_{X'}$ is an isomorphism. By 5.5.1, this is the same as the map $f^{-1}\overline{f_*\mathcal{M}}_{X'} \to \overline{\mathcal{M}}_{X'}$, so it suffices to prove *this* map is an isomorphism. By assumption (2), it suffices to prove $\overline{f_*\mathcal{M}}_{X'} \to f_*\overline{\mathcal{M}}_{X'}$ is an isomorphism.

To prove this map is injective, we first consider the commutative diagram below.

$$\begin{array}{ccc} \overline{f_*\mathcal{M}}_{X'} & \longrightarrow & f_*\overline{\mathcal{M}}_{X'} \\ \downarrow & & \downarrow \\ \overline{f_*\mathcal{M}}^{\mathrm{gp}}_{X'} & \longrightarrow & f_*(\overline{\mathcal{M}}^{\mathrm{gp}}_{X'}) \end{array}$$

The vertical arrows are monic by integrality, so to prove our map is monic, it suffices to prove the bottom horizontal arrow is monic. This in turn follows from a diagram chase (Snake Lemma) in the (exact) diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{O}_X^* & \longrightarrow & (f_*\mathcal{M}_{X'})^{\mathrm{gp}} & \longrightarrow & \overline{f_*\mathcal{M}}_{X'} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{O}_X^* & \longrightarrow & f_*(\mathcal{M}^{\mathrm{gp}}_{X'}) & \longrightarrow & f_*(\overline{\mathcal{M}}^{\mathrm{gp}}_{X'}) & & \end{array}$$

of (groupified) characteristic sequences, together with the fact that

$$(f_*\mathcal{M}_{X'})^{\mathrm{gp}} \to f_*(\mathcal{M}^{\mathrm{gp}}_{X'})$$

is monic by Lemma 5.3.6.

To prove surjectivity of $\overline{f_* \mathcal{M}_{X'}} \to f_* \overline{\mathcal{M}}_{X'}$, suppose

$$s \in (f_* \overline{\mathcal{M}}_{X'})(U) = \overline{\mathcal{M}}_{X'}(f(U))$$

is a section over some object U of X. Let $\mathscr{F}$ be the $\mathscr{O}^*_{X'}|_{f(U)}$-torsor representing

$$V \mapsto \{m \in \mathcal{M}_{X'}(V) : \overline{m} = s\}.$$

That is, define $\mathscr{F}$ by the cartesian diagram

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[d] & \mathcal{M}_{X'} \arrow[d] \\ f(U) \arrow[r, "s"] & \overline{\mathcal{M}}_{X'} \end{tikzcd}$$

in $(X'/f(U))^\sim$. By assumption (3), $f_*\mathscr{F}$ is an $\mathscr{O}^*_X|_U$-torsor, so, at least after passing to a cover, the set

$$\begin{aligned} (f_*\mathscr{F})(U) &\cong \mathscr{F}(f(U)) \\ &\cong \{m \in \mathcal{M}_{X'}(f(U)) : \overline{m} = s\} \\ &\cong \{m \in (f_*\mathcal{M}_{X'})(U) : \overline{m} = s\} \end{aligned}$$

is non-empty. Since we already proved injectivity the existence of a preimage of s on a cover of U is enough to conclude that s has a preimage on U.

□

7.2. **Descent criterion.** In this section, we consider a pseudo-commutative diagram of the form

$$\begin{tikzcd} & X'' \arrow[r, "p_1"] \arrow[dr, "p_2"] & X' \arrow[dr, "f"] & \\ X''' \arrow[ur, "p_{12}"] \arrow[r, "p_{13}"] \arrow[dr, "p_{23}"'] & X'' \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & X' \arrow[r, "f"] & X \\ & X'' \arrow[ur, "p_1"] \arrow[r, "p_2"'] & X' \arrow[ur, "f"'] & \end{tikzcd}$$

in **TopAn**.

A *descent datum* for an object $\mathscr{F}$ of X' is an isomorphism $\phi : p_1^{-1}\mathscr{F} \to p_2^{-1}\mathscr{F}$ of objects of X'' satisfying $p_{13}^{-1}\phi = p_{23}^{-1}\phi \circ p_{12}^{-1}\phi$ (we suppress notation for the pseudocommutativity 2-isomorphisms) on X'''. A *morphism* of objects with descent data $(\mathscr{F}, \phi) \to (\mathscr{G}, \psi)$ is an X' morphism $h : \mathscr{F} \to \mathscr{G}$ making the diagram

$$\begin{tikzcd} p^{-1}\mathscr{F} \arrow[r, "\phi"] \arrow[d, "p_1^{-1}h"'] & p_2^{-1}\mathscr{F} \arrow[d, "p_2^{-1}h"] \\ p_1^{-1}\mathscr{G} \arrow[r, "\psi"] & p_2^{-1}\mathscr{G} \end{tikzcd}$$

commute in X''. Let $\mathbf{DD}_{X'}$ be category of objects of X' with descent datum.

If $\mathscr{F}$ is an object of X, then pseudo-commutativity of the diagram provides the object $f^{-1}\mathscr{F}$ of X' with a *tautological descent datum*. This defines a functor $X \to \mathbf{DD}_{X'}$. We say an object with descent datum $(\mathscr{F}, \phi)$ is *effective* if it is in the essential image of this functor. We say the diagram *has effective descent of objects* if this functor is an equivalence of categories.

We define descent data for modules over the structure sheaves, log structures, etc. by replacing the pullback f^{-1} of objects with the appropriate pullback f^*. We similarly define categories $\mathbf{ModDD}_{X'}, \mathbf{LogDD}_{X'}$ of modules with descent data, log structures with descent data, etc. as well as notions of *effective descent of modules, log structures, etc.*

Although we only adorn the notation with the subscript X', the categories $\mathbf{DD}_{X'}$, etc. depend on the data of the entire diagram. In practice, $f : X' \to X$ with be a morphism of schemes, topological spaces, analytic spaces, etc., X'', X''' will be the cartesian products $X' \times_X X'$, $X' \times_X X' \times_X X'$, and the diagram will be the diagram of natural projections.

7.3. **The case of schemes.** As in Section 5.1, we let X_t denote one of X_{Zar}, $X_{\text{ét}}$, or X_{Nis} for a scheme X. For a sheaf $\mathscr{F}$ on X_t, we write $\mathscr{F}^{\mathrm{fppf}}$ for the corresponding sheaf on X_{fppf} (the inverse image of $\mathscr{F}$ under the map of topoi $X_{\mathrm{fppf}}^{\sim} \to X_t^{\sim}$). We write $\mathscr{F}|_{X_t}$ for the restriction (direct image) of an fppf sheaf on X to one of these sites. Similarly, for a log structure $\mathcal{M}_X$ on one of the X_t, we write $\mathcal{M}_X^{\mathrm{fppf}}$ for its pullback to the fppf site, and we write $\mathcal{M}_X|_{X_t}$ for the restriction of an fppf log structure to a log structure on X_t. There is no difference between the direct image log structure and the direct image sheaf because the map $\mathscr{O}_{X_t} \to \mathscr{O}_{X_{\mathrm{fppf}}}|_{X_t}$ defining the morphism of ringed sites is an isomorphism by definition of the sheaves of rings in question.

Introduce the notation $\mathbf{IntLog}_X$ for the category of integral log structures on a ringed topos X.

Proposition 7.3.1. *For a scheme X and an integral log structure $\mathcal{M}_X$ on one of the X_t, the log structure $\mathcal{M}_X^{\mathrm{fppf}}$ is given by*

$$(f : X' \to X) \mapsto \Gamma(X', f^*\mathcal{M}_X).$$

In particular, the adjunction morphism $\mathcal{M}_X \to \mathcal{M}_X^{\mathrm{fppf}}|_{X_t}$ is an isomorphism.

Proof. Certainly $\mathcal{M}^{\mathrm{fppf}}$ is the sheaf associated to the presheaf

$$(f : X' \to X) \mapsto \Gamma(X', f^*\mathcal{M}_X)$$

on X_{fppf}. We must prove this is already a sheaf. That is, we must prove that for any cover f as in the diagram

$$\begin{tikzcd} X' \arrow[rr, "f"] \arrow[dr, "k"'] & & U \arrow[dl, "h"] \\ & X & \end{tikzcd}$$

in X_{fppf}, the diagram

(7.3.1.1) $$\Gamma(U, h^*\mathcal{M}_X) \to \Gamma(U', k^*\mathcal{M}_X) \rightrightarrows \Gamma(X' \times_U X', g^*\mathcal{M}_X)$$

is an equalizer diagram. Here $g = \pi_1 f = \pi_2 f$. Replacing U by X and $\mathcal{M}_X$ by $h^*\mathcal{M}_X$, we may assume $U = X$. The diagram (7.3.1.1) is obtained by the evaluating the diagram

$$\mathcal{M}_X \to f_* f^* \mathcal{M}_X \rightrightarrows g_* g^* \mathcal{M}_X$$

on the terminal object X of X_{fppf} (i.e. by applying the left exact functor $\mathrm{Hom}_X(X, \_\,)$), so it suffices to prove this sequence of sheaves is exact.

