Tôhoku Math. J.
48 (1996), 317–354

LOG SMOOTH DEFORMATION THEORY

FUMIHARU KATO

(Received June 13, 1994, revised May 14, 1996)

Abstract. This paper lays a foundation for log smooth deformation theory. We study the infinitesimal liftings of log smooth morphisms and show that the log smooth deformation functor has a representable hull. This deformation theory gives, for example, the following two types of deformations: (1) relative deformations of a certain kind of a pair of an algebraic variety and a divisor on it, and (2) global smoothings of normal crossing varieties. The former is a generalization of the relative deformation theory introduced by Makio and others, and the latter coincides with the logarithmic deformation theory introduced by Kawamata and Namikawa.

1. Introduction. In this article, we formulate and develop the theory of *log smooth deformations*. Here, log smoothness (more precisely, logarithmic smoothness) is a concept in *log geometry* which is a generalization of "usual" smoothness of morphisms of algebraic varieties. Log geometry is a beautiful geometric theory which successfully generalizes and unifies the scheme theory and the theory of toric varieties. This theory was initiated by Fontaine and Illusie, based on their idea of *log structures* on schemes, and further developed by Kazuya Kato [5]. Recently, the importance of log geometry has come to be recognized by many geometers and applied to various fields of algebraic and arithmetic geometry. One of such applications can be seen in the recent work of Steenbrink [12]. In the present paper, we attempt to apply log geometry to extend the usual smooth deformation theory by using the concept of log smoothness.

Log smoothness is one of the most important concepts in log geometry, and is a log geometric generalization of usual smoothness. For example, varieties with toric singularities or normal crossing varieties may become log smooth over certain logarithmic points. Kazuya Kato [5] showed that any log smooth morphism is written étale locally as the composite of a usual smooth morphism and a morphism induced by a homomorphism of monoids which essentially determines the log structures (Theorem 4.1). On the other hand, log smoothness is described in terms of *log differentials* and *log derivations* similarly to usual smoothness in terms of differentials and derivations. Hence if we consider log smooth deformations by analogy with usual smooth deformations, it is expected that the first order deformations are controlled by the sheaf of log derivations. This intuition motivated this work and we shall see later that this is, in fact, the case.

In the present paper, we construct log smooth deformation functor by the concept

1991 *Mathematics Subject Classification*. Primary 13D10; Secondary 14D15, 14M25, 16S80.

of infinitesimal log smooth lifting. The goal of this paper is to show that this functor has a representable hull in the sense of Schlessinger [11], under certain conditions (Theorem 8.7). At the end of this paper, we give two examples of our log smooth deformation theory, which are summarized as follows:

1. Deformations with divisors (§10): Let X be a variety over a field k. Assume that the variety X has an étale covering $\{U_i\}_{i\in I}$ and a divisor D such that

(a) there exists a smooth morphism $h_i\colon U_i \to V_i$ where V_i is an affine toric variety over k for each $i \in I$,

(b) the divisor $U_i \times_X D$ on U_i is the pull-back of the union of the closure of codimension one torus orbits of V_i by h_i for each $i \in I$.

Then, there exists a log structure $\mathscr{M}$ on X such that the log scheme $(X, \mathscr{M})$ is log smooth over k with trivial log structure. (The converse is also true in a certain excellent category of log schemes.) In this case, a log smooth deformation in our sense is a deformation of the piar (X, D). If X itself is smooth and D is a smooth divisor on X, our deformations coincides with the relative deformations studied by Makio [9] and others.

2. Smoothings of normal crossing varieties (§11): If a scheme of finite type X over a field k is, étale locally, isomorphic to an affine normal crossing variety $\operatorname{Spec} k[z_1, \ldots, z_n]/(z_1 \cdots z_l)$, then we call X a normal crossing variety over k. If X is d-semistable (cf. [1]), then there exists a log structure $\mathscr{M}$ on X of semistable type (Definition 11.6) and $(X, \mathscr{M})$ is log smooth over a standard log point $(\operatorname{Spec} k, \boldsymbol{N})$ (Theorem 11.7). Then, a log smooth deformation in our sense is nothing but a smoothing of X. If the singular locus of X is connected, our deformation theory coincides with the one introduced by Kawamata and Namikawa [6].

The organization of this paper is as follows. We recall some basic notions in log geometry in the next section, and review the definition and basic properties of log smoothness in Section 3. In Section 4, we study the characterization of log smoothness by means of the theory of toric varieties according to Illusie [3] and Kato [5]. In Section 5, we recall the definitions and basic properties of log derivations and log differentials. In Sections 6 and 7, we give the proofs of the theorems stated in Section 4. Section 8 is devoted to the formulation of log smooth deformation theory, and is the main section of this present paper. We prove the existence of a representable hull of the log smooth deformation functor in Section 9. In Sections 10 and 11, we give two examples of log smooth deformations. In Section 12, we give the proof of the theorem stated in Section 11, which generalizes the result of Kawamata and Namikawa [6, (1.1)].

The author would like to express his thanks to Professors Kazuya Kato and Yoshinori Namikawa for valuable suggestions and advice. The author is also very grateful to Professors Luc Illusie and Kenji Ueno for valuable advice on this paper. Thanks are also due to the referee for valuable comments; he pointed out some errors in the first draft of this paper. Section 11 and Section 12 resulted from discussions with Professors Sanpei Usui and Takeshi Usa and Dr. Taro Fujisawa; the author is very grateful to them for discussions and encouragements.

CONVENTION. We assume that all monoids are commutative and have neutral elements. A homomorphism of monoids is assumed to preserve neutral elements. We write the binary operations of all monoids multiplicatively except in the cases of $\boldsymbol{N}$ (the monoid of non-negative integers), $\boldsymbol{Z}$, etc., when we write them additively. All sheaves on schemes are considered with respect to the étale topology.

2. Fine saturated log schemes. In this and subsequent sections, we use the terminology of log geometry basically as in [5]. Let X be a scheme. We view the structure sheaf $\mathcal{O}_X$ of X as a sheaf of monoids under multiplication.

DEFINITION 2.1 (cf. [5, §1]). A *pre-log structure* on X is a homomorphism $\mathcal{M} \to \mathcal{O}_X$ of sheaves of monoids where $\mathcal{M}$ is a sheaf of monoids on X. A pre-log structure $\alpha: \mathcal{M} \to \mathcal{O}_X$ is said to be a *log structure* on X if α induces an isomorphism

$$\alpha: \alpha^{-1}(\mathcal{O}_X^\times) \xrightarrow{\sim} \mathcal{O}_X^\times ,$$

where $\mathcal{O}_X^\times$ is the subsheaf of invertible elements on $\mathcal{O}_X$.

Given a pre-log structure $\alpha: \mathcal{M} \to \mathcal{O}_X$, we can construct the *associated log structure* $\alpha^{\mathrm{a}}: \mathcal{M}^{\mathrm{a}} \to \mathcal{O}_X$ functorially by

$$\mathcal{M}^{\mathrm{a}} = (\mathcal{M} \oplus \mathcal{O}_X^\times)/\mathcal{P} \tag{1}$$

and

$$\alpha^{\mathrm{a}}(x, u) = u \cdot \alpha(x)$$

for $(x, u) \in \mathcal{M}^{\mathrm{a}}$, where $\mathcal{P}$ is the submonoid defined by

$$\mathcal{P} = \{(x, \alpha(x)^{-1}) \mid x \in \alpha^{-1}(\mathcal{O}_X^\times)\} .$$

Here, in general, the quotient M/P of a monoid M with respect to a submonoid P is the coset space $M/\sim$ with induced monoid structure, where the equivalence relation $\sim$ is defined by

$$x \sim y \Leftrightarrow xp = yq \quad \text{for some } p, q \in P .$$

$\mathcal{M}^{\mathrm{a}}$ has a universal mapping property: if $\beta: \mathcal{N} \to \mathcal{O}_X$ is a log structure on X and $\varphi: \mathcal{M} \to \mathcal{N}$ is a homomorphism of sheaves of monoids such that $\alpha = \beta \circ \varphi$, then there exists a unique lifting $\varphi^{\mathrm{a}}: \mathcal{M}^{\mathrm{a}} \to \mathcal{N}$. Note that the monoid $\mathcal{M}^{\mathrm{a}}$ defined by (1) is the push-out of the diagram

$$\mathcal{M} \supset \alpha^{-1}(\mathcal{O}_X^\times) \xrightarrow{\alpha} \mathcal{O}_X^\times$$

in the category of monoids, and the homomorphism α^{a} is induced by α and the inclusion $\mathcal{O}_X^\times \hookrightarrow \mathcal{O}_X$. We sometimes denote the monoid $\mathcal{M}^{\mathrm{a}}$ by $\mathcal{M} \oplus_{\alpha^{-1}(\mathcal{O}_X^\times)} \mathcal{O}_X^\times$. Note that we have a natural isomorphism

$$\mathscr{M}/\alpha^{-1}(\mathscr{O}_X^\times) \xrightarrow{\sim} \mathscr{M}^a/\mathscr{O}_X^\times . \tag{2}$$

DEFINITION 2.2. By a *log scheme*, we mean a pair $(X, \mathscr{M})$ with a scheme X and a log structure $\mathscr{M}$ on X. A *morphism* of log schemes $f: (X, \mathscr{M}) \to (Y, \mathscr{N})$ is a pair $f = (f, \varphi)$ where $f: X \to Y$ is a morphism of schemes and $\varphi: f^{-1}\mathscr{N} \to \mathscr{M}$ is a homomorphism of sheaves of monoids such that the diagram

$$\begin{tikzcd} f^{-1}\mathscr{N} \arrow[r, "\varphi"] \arrow[d] & \mathscr{M} \arrow[d] \\ f^{-1}\mathscr{O}_Y \arrow[r] & \mathscr{O}_X \end{tikzcd}$$

commutes.

DEFINITION 2.3. Let $\alpha: \mathscr{M} \to \mathscr{O}_X$ and $\alpha': \mathscr{M}' \to \mathscr{O}_X$ be log structures on a scheme X. These log structures are said to be *equivalent* if there exists an isomorphism $\varphi: \mathscr{M} \xrightarrow{\sim} \mathscr{M}'$ such that $\alpha = \alpha' \circ \varphi$, i.e., there exists an isomorphism of log schemes $(X, \mathscr{M}) \xrightarrow{\sim} (X, \mathscr{M}')$ whose underlying morphism of schemes is the identity id_X. Let $\beta: \mathscr{N} \to \mathscr{O}_Y$ and $\beta': \mathscr{N}' \to \mathscr{O}_Y$ be log structures on a scheme Y. Let $f: (X, \mathscr{M}) \to (Y, \mathscr{N})$ and $f': (X, \mathscr{M}') \to (Y, \mathscr{N}')$ be morphisms of log schemes. Then f and f' are said to be *equivalent* if there exist isomorphisms $\varphi: \mathscr{M} \xrightarrow{\sim} \mathscr{M}'$ and $\psi\ \mathscr{N} \xrightarrow{\sim} \mathscr{N}'$ such that $\alpha = \alpha' \circ \varphi$, $\beta = \beta' \circ \psi$ and the diagram

$$\begin{tikzcd} \mathscr{M} \arrow[r, "\varphi"] & \mathscr{M}' \\ f^{-1}\mathscr{N} \arrow[u] \arrow[r, "f^{-1}\psi"'] & f^{-1}\mathscr{N}' \arrow[u] \end{tikzcd}$$

commutes.

We denote the category of log schemes by **LSch**. For $(S, \mathscr{L}) \in \mathrm{Obj}(\mathbf{LSch})$, we denote the category of log schemes over $(S, \mathscr{L})$ by $\mathbf{LSch}_{(S, \mathscr{L})}$. The following examples play important roles in the sequel.

EXAMPLE 2.4. On any scheme X, we can define a log structure by the inclusion $\mathscr{O}_X^\times \hookrightarrow \mathscr{O}_X$, called the *trivial* log structure. Thus, we have an inclusion functor from the category of schemes to that of log schemes sending X to $(X, \mathscr{O}_X^\times \hookrightarrow \mathscr{O}_X)$, which we often denote simply by X.

EXAMPLE 2.5. Let A be a commutative ring. For a monoid P, we can define a log structure canonically on the scheme $\operatorname{Spec} A[P]$, where $A[P]$ denotes the monoid ring of P over A, as the log structure associated to the natural homomorphism,

$$P \xrightarrow{\alpha} A[P] .$$

This log structure is called the *canonical log structure* on $\operatorname{Spec} A[P]$. Thus we obtain a log scheme which we denote simply by $(\operatorname{Spec} A[P], P)$. A monoid homomorphism $P \to Q$ induces a morphism $(\operatorname{Spec} A[Q], Q) \to (\operatorname{Spec} A[P], P)$ of log schemes. Thus, we have a contravariant functor from the category of monoids to $\mathbf{LSch}_{\operatorname{Spec} A}$.

EXAMPLE 2.6. Let Σ be a fan in $N_{\boldsymbol{R}} = \boldsymbol{R}^d$, $N = \boldsymbol{Z}^d$, and X_Σ the toric variety determined by the fan Σ over a commutative ring A. Then, we get an induced log structure on the scheme X_Σ by gluing the log structures associated to the homomorphism

$$M \cap \sigma^\vee \longrightarrow A[M \cap \sigma^\vee]\,,$$

for each cone σ in Σ, where $M = \operatorname{Hom}_{\boldsymbol{Z}}(N, \boldsymbol{Z})$. Thus, the toric variety X_Σ is naturally viewed as a log scheme over $\operatorname{Spec} A$, which we denote by (X_Σ, Σ).

Next, we define important subcategories of **LSch**.

DEFINITION 2.7. A monoid M is said to be *finitely generated* if there exists a surjective homomorphism $\boldsymbol{N}^n \to M$ for some n. A monoid M is said to be *integral* if the natural homomorphism $M \to M^{\mathrm{gp}}$ is injective, where M^{gp} denotes the Grothendieck group associated with M. If M is finitely generated and integral, it is said to be *fine*.

DEFINITION 2.8. Let $(X, \mathscr{M}) \in \operatorname{Obj}(\mathbf{LSch})$. A *chart* of $\mathscr{M}$ is a homomorphism $P \to \mathscr{M}$ from the constant sheaf of a fine monoid P which induces an isomorphism from the associated log structure P^{a} to $\mathscr{M}$.

DEFINITION 2.9. Let $f\colon (X, \mathscr{M}) \to (Y, \mathscr{N})$ be a morphism in **LSch**. A *chart* of f is a triple $(P \to \mathscr{M}, Q \to \mathscr{N}, Q \to P)$, where $P \to \mathscr{M}$ and $Q \to \mathscr{N}$ are charts of $\mathscr{M}$ and $\mathscr{N}$, respectively, and $Q \to P$ is a homomorphism for which the diagram

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & P \arrow[d] \\ f^{-1}\mathscr{N} \arrow[r] & \mathscr{M} \end{tikzcd}$$

is commutative.

DEFINITION 2.10 (cf. [5, §2]). A log structure $\mathscr{M} \to \mathscr{O}_X$ on a scheme X is said to be *fine if* $\mathscr{M}$ has étale locally a chart $P \to \mathscr{M}$. A log scheme $(X, \mathscr{M})$ with a fine log structure $\mathscr{M} \to \mathscr{O}_X$ is called a *fine* log scheme.

We denote the category of fine log schemes by $\mathbf{LSch}^{\mathrm{f}}$. Similarly, we denote the category of fine log schemes over $(S, \mathscr{L}) \in \operatorname{Obj}(\mathbf{LSch}^{\mathrm{f}})$ by $\mathbf{LSch}^{\mathrm{f}}_{(S,\mathscr{L})}$. The category $\mathbf{LSch}^{\mathrm{f}}$ (resp. $\mathbf{LSch}^{\mathrm{f}}_{(S,\mathscr{L})}$) is a full subcategory of **LSch** (resp. $\mathbf{LSch}_{(S,\mathscr{L})}$). Both **LSch** and $\mathbf{LSch}^{\mathrm{f}}$ have fiber products (cf. [5, (1.6), (2.8)]). But the inclusion functor $\mathbf{LSch}^{\mathrm{f}} \hookrightarrow \mathbf{LSch}$ does not preserve fiber products (cf. Lemma 3.4). The inclusion functor $\mathbf{LSch}^{\mathrm{f}} \hookrightarrow \mathbf{LSch}$ has a right adjoint $\mathbf{LSch} \to \mathbf{LSch}^{\mathrm{f}}$ (cf. [5, (2.7)]). Then, the fiber product of a diagram $(X, \mathscr{M}) \to (Z, \mathscr{P}) \leftarrow (Y, \mathscr{N})$ in $\mathbf{LSch}^{\mathrm{f}}$ is the image of that in **LSch** by this adjoint functor.

Note that the underlying scheme of the fiber product of $(X, \mathscr{M}) \to (Z, \mathscr{P}) \leftarrow (Y, \mathscr{N})$ in **LSch** is $X \times_Z Y$, but this is not always the case in $\mathbf{LSch}^{\mathrm{f}}$.

Next, we introduce a more excellent subcategory of **LSch**.

DEFINITION 2.11. Let M be a monoid and P a submonoid of M. The monoid P is said to be *saturated* in M if $x \in M$ and $x^n \in P$ for some positive integer n imply $x \in P$. An integral monoid N is said to be *saturated* if N is saturated in N^{gp}.

EXAMPLE 2.12. Put $M = \boldsymbol{N}$ and $P = l \cdot M$ for an integer $l > 1$. Then P is saturated but is not saturated in M.

DEFINITION 2.13. A fine log scheme $(X, \mathscr{M}) \in \mathrm{Obj}(\mathbf{LSch}^{\mathrm{f}})$ is said to be *saturated* if the log structure $\mathscr{M}$ is a sheaf of saturated monoids.

We denote the category of fine saturated log schemes by $\mathbf{LSch}^{\mathrm{fs}}$. Similarly, we denote the category of fine saturated log schemes over $(S, \mathscr{L}) \in \mathrm{Obj}(\mathbf{LSch}^{\mathrm{fs}})$ by $\mathbf{LSch}^{\mathrm{fs}}_{(S,\mathscr{L})}$. The category $\mathbf{LSch}^{\mathrm{fs}}$ (resp. $\mathbf{LSch}^{\mathrm{fs}}_{(S,\mathscr{L})}$) is a full subcategory of $\mathbf{LSch}^{\mathrm{f}}$ (resp. $\mathbf{LSch}^{\mathrm{f}}_{(S,\mathscr{L})}$). The following lemma is an easy consequence of [5, Lemma (2.10)].

