Divisorial models of non-$\mathbb{Q}$-Gorenstein varieties

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Introduction

Let X be a normal $\mathbb{Q}$-Gorenstein ($\exists\ m \geq 1$ such that mK_X is a Cartier divisor) variety over the complex numbers, $\dim X = n$.
Let $f : Y \to X$ be a birational morphism with Y smooth variety. We define the relative canonical divisor as:

$$K_{Y/X} = K_Y - f^*(K_X)$$

and we can "measure" the singularities of X studying this divisor.
In particular, for any prime divisor $E \subset Y$, the discrepancy of f with respect to E is given by $a_E = \mathrm{val}_E(K_{Y/X})$ and we say that X is *canonical* (resp. *terminal*) if there exist a resolution of singularities $f : Y \to X$ of X such that $a_E \geq 0$ (resp. $a_E > 0$) for all exeptional divisors E over X.
There is a direct link between these singularities and the singularities that appear in the Minimal Model Program.
For example:

- {X normal, terminal, $\mathbb{Q}$-Gorenstein} is the smallest category in which MMP works.
- If X is the canonical model of a variety of general type, then X is normal, canonical, and $\mathbb{Q}$-Gorenstein.

Our aim is to investigate some surprising features of singularities of normal varieties in the non-$\mathbb{Q}$-Gorenstein case as defined by T. de Fernex and C. D. Hacon (cf. [1]).

We focus on three properties that for $\mathbb{Q}$-Gorenstein varieties are straightforward:

- The relative canonical divisor always has rational valuations (cf. [2, Theorem 92]).
- A canonical variety is always kawamata log terminal (cf. [3, Definition 2.34]).
- The canonical model, when it is defined, has canonical singularities.
- The jumping numbers are a set of rational numbers that have no accumulation points (cf. [4, Lemma 9.3.21]).

Setup

Consider a proper projective birational morphism $f : Y \to X$, Y normal variety. For any divisor D on X, the natural pullback of D to Y is

$$f^\natural D = \mathrm{div}(\mathcal{O}_X(-D) \cdot \mathcal{O}_Y),$$

so that we have the equality of reflexive sheaves, $\mathcal{O}_Y(-f^\natural D) = (\mathcal{O}_X(-D) \cdot \mathcal{O}_Y)^{\vee\vee}$.
For every $m \geq 1$, the m-th limiting relative canonical $\mathbb{Q}$-divisors $K^{\pm}_{m,Y/X}$ of Y over X are

$$K^{\pm}_{m,Y/X} := K_Y \pm \frac{1}{m} \cdot f^\natural(\mp mK_X).$$

Notation/Proposition:

$$K^-_{m,Y/X} \leq \lim K^-_{m,Y/X} = K^-_{Y/X} \leq$$
$$\leq K^+_{Y/X} = \lim K^+_{m,Y/X} \leq K^+_{m,Y/X}.$$

Note that all of those divisors coincide if X is $\mathbb{Q}$-Gorenstein, for m sufficiently divisible.
Let (X, Z) be a pair, with X a normal variety and Z a formal linear combination of proper closed subschemes. $f : Y \to X$ is a log resolution if Y is smooth and: (i) $\mathcal{O}_Y \cdot \mathscr{I}_Z$ is invertible (corresponding to a divisor E), (ii) $\mathrm{Ex}(f)$ is a divisor and (iii) $\mathrm{Supp}(E) \cup \mathrm{Supp}(\mathrm{Ex}(f))$ has simple normal crossing support.

Definition: (X, Z) is said to be:

- *log canonical* (resp. *log terminal*) if $\exists\ m_0 > 0$ such that for every F prime over X, $a^-_{m,F} := \mathrm{ord}_F(K^-_{m,Y/X}) + 1 - \mathrm{val}_F(Z) \geq 0$ (resp. > 0)
- *canonical* (resp. *terminal*) if for every F prime exceptional over X, $a^+_F := \mathrm{ord}_F(K^+_{Y/X}) + 1 - \mathrm{val}_F(Z) \geq 1$ (resp. > 1).