To do this, we need only verify the hypotheses of Proposition 7.1.1. The first diagram in Proposition 7.1.1 is an equalizer diagram because the fppf topology is subcanonical. We already mentioned that the morphisms of ringed topoi in question are local (6.1.5), so the second diagram is identified with the one in Remark 7.1.4. This diagram is an equalizer diagram by generalities (when $t = \mathrm{Zar}$ this is elementary, when $t = \text{ét}$ this is [SGA4.VIII.9.2(c)], and when $t = \mathrm{Nis}$ the result follows by the same argument as in the proof of [SGA4.VIII.9.2(c)]).

$\square$

Corollary 7.3.2. *For any scheme X, the functor*

$$\begin{aligned} \mathbf{IntLog}_{X_t} &\to \mathbf{IntLog}_{X_{\mathrm{fppf}}} \\ \mathcal{M}_X &\mapsto \mathcal{M}_X^{\mathrm{fppf}} \end{aligned}$$

is fully faithful.

Proof. If $f : U \to X$ is in X_t, then $\Gamma(U, f^*\mathcal{M}_X)$ is nothing but $\mathcal{M}_X(U)$, so this follows from the proposition. $\square$

Example 7.3.3. The following pathology example demonstrates that some hypotheses are required on a log structure $\mathcal{M}$ on X_{fppf} in order to conclude that it is pulled back from $X_{\text{ét}}$. Let t be the coordinate on $\mathbb{A}^1 = \mathbb{A}^1_t$ and consider the log structure $\mathcal{M}$ on $\mathbb{A}^1_{\text{fppf}}$ given by

$$(f : X \to \mathbb{A}^1) \mapsto \{s \in \Gamma(X, \mathscr{O}_X) : s^2 = (f^\sharp t)^n u \text{ for some } n \in \mathbb{N}, u \in \Gamma(X, \mathscr{O}_X)^*\}.$$

The log structure $\mathcal{M}$ is neither integral nor quasi-coherent. To see the failure of integrality, consider the natural map

$$X := \operatorname{Spec} \mathbb{Z}[s, s_1, s_2, t]/\langle s^2 - t, s_1^2 - t, s_2^2 - t, ss_1 - ss_2\rangle \to \operatorname{Spec} \mathbb{Z}[t] = \mathbb{A}^1.$$

Then $s, s_1, s_2 \in \mathcal{M}(X)$ are distinct, but $ss_1 = ss_2$. The lack of a chart is because of the possibility of continually adjoining new square roots of t. Indeed, if $f : X \to \mathbb{A}^1$ is any fppf cover of $\mathbb{A}^1$, and $P \to \mathcal{M}(X)$ is any map of monoids, then set

$$Y := \operatorname{Spec}_X \mathscr{O}_X[s]/\langle s^2 - f^\sharp t\rangle,$$

and let $g : Y \to X$ be the natural map. Then $s \in \Gamma(Y, \mathscr{O}_Y)$ is in $\mathcal{M}(Y)$, but it is not in the image of $P^a(Y) \to \Gamma(Y, \mathscr{O}_Y)$, so there can be no isomorphism $P^a \cong \mathcal{M}_X$. The restriction $\mathcal{M}|_{X_{\text{ét}}}$ is nothing but our old friend $\operatorname{Spec}(\mathbb{N} \to \mathbb{Z}[\mathbb{N}])$, and the adjunction morphism $(\mathcal{M}|_{X_{\text{ét}}})^{\text{fppf}} \to \mathcal{M}$ is a monomorphism, but not an isomorphism (in particular because $(\mathcal{M}|_{X_{\text{ét}}})^{\text{fppf}}$ is manifestly fine, while $\mathcal{M}$ is not even quasi-coherent.

8. Log Smoothness

In this section we cover the important concept of log smoothness, and the related concept of log étaleness. Throughout this section, we will work with étale log structures. In light of the general results of Section 7, we may always start with a Zariski or Nisnevich log structure and take the inverse image log structure on the étale site. Restricting this log structure back to the Zariski/Nisnevich site then gives us back the log structure we started with. However, the best results require one to work étale locally.

8.1. **The lifting criterion.** A morphism of log schemes $f : X \to Y$ is called *formally log smooth* iff it has the local left lifting property with respect to any strict square-zero thickening of log schemes. That is, f is log smooth, iff, in any commutative diagram of solid arrows

(8.1.0.1)
$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Z \arrow[r] \arrow[ur, dotted, "g"] & Y \end{tikzcd}$$

where $\underline{W} \to \underline{Z}$ is a closed embedding with square-zero ideal I and $W \to Z$ is strict, there is a completion g étale locally on $\underline{Z}$.

A formally log smooth morphism is called *formally log étale* iff, étale locally on $\underline{Z}$, there is a *unique* dotted arrow g completing (8.1.0.1). (We will see later that under mild coherence assumptions this is equivalent to the existence of a unique *global* lift g.) A morphism is called *log smooth* (resp. *log étale*) iff it is formally log smooth (resp. formally log étale) and the underlying morphism of schemes is locally of finite presentation. These terms are sometimes reserved for fine log schemes, where we will be able to obtain a relatively simple characterization of log smoothness and log étaleness in terms of local charts for f.

Let us make a few remarks about the relationship between the log structures $\mathcal{M}_W$ and $\mathcal{M}_Z$ in this situation. The square zero thickening $\underline{W} \to \underline{Z}$ is an isomorphism on the level of spaces and étale toposes [SGA4 VIII.1.1], so it is harmless to suppress notation for pullback along this map. We have an exact sequence

$$0 \to I \to \mathscr{O}_Z \to \mathscr{O}_W \to 0$$

of $\mathscr{O}_Z$ modules with $I^2 = 0$. From this, we obtain an exact sequence of sheaves of abelian groups

$$0 \longrightarrow I \longrightarrow \mathscr{O}_Z^* \longrightarrow \mathscr{O}_W^* \longrightarrow 0 \tag{8.1.0.2}$$

where the left map is given by $i \mapsto 1 + i$. Since $W \to Z$ is strict, we know $\mathcal{M}_W$ is the log structure associated to the prelog structure on $\mathscr{O}_W$ given by the composition α of $\mathcal{M}_Z \to \mathscr{O}_Z \to \mathscr{O}_W$. That is, $\mathcal{M}_W = \mathcal{M}_Z \oplus_{\alpha^{-1}\mathscr{O}_W^*} \mathscr{O}_W^*$. But the square zero surjection $\mathscr{O}_Z \to \mathscr{O}_W$ is clearly exact in the sense that a local section of $\mathscr{O}_Z$ maps to a unit in $\mathscr{O}_W$ iff it was already a unit in $\mathscr{O}_Z$, so $\alpha^{-1}\mathscr{O}_W^* \subseteq \mathcal{M}_Z$ is just the usual inclusion $\mathscr{O}_Z^* \subseteq \mathcal{M}_Z$, and we see that (8.1.0.2) is part of a larger diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, equal] & \mathscr{O}_Z^* \arrow[r] \arrow[d] & \mathscr{O}_W^* \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I \arrow[r] & \mathcal{M}_Z \arrow[r] & \mathcal{M}_W \arrow[r] & 0 \end{tikzcd} \tag{8.1.0.3}$$

Proposition 8.1.1. *(Formally) log smooth morphisms are stable under base change and composition.*

Proof. This is a formal consequence of the definition and the fact that the analogous statements are true of morphisms of locally finite presentation. □

Lemma 8.1.2. *A strict morphism of fine log schemes $f : X^\dagger \to Y^\dagger$ is (formally) log smooth iff the underlying morphism of schemes is (formally) smooth.*

8.2. **Log differentials.** Throughout this section, we will consider a morphism

$$\begin{tikzcd} P \arrow[r, "\alpha"] & B \\ Q \arrow[u, "h"] \arrow[r, "\beta"] & A \arrow[u, "f"'] \end{tikzcd} \tag{8.2.0.1}$$

of prelog rings (or prelog ring objects of a topos). Let $g : M \to N$ be a morphism of B-modules. An A-linear *log derivation* (d, dlog) with values in $(g : M \to N)$ consists of an A-linear derivation $d : B \to M$, together with a map of B-modules

$$\mathrm{dlog} : B \otimes_{\mathbb{Z}} P^{\mathrm{gp}} \to N$$

satisfying:

$$\mathrm{dlog}(1 \otimes h(q)) = 0 \quad \text{for all } q \in Q. \tag{8.2.0.2}$$

$$\alpha(p)\,\mathrm{dlog}(1 \otimes p) = g(d\alpha(p)) \quad \text{for all } p \in P. \tag{8.2.0.3}$$

When we speak of a log derivation with values in a B module M, we mean a log derivation with values in $\mathrm{Id} : M \to M$ in the above sense.