LEMMA 2.14. *Let $f\colon (X, \mathscr{M}) \to (Y, \mathscr{N})$ be a morphism in $\mathbf{LSch}^{\mathrm{fs}}$, and $Q \to \mathscr{N}$ a chart of $\mathscr{N}$, where Q is a fine saturated monoid. Then there exists étale locally a chart $(P \to \mathscr{M}, Q \to \mathscr{N}, Q \to P)$ of f extending $Q \to \mathscr{N}$ such that the monoid P is also fine and saturated.*

LEMMA 2.15. *The inclusion functor $\mathbf{LSch}^{\mathrm{fs}} \hookrightarrow \mathbf{LSch}^{\mathrm{f}}$ has a right adjoint.*

PROOF. Let M be an integral monoid. Define

$$M^{\mathrm{sat}} = \{x \in M^{\mathrm{gp}} \mid x^n \in M \text{ for some positive integer } n\}.$$

Then M^{sat} is an integral saturated monoid. For any integral saturated monoid N and homomorphism $M \to N$, there exists a unique lifting $M^{\mathrm{sat}} \to N$. In this sense, M^{sat} is the universal saturated monoid associated with M. Let $(X, \mathscr{M})$ be a fine log scheme. Then we have étale locally a chart, $P \to \mathscr{M}$. This cahrt defines a morphism $X \to \operatorname{Spec} \boldsymbol{Z}[P]$ étale locally. Let $X' = X \times_{\operatorname{Spec} \boldsymbol{Z}[P]} \operatorname{Spec} \boldsymbol{Z}[P^{\mathrm{sat}}]$. Then $X' \to \operatorname{Spec} \boldsymbol{Z}[P^{\mathrm{sat}}]$ induces a log structure $\mathscr{M}'$ by the associated log structure of $P^{\mathrm{sat}} \to \boldsymbol{Z}[P^{\mathrm{sat}}] \to \mathscr{O}_{X'}$. It is easy to show that we can glue those log schemes $(X', \mathscr{M}')$ and get a fine saturated log scheme. This procedure defines a functor $\mathbf{LSch}^{\mathrm{f}} \to \mathbf{LSch}^{\mathrm{fs}}$. It is easy to see that this functor is the right adjoint of the inclusion functor $\mathbf{LSch}^{\mathrm{fs}} \hookrightarrow \mathbf{LSch}^{\mathrm{f}}$. □

COROLLARY 2.16. *$\mathbf{LSch}^{\mathrm{fs}}$ has fiber products. More precisely, the fiber product of morphisms $(X, \mathscr{M}) \to (Z, \mathscr{P}) \leftarrow (Y, \mathscr{N})$ in $\mathbf{LSch}^{\mathrm{fs}}$ is the image of that in $\mathbf{LSch}^{\mathrm{f}}$ by the right adjoint functor of $\mathbf{LSch}^{\mathrm{fs}} \hookrightarrow \mathbf{LSch}^{\mathrm{f}}$.*

3. Log smooth morphisms. In this section, we review the definition and basic properties of log smoothness (cf. [5]).

DEFINITION 3.1. Let $f: X \to Y$ be a morphism of schemes, and $\mathcal{N}$ a log structure on Y. Then the *pull-back* of $\mathcal{N}$, denoted by $f^*\mathcal{N}$, is the log structure on X associated with the pre-log structure $f^{-1}\mathcal{N} \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. A morphism of log schemes $f: (X, \mathcal{M}) \to (Y, \mathcal{N})$ is said to be *strict* if the induced homomorphism $f^*\mathcal{N} \to \mathcal{M}$ is an isomorphism. A morphism of log schemes $f: (X, \mathcal{M}) \to (Y, \mathcal{N})$ is said to be an *exact closed immersion* if it is strict and $f: X \to Y$ is a closed immersion in the usual sense.

Exact closed immersions are stable under base change in $\mathbf{LSch}^{\mathrm{f}}$ (cf. [5, (4.6)]).

LEMMA 3.2. *Let $\alpha: \mathcal{M} \to \mathcal{O}_X$ and $\alpha': \mathcal{M}' \to \mathcal{O}_X$ be fine log structures on a scheme X with a homomorphism $\varphi: \mathcal{M} \to \mathcal{M}'$ of monoids such that $\alpha = \alpha' \circ \varphi$. Then, φ is an isomorphism if and only if $(\varphi \bmod \mathcal{O}_X^\times): \mathcal{M}/\mathcal{O}_X^\times \to \mathcal{M}'/\mathcal{O}_X^\times$ is an isomorphism.*

The proof is straightforward.

LEMMA 3.3. *Let $f: (X, \mathcal{M}) \to (X, \mathcal{N})$ be a morphism of fine log schemes. Then, we have a natural isomorphism*

$$f^{-1}(\mathcal{N}/\mathcal{O}_Y^\times) \xrightarrow{\sim} f^*\mathcal{N}/\mathcal{O}_X^\times .$$

In particular, f is strict if and only if the induced morphism is an isomorphism

$$f^{-1}(\mathcal{N}/\mathcal{O}_Y^\times) \xrightarrow{\sim} \mathcal{M}/\mathcal{O}_X^\times .$$

PROOF. The first part is easy to see. As for the second part, apply (2) and Lemma 3.2. □

LEMMA 3.4 (cf. [4, (1.7)]). *Let*

$$(X, \mathcal{M}) \longrightarrow (Z, \mathcal{P}) \longleftarrow (Y, \mathcal{N}) \tag{3}$$

be morphisms in $\mathbf{LSch}^{\mathrm{fs}}$. If $(Y, \mathcal{N}) \to (Z, \mathcal{P})$ is strict, then the fiber product of (3) in $\mathbf{LSch}^{\mathrm{fs}}$ is isomorphic to that in $\mathbf{LSch}$. In particular, the underlying scheme of the fiber product of (3) in $\mathbf{LSch}^{\mathrm{fs}}$ is isomorphic to $X \times_Z Y$.

PROOF. We may work étale locally. Let $P \to \mathcal{P}$ be a chart of $\mathcal{P}$, where P is a fine saturated monoid. Since $(Y, \mathcal{N}) \to (Z, \mathcal{P})$ is strict, $P \to \mathcal{P} \to \mathcal{N}$ is a chart of $\mathcal{N}$ by Lemmas 3.2 and 3.3, and (2). Take a chart

$$\begin{array}{ccc} P & \longrightarrow & M \\ \downarrow & & \downarrow \\ \mathcal{P} & \longrightarrow & \mathcal{M} \end{array}$$

of $(X, \mathcal{M}) \to (Z, \mathcal{P})$ extending $P \to \mathcal{P}$. Set $W = X \times_Z Y$. There exists an induced homomorphism $M \to \mathcal{O}_W$. Define a log structure on W by this homomorphism. Then this log scheme $(W, M \to \mathcal{O}_W)$ is the fiber product of (3) in $\mathbf{LSch}$ (cf. [5, (1.6)]). Since the associated log structure $\mathcal{M}_W$ of $M \to \mathcal{O}_W$ is fine and saturated, $(W, \mathcal{M}_W)$ is indeed the fiber product of (3) in $\mathbf{LSch}^{\mathrm{fs}}$. □

DEFINITION 3.5. The exact closed immersion $t\colon (T', \mathscr{L}') \to (T, \mathscr{L})$ is said to be a *thickening of order* $\leq n$, if $\mathscr{I} = \mathrm{Ker}(\mathcal{O}_T \to \mathcal{O}_{T'})$ is a nilpotent ideal such that $\mathscr{I}^{n+1} = 0$.

LEMMA 3.6 (cf. [3]). *Let $(T, \mathscr{L})$ and $(T', \mathscr{L}')$ be fine log schemes. If $(t, \theta)\colon (T', \mathscr{L}') \to (T, \mathscr{L})$ is a thickening of order $\leq n$, then there exists a commutative diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & 1+\mathscr{I} & \hookrightarrow & t^{-1}\mathscr{L} & \xrightarrow{\theta} & \mathscr{L}' & \longrightarrow & 1 \\ & & \| & & \cap & & \cap & & \\ 1 & \longrightarrow & 1+\mathscr{I} & \longrightarrow & t^{-1}\mathscr{L}^{\mathrm{gp}} & \xrightarrow[\theta^{\mathrm{gp}}]{} & (\mathscr{L}')^{\mathrm{gp}} & \longrightarrow & 1, \end{array}$$

with exact rows and $\mathscr{I} = \mathrm{Ker}(\mathcal{O}_T \to \mathcal{O}_{T'})$, such that the square on the right hand side is Cartesian.

The proof is straightforward. Note that the multiplicative monoid $1+\mathscr{I}$ can be identified with the additive monoid $\mathscr{I}$ by $1+x \mapsto x$ if $\mathscr{I}^2 = 0$.

DEFINITION 3.7 (cf. [5, (3.3)]). Let $f\colon (X, \mathscr{M}) \to (Y, \mathscr{N})$ be a morphism in $\mathbf{LSch}^{\mathrm{f}}$. f is said to be *log smooth* if the following conditions are satisfied:

1. The underlying morphism f of schemes is locally of finite presentation.
2. For any commutative diagram

$$\begin{array}{ccc} (T', \mathscr{L}') & \xrightarrow{s'} & (X, \mathscr{M}) \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle f} \\ (T, \mathscr{L}) & \xrightarrow[s]{} & (Y, \mathscr{N}) \end{array}$$

   in $\mathbf{LSch}^{\mathrm{f}}$, where t is a thickening of order one, there exists étable locally a morphism $g\colon (T, \mathscr{L}) \to (X, \mathscr{M})$ such that $s' = g \circ t$ and $s = f \circ g$.

The proofs of the following two propositions are straightforward and are left to the reader.

PROPOSITION 3.8. *Let $f\colon (X, \mathscr{M}) \to (Y, \mathscr{N})$ be a morphism in $\mathbf{LSch}^{\mathrm{f}}$. If f is strict, then f is log smooth if and only if the underlying morphism f of schemes is smooth in the usual sense.*

PROPOSITION 3.9. *For $(S, \mathscr{L}) \in \mathrm{Obj}(\mathbf{LSch}^{\mathrm{f}})$ and $(X, \mathscr{M}), (Y, \mathscr{N}) \in \mathrm{Obj}(\mathbf{LSch}^{\mathrm{f}}_{(S,\mathscr{L})})$, let $f\colon (X, \mathscr{M}) \to (Y, \mathscr{N})$ be a morphism in $\mathbf{LSch}^{\mathrm{f}}_{(S,\mathscr{L})}$. Assume that f is log smooth. If $(S', \mathscr{L}')$ is a log scheme over $(S, \mathscr{L})$, then the induced morphism*

$$(X, \mathscr{M}) \times_{(S,\mathscr{L})} (S', \mathscr{L}') \to (Y, \mathscr{N}) \times_{(S,\mathscr{L})} (S', \mathscr{L}')$$

is also log smooth.

We conclude this section by introducing *integral morphisms* of fine log schemes.

DEFINITION 3.10 (cf. [5, (4.1), (4.3)]. Let $f\colon (X, \mathscr{M}) \to (Y, \mathscr{N})$ be a morphism in $\mathbf{LSch}^{\mathrm{f}}$. We say f to be *integral* if for any $x \in X$, setting $Q = f^{-1}(\mathscr{N}/\mathscr{O}_Y^{\times})_{\bar{x}}$ and $P = (\mathscr{M}/\mathscr{O}_X^{\times})_{\bar{x}}$, the ring homomorphism $\boldsymbol{Z}[Q] \to \boldsymbol{Z}[P]$ induced by f is flat, where $\bar{x}$ denotes the separable closure of x.

PROPOSITION 3.11 (cf. [5, (4.4)]). *Let $f\colon (X, \mathscr{M}) \to (Y, \mathscr{N})$ be a morphism in* $\mathbf{LSch}^{\mathrm{f}}$. *Then, f is integral in each of the following cases*:

1. *f is strict.*
2. *For any $y \in Y$, the monoid $(\mathscr{N}/\mathscr{O}_Y^{\times})_{\bar{y}}$ is generated by one element, where $\bar{y}$ denotes the separable closure of y.*

4. Toroidal characterization of log smoothness. The following theorem is due to Kazuya Kato [5], and we prove it in §6 for the reader's convenience.

THEOREM 4.1 ([5, (3.5), (4.5)]). *Let $f\colon (X, \mathscr{M}) \to (Y, \mathscr{N})$ be a morphism in* $\mathbf{LSch}^{\mathrm{f}}$ *(resp.* $\mathbf{LSch}^{\mathrm{fs}}$*) and $Q \to \mathscr{N}$ a chart of $\mathscr{N}$ (resp. with Q saturated). Then the following conditions are equivalent*:

1. *f is log smooth.*
2. *There exists étale locally a chart $(P \to \mathscr{M}, Q \to \mathscr{N}, Q \to P)$ of f extending $Q \to \mathscr{N}$ (resp. with P saturated), such that*
   (a) $\mathrm{Ker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}})$ *and the torsion part of* $\mathrm{Coker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}})$ *are finite groups of orders invertible on X,*
   (b) $X \to Y \times_{\mathrm{Spec}\, \boldsymbol{Z}[Q]} \mathrm{Spec}\, \boldsymbol{Z}[P]$ *is smooth (in the usual sense).*

Moreover, if f is a log smooth and integral morphism in $\mathbf{LSch}^{\mathrm{f}}$ *(resp.* $\mathbf{LSch}^{\mathrm{fs}}$*) and $Q \to \mathscr{N}$ is a chart of $\mathscr{N}$ (resp. with Q saturated), then there exists a chart $(P \to \mathscr{M}, Q \to \mathscr{N}, Q \to P)$ of f as above such that the ring homomorphism $\boldsymbol{Z}[Q] \to \boldsymbol{Z}[P]$ induced by $Q \to P$ is flat.*

REMARK 4.2. The proof of Theorem 4.1 in §6 shows that we can require in the condition (a) that $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is injective without changing the conclusion. Moreover, we can replace the smoothness in the condition (b) by the étaleness without changing the conclusion (cf. [5, (3.6)]).

COROLLARY 4.3 (cf. [5, (4.5)]). *Let $f\colon (X, \mathscr{M}) \to (Y, \mathscr{N})$ be a log smooth and integral morphism in* $\mathbf{LSch}^{\mathrm{f}}$. *Then the underlying morphism $X \to Y$ of schemes is flat.*

We give some important examples of log smooth morphisms in the following. Let k be a field.

DEFINITION 4.4. A log structure on $\mathrm{Spec}\, k$ is called a log structure of a *logarithmic point* if it is equivalent (cf. Definition 2.3) to the associated log structure of $\alpha\colon Q \to k$, where Q is a monoid having no invertible element other than 1 and α is a homomorphism defined by

$$\alpha(x)=\begin{cases}1 & \text{if } x=1\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

Note that this log structure is equivalent to $Q\oplus k^{\times}\to k$. We denote the log scheme obtained in this way by $(\operatorname{Spec} k, Q)$. The log scheme $(\operatorname{Spec} k, Q)$ is called a *logarithmic point*. Especially, if $Q=\boldsymbol{N}$, the logarithmic point $(\operatorname{Spec} k, \boldsymbol{N})$ is said to be the *standard point*.

If k is algebraically closed, any log structure on $\operatorname{Spec} k$ is equivalent to the log structure of a logarithmic point (cf. [3] and [5, (2.5), (2)]). Note that if we set $Q=\{1\}$, then the log structure of the logarithmic point induced by Q is the trivial log structure (cf. Example 2.4).

EXAMPLE 4.5. Let P be a submonoid of a group $M=\boldsymbol{Z}^d$ such that $P^{\mathrm{gp}}=M$ and that P is saturated. Let Q be a submonoid of P, which is saturated but is not necessarily saturated in P. We assume the following:

1. The monoid Q has no invertible element other than 1.
2. The order of the torsion part of M/Q^{gp} is invertible in k.

Let $R=\boldsymbol{Z}[1/N]$ where N is the order of the torsion part of M/Q^{gp}. The latter assumption implies by Theorem 4.1 that $(\operatorname{Spec} R[P], P)\to(\operatorname{Spec} R[Q], Q)$ (see Example 2.5) is log smooth. Define $\operatorname{Spec} k\to\operatorname{Spec} R[Q]$ by $\alpha\colon Q\to k$ as in Definition 4.4. Let X be a scheme over k which is smooth over $\operatorname{Spec} k\times_{\operatorname{Spec} R[Q]}\operatorname{Spec} R[P]$. Then we have a diagram

$$\begin{tikzcd} X \arrow[d] & \\ \operatorname{Spec} k\times_{\operatorname{Spec} R[Q]}\operatorname{Spec} R[P] \arrow[r] \arrow[d] & \operatorname{Spec} R[P] \arrow[d] \\ \operatorname{Spec} K \arrow[r] & \operatorname{Spec} R[Q]\,. \end{tikzcd}$$

Define a log structure $\mathscr{M}$ on X by the pull-back of the canonical log structure on $\operatorname{Spec} R[P]$. Then we have a morphism

$$f\colon (X,\mathscr{M})\longrightarrow(\operatorname{Spec} k, Q)$$

of fine saturated log schemes. This morphism f is log smooth by Proposition 3.8 and Proposition 3.9. We denote this log scheme $(X,\mathscr{M})$ simply by (X, P).

EXAMPLE 4.6. (Toric varieties.) In this and the following examples, we use the notation appearing in Example 4.5. Let σ be a cone in $N_{\boldsymbol{R}}=\boldsymbol{R}^d$ and $\sigma^{\vee}$ its dual cone in $M_{\boldsymbol{R}}=\boldsymbol{R}^d$. Set $P=M\cap\sigma^{\vee}$ and $Q=\{0\}\subset P$. Then, $\operatorname{Spec} k\times_{\operatorname{Spec}\boldsymbol{Z}[Q]}\operatorname{Spec}\boldsymbol{Z}[P]$ is k-isomorphic to $\operatorname{Spec} k[P]$ which is nothing but an affine toric variety. Let $X\to\operatorname{Spec} k[P]$ be a smooth morphism. Then $(X, P)\to\operatorname{Spec} k$ is log smooth. Note that this morphism is integral by Proposition 3.11.

EXAMPLE 4.7. (Variety with normal crossings.) Let σ be the cone in $M_{\boldsymbol{R}}=\boldsymbol{R}^d$ generated by $e_1, \dots, e_d$, where $e_i=(0, \dots, 0, 1, 0, \dots, 0)$ (1 at the i-th entry), $1 \leq i \leq d$. Let τ be the subcone generated by $a_1 e_1 + \cdots + a_d e_d$ with positive integers a_j for $j=1, \dots, d$. We assume that $\mathrm{GCD}(a_1, \dots, a_d)$ $(=N)$ is invertible in k. Set $R=\boldsymbol{Z}[1/N]$. Then, by setting $P=M \cap \sigma$ and $Q=M \cap \tau$, we see that $\operatorname{Spec} k \times_{\operatorname{Spec} R[Q]} \operatorname{Spec} R[P]$ is k-isomorphic to $\operatorname{Spec} k[z_1, \dots, z_d]/(z_1^{a_1} \cdots z_d^{a_d})$ and f is induced by

$$\begin{array}{ccc} \boldsymbol{N}^d & \longrightarrow & k[z_1, \dots, z_d]/(z_1^{a_1} \cdots z_d^{a_d}) \\ {\scriptstyle\varphi}\uparrow & & \uparrow \\ \boldsymbol{N} & \longrightarrow & k, \end{array}$$

where the morphism in the first row is defined by $e_i \mapsto z_i$, $(1 \leq i \leq d)$, and φ is defined by $\varphi(1)=a_1 e_1 + \cdots + a_d e_d$. Let $X \to \operatorname{Spec} k[z_1, \dots, z_d]/(z_1^{a_1} \cdots z_d^{a_d})$ be a smooth morphism. Then, $(X, \boldsymbol{N}^d) \to (\operatorname{Spec} k, \boldsymbol{N})$ is log smooth. Note that this morphism is integral by Proposition 3.11.

The following theorem is an application of Theorem 4.1 and will be proved in §7.