In particular we have a strict connection with the classical definitions using boundaries via the following (cf. [1, Thm 5.4]):

Theorem: Every effective pair (X, Z) admits m-compatible boundaries for $m \geq 2$, where, a boundary Δ is said to be m-compatible if:

i) $m\Delta$ is integral and $\lfloor \Delta \rfloor = 0$.
ii) No component of Δ is contained in the support of Z.
iii) f is a log resolution for the log pair $((X, \Delta); Z + \mathcal{O}_X(-mK_X))$.
iv) $K^{\Delta}_{Y/X} = K_{m,Y/X}$.

Main results

- We show that if X is canonical in the sense of [1], then the canonical ring is finitely generated. In particular the relative canonical divisor has rational valuations. We give an example of a (non canonical) variety X with an irrational valuation and of an irrational jumping number.
- We give an example of a variety with canonical but not klt singularities. We show that a canonical variety whose anticanonical ring is finitely generated, has klt singularities.
- We show that the finite generation of the canonical ring implies the existence of a canonical model with canonical singularities.
- We show that for a normal variety whose singularities are either klt or isolated, it is never possible to have accumulation points for the jumping numbers and we conjecture that this is the case for every normal variety.

Irrational valuations

We constructed an example of a threefold whose relative canonical divisor has an irrational valuation. The example is given by the resolution of a cone singularity over an abelian surface $X = E \times E$ where E is an elliptic curve.
Consider a double cover W of this surface ramified over a general very ample divisor $H \in |2\mathcal{L}|$ where $\mathcal{L}$ is an ample line bundle. $W = \mathrm{Spec}_X(\mathcal{O}_X \oplus \mathcal{L}^\vee)$ with projection $p : W \to X$. In particular

$$\omega_W = p^*(\omega_X \otimes \mathcal{L}).$$

Let L be an ample divisor on X such that p^*L defines an embedding $W \subset \mathbb{P}^n$. Let $C \subset \mathbb{P}^{n+1}$ be the projective cone over W.
Theorem: With the above construction, if $H \sim 6(f_1 + f_2)$ and $L \sim (3f_1 + 6f_2 + 6\delta)$, then the relative canonical divisor for a log resolution of C has an irrational valuation.
Theorem: With the above construction, if $H \sim 4(f_1 - 4\delta)$ and $(p^*L) \sim p^*(4f_1 + 4f_2 + 4\delta)$, the cone singularity is canonical but not klt.

Jumping numbers

Let (X, Z) be an effective pair. The multiplier ideal sheaf of (X, Z), denoted by $\mathcal{J}(X, Z)$, is the unique maximal element of $\{\mathcal{J}_m(X, Z)\}_{m \geq 1}$, where

$$\mathcal{J}_m(X, Z) := f_{m_*}\mathcal{O}_{Y_m}(\lceil K_{m,Y_m/X} - f_m^{-1}(Z) \rceil),$$

with $f_m : Y_m \to X$ a log resolution of the pair $(X, Z + \mathcal{O}_X(-mK_X))$. A number $\mu \in \mathbb{R}_{>0}$ is a *jumping number* of an effective pair (X, Z) if $\mathcal{J}(X, \lambda \cdot Z) \neq \mathcal{J}(X, \mu \cdot Z)$ for all $0 \leq \lambda < \mu$.
Theorem: The previous irrational valuation induces irrational jumping numbers.
Theorem: If (X, Z) is an effective pair such that X is a projective normal variety with either log terminal or isolated singularities, then the set of jumping numbers has no accumulation points.
The key step in the proof is the following:
Proposition: Let X be as above. Then, for any divisor $D \in \mathrm{WDiv}_{\mathbb{Q}}(X)$, there exists a very ample divisor A such that $\mathcal{O}_X(mD) \otimes \mathcal{O}_X(A)^{\otimes m}$ is globally generated for every $m \geq 1$.
Remark: It seems that it is not known if the proposition holds for any divisor $D \in \mathrm{WDiv}_{\mathbb{Q}}(X)$ on any projective normal variety (regardless of the singularity). We conjecture that this is the case. Note that by the Proposition this conjecture holds for surfaces.

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