Suppose $(\alpha : P \to B)$ is a log structure, so we may regard B^* as a submonoid of P. Then the restriction

$$\mathrm{dlog}\,|_{B\otimes_{\mathbb{Z}}B^*} : B \otimes_{\mathbb{Z}} B^* \to N$$

is completely determined by (8.2.0.3): we must have

$$\mathrm{dlog}(b \otimes u) = b\frac{du}{u}.$$

(This is supposed to justify the terminology.) In particular, if B has the trivial log structure, then any log derivation is uniquely determined from the derivation d by the above rule. Conversely, the above rule determines a log derivation. That is, it satisfies (8.2.0.2) because $dh(q) = df(\beta(q)) = 0$.

Remark 8.2.1. A B-module map $\mathrm{dlog} : B \otimes_{\mathbb{Z}} P^{\mathrm{gp}} \to N$ satisfying the above properties is the same thing as a monoid homomorphism $\mathrm{dlog} : P \to N$ satisfying:

$$\alpha(p)\,\mathrm{dlog}\,p = g(d\beta(p)) \tag{8.2.1.1}$$

$$\mathrm{dlog}\,h(q) = 0 \quad \text{for all } q \in Q. \tag{8.2.1.2}$$

We typically think of dlog via the second description.

Proposition 8.2.2. *There is a B-module $\Omega^{\dagger}_{B/A}$, a B-module map $\Omega_{B/A} \to \Omega^{\dagger}_{B/A}$, and a log derivation $(d_{B/A}, \mathrm{dlog}_{B/A})$ with values in $(\Omega_{B/A} \to \Omega^{\dagger}_{B/A})$ which is universal in the sense that any log derivation with values in $(g : M \to N)$ is obtained from the universal one by composing with the horizontal arrows in a unique commutative diagram*

$$\begin{tikzcd} \Omega_{B/A} \arrow[r] \arrow[d] & M \arrow[d] \\ \Omega^{\dagger}_{B/A} \arrow[r] & N \end{tikzcd}$$

of B-module maps.

Remark 8.2.3. This property determines $\Omega_{B/A} \to \Omega^{\dagger}_{B/A}$ up to unique isomorphism in the arrow category of B-modules. We suppress the subscripts "B/A" if the prelog rings are clear from context. As in Remark 8.2.1, we usually think of dlog as a monoid homomorphism $\mathrm{dlog} : P \to \Omega^{\dagger}_{B/A}$ satisfying:

$$\alpha(p)\,\mathrm{dlog}\,p = d\alpha(p) \quad \text{for all } p \in P \tag{8.2.3.1}$$

$$\mathrm{dlog}\,h(q) = 0 \quad \text{for all } q \in Q \tag{8.2.3.2}$$

(suppressing the natural map $\Omega_{B/A} \to \Omega^{\dagger}_{B/A}$ on the right side of (8.2.3.1)).

Proof. We just force this to be true by taking $\Omega^{\dagger}_{B/A}$ to be the quotient of the B-module

$$\Omega_{B/A} \oplus (B \otimes_{\mathbb{Z}} P^{\mathrm{gp}})$$

by the following relations:

$$(d\alpha(p), 0) - (0, \alpha(p) \otimes p) \quad \text{for all } p \in P \tag{8.2.3.3}$$

$$(0, 1 \otimes h(q)) \quad \text{for all } q \in Q. \tag{8.2.3.4}$$

The map $\Omega_{B/A} \to \Omega^{\dagger}_{B/A}$ is induced by inclusion in the first factor, and the universal log derivation is $(d_{B/A}, \mathrm{dlog}_{B/A})$, where we define

$$\begin{array}{rcl} \mathrm{dlog}_{B/A} : B \otimes_{\mathbb{Z}} P^{\mathrm{gp}} & \to & \Omega^{\dagger}_{B/A} \\ b \otimes p & \mapsto & [0, b \otimes p]. \end{array}$$

Given any A-linear log derivation (d, dlog), we get a commutative diagram as in the statement of the theorem by taking $d_{B/A}b \mapsto db$ for the top horizontal arrow (as usual), and

$$[d_{B/A}b, c \otimes p] \mapsto g(db) + \mathrm{dlog}(c \otimes p)$$

for the bottom arrow. This is well defined (kills the relations) by definition of a log derivation. Uniqueness follows from the fact that

$$\Omega_{B/A} \oplus (B \otimes_{\mathbb{Z}} P^{\mathrm{gp}}) \to \Omega^{\dagger}_{B/A}$$

is surjective by construction. □

We should really write $\Omega^{\dagger}_{(P \to B)/(Q \to A)}$ instead of $\Omega^{\dagger}_{B/A}$ to be clear about the prelog structures, but we continue using the simplified notation $\Omega^{\dagger}_{B/A}$ if the prelog structures on A, B are clear from context.

Lemma 8.2.4. *Formation of $\Omega^{\dagger}_{B/A}$ commutes with taking associated log structures on both A and B. That is, we have natural isomorphisms*

$$\begin{array}{rcl} \Omega^{\dagger}_{(P^a \to B)/(Q \to A)} & = & \Omega^{\dagger}_{(P \to B)/(Q \to A)} \\ \Omega^{\dagger}_{(P^a \to B)/(Q^a \to A)} & = & \Omega^{\dagger}_{(P^a \to B)/(Q \to A)}. \end{array}$$

Proof. Note $P^a = P \oplus_{\alpha^{-1}B^*} B^*$. It is routine to check that

$$\begin{array}{rcl} \Omega^{\dagger}_{(P^a \to B)/(Q \to A)} & \to & \Omega^{\dagger}_{(P \to B)/(Q \to A)} \\ [db, c \otimes [p, u]] & \mapsto & [db + cu^{-1}du, c \otimes p] \end{array}$$

is a well-defined inverse to the natural map

$$\begin{array}{rcl} \Omega^{\dagger}_{(P \to B)/(Q \to A)} & \to & \Omega^{\dagger}_{(P^a \to B)/(Q \to A)} \\ [db, c \otimes p] & \mapsto & [db, c \otimes [p, 1]]. \end{array}$$

For the second isomorphism, note that the only Q dependence in the definition of $\Omega^{\dagger}_{B/A}$ is in the relations (8.2.3.4). Obviously when Q is replaced with $Q^a = Q \oplus_{\beta^{-1}A^*} A^*$, we still have all the "old" relations (8.2.3.4). It remains to show that $[1 \otimes h^a([q, v])]$ is already zero in $\Omega^{\dagger}_{(P^a \to B)/(Q \to A)}$. Indeed, we compute using (8.2.3.3) that

$$\begin{array}{rcl} [0, 1 \otimes h^a([q, v])] & = & [0, 1 \otimes h(q)] + [0, 1 \otimes f(v)] \\ & = & 0 + f(v^{-1})[0, f(v) \otimes f(v)] \\ & = & f(v^{-1})[df(v), 0] \\ & = & 0. \end{array}$$

In writing $1 \otimes f(v) \in B \otimes_{\mathbb{Z}} (P^a)^{\mathrm{gp}}$ on the first line, we are using the fact that the log structure P^a contains the unit $f(v) \in B^*$. □

Lemma 8.2.5. *If $h : Q \to P$ induces an isomorphism $Q^a \cong P^a$ of associated log structures on B, then the natural map $\Omega_{B/A} \to \Omega^\dagger_{B/A}$ is an isomorphism.*

Proof. In light of Lemma 8.2.4, we can assume $P = P^a$. To say that $h^a : Q^a \to P$ is an isomorphism is to say that, for each $p \in P$, there is some $q \in Q$ and $u \in B^*$ with $p = h(q)u$, and that this choice is unique up to $(q, u) \mapsto (q + q', uf(\alpha(q'))^{-1})$ when $f(\alpha(q')) \in B^*$. It is then straighforward to check that the map

$$\begin{array}{rcl} \Omega^\dagger_{B/A} & \to & \Omega_{B/A} \\ [db, c \otimes p] & \mapsto & db + cu^{-1}du \end{array}$$

is well defined (does not depend on the choice made in writing $p = h(q)u$ and kills the relations (8.2.3.3), (8.2.3.4)), and this map clearly provides an inverse to the natural map $\Omega_{B/A} \to \Omega^\dagger_{B/A}$. ☐

If $X^\dagger \to Y^\dagger$ is a morphism of prelog ringed topoi, then we define $\Omega^\dagger_{X/Y}$ to be the sheaf of log differentials (on X) associated to the morphism

$$\begin{tikzcd} \mathcal{M}_X \arrow[r] & \mathcal{O}_X \\ f^{-1}\mathcal{M}_Y \arrow[r] \arrow[u, "f^\dagger"] & f^{-1}\mathcal{O}_Y \arrow[u, "f^\sharp"'] \end{tikzcd}$$

of prelog ring objects of X. In particular, we can define $\Omega^\dagger_{X/Y}$ in this manner for a morphism of log schemes $X^\dagger \to Y^\dagger$. In light of Lemma 8.2.4, we could equivalently define $\Omega^\dagger_{X/Y}$ to be the sheaf of log differentials associated to the morphism

$$\begin{tikzcd} \mathcal{M}^a_X \arrow[r] & \mathcal{O}_X \\ f^{-1}\mathcal{M}^a_Y \arrow[r] \arrow[u, "f^\dagger"] & f^{-1}\mathcal{O}_Y \arrow[u, "f^\sharp"'] \end{tikzcd}$$

of log ring objects of X.