THEOREM 4.8. *Let X be an algebraic scheme over a field k, and $\mathscr{M} \to \mathcal{O}_X$ a fine saturated log structure on X. Then the log scheme $(X, \mathscr{M})$ is log smooth over* Spec k *with trivial log structure if and only if there exist an open étale covering $\mathscr{U}=\{U_i\}_{i \in I}$ of X and a divisor D on X such that*

1. *there exists a smooth morphism $h_i \colon U_i \to V_i$ where V_i is an affine toric variety over k for each $i \in I$,*
2. *the divisor $U_i \times_X D$ on U_i is the pull-back of the union of the closure of codimension one torus orbits of V_i by h_i for each $i \in I$,*
3. *the log structure $\mathscr{M} \to \mathcal{O}_X$ is equivalent to the log structure $\mathcal{O}_X \cap j_* \mathcal{O}_{X \setminus D}^{\times} \hookrightarrow \mathcal{O}_X$ where $j \colon X \setminus D \hookrightarrow X$ is the inclusion.*

COROLLARY 4.9. *Let X be a smooth algebraic variety over a field k, and $\mathscr{M} \to \mathcal{O}_X$ a fine saturated log structure on X. Then, the log scheme $(X, \mathscr{M})$ is log smooth over* Spec k *with trivial log structure if and only if there exists a reduced normal crossing divisor D on X such that the log structure $\mathscr{M} \to \mathcal{O}_X$ is equivalent to the log structure $\mathcal{O}_X \cap j_* \mathcal{O}_{X \setminus D}^{\times} \hookrightarrow \mathcal{O}_X$ where $j \colon X \setminus D \hookrightarrow X$ is the inclusion.*

5. Log differentials and log derivations. In this section, we are going to discuss the log differentials and log derivations, which are closely related with log smoothness, and play important roles in the sequel. To begin with, let us introduce some abbreviated notation in order to avoid complications. Let $(X, \mathscr{M})$ be a log scheme. If we like to omit writing the log structure $\mathscr{M}$, we write this log scheme by $X^\dagger$ to distinguish it from the underlying scheme X.

DEFINITION 5.1 (cf. [5], in different notation). Let $X^\dagger=(X, \mathscr{M})$ and $Y^\dagger=(Y, \mathscr{N})$

be fine log schemes, and $(f, \varphi)\colon X^\dagger \to Y^\dagger$ a morphism, where $\varphi\colon f^{-1}\mathcal{N} \to \mathcal{M}$ is a homomorphism of sheaves of monoids.

1. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module. The sheaf of *log derivations* $\mathcal{D}er_{Y^\dagger}(X^\dagger, \mathcal{E})$ of $X^\dagger$ to $\mathcal{E}$ over $Y^\dagger$ is the sheaf of germs of pairs $(D, D\log)$ with $D \in \mathcal{D}er_Y(X, \mathcal{E})$ and $D\log\colon \mathcal{M} \to \mathcal{E}$ such that the following conditions are satisfied:
   (a) $D\log(ab) = D\log(a) + D\log(b)$, for $a, b \in \mathcal{M}$,
   (b) $\alpha(a)D\log(a) = D(\alpha(a))$, for $a \in \mathcal{M}$,
   (c) $D\log(\varphi(c)) = 0$ for $c \in f^{-1}\mathcal{N}$.
2. The sheaf of *log differentials* of $X^\dagger$ over $Y^\dagger$ is the $\mathcal{O}_X$-module defined by

$$\Omega^1_{X^\dagger/Y^\dagger} = [\Omega^1_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbf{Z}} \mathcal{M}^{\mathrm{gp}})]/\mathcal{K}\ ,$$

   where $\mathcal{K}$ is the $\mathcal{O}_X$-submodule generated by

$$(d\alpha(a), 0) - (0, \alpha(a) \otimes a) \quad \text{and} \quad (0, 1 \otimes \varphi(b))\ ,$$

   for all $a \in \mathcal{M}$, $b \in f^{-1}\mathcal{N}$.

These are coherent $\mathcal{O}_X$-modules if Y is locally Noetherian and X locally of finite type over Y (cf. [3]).

PROPOSITION 5.2 (cf. [5, §1]). *Given a Cartesian diagram of fine log schemes*

$$\begin{array}{ccc} X^\dagger & \xrightarrow{g} & \tilde{X}^\dagger \\ \downarrow & & \downarrow \\ Y^\dagger & \longrightarrow & \tilde{Y}^\dagger\ , \end{array}$$

we have an isomorphism

$$g^*\Omega^1_{\tilde{X}^\dagger/\tilde{Y}^\dagger} \xrightarrow{\sim} \Omega^1_{X^\dagger/Y^\dagger}\ .$$

The proofs of the following three propositions are found in [5, §3].

PROPOSITION 5.3. *Let $X^\dagger$, $Y^\dagger$, f, and $\mathcal{E}$ be the same as in Definition* 5.1. *Then there is a natural isomorphism*

$$\mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X^\dagger/Y^\dagger}, \mathcal{E}) \xrightarrow{\sim} \mathcal{D}er_{Y^\dagger}(X^\dagger, \mathcal{E})\ ,$$

by $u \mapsto (u \circ d, u \circ d\log)$, where d and $d\log$ are defined by

$$d\colon \mathcal{O}_X \longrightarrow \Omega^1_{X/Y} \longrightarrow \Omega^1_{X^\dagger/Y^\dagger}$$

and

$$d\log\colon \mathcal{M} \longrightarrow \mathcal{O}_X \otimes_{\mathbf{Z}} \mathcal{M}^{\mathrm{gp}} \longrightarrow \Omega^1_{X^\dagger/Y^\dagger}\ .$$

PROPOSITION 5.4. *Let $X^\dagger \xrightarrow{f} Y^\dagger \xrightarrow{g} Z^\dagger$ be morphisms of fine log schemes.*

1. *There exists an exact sequence*

$$f^*\Omega^1_{Y^\dagger/Z^\dagger} \longrightarrow \Omega^1_{X^\dagger/Z^\dagger} \longrightarrow \Omega^1_{X^\dagger/Y^\dagger} \longrightarrow 0\,.$$

2. *If f is log smooth, then*

$$0 \longrightarrow f^*\Omega^1_{Y^\dagger/Z^\dagger} \longrightarrow \Omega^1_{X^\dagger/Z^\dagger} \longrightarrow \Omega^1_{X^\dagger/Y^\dagger} \longrightarrow 0 \tag{4}$$

is exact.

3. *If $g\circ f$ is log smooth and* (4) *is exact and splits locally, then f is log smooth.*

PROPOSITION 5.5. *If $f: X^\dagger \to Y^\dagger$ is log smooth, then $\Omega^1_{X^\dagger/Y^\dagger}$ is a locally free $\mathcal{O}_X$-module of finite rank.*

EXAMPLE 5.6 (cf. [10, Chap. 3, §(3.1)]). Let X_Σ be a toric variety over a field k determined by a fan Σ on $N_{\boldsymbol{R}}$ with $N = \boldsymbol{Z}^d$. Consider the log scheme (X_Σ, Σ) (cf. Example 2.6) over $\operatorname{Spec} k$. Then we have isomorphisms of $\mathcal{O}_X$-modules

$$\mathcal{D}er_k(X^\dagger, \mathcal{O}_X) \xrightarrow{\sim} \mathcal{O}_X \otimes_{\boldsymbol{Z}} N \quad \text{and} \quad \Omega^1_{X^\dagger/k} \xrightarrow{\sim} \mathcal{O}_X \otimes_{\boldsymbol{Z}} M\,,$$

where $M = \operatorname{Hom}_{\boldsymbol{Z}}(N, \boldsymbol{Z})$.

EXAMPLE 5.7. For $X = \operatorname{Spec} k[z_1, \ldots, z_n]/(z_1 \cdots z_l)$, let $f: (X, \mathcal{M}) \to (\operatorname{Spec} K, \boldsymbol{N} \to k)$ be the log smooth morphism defined in Example 4.7. Then $\mathcal{D}er_{k^\dagger}(X^\dagger, \mathcal{O}_X)$ is a free $\mathcal{O}_X$-module generated by

$$z_1 \frac{\partial}{\partial z_1}, \ldots, z_l \frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_{l+1}}, \ldots, \frac{\partial}{\partial z_n}$$

with a relation

$$z_1 \frac{\partial}{\partial z_1} + \cdots + z_l \frac{\partial}{\partial z_l} = 0\,.$$

Then sheaf $\Omega^1_{X^\dagger/k^\dagger}$ is a free $\mathcal{O}_X$-module generated by the *logarithmic differentials*

$$\frac{dz_1}{z_1}, \ldots, \frac{dz_l}{z_l}, dz_{l+1}, \ldots, dz_n$$

with a relation

$$\frac{dz_1}{z_1} + \cdots + \frac{dz_l}{z_l} = 0\,.$$

In the complex analytic case, the sheaf $\Omega^1_{X^\dagger/k^\dagger}$ is nothing but the sheaf of *relative logarithmic differentials* introduced, for example, in [1, §3] and [6, §2].

6. The proof of Theorem 4.1. In this section, we give a proof of Theorem 4.1 due to Kazuya Kato [5]. Before proving the general case, we prove the following proposition.

PROPOSITION 6.1. *Let A be a commutative ring and $h\colon Q \to P$ a homomorphism of fine monoids. The homomorphism h induces a morphism of log schemes*

$$f\colon X^{\dagger}=(\operatorname{Spec} A[P], P) \longrightarrow Y^{\dagger}=(\operatorname{Spec} A[Q], Q).$$

We set $K=\operatorname{Ker}(h^{\mathrm{gp}}\colon Q^{\mathrm{gp}} \to P^{\mathrm{gp}})$ and $C=\operatorname{Coker}(h^{\mathrm{gp}}\colon Q^{\mathrm{gp}} \to P^{\mathrm{gp}})$, and denote the torsion part of C by C_{tor}. If both K and C_{tor} are finite groups of order invertible in A, then f is log smooth.

PROOF. Suppose we have a commutative diagram

$$\begin{tikzcd}
(T', \mathscr{L}') \arrow[r, "s'"] \arrow[d, "t"'] & X^{\dagger}=(\operatorname{Spec} A[P], P) \arrow[d, "f"] \\
(T, \mathscr{L}) \arrow[r, "s"'] & Y^{\dagger}=(\operatorname{Spec} A[Q], Q)
\end{tikzcd}$$

in $\mathbf{LSch}^{\mathrm{f}}$, where the morphism t is a thickening of order one. Since we may work étale locally, we may assume that T is affine. Set $\mathscr{I}=\operatorname{Ker}(\mathcal{O}_T \to \mathcal{O}_{T'})$. Since the morphism t is a thickening of order one, by Lemma 3.6, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & 1+\mathscr{I} \arrow[r, hook] \arrow[d, equal] & \mathscr{L} \arrow[r, "t^*"] \arrow[d, phantom, "\cap"] & \mathscr{L}' \arrow[r] \arrow[d, phantom, "\cap"] & 1 \\
1 \arrow[r] & 1+\mathscr{I} \arrow[r] & \mathscr{L}^{\mathrm{gp}} \arrow[r, "(t^*)^{\mathrm{gp}}"'] & (\mathscr{L}')^{\mathrm{gp}} \arrow[r] & 1\,.
\end{tikzcd}$$

Note that the square on the right hand side is Cartesian.

First, consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r] \arrow[d, "u"] & Q^{\mathrm{gp}} \arrow[r, "h^{\mathrm{gp}}"] \arrow[d, "v"] & P^{\mathrm{gp}} \arrow[r] \arrow[d, "w"] & C \arrow[r] & 1 \\
1 \arrow[r] & 1+\mathscr{I} \arrow[r] & \mathscr{L}^{\mathrm{gp}} \arrow[r, "(t^*)^{\mathrm{gp}}"'] & (\mathscr{L}')^{\mathrm{gp}} \arrow[r] & 1\,. &
\end{tikzcd}$$

The multiplicative monoid $1+\mathscr{I}$ is isomorphic to the additive monoid $\mathscr{I}$ by $1+x \mapsto x$ since $\mathscr{I}^2=0$. If the order of K is invertible in A, then we have $u=1$, and hence there exists a morphism $a'\colon R \to \mathscr{L}^{\mathrm{gp}}$ with $R=\operatorname{Image}(h^{\mathrm{gp}}\colon Q^{\mathrm{gp}} \to P^{\mathrm{gp}})$ such that $a' \circ h^{\mathrm{gp}}=v$ and $(t^*)^{\mathrm{gp}} \circ a'=w$.

Next, we consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & R \arrow[r, "i"] \arrow[d, "a'"'] & P^{\mathrm{gp}} \arrow[r] \arrow[d, "w"] & C \arrow[r] & 1 \\
0 \arrow[r] & \mathscr{I} \arrow[r] & \mathscr{L}^{\mathrm{gp}} \arrow[r, "(t^*)^{\mathrm{gp}}"'] & (\mathscr{L}')^{\mathrm{gp}} \arrow[r] & 1\,.
\end{tikzcd}$$

We show that there exists a homomorphism $a'': P^{gp} \to \mathscr{L}^{gp}$ such that $a'' \circ t = a'$ and $(t^*)^{gp} \circ a'' = w$. The obstruction for the existence of a'' lies in $\mathrm{Ext}^1(C, \mathscr{I})$. In general, if a positive integer n is invertible in A then we have $\mathrm{Ext}^1(\boldsymbol{Z}/n\boldsymbol{Z}, \mathscr{I}) = 0$. Combining this with $\mathrm{Ext}^1(\boldsymbol{Z}, \mathscr{I}) = 0$, we have $\mathrm{Ext}^1(C, \mathscr{I}) = 0$ since the order of the torsion part of C is invertible in A. Hence a homomorphism a'' exists. Since the diagram

$$\begin{array}{ccc} \mathscr{L} & \xrightarrow{t^*} & \mathscr{L}' \\ \cap & & \cap \\ \mathscr{L}^{gp} & \xrightarrow[(t^*)^{gp}]{} & (\mathscr{L}')^{gp} \end{array}$$

is Cartesian, there exists a homomorphism $a: P \to \mathscr{L}$ such that $t^* \circ a = (s')^*$ and $a \circ h = s^*$. Using this a, we can construct a morphism of log schemes $g: (T, \mathscr{L}) \to X^\dagger = (\mathrm{Spec}\, A[P], P)$ such that $g \circ t = s'$ and $s \circ g = f$. □

Now, let us prove Theorem 4.1. First, we prove the implication $2 \Rightarrow 1$. Let $R = \boldsymbol{Z}[1/(N_1 \cdot N_2)]$ where N_1 is the order of $\mathrm{Ker}(Q^{gp} \to P^{gp})$ and N_2 the order of the torsion part of $\mathrm{Coker}(Q^{gp} \to P^{gp})$. By the assumption (a), we have

$$Y \times_{\mathrm{Spec}\, \boldsymbol{Z}[Q]} \mathrm{Spec}\, \boldsymbol{Z}[P] \xrightarrow{\sim} Y \times_{\mathrm{Spec}\, R[Q]} \mathrm{Spec}\, R[P]\,.$$

Since $X \to Y \times_{\mathrm{Spec}\, R[Q]} \mathrm{Spec}\, R[P]$ is smooth by (b), f is log smooth by Propositions 3.8, 3.9 and 6.1.

Next, let us prove the converse. Assume that the morphism f is log smooth. Then, the sheaf $\Omega^1_{X^\dagger/Y^\dagger}$ is a locally free $\mathcal{O}_X$-module of finite rank (cf. Proposition 5.5). Take any point $x \in X$. We denote by $\bar{x}$ the separable closure of x.

Step 1. Consider the morphism of $\mathcal{O}_X$-modules

$$1 \otimes d\log: \mathcal{O}_X \otimes_{\boldsymbol{Z}} \mathscr{M}^{gp} \longrightarrow \Omega^1_{X^\dagger/Y^\dagger}\,,$$

which is surjective by the definition of $\Omega^1_{X^\dagger/Y^\dagger}$. Then we can take elements $t_1, \dots, t_r \in \mathscr{M}_{\bar{x}}$ in such a way that the system $\{d\log t_i\}_{1 \le i \le r}$ is an $\mathcal{O}_{X,\bar{x}}$-base of $\Omega^1_{X^\dagger/Y^\dagger,\bar{x}}$. Consider the homomorphism $\psi: \boldsymbol{N}^r \to \mathscr{M}_{\bar{x}}$ defined by

$$\boldsymbol{N}^r \ni (n_1, \dots, n_r) \longmapsto t_1^{n_1} \cdots t_1^{n_r} \in \mathscr{M}_{\bar{x}}\,.$$

Combining this ψ with the homomorphism $Q \to f^{-1}(\mathscr{N})_{\bar{x}} \to \mathscr{M}_{\bar{x}}$, we have a homomorphism $\varphi: H = \boldsymbol{N}^r \oplus Q \to \mathscr{M}_{\bar{x}}$.

Step 2. Let $k(\bar{x})$ denote the residue field at $\bar{x}$. We have a homomorphism

$$k(\bar{x}) \otimes_{\boldsymbol{Z}} \boldsymbol{Z}^r \longrightarrow k(\bar{x}) \otimes_{\boldsymbol{Z}} \mathrm{Coker}(f^{-1}(\mathscr{N}^{gp}/\mathcal{O}_Y^\times)_{\bar{x}} \longrightarrow \mathscr{M}_{\bar{x}}^{gp}/\mathcal{O}_{X,\bar{x}}^\times) \tag{5}$$

by $k(\bar{x}) \otimes_{\boldsymbol{Z}} \psi^{gp}: k(\bar{x}) \otimes_{\boldsymbol{Z}} \boldsymbol{Z}^r \to k(\bar{x}) \otimes_{\boldsymbol{Z}} \mathscr{M}_{\bar{x}}^{gp}$ and the canonical projections $\mathscr{M}_{\bar{x}}^{gp} \to \mathscr{M}_{\bar{x}}^{gp}/\mathcal{O}_{X,\bar{x}}^\times \to \mathrm{Coker}(f^{-1}(\mathscr{N}^{gp}/\mathcal{O}_Y^\times)_{\bar{x}} \to \mathscr{M}_{\bar{x}}^{gp}/\mathcal{O}_{X,\bar{x}}^\times)$. We claim that this morphism (5) is surjective. Indeed, this morphism coincides with the composite morphism

$$k(\bar{x}) \otimes_{\boldsymbol{Z}} \boldsymbol{Z}^r \longrightarrow k(\bar{x}) \otimes_{\mathcal{O}_{X,\bar{x}}} \Omega^1_{X^\dagger/Y^\dagger,\bar{x}} \longrightarrow k(\bar{x}) \otimes_{\boldsymbol{Z}} \mathrm{Coker}(f^{-1}(\mathscr{N}^{gp}/\mathcal{O}_Y^\times)_{\bar{x}} \longrightarrow \mathscr{M}_{\bar{x}}^{gp}/\mathcal{O}_{X,\bar{x}}^\times)\,,$$

where the first morphism is induced by $d\log\circ\psi$ and the second one by the canonical projection, and these morphisms are clearly surjective. Hence the morphism (5) is surjective. On the other hand, the homomorphism $Q^{\mathrm{gp}}\to f^{-1}(\mathscr{N}/\mathcal{O}_Y^\times)_{\bar{x}}$ is surjective since $Q\to\mathscr{N}$ is a chart of $\mathscr{N}$. Hence, the homomorphism

$$k(\bar{x})\otimes_{\mathbf{Z}}Q^{\mathrm{gp}}\longrightarrow k(\bar{x})\otimes_{\mathbf{Z}}f^{-1}(\mathscr{N}/\mathcal{O}_Y^\times)_{\bar{x}}$$

is surjective, and then, the homomorphism

$$1\otimes_{\mathbf{Z}}\varphi^{\mathrm{gp}}\colon k(\bar{x})\otimes_{\mathbf{Z}}H^{\mathrm{gp}}\longrightarrow k(\bar{x})\otimes_{\mathbf{Z}}(\mathscr{M}_{\bar{x}}/\mathcal{O}_{X,\bar{x}}^\times)$$

is surjective. This shows that the cokernel $C=\mathrm{Coker}(\varphi^{\mathrm{gp}}\colon H^{\mathrm{gp}}\to\mathscr{M}_{\bar{x}}^{\mathrm{gp}}/\mathcal{O}_{X,\bar{x}}^\times)$ is annihilated by an integer N invertible in $\mathcal{O}_{X,\bar{x}}$.