Proposition 8.2.6. *If $f : X^\dagger \to Y^\dagger$ is a locally finite type (resp. locally finitely presented) morphism of fine log schemes, then $\Omega^\dagger_{X/Y}$ is a locally finitely generated (resp. locally finitely presented) $\mathcal{O}_X$-module.*

Proof. The question is étale local, and we know f has a chart étale locally, so we reduce to the case of a map (8.2.0.1) of prelog rings where B is finite type (resp. finitely presented) over A and P, Q are finitely generated monoids. In this case, the usual module $\Omega_{B/A}$ of differentials is a finitely generated (resp. finitely presented) B module, and $B \otimes_{\mathbb{Z}} P^{\text{gp}}$ is a finitely presented B module because the finitely generated abelian group P^{gp} is a finitely presented $\mathbb{Z}$ module. Evidently then,

$$\Omega_{B/A} \oplus (B \otimes_{\mathbb{Z}} P^{\text{gp}})$$

is a finitely generated (resp. finitely presented) B module. Since $\Omega^\dagger_{B/A}$ is the quotient of this module by the relations (8.2.3.3),(8.2.3.4) we reduce to showing that these relations can be generated by finitely many such relations. But it is clearly enough to impose (8.2.3.3),(8.2.3.4) as p, q run over a set of generators for P, Q. $\square$

Lemma 8.2.7. *Let*

$$\begin{tikzcd} P \arrow[r, "\alpha"] & B \\ Q \arrow[u, "h"] \arrow[r, "\beta"] & A \arrow[u, "f"'] \end{tikzcd}$$

be a morphism of prelog rings. Suppose that B is a local ring with residue field k and $\alpha : P \to B$ is a log structure. Then:

(1) The map

$$\begin{array}{rcl} B \otimes_{\mathbb{Z}} P^{\mathrm{gp}} & \to & \Omega^{\dagger}_{B/A} \\ a \otimes b & \mapsto & [0, a \otimes b] = a \cdot \operatorname{dlog} b \end{array}$$

is surjective.

(2) Let $R := P/(\alpha^{-1}B^* + h(Q))$ denote the relative characteristic monoid and let $\overline{p}$ denote the image of $p \in P^{\mathrm{gp}}$ in R^{gp}. There is a natural surjection

$$\begin{array}{rcl} k \otimes_B \Omega^{\dagger}_{B/A} & \to & k \otimes_{\mathbb{Z}} R^{\mathrm{gp}} \\ a \otimes [db, c \otimes p] & \mapsto & a\overline{c} \otimes \overline{p}, \end{array}$$

where $\overline{c} \in k$ denotes the image of $c \in B$ under $B \to k$.

Proof. (1) For the surjectivity of the first map, it is enough to show that a typical element $[df, a \otimes b]$ is in the image. But we can assume $f \in B^*$ since B is local, hence $1 + f \in B^*$ if f is not in B^*, and $d(1 + f) = df$. Then we have

$$[df, a \otimes b] = [0, a \otimes b] + [0, f \otimes f],$$

using the fact that P is a log structure, so the unit f "is in" P (by minor abuse of notation, we write $f \in P$ for the unique $p \in P$ with $\alpha(p) = f$). The terms $[0, a \otimes b]$ and $[0, f \otimes f]$ are clearly in the image of our map, hence so is $[df, a \otimes b]$.

(2) The only issue is to prove that the map is well-defined. That is, we want to show that the map

$$\begin{array}{rcl} F : k \otimes_B (\Omega_{B/A} \oplus (B \otimes_{\mathbb{Z}} P^{\mathrm{gp}}) & \to & k \otimes_{\mathbb{Z}} R^{\mathrm{gp}} \\ a \otimes (db, c \otimes p) & \to & a\overline{c} \otimes \overline{p} \end{array}$$

kills $k \otimes$ the relations (8.2.3.3), (8.2.3.4). To see that it kills $k \otimes$ (8.2.3.3) we need to prove that $a\overline{\alpha(p)} \otimes \overline{p}$ is zero for any $p \in P$, $a \in k$. If $\alpha(p)$ is in the maximal ideal of B, then $\overline{\alpha(p)} = 0$, so we're okay. Otherwise, $\alpha(p)$ is a unit in the local ring B, so $\overline{p}$ is zero, and we're okay. Certainly the relations $k \otimes$ (8.2.3.4) are killed since $\overline{h(q)} = 0$ in R for any $q \in Q$ by definition of R. $\square$

The sheaf of log differentials enjoys a property very much analogous to that of usual Kähler differentials:

Proposition 8.2.8. *Consider a commutative diagram of log rings*

$$\begin{tikzcd} (P \to B) \arrow[r] & (Q \to C) \\ (P' \to B') \arrow[r] \arrow[u] & (R \to D) \arrow[u] \end{tikzcd}$$

(in a topos) where $B \to B'$ is surjective with square zero kernel I and $(P' \to B) \to (P \to B)$ is strict, so we have an exact sequence

$$0 \to I \to P' \to P \to 0.$$

Then the set of maps g completing the diagram is a pseudo-torsor under $\operatorname{Hom}_B(\Omega^\dagger_{C/D} \otimes_C B, I)$.

Proof. □

8.3. **The chart criterion.** Because the following situation will arise frequently, it will be convenient to introduce an abbreviated notation. We agree to say that a commutative diagram of rings

$$\begin{tikzcd} B \arrow[r] & D \\ A \arrow[r] \arrow[u] & C \arrow[u] \end{tikzcd}$$

is *cocartesian up to an étale map* if the map $B \otimes_A C \to D$ is an étale map of rings.

In this section we will work only with fine, affine charts, so a *chart for a log scheme $X^\dagger$* is a prelog ring $P \to A$ with P a fine monoid, together with a strict map

$$h : \operatorname{Spec}(P \to A) \to X$$

of log schemes, so that $\operatorname{Spec} A \to X$ is an étale map of schemes. Strictness of the map means $h^\dagger : P \to \mathcal{M}_X(\operatorname{Spec} A)$ induces an isomorphism $P^a \cong \mathcal{M}_X|_{\operatorname{Spec} A}$. By definition, $X^\dagger$ is a fine log scheme iff it can be covered by such charts.

Similarly, a *chart for a morphism $f : X^\dagger \to Y^\dagger$* of log schemes is a morphism of prelog rings

$$\begin{tikzcd} P \arrow[r] & B \\ Q \arrow[r] \arrow[u] & A \arrow[u] \end{tikzcd}$$

such that there is a commutative diagram of log schemes

$$\begin{tikzcd} \operatorname{Spec}(Q \to A) \arrow[r] \arrow[d] & X^\dagger \arrow[d, "f"] \\ \operatorname{Spec}(P \to B) \arrow[r] & Y^\dagger \end{tikzcd}$$

where the top (resp. bottom) horizontal arrow is a chart for $X^\dagger$ (resp. $Y^\dagger$).

The following theorem gives a practical criterion for log étaleness/smoothness.

Theorem 8.3.1. *Let $f : X^\dagger \to Y^\dagger$ be a morphism of fine log schemes. The following are equivalent:*

(1) f is log étale (resp. log smooth).
(2) Any chart $\mathrm{Spec}(Q \to A) \to Y^\dagger$ *for $Y^\dagger$ lifts, locally on X, to a chart*

$$\begin{tikzcd} P \arrow[r] & B \\ Q \arrow[u, "h"] \arrow[r] & A \arrow[u, "f"'] \end{tikzcd}$$

for f satisfying the conditions:
(a) h is injective.
(b) $\mathrm{Cok}\, h^{\mathrm{gp}}$ *(resp. the torsion part of* $\mathrm{Cok}\, h^{\mathrm{gp}}$*) has order invertible in B.*
(c) The commutative diagram of rings

$$\begin{tikzcd} \mathbb{Z}[P] \arrow[r] & B \\ \mathbb{Z}[Q] \arrow[u, "\mathbb{Z}[h]"] \arrow[r] & A \arrow[u, "f"'] \end{tikzcd}$$

is cocartesian up to an étale map.

If, furthermore, f is an integral morphism, then it can be arranged that h is an integral morphism.

Proof. □

Corollary 8.3.2. *If $f : X^\dagger \to Y^\dagger$ is a log smooth, integral morphism of fine log schemes, then the underlying morphism of schemes $f : X \to Y$ is flat.*

Proof. Flatness is étale local, so by the theorem it is enough to prove that a ring map $A \to B$ is flat when there is a cocartesian diagram of rings

$$\begin{tikzcd} \mathbb{Z}[P] \arrow[r] & B \\ \mathbb{Z}[Q] \arrow[u, "\mathbb{Z}[h]"] \arrow[r] & A \arrow[u, "f"'] \end{tikzcd}$$

with $h : Q \to P$ an integral monomorphism of integral monoids. This follows from the fact that flatness is preserved under pushouts (tensor products) of rings, and the fact that $\mathbb{Z}[Q] \to \mathbb{Z}[P]$ is flat by Proposition 1.10.4. □

9. Rounding

9.1. **Kato-Nakayama spaces.** In this section, we work in the category **CLRS** of "continuous locally ringed spaces over $\mathbb{C}$". By definition, **CLRS** is the full subcategory of the category **LRS**/$\mathbb{C}$ of locally ringed spaces over $\mathbb{C}$ whose objects are those $(X, \underline{\mathbb{C}}_X \to \mathscr{O}_X)$ satisfying the following two conditions:

(1) Every point of x is a $\mathbb{C}$-point, in the sense that the composition $\mathbb{C} \to \mathscr{O}_{X,x} \to k(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x$ yields a natural isomorphism $k(x) = \mathbb{C}$.