Step 3. Take elements $a_1,\dots,a_d\in\mathscr{M}_{\bar{x}}^{\mathrm{gp}}$ which generate C. Then we can write $a_i^N=u_i\varphi(b_i)$ for $u_i\in\mathcal{O}_{X,\bar{x}}^\times$ and $b_i\in H^{\mathrm{gp}}$, for $i=1,\dots,d$. Since $\mathcal{O}_{X,\bar{x}}^\times$ is N-divisible, we can write $u_i=v_i^N$ for $v_i\in\mathcal{O}_{X,\bar{x}}^\times$, for $i=1,\dots,d$, and hence we may suppose $a_i^N=\varphi(b_i)$, replacing a_i by a_i/v_i for $i=1,\dots,d$. Let G be the push-out of the diagram

$$H^{\mathrm{gp}}\longleftarrow\mathbf{Z}^d\longrightarrow\mathbf{Z}^d,$$

where $\mathbf{Z}^d\to H^{\mathrm{gp}}$ is defined by $e_i\mapsto b_i$, and $\mathbf{Z}^d\to\mathbf{Z}^d$ is defined by $e_i\mapsto Ne_i$ for $i=1,\dots,d$. Then $\varphi^{\mathrm{gp}}\colon H^{\mathrm{gp}}\to\mathscr{M}_{\bar{x}}^{\mathrm{gp}}$ and $\mathbf{Z}^d\to\mathscr{M}_{\bar{x}}^{\mathrm{gp}}$, defined by $e_i\mapsto a_i$ for $i=1,\dots,d$, induce a homomorphism

$$\phi\colon G\longrightarrow\mathscr{M}_{\bar{x}}^{\mathrm{gp}}$$

which maps G surjectively onto $\mathscr{M}_{\bar{x}}^{\mathrm{gp}}/\mathcal{O}_{X,\bar{x}}^\times$. Then $P:=\phi^{-1}(\mathscr{M}_{\bar{x}})$ defines a chart of $\mathscr{M}$ on some neighborhood of $\bar{x}$ (cf. [5, Lemma 2.10]). If $\mathscr{M}$ is saturated, then so is P. There exists an induced map $Q\to P$ which defines a chart of f in some neighborhood of $\bar{x}$. Since $H^{\mathrm{gp}}\to P^{\mathrm{gp}}$ is injective, so is $Q^{\mathrm{gp}}\to P^{\mathrm{gp}}$. The cokernel $\mathrm{Coker}(H^{\mathrm{gp}}\to P^{\mathrm{gp}})$ is annihilated by N, hence $\mathrm{Coker}(Q^{\mathrm{gp}}\to P^{\mathrm{gp}})_{\mathrm{tor}}$ is finite and annihilated by N.

Step 4. Set $X'=Y\times_{\mathrm{Spec}\,\mathbf{Z}[Q]}\mathrm{Spec}\,\mathbf{Z}[P]$ and $g\colon X\to X'$. We need to show that the morphism g is smooth in the usual sense. Since $X^\dagger$ has the log structure induced by g from $X'^\dagger=(X',P)$, it suffices to show that g is log smooth (cf. Proposition 3.8). Since $k(\bar{x})\otimes_{\mathbf{Z}}(P^{\mathrm{gp}}/Q^{\mathrm{gp}})\xrightarrow{\sim}k(\bar{x})\otimes_{\mathbf{Z}}\mathbf{Z}^d\xrightarrow{\sim}k(\bar{x})\otimes_{\mathcal{O}_{X,\bar{x}}}\Omega^1_{X^\dagger/Y^\dagger,\bar{x}}$ and $\Omega^1_{X^\dagger/Y^\dagger}$ is locally free, we have $\Omega^1_{X^\dagger/Y^\dagger}\xrightarrow{\sim}\mathcal{O}_X\otimes_{\mathbf{Z}}(P^{\mathrm{gp}}/Q^{\mathrm{gp}})$ in some neighborhood of $\bar{x}$. On the other hand, by direct calculation, one sees that $\Omega^1_{X'^\dagger/Y^\dagger}\cong\mathcal{O}_{X'}\otimes_{\mathbf{Z}[P]}\Omega^1_{\mathbf{Z}[P]/\mathbf{Z}[Q]}\xrightarrow{\sim}\mathcal{O}_{X'}\otimes_{\mathbf{Z}}(P^{\mathrm{gp}}/Q^{\mathrm{gp}})$. Hence we have $g^*\Omega^1_{X'^\dagger/Y^\dagger}\xrightarrow{\sim}\Omega^1_{X^\dagger/Y^\dagger}$. This implies that g is log smooth by Proposition 5.4 (in fact, g is *log étale* (cf. [5])).

Step 5. Finally, we need to show that the ring homomorphism $\mathbf{Z}[Q]\to\mathbf{Z}[P]$ induced by $h\colon Q\to P$ is flat in case f is integral. By [5, (4.1)], the ring homomorphism $\mathbf{Z}[Q]\to\mathbf{Z}[P]$ is flat if and only if for any $a_1,a_2\in Q$, $b_1,b_2\in P$ with $h(a_1)b_1=h(a_2)b_2$, there exist $a_3,a_4\in Q$ and $b\in P$ such that $b_1=h(a_3)b$, $b_2=h(a_4)b$, and $a_1a_3=a_2a_4$. Set $M=(\mathscr{M}/\mathcal{O}_X^\times)_{\bar{x}}$ and $N=(\mathscr{N}/\mathcal{O}_Y^\times)_{\overline{f(x)}}$. Let $\tilde{h}\colon N\to M$ be the homomorphism induced by f. Then, we have a commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{\tilde{h}} & M \\ {\scriptstyle\psi}\uparrow & & \uparrow{\scriptstyle\phi} \\ Q & \xrightarrow[h]{} & P. \end{array}$$

Since f is integral, the ring homomorphism $\boldsymbol{Z}[N] \to \boldsymbol{Z}[M]$ induced by $\tilde{h}$ is flat, and hence $\tilde{h}\colon N \to M$ satisfies the above condition. Recall that the morphism ϕ maps P surjectively onto M, and $P/R \xrightarrow{\sim} M$ by ϕ where R is a subgroup in P. Similarly, ψ maps Q surjectively onto N, and $Q/S \xrightarrow{\sim} N$ by ψ where S is a submonoid in Q, since $Q \to \mathscr{N}$ is a chart of $\mathscr{N}$. For any $a_1, a_2 \in Q$, $b_1, b_2 \in P$ with $h(a_1)b_1 = h(a_2)b_2$, there exist $\tilde{a}_3, \tilde{a}_4 \in N$ and $\tilde{b} \in M$ such that $\phi(b_1) = \tilde{h}(\tilde{a}_3)\tilde{b}$, $\phi(b_2) = \tilde{h}(\tilde{a}_4)\tilde{b}$, and $\psi(a_1)\tilde{a}_3 = \psi(a_2)\tilde{a}_4$. Take $a_3, a_4 \in Q$ and $b' \in P$ such that $\psi(a_3) = \tilde{a}_3$, $\psi(a_4) = \tilde{a}_4$, and $\phi(b') = \tilde{b}$. We may assume $a_1a_3 = a_2a_4$. Then we have $b_1 = h(a_3)b'c_1$, $b_2 = h(a_4)b'c_2$ for some $c_1, c_2 \in R$. Since $h(a_1)b_1 = h(a_2)b_2$, $a_1a_3 = a_2a_4$, and P is fine, we have $c_1 = c_2$. Then, by setting $b = b'c_1 = b'c_2$, we have the desired result. □

7. The proof of Theorem 4.8. In this section, we give a proof of Theorem 4.8. If $V = \operatorname{Spec} k[P]$ is an affine toric variety, then it is easy to see that the log structure associated to $P \to k[P]$ is equivalent to the log structure $\mathcal{O}_X \cap j_* \mathcal{O}^\times_{V \setminus D} \hookrightarrow \mathcal{O}_X$ where D is the union of the closure of codimension one torus orbits of V and $j\colon V \setminus D \hookrightarrow V$ is the inclusion. Hence, the "if" part of Theorem 4.8 is easy to see. Let us prove the converse. Let $(X, \mathscr{M})$ be as in the assumption of Theorem 4.8 and $f\colon (X, \mathscr{M}) \to \operatorname{Spec} k$ the structure morphism. The key lemma is the following.

LEMMA 7.1. *We can take étale locally a chart $P \to \mathscr{M}$ of $\mathscr{M}$ such that*

1. *the chart $(P \to \mathscr{M}, 1 \to k^\times, 1 \to P)$ of f satisfies the conditions* (a) *and* (b) *in Theorem* 4.1,
2. *P is a fine saturated monoid, and has no torsion element.*

Here, by a torsion element, we mean an element $x \neq 1$ such that $x^n = 1$ for some positive integer n.

First, we are going to show that the theorem follows from the above lemma. Since the monoid P has no torsion element, P is the saturated submonoid of a finitely generated free abelian group P^{gp}. Hence, X is étale locally smooth over an affine toric variety, and the log structure $\mathscr{M}$ on X is étale locally equivalent to the pull-back of the log structure induced by the union of the closure of codimension one torus orbits. Since these log structures glue to the log structure $\mathscr{M}$ on X, the pull-back of the union of the closure of codimension one torus orbits glue to a divisor on X. In fact, this divisor is the complement of the largest open subset U such that $\mathscr{M}|_U$ is trivial, with the reduced scheme structure. Hence our assertion is proved.

Now, we are going to prove Lemma 7.1. We may work étale locally. Take a chart

$(P \to \mathscr{M}, 1 \to k^{\times}, 1 \to P)$ of f as in Theorem 4.1. We may assume that P is saturated. Then

$$P_{\text{tor}} := \{x \in P \mid x^n = 1 \text{ for some } n\}$$

is a subgroup in P. Take a decomposition $P^{\text{gp}} = G_{\text{f}} \oplus G_{\text{tor}}$ of the finitely generated abelian goup P^{gp}, where G_{f} (resp. G_{tor}) is a free (resp. torsion) subgroup of P^{gp}. Then we have $P_{\text{tor}} = P \cap G_{\text{tor}} = G_{\text{tor}}$ since P is saturated. Define a submonoid by $P_{\text{f}} = P \cap G_{\text{f}}$.

CLAIM 1. $P = P_{\text{f}} \oplus P_{\text{tor}}$.

PROOF. Take $x \in P$. Decompose $x = yz$ in P^{gp} so that $y \in G_{\text{f}}$ and $z \in G_{\text{tor}} = P_{\text{tor}}$. Since $y^n = (xz^{-1})^n = x^n \in P$ for n large, we have $y \in P$. Hence $y \in P_{\text{f}}$. □

CLAIM 2. *The homomorphism* $\alpha_{\text{f}}\colon P_{\text{f}} \hookrightarrow P \xrightarrow{\alpha} \mathcal{O}_X$ *defines a log structure equivalent to* $\mathscr{M}$.

PROOF. If $x \in P_{\text{tor}}$, then $\alpha(x) \in \mathcal{O}_X^{\times}$ since $\alpha(x)^n = 1$ for n large. Hence $\alpha(P_{\text{tor}}) \subset \mathcal{O}_X^{\times}$, which implies that the associated log structure of P_{f} is equivalent to that of P. □

Hence, the morphism f is equivalent to the morphism induced by the diagram

$$\begin{array}{ccc} P_{\text{f}} & \xrightarrow{\alpha_{\text{f}}} & \mathcal{O}_X \\ {\scriptstyle\varphi_{\text{f}}}\uparrow & & \uparrow \\ 1 & \xrightarrow[\lambda]{} & k\,. \end{array}$$

Then we have to check the conditions (a) and (b) in Theorem 4.1. The condition (a) is easy to verify. Let us check the condition (b). We need to show that the morphism

$$X \longrightarrow \operatorname{Spec} k[P_{\text{f}}]$$

induced by $X \to \operatorname{Spec} \boldsymbol{Z}[P] \to \operatorname{Spec} \boldsymbol{Z}[P_{\text{f}}]$ is smooth.

CLAIM 3. *The morphism*

$$\operatorname{Spec} k[P] \longrightarrow \operatorname{Spec} k[P_{\text{f}}] \tag{6}$$

induced by $P_{\text{f}} \hookrightarrow P$ *is étale.*

PROOF. Since $P = P_{\text{f}} \oplus P_{\text{tor}}$, we have $k[P] = k[P_{\text{f}}] \otimes_k k[P_{\text{tor}}]$. Since every element in P_{tor} is a root of unity, and since the order of P_{tor} is invertible in k, the morphism $k \hookrightarrow k[P_{\text{tor}}]$ is a finite separable extension of the field k. This shows that the morphism (6) is étale. □

Now we have proved Lemma 7.1, and hence, Theorem 4.8.

8. Formulation of log smooth deformations. From now on, we fix the following notation. Let k be a filed and Q a fine saturated monoid having no invertible element other than 1. Then we have a logarithmic point (cf. Definition 4.4) $k^\dagger = (\operatorname{Spec} k, Q)$. Let $(f, \varphi)\colon X^\dagger = (X, \mathscr{M}) \to k^\dagger = (\operatorname{Spec} k, Q)$ be a log smooth morphism in $\mathbf{LSch}^{\mathrm{fs}}$. We often denote this morphism of log schemes simply by f.

Let Λ be a complete Noetherian local ring with residue field k. For example, Λ is k or the ring of Witt vectors with entries in k when k is perfect. We denote by $\Lambda[\![Q]\!]$ the completion of the monoid ring $\Lambda[Q]$ along the maximal ideal $\mu + \Lambda[Q \setminus \{1\}]$ where μ denotes the maximal ideal of Λ. The completion $\Lambda[\![Q]\!]$ is a complete local Λ-algebra and is Noetherian since Q is finitely generated. If the monoid Q is isomorphic to $\boldsymbol{N}$, then the ring $\Lambda[\![Q]\!]$ is isomorphic to $\Lambda[\![t]\!]$ as a local Λ-algebra. Let $\mathscr{C}_{\Lambda[\![Q]\!]}$ be the category of Artinian local $\Lambda[\![Q]\!]$-algebras with residue field k, and $\hat{\mathscr{C}}_{\Lambda[\![Q]\!]}$ the category of pro-objects of $\mathscr{C}_{\Lambda[\![Q]\!]}$ (cf. [11]). For $A \in \operatorname{Obj}(\hat{\mathscr{C}}_{\Lambda[\![Q]\!]})$, we define a log structure on the scheme $\operatorname{Spec} A$ by the log structure

$$Q \oplus A^\times \longrightarrow A$$

associated to the homomorphism $Q \to \Lambda[\![Q]\!] \to A$. We denote by $(\operatorname{Spec} A, Q)$ the log scheme obtained in this way.

Definition 8.1. For $A \in \operatorname{Obj}(\mathscr{C}_{\Lambda[\![Q]\!]})$, a *log smooth lifting* of $f\colon (X, \mathscr{M}) \to (\operatorname{Spec} k, Q)$ on A is a log smooth morphism $\tilde{f}\colon (\tilde{X}, \tilde{\mathscr{M}}) \to (\operatorname{Spec} A, Q)$ in $\mathbf{LSch}^{\mathrm{fs}}$ together with a Cartesian diagram

$$\begin{array}{ccc} (X, \mathscr{M}) & \longrightarrow & (\tilde{X}, \tilde{\mathscr{M}}) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{f}} \\ (\operatorname{Spec} k, Q) & \longrightarrow & (\operatorname{Spec} A, Q) \end{array}$$

in $\mathbf{LSch}^{\mathrm{fs}}$. Two log smooth liftings are said to be *isomorphic* if they are isomorphic in $\mathbf{LSch}^{\mathrm{fs}}_{(\operatorname{Spec} A, Q)}$.

Note that $(\operatorname{Speck} k, Q) \to (\operatorname{Spec} A, Q)$ is an exact closed immersion, and hence the above diagram is Cartesian in $\mathbf{LSch}^{\mathrm{fs}}$ if and only if so is it in $\mathbf{LSch}$ (cf. Lemma 3.4). In particular, the underlying morphisms of log smooth liftings are (not necessarily flat) liftings in the usual sense. Moreover, since exact closed immersions are stable under base change, $(X, \mathscr{M}) \to (\tilde{X}, \tilde{\mathscr{M}})$ is also an exact closed immersion. If either $Q = \{1\}$ or $Q = \boldsymbol{N}$, the underlying morphisms of any log smooth liftings of f are flat since these morphisms of log schemes are integral (cf. Proposition 3.11). Hence in this case the underlying morphisms of log smooth liftings of f are flat liftings of f.

Remark 8.2. In Definition 8.1, we assume that any log smooth lifting is log smooth. This assumption is crucial since any lifting of a log smooth morphism is not necessarily log smooth. Here is an example due to the referee. Set $Q = \boldsymbol{N}$ and $X = \operatorname{Spec} k[x]$. The morphism $X \to \operatorname{Speck} k$ of schemes induces a log structure $\mathscr{M}$ on X such that the morphism $f\colon (X, \mathscr{M}) \to (\operatorname{Spec} k, \boldsymbol{N})$ is strict, and hence log smooth (cf.

Proposition 3.8). The strict morphism

$$\tilde{f}\colon (\operatorname{Spec} k[x, \varepsilon]/(x\varepsilon, \varepsilon^2), \tilde{\mathscr{M}}) \longrightarrow (\operatorname{Spec} k[\varepsilon]/(\varepsilon^2), \boldsymbol{N})$$

induced by the morphism Speck $k[x, \varepsilon]/(x\varepsilon, \varepsilon^2) \to \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$ of schemes gives a lifting of f to (Speck $k[\varepsilon]/(\varepsilon^2)$, $\boldsymbol{N}$), where $\boldsymbol{N} \to k[\varepsilon]/(\varepsilon^2)$ is defined by $1 \mapsto \varepsilon$. Then $\tilde{f}$ is not log smooth since $\tilde{f}$ is strict but the underlying morphism Spek $k[x, \varepsilon]/(x\varepsilon, \varepsilon^2) \to$ Speck $k[\varepsilon]/(\varepsilon^2)$ of schemes is not flat (cf. Proposition 3.11, Corollary 4.3).

Take a local chart $(P \to \mathscr{M}, Q \to Q \oplus k^\times, Q \to P)$ of f extending the given $Q \to k$ as in Theorem 4.1 such that $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is injective and the induced morphism $X \to \operatorname{Spec} k \times_{\operatorname{Spec} \boldsymbol{Z}[Q]} \operatorname{Spec} \boldsymbol{Z}[P]$ is étale (cf. Remark 4.2). Then f factors through $\operatorname{Spec} k \times_{\operatorname{Spec} \boldsymbol{Z}[Q]} \operatorname{Spec} \boldsymbol{Z}[P]$ by the étale morphism $X \to \operatorname{Spec} k \times_{\operatorname{Spec} \boldsymbol{Z}[Q]} \operatorname{Spec} \boldsymbol{Z}[P]$ and the natural projection étale locally. For $A \in \operatorname{Obj}(\mathscr{C}_{\Lambda[\![Q]\!]})$, an étale lifting

$$\tilde{X} \longrightarrow \operatorname{Spec} A \times_{\operatorname{Spec} \boldsymbol{Z}[Q]} \operatorname{Spec} \boldsymbol{Z}[P] \tag{7}$$

of $X \to \operatorname{Spec} k \times_{\operatorname{Spec} \boldsymbol{Z}[Q]} \operatorname{Spec} \boldsymbol{Z}[P]$, with the naturally induced log structure, gives a local log smooth lifting of f. Conversely, suppose $\tilde{f}\colon (\tilde{X}, \tilde{\mathscr{M}}) \to (\operatorname{Spec} A, Q)$ is a local log smooth lifting of f on A.

LEMMA 8.3. *The local chart $(P \to \mathscr{M}, Q \to Q \oplus k^\times, Q \to P)$ of f lifts to a local chart $(P \to \tilde{\mathscr{M}}, Q \to Q \oplus A^\times, Q \to P)$ of $\tilde{f}$.*

PROOF. The proof is done by induction on the length of A. Take $A' \in \operatorname{Obj}(\mathscr{C}_{\Lambda[\![Q]\!]})$ with a surjective morphism $A \to A'$ such that $I = \operatorname{Ker}(A \to A') \neq 0$ and $I^2 = 0$. Let $\tilde{f}'\colon (\tilde{X}', \tilde{\mathscr{M}}') \to (\operatorname{Spec} A', Q)$ be a pull-back of $\tilde{f}$. Then $\tilde{f}'$ is a log smooth lifting of f to A'. By induction, we have a lifted local chart $(P \to \tilde{\mathscr{M}}', Q \to Q \oplus (A')^\times, Q \to P)$ of $\tilde{f}'$. The morphism $(\tilde{X}', \tilde{\mathscr{M}}') \to (\tilde{X}, \tilde{\mathscr{M}})$ is a thickening of order one. Let $R = \boldsymbol{Z}[1/N]$, where N is the order of the torsion part of $P^{\mathrm{gp}}/Q^{\mathrm{gp}}$. Consider the commutative diagram

$$\begin{array}{ccc}
(\tilde{X}', \tilde{\mathscr{M}}') & \longrightarrow & (\operatorname{Spec} R[P], P) \\
\downarrow & & \downarrow \\
(\tilde{X}, \tilde{\mathscr{M}}) & \longrightarrow & (\operatorname{Spec} R[Q], Q)
\end{array} .$$

Since $(\operatorname{Spec} R[P], P) \to (\operatorname{Spec} R[Q], Q)$ is log smooth, $(\tilde{X}, \tilde{\mathscr{M}}) \to (\operatorname{Spec} R[Q], Q)$ factors through $(\operatorname{Spec} R[P], P)$ by a morphism $g\colon (\tilde{X}, \tilde{\mathscr{M}}) \to (\operatorname{Spec} R[P], P)$. This morphism g defines a homomorphism $P \to \tilde{\mathscr{M}}$ of sheaves of monoids on $\tilde{X}$ such that the diagram

$$\begin{array}{ccc}
\mathscr{P} & \longrightarrow & \tilde{\mathscr{M}} \\
\uparrow & & \uparrow \\
Q & \longrightarrow & Q \oplus A^\times
\end{array}$$

is commutative. Since $\tilde{\mathscr{M}}/\mathscr{O}_{\tilde{X}}^\times \xrightarrow{\sim} \tilde{\mathscr{M}}'/\mathscr{O}_{\tilde{X}'}^\times$, we can easily show that the morphism $P \to \tilde{\mathscr{M}}$

defines a chart (cf. Lemma 3.3). □

Then $\tilde{f}$ factors through $\operatorname{Spec} A \times_{\operatorname{Spec} \boldsymbol{Z}[Q]} \operatorname{Spec} \boldsymbol{Z}[P]$ by the induced morphism $\tilde{X} \to \operatorname{Spec} A \times_{\operatorname{Spec} \boldsymbol{Z}[Q]} \operatorname{Spec} \boldsymbol{Z}[P]$ and the natural projection, and we have the following commutative diagram

$$\begin{array}{ccc}
X & \longrightarrow & \tilde{X} \\
\downarrow & & \downarrow \\
\operatorname{Spec} k \times_{\operatorname{Spec} \boldsymbol{Z}[Q]} \operatorname{Spec} \boldsymbol{Z}[P] & \longrightarrow & \operatorname{Spec} A \times_{\operatorname{Spec} \boldsymbol{Z}[Q]} \operatorname{Spec} \boldsymbol{Z}[P] \\
\downarrow & & \downarrow \\
\operatorname{Spec} k & \longrightarrow & \operatorname{Spec} A\,,
\end{array}$$

such that each square is Cartesian. We need to show that $\tilde{X} \to \operatorname{Spec} A \times_{\operatorname{Spec} \boldsymbol{Z}[Q]} \operatorname{Spec} \boldsymbol{Z}[P]$ is étale. Set $\tilde{Y} = \operatorname{Soec} A \times_{\operatorname{Spec} \boldsymbol{Z}[Q]} \operatorname{Spec} \boldsymbol{Z}[P]$. Since $\tilde{X} \to \operatorname{Spec} A$ is log smooth, $\Omega^1_{\tilde{X}^\dagger/A^\dagger}$ is a locally free $\mathcal{O}_{\tilde{X}}$-module. By Proposition 5.2, we have $\Omega^1_{\tilde{X}^\dagger/A^\dagger} \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_X \xrightarrow{\sim} \Omega^1_{X^\dagger/k^\dagger}$ ($\xrightarrow{\sim} \mathcal{O}_X \otimes_{\boldsymbol{Z}} (P^{\mathrm{gp}}/Q^{\mathrm{gp}})$), and hence we have $\Omega^1_{\tilde{X}^\dagger/A^\dagger} \xrightarrow{\sim} \mathcal{O}_{\tilde{X}} \otimes_{\boldsymbol{Z}} (P^{\mathrm{gp}}/Q^{\mathrm{gp}})$ étale locally. On the other hand, by direct calculation, we have $\Omega^1_{\tilde{Y}^\dagger/A^\dagger} \xrightarrow{\sim} \mathcal{O}_{\tilde{Y}} \otimes (P^{\mathrm{gp}}/Q^{\mathrm{gp}})$. Hence, we have $\Omega^1_{\tilde{Y}^\dagger/A^\dagger}|_{\tilde{X}} \xrightarrow{\sim} \Omega^1_{\tilde{X}^\dagger/A^\dagger}$. By [5, (3.8), (3.12)], $\tilde{X} \to \tilde{Y}$ is étale. Therefore, we have proved the following proposition.

PROPOSITION 8.4 (cf. [5, (3.14)]). *For $A \in \mathscr{C}_{\Lambda[\![Q]\!]}$, a log smooth lifting of f: $(X, \mathscr{M}) \to (\operatorname{Spec} k, Q)$ on A exists étale locally, and is unique up to isomorphism. In particular, log smooth liftings of an integral and log smooth morphism are integral.*

REMARK 8.5. In Definition 8.1, we assume that any log smooth lifting is log smooth. Without this assumption, Proposition 8.4 is false. For example, the log smooth morphism $f: (X, \mathscr{M}) \to (\operatorname{Spec} k, \boldsymbol{N})$ in Remark 8.2 has at least two different liftings to $(\operatorname{Spec} k[\varepsilon]/(\varepsilon^2), \boldsymbol{N})$, one of which is log smooth while the other is not log smooth.

Let $\tilde{f}: (\tilde{X}, \tilde{\mathscr{M}}) \to (\operatorname{Spec} A, Q)$ be a log smooth lifting of f to A, and $u: A' \to A$ a surjective homomorphism in $\mathscr{C}_{\Lambda[\![Q]\!]}$ such that $I^2 = 0$ where $I = \operatorname{Ker}(u)$. Suppose $\tilde{f}': (\tilde{X}', \tilde{\mathscr{M}}') \to (\operatorname{Spec} A', Q)$ is a log smooth lifting of f to A' which is also a lifting of $\tilde{f}$. Let $(P \to \tilde{\mathscr{M}}', Q \to Q \oplus (A')^\times, Q \to P)$ be a local chart of $\tilde{f}'$ which is a lifting of $(P \to \mathscr{M}, Q \to Q \oplus k^\times, Q \to P)$. Define a local chart $(P \to \tilde{\mathscr{M}}, Q \to Q \oplus A^\times, Q \to P)$ of $\tilde{f}$ by $P \to \mathscr{M}' \to \mathscr{M}$ and $Q \to Q \oplus (A')^\times \to Q \oplus A^\times$. An automorphism $\Theta: (\tilde{X}', \tilde{\mathscr{M}}') \xrightarrow{\sim} (\tilde{X}', \tilde{\mathscr{M}}')$ over $(\operatorname{Spec} A', Q)$, which is the identity on $(\tilde{X}, \tilde{\mathscr{M}})$, induces an automorphism $\theta: (\tilde{\mathscr{M}}')^{\mathrm{gp}} \xrightarrow{\sim} (\tilde{\mathscr{M}}')^{\mathrm{gp}}$. Consider the diagram

$$\begin{array}{ccccccccc}
 & & & & P^{\mathrm{gp}} & = & P^{\mathrm{gp}} & & \\
 & & & & {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle \alpha} & & \\
1 & \longrightarrow & 1+\mathscr{I} & \longrightarrow & (\tilde{\mathscr{M}}')^{\mathrm{gp}} & \longrightarrow & \tilde{\mathscr{M}}^{\mathrm{gp}} & \longrightarrow & 1\,.
\end{array}$$

For $a \in P^{\mathrm{gp}}$, the element $\alpha'(a) \cdot [\theta \circ \alpha'(a)]^{-1}$ is in $1+\mathscr{I}$. Then we have a morphism $\Delta: P^{\mathrm{gp}} \to \mathscr{I} = I \cdot \mathscr{O}_{\tilde{X}'} \xrightarrow{\sim} I \otimes_A \mathscr{O}_{\tilde{X}}$ by $\Delta(a) = \alpha'(a) \cdot [\theta \circ \alpha'(a)]^{-1} - 1$. The morphism Δ lifts to a morphism $\Delta: P^{\mathrm{gp}}/Q^{\mathrm{gp}} \to I \otimes_A \mathscr{O}_{\tilde{X}}$ and defines a morphism of $\mathscr{O}_{\tilde{X}}$-modules

$$\mathscr{O}_{\tilde{X}} \otimes_{\mathbf{Z}} (P^{\mathrm{gp}}/Q^{\mathrm{gp}}) \to I \otimes_A \mathscr{O}_{\tilde{X}}\,.$$

Since $\Omega^1_{\tilde{X}^\dagger/A^\dagger} \xrightarrow{\sim} \mathscr{O}_{\tilde{X}} \otimes_{\mathbf{Z}} (P^{\mathrm{gp}}/Q^{\mathrm{gp}})$ étale locally, this defines a local section of

$$\mathscr{H}\!om_{\mathscr{O}_{\tilde{X}}}(\Omega^1_{\tilde{X}^\dagger/A^\dagger}, I \otimes_A \mathscr{O}_{\tilde{X}}) \xrightarrow{\sim} \mathscr{D}er(\tilde{X}^\dagger, \mathscr{O}_{\tilde{X}}) \otimes_A I\,.$$

Conversely, for a local section $(D, D\log) \in \mathscr{D}er(\tilde{X}^\dagger, \mathscr{O}_{\tilde{X}}) \otimes_A I$, D induces an automorphism of $\mathscr{O}_{\tilde{X}'}$ and $D\log$ induces an automorphism of $\tilde{\mathscr{M}}'$, and then, induces an automorphism of $(\tilde{X}', \tilde{\mathscr{M}}')$. By this, applying the argument in SGA I [2, Exposé 3], we get the following proposition.

PROPOSITION 8.6 (cf. [5, (3.14)]). *Let $\tilde{f}: (\tilde{X}, \tilde{\mathscr{M}}) \to (\operatorname{Spec} A, Q)$ be a log smooth lifting of f to A, and $u: A' \to A$ a surjective homomorphism in $\mathscr{C}_{\Lambda[\![Q]\!]}$ such that $I^2 = 0$, where $I = \operatorname{Ker}(u)$ (i.e., $(\operatorname{Spec} A, Q) \to (\operatorname{Spec} A', Q)$ is a thickening of order ≤ 1).*

1. *The sheaf of germs of lifting automorphisms of $\tilde{f}$ to A' is*

$$\mathscr{D}er_{A^\dagger}(\tilde{X}^\dagger, \mathscr{O}_{\tilde{X}}) \otimes_A I\,.$$

2. *Any log smooth lifting of f to A' which lifts $\tilde{f}$ canonically induces an isomorphism from the set of all isomorphism classes of such liftings to*

$$\mathrm{H}^1(\tilde{X}, \mathscr{D}er_{A^\dagger}(\tilde{X}^\dagger, \mathscr{O}_{\tilde{X}})) \otimes_A I\,,$$

as pointed sets.

3. *The obstructions for lifting $\tilde{f}$ to A' are in*

$$\mathrm{H}^2(\tilde{X}, \mathscr{D}er_{A^\dagger}(\tilde{X}^\dagger, \mathscr{O}_{\tilde{X}})) \otimes_A I\,.$$

Define the *log smooth deformation functor* $\mathbf{LD} = \mathbf{LD}_{X^\dagger/k^\dagger}$ by letting $\mathbf{LD}_{X^\dagger/k^\dagger}(A)$ to be the set of isomorphism classes of log smooth liftings of $f: X^\dagger \to k^\dagger$ to A for $A \in \operatorname{Obj}(\mathscr{C}_{\Lambda[\![Q]\!]})$. This is a covariant functor from $\mathscr{C}_{\Lambda[\![Q]\!]}$ to the category $\mathbf{Ens}$ of sets such that $\mathbf{LD}_{X^\dagger/k^\dagger}(k)$ consists of one point. We shall prove the following theorem in the next section.

THEOREM 8.7. *The log smooth deformation functor $\mathbf{LD}_{X^\dagger/k^\dagger}$ has a representable hull (cf. [11]) if f is integral and X is proper over k.*

9. The proof of Theorem 8.7. In this section, we prove Theorem 8.7 by checking Schlessinger's criterion ([11, Theorem 2.11]) for $\mathbf{LD}$. Let $u_1: A_1 \to A_0$ and $u_2: A_2 \to A_0$ be morphisms in $\mathscr{C}_{\Lambda[\![Q]\!]}$. Consider the map

$$\mathbf{LD}(A_1 \times_{A_0} A_2) \longrightarrow \mathbf{LD}(A_1) \times_{\mathbf{LD}(A_0)} \mathbf{LD}(A_2)\,. \tag{8}$$

Then we need to check the following conditions.

(H1) *The map* (8) *is surjective whenever* $u_2: A_2 \to A_0$ *is surjective.*
(H2) *The map* (8) *is bijective when* $A_0 = k$ *and* $A_2 = k[\varepsilon]$, *where* $k[\varepsilon] = k[E]/(E^2)$.
(H3) $\dim_k(t_{\mathbf{LD}}) < \infty$, *where* $t_{\mathbf{LD}} = \mathbf{LD}(k[\varepsilon])$.

Suppose (H1) and (H2) are valid. Then by Proposition 8.6, we have an isomorphism

$$t_{\mathbf{LD}} \xrightarrow{\sim} \mathrm{H}^1(X, \mathscr{D}er_{k^\dagger}(X^\dagger, \mathcal{O}_X))$$

of k-linear spaces. Our assumption implies that $t_{\mathbf{LD}}$ is a finite dimensional vector space, since $\mathscr{D}er_{k^\dagger}(X^\dagger, \mathcal{O}_X)$ is a coherent $\mathcal{O}_X$-module. Thus, (H3) follows. Hence, we need to check (H1) and (H2). Set $B = A_1 \times_{A_0} A_2$. Let $v_i: B \to A_i$ be the natural map for $i = 1, 2$. We denote the morphisms of schemes associated to u_i and v_i also by u_i: $\operatorname{Spec} A_0 \to \operatorname{Spec} A_i$ and v_i: $\operatorname{Spec} A_i \to \operatorname{Spec} B$ for $i = 1, 2$, respectively.