(2) Sections of $\mathscr{O}_X$ are continuous in the sense that for any open $U \subseteq X$ and any $f \in \mathscr{O}_X(U)$, the function $U \to \mathbb{C}$ given by $x \mapsto f(x)$ (here $f(x)$ denotes the image of the stalk f_x in the fiber $k(x) = \mathbb{C}$ using the identification from (1)) is continuous when $\mathbb{C}$ is given the metric topology.

One can prove the following[8]:

(1) The fibered product of **CLRS** morphisms $f_i : X_i \to Y$ $(i = 1, 2)$, taken in **LRS**, yields a locally ringed space $X_1 \times_Y X_2$ over $\mathbb{C}$ which is in **CLRS** and is hence the fibered product in **CLRS** as well.
(2) The formation of fibered products in **CLRS** commutes with the forgetful functor from **CLRS** to topological spaces.

The category **CLRS** contains the category of analytic spaces as a full subcategory. Let **LogCLRS** denote the category whose objects are pairs $(\underline{X}, \mathcal{M}_X)$ where $\underline{X} \in$ **CLRS** and $\mathcal{M}_X$ is a log structure on $\underline{X}$. To save notation, we will typically write X for the pair $(\underline{X}, \mathcal{M}_X)$ and reserve the notation $\underline{X}$ for the underlying $\underline{X} \in$ **CLRS**. By definition, a morphism $f = (\underline{f}, f^\dagger) : X \to Y$ in **LogCLRS** is a pair consisting of a **CLRS** morphism $\underline{f} : \underline{X} \to \underline{Y}$ and a morphism $f^\dagger : \underline{f}^* \mathcal{M}_Y \to \mathcal{M}_X$ of log structures on $\underline{X}$.

The category **LogCLRS** has fibered products (in fact, it has all finite inverse limits), and the fibered product of $f_i : X_i \to Y$ $(i = 1, 2)$ is constructed in the following "obvious" manner: We first let $\underline{X_1} \times_{\underline{Y}} \underline{X_2}$ be the fibered product of $\underline{f}_1, \underline{f}_2$ in **CLRS** (which coincides with their fibered product in locally ringed spaces), and we let $\underline{\pi}_i : \underline{X_1} \times_{\underline{Y}} \underline{X_2} \to \underline{X_i}$ be the natural projections. Then we set $g := \underline{f}_1 \underline{\pi}_1 = \underline{f}_2 \underline{\pi}_2$, and we endow $\underline{X_1} \times_{\underline{Y}} \underline{X_2}$ with the log structure $\mathcal{N}$ defined by the pushout diagram:

$$\begin{tikzcd} g^* \mathcal{M}_Y \arrow[r, "\underline{\pi}_2^* f_2^\dagger"] \arrow[d, "\underline{\pi}_1^* f_1^\dagger"'] & \underline{\pi}_2^* \mathcal{M}_{X_2} \arrow[d] \\ \underline{\pi}_1^* \mathcal{M}_{X_1} \arrow[r] & \mathcal{N} \end{tikzcd}$$

Then it is easy to see that $X_1 \times_Y X_2 := (\underline{X}_1 \times_{\underline{Y}} \underline{X}_2, \mathcal{N})$ is the fibered product of f_1, f_2 in **LogCLRS**.

Given $X \in$ **LogCLRS**, the *Kato-Nakayama space* X^{KN} of X is, at the very least, a map of topological space $\tau : X^{\mathrm{KN}} \to X$. Here we write X as abuse of notation for the topological space underlying $\underline{X}$. We will discuss various additional structure of τ later. The Kato-Nakayama space X^{KN} is constructed as follows. As a set, X^{KN} is the set of pairs (x, f) where $x \in \underline{X}$, and $f : \mathcal{M}_{X,x} \to S^1$ is a monoid homomorphism satisfying:

9.1.1. For every $u \in \mathscr{O}_{X,x}^* \subseteq \mathcal{M}_{X,x}$, $f(u) = u(x)/|u(x)|$, where $u(x) \in \mathbb{C}^*$ is the image of the stalk $u_x \in \mathscr{O}_{X,x}$ in the residue field $k(x) = \mathbb{C}$ as in (2).

Remark 9.1.1. We can define the set X^{KN} even when $\alpha_X : \mathcal{M}_X \to \mathscr{O}_X$ is merely a *prelog* structure by replacing the condition (9.1.1) with the condition: For every $m \in \mathcal{M}_{X,x}$ with $\alpha_{X,x}(m) \in \mathscr{O}_{X,x}^*$, we have $f(m) = (\alpha_{X,x}(m))(x)/|(\alpha_{X,x}(m))(x)|$. If we pass to the associated log structure $\mathcal{M}_X^a = \mathcal{M}_X \oplus_{\alpha_X^{-1}\mathscr{O}_X^*} \mathscr{O}_X^*$, we obtain a natural bijection

[8]See, for example, Remark 10 in my article "Localization of ringed spaces."

$(X, \mathcal{M}_X)^{\mathrm{KN}} = (X, \mathcal{M}_X^a)$ as follows: We have $\mathcal{M}^a_{X,x} = \mathcal{M}_{X,x} \oplus_{\alpha^{-1}_{X,x}\mathscr{O}^*_{X,x}} \mathscr{O}^*_{X,x}$, so, given $f : \mathcal{M}_{X,x} \to S^1$ satisfying the condition above, we obtain $f^a : \mathcal{M}^a_{X,x} \to S^1$ satisfying (9.1.1) by setting $f^a([m,u]) := u(x)f(m)$ using the universal property of the pushout. The map is bijective because and $g = (f,t) : \mathcal{M}^a_{X,x} \to S^1$ satisfying (9.1.1) must be of this form (i.e. $t : \mathscr{O}^*_{X,x} \to S^1$ must be the map $t(u) = u(x)/|u(x)|$).

Remark 9.1.2. Consider the (non-integral) *Kato-Nakayama* log structure $S^1 \times_{\mathbb{R}_{\geq 0}} \to \mathbb{C}$ given by the multiplication map, where we view S^1 as the subset of $\mathbb{C}$ consisting of complex numbers of magnitude 1. The set X^{KN} can be interpreted as the set of maps of log locally ringed spaces over $\mathbb{C}$ from $\operatorname{Spec}\mathbb{C}$ with the Kato-Nakayama log structure to X. Indeed, such a map is the same thing as a choice of point $x \in X$, and monoid homomorphism

$$(f,g) : \mathcal{M}_{X,x} \to S^1 \times \mathbb{R}_{\geq 0}$$

commuting with the natural maps to $\mathbb{C}$. This forces g to be given by $m \mapsto |\alpha_X(m)(x)|$, and the data of such a monoid homomorphism is equivalent to a monoid homomorphism f satisfying (9.1.1). This also makes it clear that X^{KN} can be defined equally well when $\alpha_X : \mathcal{M}_X \to \mathscr{O}_X$ is merely a prelog structure as in the previous remark.

We have an obvious forgetful map

$$\begin{array}{rcl} \tau : X^{\mathrm{KN}} & \to & X \\ (x,f) & \mapsto & x. \end{array}$$

Given any open $U \subseteq X$ and any $m \in \mathcal{M}_X(U)$, we have an obvious function, named by abuse of notation,

$$\begin{array}{rcl} m : U^{\mathrm{KN}} & \to & S^1 \\ (x,f) & \mapsto & f(m_x). \end{array} \tag{9.1.2.1}$$

Observe that if $m \in \mathcal{M}_X(U)$ happens to be in $\mathscr{O}^*_X(U) \subseteq \mathcal{M}_X(U)$, then the function m of (9.1.2.1) is the composition of:

(1) $\tau : U^{\mathrm{KN}} \to U$,
(2) the function $m : U \to \mathbb{C}^*$ given by $x \mapsto m(x)$ (which is continuous because $X \in \mathbf{CLRS}$) and
(3) the continuous function $\mathbb{C}^* \to S^1$ given by $z \mapsto z/|z|$.

We give X^{KN} the smallest topology making the maps τ and m continuous. This topology can be explicitly described as follows: Given an open subset $U \subseteq X$, a list of sections $\overline{m} = (m_1, \dots, m_n) \in \mathcal{M}_X(U)^n$ (for some $n > 0$), and a basic open subset $V = V_1 \times \cdots \times V_n$ of $(S^1)^n$, we let

$$\mathbf{U}(U, \overline{m}, V) \;\; := \;\; \{(x,f) \in X^{\mathrm{KN}} : x \in U, f(\overline{m}_x) \in V\}.$$

If we view $\overline{m}$ as a map

$$\begin{array}{rcl} U^{\mathrm{KN}} & \to & (S^1)^n \\ x & \mapsto & f(\overline{m}_x) \end{array}$$

(the product of the maps (9.1.2.1)), then

$$\mathbf{U}(U, \overline{m}, V) \;\; = \;\; \overline{m}^{-1}(V).$$

Notice that

$$\mathbf{U}(U,\overline{m},V)\cap \mathbf{U}(U',\overline{m}',V') \quad = \quad \mathbf{U}(U\cap U',\overline{m}|_{U\cap U'}\overline{m}'|_{U\cap U'},V\times V')$$

so these $\mathbf{U}(U,\overline{m},V)$ form a basis for the topology of X^{KN}.

Example 9.1.3. Suppose P is a (finitely generated) monoid, and X is the log analytic space associated to the finite type log scheme $\operatorname{Spec}(P\to\mathbb{C}[P])$ over $\mathbb{C}$. The points of X are the $\mathbb{C}$ points of $\operatorname{Spec}\mathbb{C}[P]$, so we have a natural bijection $X = \operatorname{Hom}_{\mathbf{Mon}}(\mathbb{C})$. Recall (Remark 9.1.2) the Kato-Nakayama log structure $\alpha : S^1\times\mathbb{R}_{\geq 0}\to\mathbb{C}$. The space X^{KN} is naturally identified with $\operatorname{Hom}_{\mathbf{Mon}}(P,S^1\times\mathbb{R}_{\geq 0})$, and the natural map $\tau : X^{\mathrm{KN}}\to X$ is naturally identified with

$$\alpha_* : \operatorname{Hom}_{\mathbf{Mon}}(P,S^1\times\mathbb{R}_{\geq 0})\to\operatorname{Hom}_{\mathbf{Mon}}(P,\mathbb{C}).$$

Suppose $\mathcal{M}_X$ has a chart $h : P\to\mathcal{M}_X(X)$ with P a finitely generated monoid. Choose generators $p_1,\dots,p_k\in P$. Then it is easy to see that any monoid homomorphism $f:\mathcal{M}_{X,x}\to S^1$ satisfying (9.1.1) is determined by $f((p_1)_x),\dots,f((p_k)_x)$ (we suppress notation for h throughout), so we obtain a monic set map

$$\begin{array}{rcl} e : X^{\mathrm{KN}} & \to & X\times(S^1)^k \\ (x,f) & \mapsto & (x,f((p_1)_x),\dots,f((p_k)_x)). \end{array}$$

I claim that this map is a closed embedding. It is certainly continuous since $e^{-1}(V) = \mathbf{U}(X,\overline{p},V)$, where $\overline{p}=(p_1,\dots,p_k)\in\mathcal{M}_X(X)^k$. To see that e is an embedding, consider a basic open set $\mathbf{U}(U,\overline{m},V)$ of X^{KN}. For each point $x\in U$, the fact that h is a chart means we can find a neighborhood U_x of x (contained in U) where $m_i|U_x = u_i\overline{p}^{\overline{e}_i}$ for some $\overline{e}_i\in\mathbb{N}^k$ and some $u_i\in\mathscr{O}_X^*(U_x)\subseteq\mathcal{M}_X(U_x)$. There are only finitely many m_i, so we can assume, as the notation suggests, that there is a single U_x where we can do this for $m_1,\dots,m_n$. The function

$$\begin{array}{rcl} u_x : U_x\times(S^1)^k & \to & U_x\times(S^1)^n \\ (x,\overline{\lambda}) & \mapsto & (x,u_1(x)\overline{\lambda}^{\overline{e}_1},\dots,u_n(x)\overline{\lambda}^{\overline{e}_n})) \end{array}$$

is clearly continuous, and, since the maps $f:\mathcal{M}_{X,x}\to S^1$ in the definition of X^{KN} are monoid homomorphisms satisfying (9.1.1), the diagram

$$\begin{tikzcd} U_x^{\mathrm{KN}} \arrow[rr, "e|U_x"] \arrow[dr, "\mathrm{Id}\times\overline{m}|U_x"'] & & U_x\times(S^1)^k \arrow[dl, "u_x"] \\ & U_x\times(S^1)^n & \end{tikzcd}$$

commutes. We then compute

$$\begin{aligned}
\mathbf{U}(U,\overline{m},V) &= \bigcup_{x\in U} \mathbf{U}(U_x,\overline{m}|U_x,V) \\
&= \bigcup_{x\in U} (\mathrm{Id}\times\overline{m}|U_x)^{-1}(U_x\times V) \\
&= \bigcup_{x\in U} (e|U_x)^{-1}(u_x^{-1}(U_x\times V)) \\
&= \bigcup_{x\in U} e^{-1}(u_x^{-1}(U_x\times V)) \\
&= e^{-1}\left(\bigcup_{x\in U} u_x^{-1}(U_x\times V)\right),
\end{aligned}$$

which shows that the basic open set $\mathbf{U}(U,\overline{m},V)$ is the preimage, under e, of an open subset of $X\times(S^1)^k$. This proves that e is an embedding. The map e is then a *closed* embedding because a point $(x,\overline{y})$ arises as $e(x,f)$ for some (necessarily unique) f iff $f((p_i)_x) := y_i$ actually defines a monoid homomorphism $f:\mathcal{M}_{X,x}\to S^1$ satisfying (9.1.1), and this is an intersection of closed conditions on the y_i (namely, the conditions that f should respect the relations among the p_i and behave properly on units). This basically proves:

Theorem 9.1.4. *If $X\in$ **CLRS** has a coherent log structure $\mathcal{M}_X$, then $\tau: X^{\mathrm{KN}}\to X$ is a proper map of topological spaces.*

Proof. The question is local on X, to we can assume $\mathcal{M}_X$ has a chart $h:P\to\mathcal{M}_X(X)$ with P finitely generated. But then the discussion above shows that τ factors as a closed embedding followed by the projection $\pi_1: X\times(S^1)^k\to X$, hence τ is proper. $\square$

The construction of X^{KN} is functorial in $X\in$ **LogCLRS**, so we may view it as a functor

$$\begin{aligned}
\mathrm{KN}:\mathbf{LogCLRS} &\to \mathbf{Top} \\
X &\mapsto X^{\mathrm{KN}} \\
(f:X\to Y) &\mapsto (f^{\mathrm{KN}}:X^{\mathrm{KN}}\to Y^{\mathrm{KN}})
\end{aligned}$$

defined as follows. Given a **LogCLRS** morphism $f:X\to Y$, we have an obvious map of topological spaces $f^{\mathrm{KN}}:X^{\mathrm{KN}}\to Y^{\mathrm{KN}}$ by setting $f^{\mathrm{KN}}(x,f):=(f(x),f\circ f_x^\dagger)$, using the natural map $f_x:\mathcal{M}_{Y,f(x)}\to\mathcal{M}_{X,x}$. To see that f^{KN} is continuous, it is enough to check that $(f^{\mathrm{KN}})^{-1}(\mathbf{U}(U,\overline{m},V))$ is open in X for a basic open subset $\mathbf{U}(U,\overline{m},V)$ of Y. But it is clear from the definitions that

$$(f^{\mathrm{KN}})^{-1}(\mathbf{U}(U,\overline{m},V)) = \mathbf{U}(f^{-1}(U),f^\dagger(f^{-1}\overline{m}),V),$$

where $f^{-1}\overline{m}$ is slight abuse of notation for the image of $f^{-1}\overline{m}\in(f^{-1}\mathcal{M}_Y)(f^{-1}U)^n$ in $(f^*\mathcal{M}_Y)(f^{-1}U)^n$.

Proposition 9.1.5. *The functor* KN *preserves finite inverse limits.*

Proof. Let $i\mapsto X_i$ be a finite inverse limit system in **LogCLRS** with inverse limit X and structure maps $\pi_i:X\to X_i$. For $x\in\underline{X}$, set $x_i:=\underline{\pi}_i(x)\in\underline{X}_i$ to save notation. Note that the log structure $\mathcal{M}_X$ is the direct limit (in the category of log structures on $\mathscr{O}_X$) of the $\underline{\pi}_i^*\mathcal{M}_{X_i}$, and $\underline{\pi}_i^*\mathcal{M}_{X_i}$ is the log structure associated to the prelog structure

$$\pi_i^\sharp \circ \pi_i^{-1}\alpha_{X_i} : \pi_i^{-1}\mathcal{M}_{X_1} \to \mathscr{O}_X.$$