PROOF OF (H1). Suppose the homomorphism $u_2: A_2 \to A_0$ is surjective. Take an element $(\eta_1, \eta_2) \in \mathbf{LD}(A_1) \times_{\mathbf{LD}(A_0)} \mathbf{LD}(A_2)$ where η_i is an isomorphism class of a log smooth lifting f_i: $(X_i, \mathscr{M}_i) \to (\operatorname{Spec} A_i, Q)$ for each $i = 1, 2$. Note that the underlying morphism of f_i's are flat since f_i's are integral (cf. Proposition 8.4). The equality $\mathbf{LD}(u_1)(\eta_1) = \mathbf{LD}(u_2)(\eta_2)$ $(= \eta_0)$ implies that there exists an isomorphism $(u_2)^*(X_2, \mathscr{M}_2) \xrightarrow{\sim} (u_1)^*(X_1, \mathscr{M}_1)$ over $(\operatorname{Spec} A_0, Q)$. Here, $(u_i)^*(X_i, \mathscr{M}_i)$ is the pull-back of $(X_i, \mathscr{M}_i)$ by u_i: $\operatorname{Spec} A_0 \to \operatorname{Spec} A_i$ for $i = 1, 2$. Set $(X_0, \mathscr{M}_0) = (u_1)^*(X_1, \mathscr{M}_1)$. We denote the induced morphism of log schemes $(X_0, \mathscr{M}_0) \to (X_i, \mathscr{M}_i)$ by $u_i^\dagger$ for $i = 1, 2$. Then we have the following commutative diagram:

$$\begin{array}{ccccc}
(X_1, \mathscr{M}_1) & \xleftarrow{u_1^\dagger} & (X_0, \mathscr{M}_0) & \xrightarrow{u_2^\dagger} & (X_2, \mathscr{M}_2) \\
{\scriptstyle f_1}\downarrow & & {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_2} \\
(\operatorname{Spec} A_1, Q) & \xleftarrow{u_1} & (\operatorname{Spec} A_0, Q) & \xrightarrow{u_2} & (\operatorname{Spec} A_2, Q)\ .
\end{array}$$

We have to find an element $\xi \in \mathbf{LD}(B)$, which represents a lifting of f to B, such that $\mathbf{LD}(v_i)(\xi) = \eta_i$ for $i = 1, 2$. Consider the scheme $Z = (|X|, \mathcal{O}_{X_1} \times_{\mathcal{O}_{X_0}} \mathcal{O}_{X_2})$ over $\operatorname{Spec} B$. Define a log structure on Z by the natural homomorphism

$$\mathscr{N} = \mathscr{M}_1 \times_{\mathscr{M}_0} \mathscr{M}_2 \longrightarrow \mathcal{O}_Z = \mathcal{O}_{X_1} \times_{\mathcal{O}_{X_0}} \mathcal{O}_{X_2}\,.$$

It is easy to verify that this homomorphism is a log structure. Since the diagram

$$\begin{array}{ccccc}
 & & \mathscr{N} & \longrightarrow & \mathcal{O}_Z \\
 & & \uparrow & & \uparrow \\
Q & \xrightarrow{\sim} & Q \times_Q Q & \longrightarrow & B
\end{array}$$

is commutative, we have a morphism $g: (Z, \mathscr{N}) \to (\operatorname{Spec} B, Q)$ of log schemes. By construction, we have a morphism v_i: $(X_i, \mathscr{M}_i) \to (Z, \mathscr{N})$ for $i = 1, 2$ such that the diagram

$$\begin{array}{ccc}(X_1, \mathscr{M}_1) & \xrightarrow{v_1^\dagger} & (Z, \mathscr{N}) \\ {\scriptstyle u_1^\dagger}\uparrow & & \uparrow{\scriptstyle v_2^\dagger} \\ (X_0, \mathscr{M}_0) & \xrightarrow[u_2^\dagger]{} & (X_2, \mathscr{M}_2)\end{array}$$

is commutative. Since $u_2: A_2 \to A_0$ is surjective, the underlying morphism $X_1 \to Z$ of $v_1^\dagger$ is a closed immersion in the classical sense. We have to show that the morphism $v_1^\dagger$ is an exact closed immersion. Take a local chart $(P \to \mathscr{M}, Q \to Q \oplus k^\times, Q \to P)$ of f as in Theorem 4.1 such that $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is injective and the induced homomorphism $\boldsymbol{Z}[Q] \to \boldsymbol{Z}[P]$ is flat. By Lemma 8.3, this local chart lifts to a local chart of f_i for each $i=0, 1, 2$. Since $u_2: A_2 \to A_0$ is surjective, we have an isomorphism

$$\mathscr{N}/\mathscr{O}_Z^\times \xrightarrow{\sim} (\mathscr{M}_1/\mathscr{O}_{X_1}^\times) \times_{(\mathscr{M}_0/\mathscr{O}_{X_0}^\times)} (\mathscr{M}_2/\mathscr{O}_{X_2}^\times) .$$

By this, one sees that $P \xrightarrow{\sim} P \times_P P \to \mathscr{N}$ is a local chart of $\mathscr{N}$. This shows that $(Z, \mathscr{N})$ is a fine saturated log scheme, and $v_1^\dagger$ is an exact closed immersion. Hence, $(Z, \mathscr{N}) \to (\operatorname{Spec} B, Q)$ is a lifting of f to $(\operatorname{Spec} B, Q)$. Since the underlying morphism of f_i of schemes is a flat lifting of that of f for each $i=0, 1, 2$, $Z \to \operatorname{Spec} B$ is also a flat lifting of f. On the other hand, since the local lifting $Z \to \operatorname{Spec} B \times_{\operatorname{Spec} \boldsymbol{Z}[Q]} \operatorname{Spec} \boldsymbol{Z}[P] \to \operatorname{Spec} B$, where $Z \to \operatorname{Spec} B \times_{\operatorname{Spec} \boldsymbol{Z}[Q]} \operatorname{Spec} \boldsymbol{Z}[P]$ is smooth, is also a flat lifting of f, these two liftings coincide. This shows that $g: (Z, \mathscr{N}) \to (\operatorname{Spec} B, Q)$ is log smooth. Hence g represents an element $\xi \in \mathbf{LD}(B)$. It is easy to verify that $\mathbf{LD}(v_i)(\xi) = \eta_i$ for $i=1, 2$ since the morphisms f_1, f_2 and g have a common local chart. □

REMARK 9.1. In the above proof, the flatness of the underlying morphisms is important. It is used to prove that the lifting $(Z, \mathscr{N}) \to (\operatorname{Spec} B, Q)$ of f to $(\operatorname{Spec} B, Q)$ is log smooth. Without assuming that f is integral, this is false in general. Here is an example due to the referee. Let $f: (X_0, \mathscr{M}_0) = (\operatorname{Spec} k[x], \boldsymbol{N}^2) \to (\operatorname{Spec} k, \boldsymbol{N}^2)$ be a log smooth morphism defined by

$$\begin{array}{ccc} k[x] & \xleftarrow{\alpha} & \boldsymbol{N}^2 \\ \uparrow & & \uparrow{\scriptstyle h} \\ k & \xleftarrow[\beta]{} & \boldsymbol{N}^2 ,\end{array}$$

where $h(1, 0) = (1, 0)$, $h(0, 1) = (1, 1)$, $\alpha(1, 0) = 0$, $\alpha(0, 1) = x$, and $\beta(1, 0) = \beta(0, 1) = 0$. Set $(\operatorname{Spec} A_1, \boldsymbol{N}^2) = (\operatorname{Spec} A_2, \boldsymbol{N}^2) := (\operatorname{Spec} k[\varepsilon]/(\varepsilon^2), \boldsymbol{N}^2)$, where $\boldsymbol{N}^2 \to k[\varepsilon]/(\varepsilon^2)$ is defined by $(1, 0) \mapsto \varepsilon$, $(0, 1) \mapsto 0$. Let $f_i: (X_i, \mathscr{M}_i) \to (\operatorname{Spec} A_i, k[\varepsilon])$ be log smooth liftings of f for $i=1, 2$. Then, we have $X_i \xrightarrow{\sim} \operatorname{Spec} k[x, \varepsilon]/(\varepsilon^2, \varepsilon x)$, and hence $Z := (|X|, \mathscr{O}_{X_1} \times_{\mathscr{O}_{X_0}} \mathscr{O}_{X_2}) \xrightarrow{\sim} \operatorname{Spec} k[x, \varepsilon, \delta]/(\varepsilon^2, \delta^2, \varepsilon\delta, \varepsilon x, \delta x)$. But a log smooth lifting to $(\operatorname{Spec} B, \boldsymbol{N}^2)$ is isomorphic to $\operatorname{Spec} k[x, \varepsilon, \delta]/(\varepsilon^2, \delta^2, \varepsilon\delta)$, and is not isomorphic to Z.

PROOF OF (H2). We continue to use the same notation as above. We need the following lemma:

LEMMA 9.2. *Let* $g' : (Z', \mathscr{N}') \to (\operatorname{Spec} B, Q)$ *be a log smooth lifting of* f *with a commutative diagram*

$$\begin{tikzcd} (X_1, \mathscr{M}_1) \arrow[r] & (Z', \mathscr{N}') \\ (X_0, \mathscr{M}_0) \arrow[u, "u_1^\dagger"] \arrow[r, "u_2^\dagger"'] & (X_2, \mathscr{M}_2) \arrow[u] \end{tikzcd}$$

of liftings such that $(v_i)^*(Z', \mathscr{N}') \xleftarrow{\sim} (X_i, \mathscr{M}_i)$ *over* $(\operatorname{Spec} A_i, Q)$ *for* $i = 1, 2$. *Then the natural morphism* $(Z, \mathscr{N}) \to (Z', \mathscr{N}')$ *is an isomorphism.*

PROOF. We may work étale locally. By Lemma 8.3, the local chart $(P \to \mathscr{M}, Q \to Q \oplus k^\times, Q \to P)$ of f lifts to a local chart of g'. Take a local chart $(P \to \mathscr{N}, Q \to Q \oplus B^\times, Q \to P)$ of g by $P \to \mathscr{N}' \to \mathscr{N}$. Then, the schemes Z and Z' are smooth liftings of $X \to \operatorname{Spec} k \times_{\operatorname{Spec} \boldsymbol{Z}[Q]} \operatorname{Spec} \boldsymbol{Z}[P]$ to $\operatorname{Spec} B \times_{\operatorname{Spec} \boldsymbol{Z}[Q]} \operatorname{Spec} \boldsymbol{Z}[P]$. Hence we have only to show that the natural morphism $Z \to Z'$ of underlying schemes is an isomorphism. This follows from classical theory [11, Corollary 3.6] since each X_i is a smooth lifting of $X \to \operatorname{Spec} k \times_{\operatorname{Spec} \boldsymbol{Z}[Q]} \operatorname{Spec} \boldsymbol{Z}[P]$ to $\operatorname{Spec} A_i \times_{\operatorname{Spec} \boldsymbol{Z}[Q]} \operatorname{Spec} \boldsymbol{Z}[P]$ for $i = 0, 1, 2$. □

Let $g' : (Z', \mathscr{N}') \to (\operatorname{Spec} B, Q)$ be a log smooth lifting of f which represents a class $\xi' \in \mathbf{LD}(B)$. Suppose that the class ξ' is mapped to (η_1, η_2) by (8). Then,

$$(X_0, \mathscr{M}_0) \xrightarrow{\sim} (v_1 \circ u_1)^*(Z', \mathscr{N}') \xrightarrow{\sim} (v_2 \circ u_2)^*(Z', \mathscr{N}') \xleftarrow{\sim} (X_0, \mathscr{M}_0)$$

defines an automorphism θ of the lifting $(X_0, \mathscr{M}_0)$. If this automorphism θ lifts to an automorphism θ' of the lifting $(X_1, \mathscr{M}_1)$ such that $\theta' \circ u_1^\dagger = u_1^\dagger \circ \theta$, then replacing $(X_1, \mathscr{M}_1) \to (Z', \mathscr{N}')$ by $(X_1, \mathscr{M}_1) \xrightarrow{\theta'} (X_1, \mathscr{M}_1) \to (Z', \mathscr{N}')$, we have a commutative diagram as in Lemma 9.2, and then we have $\xi = \xi'$. Now if $A_0 = k$ (so that $(X_0, \mathscr{M}_0) = (X, \mathscr{M})$, $\theta = \mathrm{id}$), then θ' exists. □

10. Example 1: Log smooth deformations over a trivial base.

As we have seen in Theorem 4.8, any log scheme $(X, \mathscr{M})$ which is log smooth over $\operatorname{Spec} k$ with the trivial log structure is smooth over an affine toric variety étale locally.

EXAMPLE 10.1. (Usual smooth deformations.) Let X be a smooth algebraic variety over a field k. Then X with the trivial log structure is log smooth over $\operatorname{Spec} k$ (this is the case $D = 0$ in Corollary 4.9), and a log smooth deformation of X in our sense is nothing but a usual smooth deformation of X.

EXAMPLE 10.2. (Generalized relative deformations.) Let X be an algebraic variety over a field k. Assume that there exists a fine saturated log structure $\mathcal{M}$ on X such that $f\colon (X,\mathcal{M})\to \operatorname{Spec} k$ is log smooth. Then by Theorem 4.8, X is covered by étale open sets which are smooth over affine toric varieties, and the log structure $\mathcal{M}\to\mathcal{O}_X$ is equivalent to the log structure defined by

$$\mathcal{M}=j_*\mathcal{O}^\times_{X\setminus D}\cap\mathcal{O}_X \tag{9}$$

for some divisor D on X, where j is the inclusion $X\setminus D\hookrightarrow X$. In this situation, our log smooth deformation of f is a deformation of the pair (X, D). If X is smooth over k, then D is a reduced normal crossing divisor (cf. Corollary 4.9). Assume that X is smooth over k. Then we have an exact sequence

$$0\longrightarrow \mathcal{D}er_k(X^\dagger,\mathcal{O}_X)\longrightarrow \mathcal{D}er_k(X,\mathcal{O}_X)(=\Theta_X)\longrightarrow \mathcal{N}\longrightarrow 0\,,$$

where $\mathcal{N}$ is an $\mathcal{O}_X$-module written locally as

$$(\mathcal{N}_{D_1|X}\otimes\mathcal{O}_{D_1})\oplus\cdots\oplus(\mathcal{N}_{D_d|X}\otimes\mathcal{O}_{D_d})\,,$$

where $D_1,\ldots,D_d$ are local components of D, and $\mathcal{N}_{D_i|X}$ is the normal bundle of D_i in X for $i=1,\ldots,d$. Then, we have an exact sequence

$$\mathrm{H}^0(\mathcal{D},\mathcal{N})\longrightarrow t_{\mathbf{LD}}\longrightarrow \mathrm{H}^1(X,\Theta_X)\longrightarrow \mathrm{H}^1(D,\mathcal{N})\,.$$

In this sequence, $\mathrm{H}^0(D,\mathcal{N})$ is viewed as the set of isomorphism classes of locally trivial deformations of D in X, and $\mathrm{H}^1(D,\mathcal{N})$ is viewed as the set of obstructions for deformations of D in X. Hence this sequence explains the relation between the notion of log smooth deformations and that of usual smooth deformations. Note that, if D is a smooth divisor on X, the log smooth deformation is nothing but the *relative deformation* of the pair (X, D) studied by Makio [9] and others.

EXAMPLE 10.3. (Toric varieties.) Let X_Σ be a complete toric variety over a field k defined by a fan Σ in $N_{\mathbf{R}}$, and consider the log scheme $X^\dagger=(X_\Sigma,\Sigma)$ (cf. Example 2.6) over $\operatorname{Spec} k$. We have seen in Example 5.6 that

$$\mathcal{D}er_k(X^\dagger,\mathcal{O}_X)\xrightarrow{\sim}\mathcal{O}_X\otimes_{\mathbf{Z}}N\,,$$

and hence is a globally free $\mathcal{O}_X$-module. Since $\mathrm{H}^1(X,\mathcal{D}er_k(X^\dagger,\mathcal{O}_X))=0$, any toric variety is infinitesimally rigid with respect to log smooth deformations. Note that toric varieties without log structures need not be rigid with respect to usual smooth deformations.

11. Example 2: Smoothings of normal crossing varieties. Let $n>0$ be an integer. Let us write the n-dimensional affine space over a field F by $A^n_F=\operatorname{Spec} F[T_1,\ldots,T_n]$. For $1\leq i\leq n$, the hyperplane in A^n_F defined by the ideal (T_i) is denoted by $H_{i,F}$. We denote by 0 the origin of A^n_F.

Let k be a field. In this and the next sections, we mean, by a *normal crossing variety* over k of dimension $n-1$, a scheme X of finite type over k enjoying the following condition: For any closed point $x\in X$, there exist a scheme U and a point $y\in U$ together with étale morphisms

$$\varphi : U \to X \quad \text{and} \quad \phi : U \to \bigcup_{i=1}^{l_x} H_{i,k'}$$

such that $\varphi(y)=x$ and $\phi(y)=0$, where k' is a finite extension of k, depending on x, and $1 \le l_x \le n$. Clearly, the integer l_x depends only on the closed point x. We call it the *multiplicity* of X at x. A normal crossing variety X is said to be *simple* if every irreducible component of X is regular. It is easy to see that any scheme étale over a normal crossing variety is again a normal crossing variety. In this section, we consider a certain type of log structures on a normal crossing variety X, and discuss the deformations of the resulting log schemes.

Before discussing log structures on normal crossing varieties, we should consider their étale local structure in more detail.

LEMMA 11.1. *Let S and S' be Noetherian schemes and $\phi : S' \to S$ an étale morphism. Assume that every irreducible component of S is regular, and any union of irreducible components of S' is connected. Then, the morphism ϕ is injective in codimension zero, i.e., for any irreducible component T of S, there exists at most one irreducible component T' of S' such that $\phi(T') \subseteq T$. Moreover, in this case, we have $T' \cong T \times_S S'$, and hence, every irreducible component of S' is regular.*

PROOF. Let T be an irreducible component of S and η the generic point of T. If there exists a point $\eta' \in S'$ such that $\phi(\eta')=\eta$, then η' must be the generic point of an irreducible component of S'. Hence, one finds that $T \times_S S'$ is isomorphic to a finite union of irreducible components of S'. Since $T \times_S S'$ is regular and connected, it is irreducible. Therefore, whenever there exists $\eta' \in S'$ as above, we have $\overline{\{\eta'\}} \cong T \times_S S'$. □

DEFINITION 11.2. Let X be a normal crossing variety and $x \in X$ a closed point. An étale morphism $\varphi : U \to X$, with a point $y \in U$ such that $\varphi(y)=x$, is called a *local chart* around x if there exists a diagram

$$\begin{tikzcd} X & U \arrow[l, "\varphi"'] \arrow[r, hook, "\iota"] \arrow[d, "\phi"'] & V \arrow[d, "\Phi"] \\ & \bigcup_{i=1}^{l_x} H_{i,k'} \arrow[r, hook] & A^n_{k'} \end{tikzcd} \quad \text{sending} \quad \begin{tikzcd} x & y \arrow[l, mapsto, "\varphi"'] \arrow[d, mapsto, "\phi"] \\ & 0 \end{tikzcd}$$

with the square Cartesian, where k' is a finite extension of k and the lower horizontal arrow is the canonical closed immersion, such that

(a) $V = \operatorname{Spec} R$ is an affine scheme,
(b) Φ is an étale morphism,
(c) y is the unique point which is mapped to x by φ.

Note that, in the above definition, if we set $z_i = \iota^* \Phi^* T_i$ for $1 \le i \le l_x$, then each ideal (z_i) is prime and the irreducible components of U are precisely the closed subsets of U corresponding to the ideals $(z_1), \dots, (z_{l_x})$; these are easy consequences of Lemma 11.1.

PROPOSITION 11.3. *Let X be a normal crossing variety and $x \in X$ a closed point. Then there exists a local chart $\varphi: U \to X$ around x.*

Since any étale open set of X is again a normal crossing variety, the local charts form an open basis with respect to the étale topology.

PROOF OF PROPOSITION 11.3. Let us write $S = \bigcup_{i=1}^{l_x} H_{i,k'}$. By definition, one can take a scheme U and a point $y \in U$ with étale morphism $\varphi: U \to X$ and $\phi: U \to S$. Replacing U by a Zariski open subset, we may assume that (1) the scheme U is affine and connected, (2) y is the unique point of U which is mapped to x by φ, and (3) all the irreducible components of U contain y. Then, by Lemma 11.1, every irreducible component of U is regular, i.e., U is a simple normal crossing variety. Let $U = U_1 \cup \cdots \cup U_l$ be the decomposition of U into the union of irreducible components.

Again by Lemma 11.1, we have $l \le l_x$; however, the multiplicity l_y^U of U at y clearly satisfies $l_y^U \le l$, and since the multiplicities do not change under étale morphisms, we have $l_y^U = l_x$. Hence, we have $l = l_y^U = l_x$. Therefore, changing indices if necessary, we may assume that ϕ maps the generic point of U_i to that of the irreducible component $H_{i,k'}$ of S for $1 \le i \le l_x$.