Since formation of associated log structues commutes with direct limits, $\mathcal{M}_X$ is the log structure associated to the prelog structure $\mathcal{M}_X^{\mathrm{pre}} := \varinjlim \underline{\pi}_i^{-1}\mathcal{M}_{X_i}$, and the map $\pi_i^\dagger : \underline{\pi}_i^*\mathcal{M}_{X_i} \to \mathcal{M}_X$ is just the composition of the structure map to the direct limit and the natural map $\mathcal{M}_X^{\mathrm{pre}} \to \mathcal{M}_X$. As in Remark 9.1.1, we can define X^{KN} using $\mathcal{M}_X^{\mathrm{pre}}$ instead of $\mathcal{M}_X$. Note $\mathcal{M}_{X,x}^{\mathrm{pre}} = \varinjlim \mathcal{M}_{X_i,x_i}$, so a map $f : \mathcal{M}_{X,x}^{\mathrm{pre}} \to S^1$ is the same thing as a collection of maps $f_i : \mathcal{M}_{X,x_i} \to S^1$ respecting the transition maps in our limit system. The main thing to prove is that f satisfies the condition in Remark 9.1.1 iff this condition holds for each of the f_i; this will prove that the natural map

$$\begin{array}{rcl} X^{\mathrm{KN}} & \to & \varprojlim X_i^{\mathrm{KN}} \\ (x, f) & \mapsto & (x_i, f_i)_i \end{array}$$

is bijective. To prove this, consider the commutative diagrams:

$$\begin{tikzcd} \mathcal{M}_{X,x}^{\mathrm{pre}} \arrow[r, "f"] & S^1 \\ \mathcal{M}_{X_i,x_i} \arrow[u, "j_i"] \arrow[ur, "f_i"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{M}_{X,x}^{\mathrm{pre}} \arrow[r, "\alpha_{X,x}"] & \mathscr{O}_{X,x} \\ \mathcal{M}_{X_i,x_i} \arrow[u, "j_i"] \arrow[r, "\alpha_{X_i,x_i}"] & \mathscr{O}_{X_i,x_i} \arrow[u, "\pi_{i,x}"'] \end{tikzcd}$$

Suppose f satisfies the condition in Remark 9.1.1 and consider an element $m \in \mathcal{M}_{X_i,x_i}$ with $\alpha_{X_i,x_i}(m) \in \mathscr{O}_{X_i,x_i}^*$. We want to prove that $f_i(m) = \alpha_{X_i,x_i}(m)(x_i)/|\alpha_{X_i,x_i}(m)(x_i)|$. The commutativity of the right diagram ensures $\alpha_{X,x}(j_i(m)) \in \mathscr{O}_{X,x}^*$, so since f satisfies the condition, we know $f_i(m) = f(j_i(m)) = \alpha_{X,x}(j_i(m))(x)/|\alpha_{X,x}(j_i(m))(x)|$. But

$$\alpha_{X_i,x_i}(m)(x_i) = \alpha_{X,x}(j_i(m))(x)$$

because the right diagram commutes and $\pi_{i,x}$ is a local morphism of local $\mathbb{C}$ algebras with residue field $\mathbb{C}$. This proves that f_i satisfies the condition in Remark 9.1.1. Conversely, suppose each f_i satisfies this condition, and consider an element $m = \sum_i j_i(m_i) \in \mathcal{M}_{X,x}^{\mathrm{pre}}$ with $\alpha_{X,x}(m) \in \mathscr{O}_{X,x}^*$. Since

$$\alpha_{X,x}(m) = \prod_i \pi_{i,x}(\alpha_{X_i,x_i}(m_i))$$

it must be that each $\pi_{i,x}(\alpha_{X_i,x_i}(m_i))$ is in $\mathscr{O}_{X,x}^*$. But each $\pi_{i,x}$ is a local map (of local $\mathbb{C}$ algebras with residue field $\mathbb{C}$), so each $\alpha_{X_i,x_i}(m_i)$ is in $\mathscr{O}_{X_i,x_i}^*$, hence $f_i(m_i) = \alpha_{X_i,x_i}(m)(x_i)/|\alpha_{X_i,x_i}(m)(x_i)|$ and hence

$$\begin{array}{rcl} f(m) & = & \prod_i f_i(m_i) \\ & = & \prod_i \alpha_{X_i,x_i}(m)(x_i)/|\alpha_{X_i,x_i}(m)(x_i)| \\ & = & \alpha_{X,x}(m)(x)/|\alpha_{X,x}(m)(x)|, \end{array}$$

thus we see that f satisfies the condition of Remark 9.1.1.

It remains only to prove that the continuous bijection $X^{\mathrm{KN}} \to \varprojlim X_i^{\mathrm{KN}}$ is an open map. Note that the topological space inverse limit $\varprojlim X_i^{\mathrm{KN}}$ inherits its topology from the product topology on $\prod_i X_i^{\mathrm{KN}}$. Consider a basic open subset $\mathbf{U}(U, \overline{m}, V) \subseteq X^{\mathrm{KN}}$, $\overline{m} \in \mathcal{M}_X(U)^n$. We want to show that its image in $\varprojlim X_i^{\mathrm{KN}}$ is open. It is enough to show that, for a typical point $y = (y_i) \in U$, after possibly shrinking U to a smaller neighborhood of y, the function

$$\begin{aligned} \overline{m} : U^{\mathrm{KN}} &\to (S^1)^n \\ (x, f) &\mapsto \overline{m}(f_x) \end{aligned} \tag{9.1.5.1}$$

is the restriction of a continuous function $\prod_i U_i^{\mathrm{KN}} \to (S^1)^n$ (for the product topology) for some open neighborhoods U_i of y_i in $\underline{X}_i$. Since $\mathcal{M}_X$ is the log structure associated to the prelog structure $\varinjlim \underline{\pi}_i^{-1} \mathcal{M}_{X_i}$, we can find, after possibly shrinking U, neighborhoods U_i of y_i in $\underline{X}_i$, sections $\overline{b}_i \in \mathcal{M}_{X_i}(U_i)^n$ and units $\overline{u} \in \mathscr{O}_X^*(U)^n$ such that

$$\overline{m}|U = [\sum_i \overline{b}_i, \overline{u}]. \tag{9.1.5.2}$$

Here we are viewing $\mathcal{M}_X$ as the sheafification of

$$U \mapsto \varinjlim \underline{\pi}_i^{-1} \mathcal{M}_{X_i}(U) \oplus_{A(U)} \mathscr{O}_X^*(U),$$

where $A(U)$ consists of the sections $\varinjlim \underline{\pi}_i^{-1} \mathcal{M}_{X_i}$ over U that map to a unit in $\mathscr{O}_X^*(U)$. In (9.1.5.2) we suppress notation for passing from $\overline{b}_i \in \mathcal{M}_{X_i}(U_{x,i})^n$ to the restriction of $\underline{\pi}_i^{-1} \overline{b}_i \in (\underline{\pi}_i^{-1} \mathcal{M}_{X_i})(\underline{\pi}_i^{-1} U_{X,i})$ to $(\underline{\pi}_i^{-1} \mathcal{M}_{X_i})(U_x)$, and we suppress notation for the structure maps to the direct limit.

The function (9.1.5.1) can then be written as the composition of

$$\begin{aligned} U^{\mathrm{KN}} &\to \left(\prod_i (S^1)^n\right) \times (\mathbb{C}^*)^n \\ (x, f) &\mapsto (\overline{b}_i(f_x), \overline{u}(x)) \end{aligned}$$

and the continuous map

$$\begin{aligned} \left(\prod_i (S^1)^n\right) \times (\mathbb{C}^*)^n &\to (S^1)^n \\ ((\overline{\lambda}_i), \overline{z}) &\mapsto (\prod_i \overline{\lambda}_i)\overline{z}/|\overline{z}| \end{aligned}$$

(all operations coordinatewise in the overlined variables) so it is enough to show that the functions

$$\begin{aligned} U^{\mathrm{KN}} &\to \prod_i (S^1)^n \\ (x, f) &\mapsto \overline{b}_i(f_x) \end{aligned} \tag{9.1.5.3}$$

and

$$\begin{aligned} U^{\mathrm{KN}} &\to (\mathbb{C}^*)^n \\ x &\mapsto \overline{u}(x) \end{aligned} \tag{9.1.5.4}$$

are, after possibly shrinking U, restrictions of continuous functions on the product of neighborhoods of the y_i. The function (9.1.5.3) is just the restriction of

$$\prod_i \overline{b}_i : \prod_i U_i^{\mathrm{KN}} \to \prod_i (S^1)^n,$$

so it remains only to deal with (9.1.5.4). Now, the local ring $\mathscr{O}_{X,y}$ is not equal to $\varinjlim \mathscr{O}_{X_i,y_i}$ (this direct limit might not even be a local ring!). Rather, $\mathscr{O}_{X,y}$ is the localization of $\varinjlim \mathscr{O}_{X_i,y_i}$ at the multiplicative set consisting of those $\sum_i a_i \in \varinjlim \mathscr{O}_{X_i,y_i}$ where $\sum_i a_i(y_i) \in \mathbb{C}^*$. So, after possibly shrinking U and the U_i, we can assume that each coordinate of $\overline{u}$ takes the form

$$\frac{\sum_i a_i}{\sum_i b_i}$$

for some $a_i, b_i \in \mathscr{O}_{X_i}(U_i)$ with

$$\frac{\sum_i a_i(x_i)}{\sum_i b_i(x_i)} \in \mathbb{C}^*$$

for all $x = (x_i) \in \prod_i U_i$. Since the a_i and b_i are continous on U_i, and this coordinate of $\overline{u}$ is built from them out of continuous operations, we can express (9.1.5.4) as the restriction of a continuous function on $\prod_i U_i^{\mathrm{KN}}$ (in fact, as a continuous function pulled back from $\prod_i U_i$).