The normal crossing variety S is a normal crossing divisor in the n-dimensional affine space $A^n_{k'} = \operatorname{Spec} k'[T_1, \dots, T_n]$ over k'. Then, by [2, Exposé 1, Proposition 8.1], replacing U by a Zariski open subset, we may assume that there exists a Cartesian diagram

$$\begin{tikzcd} U \arrow[r, hook, "\iota"] \arrow[d, "\phi"'] & V \arrow[d, "\Phi"] \\ S \arrow[r, hook] & A^n_{k'} , \end{tikzcd}$$

where the horizontal arrows are closed immersions and the vertical arrows are étale. Since we may assume V is affine, we are done. □

Next, let us discuss log structures on normal crossing varieties. Let X be a normal crossing variety over a field k. Suppose that X has a closed embedding $\iota: X \hookrightarrow V$ into a smooth variety V as a normal crossing divisor. We denote the open immersion $V \setminus X \hookrightarrow V$ by j. In this situation, one can define a log structure on X as in Corollary 4.9, i.e., by

$$\iota^*(\mathcal{O}_V \cap j_* \mathcal{O}^\times_{V \setminus X}) \longrightarrow \mathcal{O}_X ,$$

where ι^* denotes the pull-back of a log structure. Let us call this the *log structure* *associated to the embedding* $\iota: X \hookrightarrow V$. For a general normal crossing variety X, we cannot define such a log structure on X, because X may not have such an embedding. But, as we have seen in Proposition 11.3, X étale locally has such an embedding. Then, we can consider the log structure of this type for a general X defined as follows:

DEFINITION 11.4 (cf. [12]). A log structure $\mathscr{M} \to \mathcal{O}_X$ is said to be of *embedding type*, if the following condition is satisfied: There exists an étale covering $\{\varphi_\lambda: U_\lambda \to X\}_{\lambda\in\Lambda}$ by local charts, with the embeddings $\iota_\lambda: U_\lambda \hookrightarrow V_\lambda$ as in Definition 11.2, such that, for each $\lambda \in \Lambda$, the restriction $\varphi_\lambda^*\mathscr{M} \to \mathcal{O}_{U_\lambda}$ is equivalent (cf. Definition 2.3) to the log structure associated to the embedding ι_λ. If $\mathscr{M} \to \mathcal{O}_X$ is a log structure of embedding type of X, we call the log scheme $(X, \mathscr{M})$ a *logarithmic embedding*.

A log structure of embedding type has étale locally a chart described explicitly as follows: Let $\varphi: U \to X$ be a local chart, and

$$\begin{array}{ccc} U & \overset{\iota}{\hookrightarrow} & V \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ \bigcup_{i=1}^{l} H_{i,k'} & \hookrightarrow & A_{k'}^n , \end{array}$$

the Cartesian diagram as in Definition 11.2. The scheme $S = \bigcup_{i=1}^{l} H_{i,k'}$ has the log structure $\mathscr{M}_S$ associated to the embedding $S \hookrightarrow A_{k'}^n$. This log structure has a chart defined by

$$N^l \longrightarrow \mathscr{M}_S \qquad \text{with} \quad e_i \longmapsto t_i$$

for $1 \le i \le l$, where t_i is the image of T_i under

$$\Gamma(A_{k'}^n, \mathcal{O}_{A_{k'}^n} \cap j_*\mathcal{O}_{A_{k'}^n \setminus S}^\times) \longrightarrow \Gamma(S, \mathscr{M}_S)$$

with $j: A_{k'}^n \setminus S \hookrightarrow A_{k'}^n$ the open immersion. Then, the log structure $\mathscr{M}_U$ on U associated to the embedding ι is equivalent to the pull-back of $\mathscr{M}_S$ by ϕ. Hence, the log structure $\mathscr{M}_U$ is equivalent to the log structure associated to the pre-log structure (cf. §2) defined by

$$N^l \longrightarrow \mathcal{O}_U \qquad \text{with} \quad e_i \longmapsto z_i , \tag{10}$$

where $z_i = \iota^*\Phi^*T_i$ for $1 \le i \le l$.

The proof of the following proposition is straightforward and left to the reader:

PROPOSITION 11.5. *For any logarithmic embedding* $(X, \mathscr{M})$, *there exists an exact sequence of abelian sheaves*

$$1 \longrightarrow \mathcal{O}_X^\times \longrightarrow \mathscr{M}^{\mathrm{gp}} \longrightarrow \nu_* \boldsymbol{Z}_{\tilde{X}} \longrightarrow 0 , \tag{11}$$

where $\nu: \tilde{X} \to X$ *is the normalization of* X.

DEFINITION 11.6 (cf. [4], [6]). A log structure of embedding type $\mathscr{M} \to \mathcal{O}_X$ is said to be of *semistable type*, if there exists a homomorphism $\mathbf{Z}_X \to \mathcal{M}^{\mathrm{gp}}$ of abelian sheaves on X such that the diagram

$$\begin{tikzcd} \mathcal{M}^{\mathrm{gp}} \arrow[r] & \nu_* \mathbf{Z}_{\tilde{X}} \\ & \mathbf{Z}_X \arrow[ul] \arrow[u, "\mathfrak{d}"'] \end{tikzcd}$$

commutes, where $\mathfrak{d}: \mathbf{Z}_X \to \nu_* \mathbf{Z}_{\tilde{X}}$ is the diagonal homomorphism.

If $\mathcal{M}$ is a log structure of semistable type and $\mathbf{Z}_X \to \mathcal{M}^{\mathrm{gp}}$ is the the homomorphism as above, then one sees that $\mathbf{Z}_X \times_{\mathcal{M}^{\mathrm{gp}}} \mathcal{M}$ is isomorphic to $\mathbf{N}_X$, and thus, one gets a morphism $\mathbf{N}_X \to \mathcal{M}$ of sheaves of monoids. This morphism defines a morphism of log schemes

$$f: (X, \mathcal{M}) \longrightarrow (\operatorname{Spec} k, \mathbf{N}) .$$

Here, $(\operatorname{Spec} k, \mathbf{N})$ is the standard point (cf. Definition 4.4). Note that this morphism is similar to that discussed in Example 4.7, i.e., writing it in terms of the local chart $\varphi: U \to X$ as above, one sees that this morphism is induced by the diagram

$$\begin{array}{ccc} \mathbf{N}^l & \longrightarrow & \Gamma(U, \mathcal{O}_U) \\ \uparrow & & \uparrow \\ \mathbf{N} & \longrightarrow & k , \end{array}$$

where the upper horizontal arrow is induced by (10) and the right vertical arrow is the diagonal homomorphism. Hence, f has étale locally a chart $(\mathbf{N}^l \to \mathcal{M}, \mathbf{N} \to \mathbf{N} \oplus k^\times, \mathbf{N} \to \mathbf{N}^l)$ with $\mathbf{N} \to \mathbf{N}^l$ the diagonal homomorphism. We call this morphism f of log schemes the *logarithmic semistable reduction*. By Theorem 4.1, logarithmic semistable reductions are log smooth.

The criteria for the existence of these log structures are stated as follows: Let X be a normal crossing variety. Let us consider the $\mathcal{O}_X$-module $\mathcal{T}_X^1 = \mathcal{E}xt^1(\Omega_X^1, \mathcal{O}_X)$, which is called the *infinitesimal normal bundle* of X. It is well-known and easily verified that $\mathcal{T}_X^1$ is, in fact, an invertible $\mathcal{O}_D$-module, where D denotes the singular locus of X considered as a scheme with the reduced structure (cf. [1]). We will prove the following theorem in the next section:

THEOREM 11.7. *Let X be a normal crossing variety with D the singular locus.*

1. *There exists a log structure of embedding type on X if and only if there exists a line bundle $\mathcal{L}$ on X such that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_D \cong \mathcal{T}_X^1$.*
2. (cf. [6, (1.1)]) *There exists a log structure of semistable type on X if and only if X is d-semistable, i.e., $\mathcal{T}_X^1 \cong \mathcal{O}_D$.*

Let X be a d-semistable normal crossing variety and $\mathcal{M}$ a log structure of semistable type. Let $f: (X, \mathcal{M}) \to (\operatorname{Spec} k, \mathbf{N})$ be the log smooth morphism constructed as above. Since f is log smooth and integral (by Proposition 3.11), the log smooth deformation functor of f has a hull (cf. Theorem 8.7). Let us consider the infinitesimal deformations of f on an Artinian local $\Lambda[\![N]\!]$-algebra A. Let τ be the image of 1 under the morphism $N \to \Lambda[\![N]\!] \to A$ of monoids. Take a suitable local chart $\varphi: U \to X$ which induces a chart $(N^l \to \mathcal{M}, N \to N \oplus k^{\times}, N \to N^l)$ of f as above, where $N \to N^l$ is the diagonal homomorphism. Let $\tilde{f}: (\tilde{X}, \tilde{\mathcal{M}}) \to (\operatorname{Spec} A, Q)$ be a local log smooth lifting of f on A and $(N^l \to \tilde{\mathcal{M}}, N \to N \oplus A^{\times}, N \to N^l)$ a lifting local chart (cf. Lemma 8.3). If we denote by $\tilde{z}_i$ the image of e_i under $N^l \to \tilde{\mathcal{M}} \to \mathcal{O}_{\tilde{X}}$ for $1 \le i \le l$, then each $\tilde{z}_i$ is mapped to z_i by $A \to k$ and we have $\tilde{z}_1 \cdots \tilde{z}_l = \tau$. Hence, if $\tau = 0$, the deformation $\tilde{f}$ is locally trivial, and if $\tau \ne 0$, it is an infinitesimal smoothing.

In the complex analytic situation, a log smooth deformation of this type is nothing but a log deformation discussed by Kawamata and Namikawa in [6].

12. The proof of Theorem 11.7. In this final section, we prove Theorem 11.7. First, we should fix ideas and notation about the infinitesimal normal bundle introduced in the previous section.

Let X be a normal crossing variety over a field k. For a local chart $\varphi: U = \operatorname{Spec} A \to X$ of X around some closed point x, we use the following notation: Let $V = \operatorname{Spec} R$ and, using the notation as in Definition 11.2, set $Z_i = \Phi^* T_i$ for $1 \le i \le l$, where $l = l_x$. We set

$$I_j = (Z_j) \quad \text{and} \quad J_j := (Z_1 \cdots \hat{Z}_j \cdots Z_l)$$

for $1 \le j \le l$ (if $l = 1$, we set $J_1 = R$ as a convention), and set

$$I = I_1 \cdots I_l \quad \text{and} \quad J = J_1 + \cdots + J_l\,.$$

Then, we have $A = R/I$, and the ideal $I_j/I \subset A$, which is prime of high zero, is generated by $z_j := (Z_j \bmod I)$. The singular locus

$$D_U = D \times_X U$$

of U is the closed subscheme defined by J. We set

$$Q = R/J\,.$$

Note that, for $1 \le j \le l$, I_j/II_j is a free A-module of rank one and is generated by $\zeta_j := (Z_j \bmod II_j)$. There exists a natural isomorphism $I_j/II_j \otimes_A Q \xrightarrow{\sim} I_j/JI_j$ of Q-modules which maps $\zeta_j \otimes 1$ to $\xi_j := (Z_j \bmod JI_j)$. Moreover, there exists a natural isomorphism

$$I/I^2 \xrightarrow{\sim} I_1/II_1 \otimes_A \cdots \otimes_A I_l/II_l \tag{12}$$

of A-modules, and hence, the A-module I/I^2 is free of rank one and is generated by $\zeta_1 \otimes \cdots \otimes \zeta_l$. We denote by π_j the natural projection $I_j/II_j \to I_j/I \subset A$.

The cotangent complex of the morphism $k \to A$ is given by

$$L^{\bullet}: 0 \longrightarrow R \otimes_R A \xrightarrow{\ \delta\ } \Omega^1_{R/k} \otimes_R A \longrightarrow 0\,,$$

where δ is defined by $R \to F \cdot R \xrightarrow{d} \Omega^1_{R/k}$ with $F = Z_1 \cdots Z_l$ (cf. [8]). Then the tangent complex of U is the complex

$$\mathrm{Hom}_A(L^{\bullet}, A)\colon 0 \longrightarrow \Theta_{R/k} \otimes_R A \xrightarrow{\delta^*} \mathrm{Hom}_A(R \otimes_R A, A) \longrightarrow 0\,,$$

where $\Theta_{R/k} = \mathrm{Hom}_R(\Omega^1_{R/k}, R)$.

We define

$$T^1_A = \mathrm{Hom}_A(R \otimes_R A, A)/\delta^*(\Theta_{R/k} \otimes_R A)\,. \tag{13}$$

LEMMA 12.1. *We have a natural isomorphism*

$$T^1_A \xrightarrow{\sim} \mathrm{Hom}_A(I/I^2, A) \otimes_A Q\,. \tag{14}$$

PROOF. Consider the exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow \Omega^1_{R/k} \otimes_R A \longrightarrow \Omega^1_{A/k} \longrightarrow 0\,.$$

Taking $\mathrm{Hom}_A(-, A)$, we get the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}_A(\Omega^1_{A/k}, A) \longrightarrow \Theta_{R/k} \otimes_R A \xrightarrow{\nu} \mathrm{Hom}_A(I/I^2, A) \xrightarrow{\mu} T^1_A \longrightarrow 0\,.$$

The A-module $\mathrm{Hom}_A(I/I^2, A)$ is isomorphic to A by

$$\mathrm{Hom}_A(I/I^2, A) \ni u \longmapsto u(\zeta_1 \otimes \cdots \otimes \zeta_l) \in A\,.$$

With this identification, one finds easily that the image of ν in A is J. This implies that T^1_A is, in fact, a Q-module. Hence, we have a morphism

$$\mu \otimes_A Q\colon \mathrm{Hom}_A(I/I^2, A) \otimes_A Q \longrightarrow T^1_A$$

of Q-modules which is, in fact, an isomorphism. □

Considering all the local charts U on X, these modules T^1_A glue to an invertible $\mathcal{O}_D$-module isomorphic to $\mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_{X/k}, \mathcal{O}_X)$, which is denoted by $\mathcal{T}^1_X$; for later purposes, we describe its gluing data explicitly in the following.

Suppose we have two local charts $\varphi\colon U \to X$ and $\varphi'\colon U' \to X$ and an étale morphism $\psi\colon U \to U'$ such that $\varphi = \varphi' \circ \psi$. Since we are interested in the singular locus, we may assume $l > 1$ and $l' > 1$. For these local charts, we use all the notation as above. (For U', we denote them by $l', A', I', J', z'_i, \zeta'_j$, etc.) Let $f\colon A' \to A$ be the ring homomorphism corresponding to ψ. We need to show that the morphism ψ naturally induces an isomorphism $T^1_{A'} \otimes_{Q'} Q \xrightarrow{\sim} T^1_A$ of Q-modules. Let U_j (resp. U'_j) be the irreducible component of U (resp. U') corresponding to the ideal I_j/I (resp. I'_j/I') for $1 \le j \le l$ (resp. $1 \le j \le l'$). Since ψ is étale and injective in codimension zero (cf. Lemma 11.1), we may assume that the generic point of U_j is mapped to that of U'_j by ψ for $1 \le j \le l$. In particular, we have $l \le l'$.

CLAIM. *In the situation as above, we have the following*:

(a) *For* $1 \leq j \leq l$, *there exists an isomorphism, naturally induced by* f, *of* Q-*modules*

$$\tau_j\colon I'_j/I'I'_j \otimes_{A'} Q \xrightarrow{\sim} I_j/II_j \otimes_A Q\,. \tag{15}$$

(b) *For* $i > l$, *the natural projection* $\pi'_i\colon I'_i/I'I'_i \to I'_i/I' \subset A'$ *and* f *induce an isomorphism of* A-*modules*

$$\tilde{\rho}_i\colon I'_i/I'I'_i \otimes_{A'} A \longrightarrow A\,. \tag{16}$$

PROOF. (a) By the proof of Lemma 11.1, one sees that $U \times_{U'} U'_j \cong U_j$ for $1 \leq j \leq l$. This implies that $A/(I_j/I) \cong (A'/(I'_j/I') \otimes_{A'} A)$ $(\cong A/((I'_j/I') \otimes_{A'} A))$, and hence,

$$I_j/I \cong (I'_j/I') \otimes_{A'} A\,, \qquad (1 \leq j \leq l)\,. \tag{17}$$

For $1 \leq j \leq l$, we can set $f(z'_j) = u_j z_j$ for some $u_j \in A$. Here, each u_j is determined modulo J_j/I. Because of (17), $u_j z_j$ generates the ideal I_j/I, and hence, u_j is a unit in $A/(J_j/I)$ (and, needless to say, in $A/(J/I)$). (Note that u_j is not necessarily a unit in A, since A is not an integral domain for $l > 1$.) Then, by $\xi'_j \mapsto (u_j \bmod J/I)\xi_j$, we get the desired isomorphism.

(b) Since ψ is injective in codimension zero, the point I'_i/I' does not belong to $\psi(U)$. Then, ψ maps $U = \operatorname{Spec} A$ to $\operatorname{Spec} A'_{(I'_i/I')}$, and this implies that the image of elements of I'_i/I' under f is invertible. Hence, $\tilde{\rho}_i(\zeta'_i \otimes 1) = f(z'_i)$ is an invertible element of A, and $\tilde{\rho}_i$ is an isomorphism. □

Set $\rho_i := \tilde{\rho}_i \otimes_A Q$. Then, these isomorphisms τ_j's and ρ_i's induce

$$\tau := \tau_1 \otimes_Q \cdots \otimes_Q \tau_l \otimes_Q \rho_{l+1} \otimes_Q \cdots \otimes_Q \rho_{l'}\colon I'/I'^2 \otimes_{A'} Q \xrightarrow{\sim} I/I^2 \otimes_A Q\,. \tag{18}$$

The Q-dual of τ is the desired isomorphism (cf. Lemma 12.1). One can easily check that this isomorphism τ does not depend on the parameters z'_j, z_j; it is canonically induced by $f\colon A' \to A$. Hence, for any sequence of étale morphisms of local charts $U \xrightarrow{\psi} U' \xrightarrow{\psi'} U''$, we obviously have $\tau'' = \tau \circ (\tau' \otimes_{Q'} Q)$, where $\tau\colon I'/I'^2 \otimes_{A'} Q \xrightarrow{\sim} I/I^2 \otimes_A Q$, $\tau'\colon I''/I''^2 \otimes_{A''} Q' \xrightarrow{\sim} I'/I'^2 \otimes_{A'} Q'$ and $\tau''\colon I''/I''^2 \otimes_{A''} Q \xrightarrow{\sim} I/I^2 \otimes_A Q$ are the isomorphisms defined as above with respect to ψ, ψ' and $\psi' \circ \psi$, respectively.

In the rest of this section, once we introduce the local chart $\varphi\colon U \to X$, we will tacitly make use of the notation as above.

CONSTRUCTION 12.2. Here, we describe the log structure of embedding type by another étale local expression. Let $\varphi\colon U = \operatorname{Spec} A \to X$ be a local chart. For $m = (m_1, \ldots, m_l) \in N^l$, define an A-module P_m by

$$P_m := (I_1/II_1)^{\otimes m_1} \otimes_A \cdots \otimes_A (I_l/II_l)^{\otimes m_l}\,.$$

Each P_m is a free A-module of rank one and $P_{(1,\ldots,1)} \cong I/I^2$. The natural projections $\pi_j\colon I_j/II_j \to I_j/I \subseteq A$ induce a natural A-homomorphism

$$\sigma_m : P_m \longrightarrow A .$$

Define a monoid

$$M := \{(m, a) \mid m \in N^l,\ a : \text{a generator of } P_m\} ,$$

and a homomorphism $\sigma : M \to A$ of monoids by $(m, a) \mapsto \sigma_m(a)$. Then, one finds that the associated log structure $\alpha_U : \mathscr{M}_U \to \mathcal{O}_U$ of the pre-log structure $M \to A$ is that associated to the embedding $\iota : U \hookrightarrow V$.