□

9.2. **Oriented real blowup.**

9.3. **The Nakayama-Ogus theorem.**

10. Log Curves

In this section, we agree that a morphism of schemes $f : C \to X$ is a *curve* if f is flat, separated, finitely presented, and every geometric fiber of f is a reduced 1-dimensional scheme. If, furthermore, every geometric fiber of f has at worst nodal singularities (étale locally $k[x,y]/xy$), then f is a *nodal curve* (or *prestable curve*). One is usually interested in the case where f is also proper with connected fibers, but we will be concerned here with local structure, so we do not need this hypothesis at the moment.

A *log curve* (in the sense of [F. Kato]) is a smooth, integral morphism of fs log schemes $f : C^\dagger \to X^\dagger$ such that the underlying morphism of schemes is a curve.

Exercise 10.0.1. Let k be a field, not of characteristic 2, let $f := y^2 - x^2(x+1) \in k[x,y]$, and let $C = Z(f)$ be the nodal cubic curve in $\mathbb{A}^2 := \mathbb{A}^2_k$. This is the standard example of a normal crossings divisor which is not normal crossings in the Zariski topology.

(1) Show that f is irreducible in the ring $k[x,y]_{(x,y)}$.
(2) Let $V = \operatorname{Spec} k[x,y,u]_{x+1}/\langle u^2 - x - 1\rangle$ and show that (Spec of)
$$k[x,y]_{x+1} \to k[x,y,u]_{x+1}/\langle u^2 - x - 1\rangle$$
is an étale cover of the complement of the line $x = -1$ in $\mathbb{A}^2$.
(3) Show that (the image of) f under the above map can be written as a product $(y + xu)(y - xu)$ in the codomain ring $\Gamma(V, \mathscr{O}_V)$, so that f is reducible in the étale local ring of $\mathbb{A}^2$ at the origin.

(4) Show that V is isomorphic to a Zariski neighborhood of the origin in $\operatorname{Spec} k[z,w]/\langle zw \rangle$.
(5) Show that $C^{\text{sing}} \cong \operatorname{Spec} k$ is the closed subscheme of C defined by the ideal $\langle x, y\rangle \subset k[x,y]/f$ (i.e. the origin), then show that the origin is a Weil divisor in C which is not Cartier. Show that it is $\mathbb{Q}$-Cartier if you want.
(6) Let $\mathcal{M}$ be the log structure on $\mathbb{A}^2_{\text{ét}}$ associated to C:

$$(g : U \to \mathbb{A}^2) \mapsto \{s \in \Gamma(U, \mathscr{O}_U) : s|_{U \setminus C} \in \Gamma(U \setminus C, \mathscr{O}_{U \setminus C})^*\}.$$

Here we write C as abuse of notation for the closed subscheme of U defined by the inverse image ideal sheaf $g^{-1}\langle f \rangle \cdot \mathscr{O}_U$ or the closed subscheme $U \times_{\mathbb{A}^2} C$ (these correspond since g is étale, hence flat). Let $U \to \mathbb{A}^2$ be the étale map from (2), which covers all of $\mathbb{A}^2$ except the line $x = -1$. Show that

$$(y + xu, y - xu) : \mathbb{N}^2 \to \mathcal{M}(U)$$

is a chart for $\mathcal{M}|_U$.
(7) Show that $(f) : \mathbb{N} \to \mathcal{M}(V)$ is a chart for $\mathcal{M}|_V$ for V equal to the complement of the origin. Conclude that $\mathcal{M}$ is fine.
(8) Show that the pullback of $\mathcal{M}$ to the origin is the log structure associated to the chart

$$\mathbb{N}^2 \xrightarrow{0} \mathbb{Z}\ .$$

(9) Let $\mathcal{N}$ denote the restriction of $\mathcal{M}$ to the Zariski site of $\mathbb{A}^2$. Show that

$$(f) : \mathbb{N} \to \mathcal{M}(\mathbb{A}^2)$$

is a global chart for $\mathcal{N}$. In particular, the pullback of $\mathcal{N}$ to the origin is the log structure associated to the chart

$$\mathbb{N} \xrightarrow{0} \mathbb{Z}\ .$$

(10) Conclude that the pullback of $\mathcal{N}$ to the étale site of $\mathbb{A}^2$ differs from $\mathcal{M}$, so that the adjunction map $\mathcal{M}|^{\text{ét}}_{\text{Zar}} \to \mathcal{M}$ is not an isomorphism.

Exercise 10.0.2. Consider the family of cubic curves

$$C = \operatorname{Spec} \mathbb{Z}[x,y,t]/y^2 - x(x+t)(x+1) \to \mathbb{A}^1_t.$$

(1) Show that $C/\mathbb{A}^1$ is a nodal curve with singular fibers at $t = 0, 1$ and that C is smooth over $\operatorname{Spec} \mathbb{Z}$.
(2) Show that the two singular fibers are normal crossings divisors in C. Let $\mathcal{M}_C$ be the corresponding log structure.
(3) Let $\mathcal{M}_{\mathbb{A}^1}$ be the log structure associated to the divisor $t(t-1)$. Show $(t, t-1) : \mathbb{N}^2 \to \mathcal{M}_{\mathbb{A}^1}(\mathbb{A}^1)$ is a global chart for $\mathcal{M}_{\mathbb{A}^1}$.
(4) Show that $(C, \mathcal{M}_C)/(\mathbb{A}^1, \mathcal{M}_{\mathbb{A}^1})$ is a log curve (i.e. log smooth), but that this

10.1. **Log curves are nodal.**

Proposition 10.1.1. *The curve underlying a log curve is a nodal curve.*

10.2. **Universal log structures.** It turns out that, for any nodal curve $f : C \to X$, there are canonical log structures $\mathcal{M}_C, \mathcal{M}_X$ on C and X making $C^\dagger/X^\dagger$ a vertical log curve. Furthermore, these canonical log structures are universal in the sense that if $\mathcal{M}'_C, \mathcal{M}'_X$ are any other such log structures with a morphism

$$(C, \mathcal{M}_C) \to (X, \mathcal{M}_X)$$

which is a vertical log curve, then there is a unique map $\mathcal{M}_X \to \mathcal{M}'_X$ such that

$$\begin{tikzcd}
(C, \mathcal{M}'_C) \arrow[r] \arrow[d] & (C, \mathcal{M}_C) \arrow[d] \\
(X, \mathcal{M}'_X) \arrow[r] & (X, \mathcal{M}_X)
\end{tikzcd}$$

is a cartesian diagram of log schemes. That is:

$$\mathcal{M}'_C = (\mathcal{M}_C \oplus_{f^*\mathcal{M}_X} f^*\mathcal{M}'_X)^a.$$

The canonical log structure has the property that $\overline{M}_{X,x} \cong \mathbb{N}^n$, where n is the number of nodes in the curve C_x. Furthermore, there is a natural bijection between the irreducible elements of $\overline{M}_{X,x}$ and the nodes of C_x, given as follows. For any irreducible $e \in \overline{M}_{X,x}$, it turns out that there is a unique point $y \in C_x$ such $\overline{f}_y(e) \in \overline{M}_{C,y}$ is reducible; this point y is the node corresponding to e.

To construct the canonical log structure, recall that, for any nodal curve $C \to X$, and any nodal point $c \in C$, we can find a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} A[x,y]/\langle xy - t\rangle \arrow[r] \arrow[d] & C \arrow[d, "f"] \\
\operatorname{Spec} A \arrow[r] & X
\end{tikzcd}$$

where the horizontal arrows are étale neighborhoods of $c, f(c)$. Locally, one then defines the canonical log structures $\mathcal{M}_X, \mathcal{M}_C$ to be those associated to the charts:

$$\begin{tikzcd}
\mathbb{N}^2 \arrow[r, "{u,v}"] & \mathscr{O}_C(U)/\langle xy - t\rangle \\
\mathbb{N} \arrow[u, "\Delta"'] \arrow[r, "t"] & \mathscr{O}_X \arrow[u, leftrightarrow, "f"']
\end{tikzcd}$$

The issue is then to prove that these glue. The key point is the following lemma.

Lemma 10.2.1. *Let $(A, \mathfrak{m})$ be a strictly Henselian local ring, $t \in \mathfrak{m}$. Let B be the strict Henselization of the ring*

$$A[x,y]/\langle xy - t\rangle$$

at the ideal $I = \langle x, y, \mathfrak{m}\rangle$. Then:

(1) Suppose x', y' are elements of B such that $I = \langle x', y', \mathfrak{m}\rangle$ and $x'y' \in A$. Then there are units $u, v \in B^*$ such that $x' = ux, y' = vy$ (after possibly exchanging x', y') and $uv \in A^*$.

(2) If $x = ux, y = vy$ for some $u, v \in R^*$ with $uv \in A^*$, then $u = v = 1$.

Index of Notation

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