CONSTRUCTION 12.3. Let $\mathscr{M}$ be a log structure of embedding type on X. Then, the exact sequence (11) induces the following commutative diagram of natural morphisms:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Z}_X}(\mathbf{Z}_X, \nu_* \mathbf{Z}_{\tilde{X}}) & \xrightarrow{\partial} & \mathrm{Ext}^1_{\mathbf{Z}_X}(\mathbf{Z}_X, \mathcal{O}_X^\times) \\ \cong \downarrow & & \cong \downarrow \\ \mathrm{H}^0(X, \nu_* \mathbf{Z}_{\tilde{X}}) & \xrightarrow[\partial]{} & \mathrm{H}^1(X, \mathcal{O}_X^\times) . \end{array}$$

Let $\mathfrak{d} : \mathbf{Z}_X \to \nu_* \mathbf{Z}_{\tilde{X}}$ be the diagonal homomorphism. Then, $\partial(\mathfrak{d})$ defines a linear equivalence class of line bundles on X. We denote this class by $\mathbf{cl}_{\mathscr{M}}$. Let $\mathbf{Res}_D : \mathrm{Pic}\, X \to \mathrm{Pic}\, D$ be the restriction morphism.

CLAIM A. *For any log structure $\mathscr{M}$ of embedding type on X, we have* $\mathbf{Res}_D(\mathbf{cl}_{\mathscr{M}}) = [(\mathscr{T}_X^1)^\vee]$.

PROOF. We will prove this claim by writing out the class $\mathbf{cl}_{\mathscr{M}}$ explicitly. For any local chart $\varphi : U \to X$, we define the constant monoid M as in Construction 12.2; then, we have a homomorphism of monoids $\sigma : M \to \mathscr{M}_U$. Let $\mathfrak{d}_U$ be the restriction of $\mathfrak{d}$ to U. One can lift $\mathfrak{d}_U$ to a homomorphism $\tilde{\mathfrak{d}}_U : \mathbf{Z}_U \to M^{\mathrm{gp}}$ by $1 \mapsto ((1, \ldots, 1), \zeta_1 \otimes \cdots \otimes \zeta_l)$.

Suppose we have two local charts $\varphi : U \to X$ and $\varphi' : U' \to X$ and an étale morphism $\psi : U \to U'$ such that $\varphi = \varphi' \circ \psi$. As in the beginning of this section, let U_j (resp. U'_j) be the irreducible component of U (resp. U') corresponding to I_j/I (resp. I'_j/I') for $1 \le j \le l$ (resp. $1 \le j \le l'$), and assume that the generic point of U_j is mapped to that of U'_j by ψ for $1 \le j \le l$. Consider the gluing morphism $\gamma : \psi^* \mathscr{M}_{U'} \xrightarrow{\sim} \mathscr{M}_U$. Let $\sigma' : M' \to \psi^* \mathscr{M}_{U'}$ be similar to σ. Then, for $1 \le j \le l$, the global section $\gamma(\sigma'(e_j, \zeta'_j))$ of $\mathscr{M}_U$ is a multiple of $\sigma(e_j, \zeta_j)$ by a section of $\mathcal{O}_U^\times$; i.e., there exists $u_j \in A^\times$ such that $\gamma(\sigma'(e_j, \zeta'_j)) = u_j \sigma(e_j, \zeta_j)$. Note that, since γ is an equivalence of log structures, we have $f(z'_j) = u_j z_j$. As for $i > l$, $\gamma(\sigma'(e_i, \zeta'_i)) = \sigma(0, v_i)$ for some $v_i \in A^\times$. Note that this v_i is nothing but $\tilde{\rho}_i(\zeta'_i \otimes 1)$, where $\tilde{\rho}_i$ is as defined in (16).

Define a cocycle $\{\Delta_\psi\}$ by

$$\Delta_\psi = u_1 \cdots u_l v_{l+1} \cdots v_n .$$

This cocycle gives a class $[\Delta_\psi]$ in $\mathrm{H}^1(X, \mathcal{O}_X^\times)$, which, by the definition of the connecting homomorphism ∂, is nothing but the class $\mathbf{cl}_{\mathcal{M}}$. Set $\delta_\psi=(\varDelta_\psi \bmod J/I)$, i.e., $[\delta_\psi]=\mathbf{Res}_D([\varDelta_\psi])$. Then, since $\xi'_1\otimes\cdots\otimes\xi'_l\mapsto\delta_\psi\xi_1\otimes\cdots\otimes\xi_l$ defines an isomorphism

$$I'/I'^2\otimes_{A'}Q\xrightarrow{\sim}I/I^2\otimes_A Q$$

which is nothing but τ defined in (15), the class $[\delta_\psi]$ is the class of $(\mathscr{T}_X^1)^\vee$. □

Then, the first part of Theorem 11.7 follows from the following claim:

CLAIM B. *Let X be a normal crossing variety. Then, the map*

$$\left\{\begin{matrix}\text{equivalence classes of log struc-}\\ \text{tures of embedding type on } X\end{matrix}\right\}\longrightarrow\{[\mathscr{L}]\in\operatorname{Pic}X \mid \mathscr{L}\otimes_{\mathcal{O}_X}\mathcal{O}_D\xrightarrow{\sim}(\mathscr{T}_X^1)^\vee\}\tag{19}$$

defined by $\mathcal{M}\mapsto\mathbf{cl}_{\mathcal{M}}$ is surjective.

In order to prove this claim, we need the following lemmas: Let $\varphi: U=\operatorname{Spec}A\to X$ be a local chart on X.

LEMMA 12.4. *The natural morphism*

$$\bigoplus_{j=1}^{l} J_j/I\longrightarrow J/I$$

of A-modules, induced by $J_j\hookrightarrow J$, is an isomorphism.

PROOF. The surjectivity is clear. Let us show the injectivity. Take $a_jZ_1\cdots\hat{Z}_j\cdots Z_l\in J_j$ for $1\le j\le l$ with $Z_1,\dots,Z_l$ as above such that

$$\sum_{j=1}^{l}a_jZ_1\cdots\hat{Z}_j\cdots Z_l=b\cdot Z_1\cdots Z_l,$$

where $a_j, b\in R$. Since R is an integral domain, a_j is divisible by Z_j, and hence, we have $a_jZ_1\cdots\hat{Z}_j\cdots Z_l\equiv 0 \pmod I$. □

Let $\pi_j\colon I_j/II_j\to I_j/I$ and $q_j: I_j/I\to I_j/JI_j$ ($\cong I_j/II_j\otimes_A Q$) be the natural projections, and set $p_j:=q_j\circ\pi_j$. Let $q\colon I/I^2\to I/JI$ ($\cong I/I^2\otimes_A Q$) be the natural projection.

LEMMA 12.5. *Let $M_1,\dots,M_l$ be free A-modules of rank one and set $M:=M_1\otimes_A\cdots\otimes_A M_l$. Suppose we are given an A-module isomorphism $\tilde{g}\colon M\xrightarrow{\sim}I/I^2$ and A-module homomorphisms $g_j\colon M_j\to I_j/I$ for $1\le j\le l$ such that*

1. *for each j, there exists a free generator δ_j of M_j satisfying $g_j(\delta_j)=z_j$,*
2. $(q_1\circ g_1)\otimes_Q\cdots\otimes_Q(q_l\circ g_l)=q\circ\tilde{g}$.

Then, there exists a unique collection $\{\tilde{g}_j\colon M_j\xrightarrow{\sim}I_j/II_j\}_{j=1}^{l}$ of A-isomorphisms such that $\pi_j\circ\tilde{g}_j=g_j$ for each j and $\tilde{g}_1\otimes_A\cdots\otimes_A\tilde{g}_l=\tilde{g}$.

PROOF. We fix the free generators δ_j of M_j as above. Then M is generated by $\delta_1\otimes\cdots\otimes\delta_l$. Set $\tilde{g}(\delta_1\otimes\cdots\otimes\delta_l)=v\zeta_1\otimes\cdots\otimes\zeta_l$ where $v\in A^\times$ (see the beginning of this section as for the notation). By the second condition, we have $v\equiv 1 \pmod{J/I}$, i.e.,

$$v=1+\sum_{j=1}^{l} a_j z_1\cdots\hat{z}_j\cdots z_l$$

for $a_j\in A$. We set $u_j=1+a_jz_1\cdots\hat{z}_j\cdots z_l$ and define $\tilde{g}_j$ by $\tilde{g}_j(\delta_j):=u_j\zeta_j$ for $1\leq j\leq l$. Then, since $v=u_1\cdots u_l$, each u_j is a unit in A and $\tilde{g}_j$ is an isomorphism. Moreover, we have $\tilde{g}_1\otimes\cdots\otimes\tilde{g}_l=\tilde{g}$ as desired. The uniqueness follows from Lemma 12.4. □

The proof of Claim B is done step by step as follows:

Step 1. Suppose that we are given a line bundle $\mathscr{L}$ on X satisfying $\mathscr{L}\otimes_{\mathcal{O}_X}\mathcal{O}_D\cong(\mathscr{T}_X^1)^\vee$. Suppose we have two local charts $\varphi\colon U\to X$ and $\varphi'\colon U'\to X$ and an étale morphism $\psi\colon U\to U'$ such that $\varphi=\varphi'\circ\psi$. Recall that $(I'_j/I')\otimes_{A'}A=I_j/I$ for $1\leq j\leq l$ (cf. (17)), and if $f(z'_j)=u_jz_j$, each u_j is determined modulo J_j/I. Giving the line bundle $\mathscr{L}$ as above is equivalent to giving a compatible system of isomorphisms

$$\tilde{\tau}\colon I'/I'^2\otimes_{A'}A\xrightarrow{\sim}I/I^2$$

for all such $U\to U'$, with $\tilde{\tau}\otimes_A Q=\tau$, where τ is defined as in (18). Then, we show that $\tilde{\tau}$ induces canonically an isomorphism of log structures $\psi^*\mathscr{M}_{U'}\xrightarrow{\sim}\mathscr{M}_U$, and prove that the log structures $\mathscr{M}_U$ glue to a log structure of embedding type on X. Moreover, since local charts form an étale open basis, we can pass through this procedure replacing U by a Zariski open subset if necessary. In particular, we may assume that each u_j as above is a unit in A, because $(u_j \bmod J/I)$ is a unit in $A/(J/I)$ (in case $l>1$).

Step 2. We construct isomorphisms

$$\tilde{\tau}_j\colon I'_j/I'I'_j\otimes_{A'}A\xrightarrow{\sim}I_j/II_j \tag{20}$$

of A-modules for $1\leq j\leq l$; this is done by considering the following three possible cases separately.

(i) If $l=l'=1$, i.e., $I_1=I$ and $I'_1=I'$, then we set $\tilde{\tau}_1\colon I'_1/I'I'_1\otimes_{A'}A\xrightarrow{\sim}I_1/II_1$ by $\tilde{\tau}_1:=\tilde{\tau}$.

(ii) If $l=1$ and $l'>1$, we define $\tilde{\tau}_1\colon I'_1/I'I'_1\otimes_{A'}A\xrightarrow{\sim}I_1/II_1$ as follows: Suppose $\tilde{\tau}$ maps $\zeta'_1\otimes\cdots\otimes\zeta'_{l'}\otimes 1$ to $v\zeta_1$, where $v\in A^\times$. Let $\tilde{\rho}_j\colon I'_i/I'I'_i\otimes_{A'}A\to A$ be as (16), for $1\leq i\leq l'$. Suppose, moreover, each $\tilde{\rho}_i$ for $i>1$ maps $\zeta'_i\otimes 1$ to $v_i\in A^\times$. Then, define $\tilde{\tau}_1$ by $\tilde{\tau}_1(\zeta'_1\otimes 1):=vv_2^{-1}\cdots v_{l'}^{-1}\zeta_1$.

(iii) Suppose $l>1$ and $l'>1$. We claim that, under the conditions

$$\pi_j\circ\tilde{\tau}_j=\tilde{\rho}_j\,,\qquad(1\leq j\leq l) \tag{21}$$

and

$$\tilde{\tau}_1\otimes_A\cdots\otimes_A\tilde{\tau}_l\otimes_A\tilde{\rho}_{l+1}\otimes_A\cdots\otimes_A\tilde{\rho}_{l'}=\tilde{\tau}\,, \tag{22}$$

the A-isomorphisms in (20) exist uniquely for $1\leq j\leq l$. Set $M_j:=I'_j/I'I'_j\otimes_{A'}A$ and $g_j:=\tilde{\rho}_j$ for $1\leq j\leq l$. Define $\tilde{g}$ by $\tilde{g}\otimes_A\tilde{\rho}_{l+1}\otimes_A\cdots\otimes_A\tilde{\rho}_{l'}=\tilde{\tau}$ (this is possible since $\tilde{\rho}_i(\zeta'_i\otimes 1)$ is a unit element in A for $i>l$), which is obviously an isomorphism. Then, since we assumed each u_j to be a unit in A, $M_j := I'_j/I'_jI' \otimes_{A'} A$, g, and g_j satisfy the conditions in Lemma 12.5. Hence, by Lemma 12.5, the isomorphisms in (20) exist uniquely.

Note that, in all cases, we have a commutative diagram

$$\tag{23} \begin{array}{ccc} I'_j/I'I'_j & \longrightarrow & I_j/II_j \\ {\scriptstyle \pi'_j}\big\downarrow & & \big\downarrow{\scriptstyle \pi_j} \\ A' & \xrightarrow[f]{} & A, \end{array}$$

for $1 \leq j \leq l$; this follows from (21) in case l, $l' > 1$, and is obvious in the other cases.

Step 3. Let us use notation as in Construction 12.2. These morphisms $\tilde{\tau}_j$ induce morphisms

$$\gamma_{m'} : P'_{m'} \longrightarrow P_m,$$

where $m=(m_1, \ldots, m_l)$ for $m'=(m_1, \ldots, m_{l'}) \in N^{l'}$. Then these $\gamma_{m'}$ induce naturally a morphism of monoids $M' \to M$ compatible with $M' \to A'$, $M \to A$ and f. By the construction of these morphisms, the induced morphism of sheaves of monoids $\gamma : \psi^*\mathscr{M}_{U'} \xrightarrow{\sim} \mathscr{M}_U$ is an isomorphism. By the commutative diagram (23), this isomorphism makes the following diagram commutative:

$$\begin{array}{ccc} \psi^*\mathscr{M}_{U'} & \xrightarrow{\sim} & \mathscr{M}_U \\ {\scriptstyle \psi^*\alpha_{U'}}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_U} \\ \mathscr{O}_U & = & \mathscr{O}_U; \end{array}$$

hence γ gives an equivalence of log structures. Our construction of the isomorphism γ is canonical in the following sense: Suppose we are given a sequence of étale morphisms $U \xrightarrow{\psi} U' \xrightarrow{\psi'} U''$ of local charts (with U and U' sufficiently small). We have $\gamma'' = \gamma \circ (\psi^*\gamma')$, where $\gamma : \psi^*\mathscr{M}_{U'} \xrightarrow{\sim} \mathscr{M}_U$, $\gamma' : \psi'^*\mathscr{M}_{U''} \xrightarrow{\sim} \mathscr{M}_{U'}$ and $\gamma'' : \psi^*\psi'^*\mathscr{M}_{U''} \xrightarrow{\sim} \mathscr{M}_U$ are the isomorphisms of log structures defined as above corresponding to ψ, ψ' and $\psi' \circ \psi$, respectively. This follows from the naturality of π_j and $\tilde{\rho}_j$, and the compatibility of $\tilde{\tau}$'s. Thus, we get a log structure $\mathscr{M}$ of embedding type on X. It is straightforward to check $\mathbf{cl}_{\mathscr{M}} = [\mathscr{L}]$. Thus, we complete the proof of Claim B, and hence, the proof of the first part of Theorem 11.7. □

Note that if X is a normal crossing divisor of a smooth variety V, and $\mathscr{M}$ is the log structure associated to the embedding $X \hookrightarrow V$, then the class $\mathbf{cl}_{\mathscr{M}}$ is nothing but the class of conormal bundle of X in V.

Next, let us prove the second part of Theorem 11.7. Suppose $\mathscr{M}$ is a log structure of semistable type. Consider the exact sequence

$$\tag{24} \mathrm{Hom}_{\mathbf{Z}_X}(\mathbf{Z}_X, \mathscr{M}^{\mathrm{gp}}) \xrightarrow{\pi} \mathrm{Hom}_{\mathbf{Z}_X}(\mathbf{Z}_X, \nu_*\mathbf{Z}_{\tilde{X}}) \xrightarrow{\partial} \mathrm{Ext}^1_{\mathbf{Z}_X}(\mathbf{Z}_X, \mathscr{O}_X^\times)$$

induced by the exact sequence (11). The morphism $\boldsymbol{Z}_X \to \mathscr{M}^{\mathrm{gp}}$ is mapped to $\mathfrak{d}$ by π. This implies that $\mathbf{cl}_{\mathscr{M}}=0$. Hence, $(\mathscr{T}_X^1)^\vee$ is a trivial line bundle on D.

Conversely, if X is d-semistable, there exists at least one log structure of embedding type on X by the first part of Theorem 11.7 proved above. Since $(\mathscr{T}_X^1)^\vee$ is trivial, we can take a log structure $\mathscr{M}$ of embedding type such that $\mathbf{cl}_{\mathscr{M}}=1$ by the natural surjection (19). Since the obstruction for the existence of a morphism $\boldsymbol{Z}_X \to \mathscr{M}^{\mathrm{gp}}$ which is mapped to $\mathfrak{d}$, is nothing but the class $\mathbf{cl}_{\mathscr{M}}$, we deduce that $\mathscr{M}$ is of semistable type. Thus, the proof of Theorem 11.7 is now completed. □

REFERENCES

[1] R. FRIEDMAN, Global smoothings of varieties with normal crossings, Ann. of Math. (2) 118 (1983), 75–114.

[2] A. GROTHENDIECK, Revêtements étales et groupe fondamentale, Lecture Notes in Math. 224, Springer-Verlag, Berlin, 1971.

[3] L. ILLUSIE, Introduction à la géométrie logarithmique, Seminar notes at Univ. of Tokyo in 1992.

[4] T. KAJIWARA, Logarithmic compactifications of the generalized Jacobian variety, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. 40 (1993), 473–502.

[5] K. KATO, Logarithmic structures of Fontaine-Illusie, in Algebraic Analysis, Geometry and Number Theory (J.-I. Igusa, ed.), Johns Hopkins Univ., 1988, 191–224.

[6] Y. KAWAMATA AND Y. NAMIKAWA, Logarithmic deformations of normal crossing varieties and smoothings of degenerate Calabi-Yau varieties, Invent. Math. 118 (1994), 395–409.

[7] K. KODAIRA AND D. C. SPENCER, On deformation of complex analytic structures, I–II, Ann. of Math. (2) 67 (1958), 328–466.

[8] S. LICHTENBAUM AND M. SCHLESSINGER, The cotangent complex of a morphism, Trans. Amer. Math. Soc. 128 (1967), 41–70.

[9] K. MAKIO, On the relative pseudo-rigidity, Proc. Japan Acad. Sect. IA 49 (1973), 6–9.

[10] T. ODA, Convex Bodies and Algebraic Geometry: An Introduction to the Theory of Toric Varieties, Ergebnisse der Math. (3) 15, Springer-Verlag, Berlin-New York, 1988.

[11] M. SCHLESSINGER, Functors of Artin rings, Trans. Amer. Math. Soc. 130 (1968), 208–222.

[12] J. H. M. STEENBRINK, Logarithmic embeddings of varieties with normal crossings and mixed Hodge structures, Math. Ann. 301 (1995), 105–118.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
KYOTO UNIVERSITY
KYOTO 606–01
JAPAN