Rationality and related problems

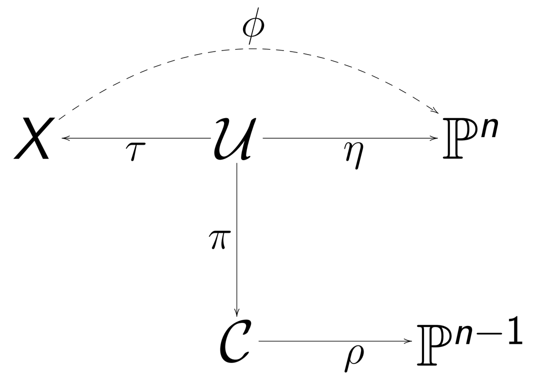
Davide Fusi
University of Utah

Introduction

In this work we follow the point of view proposed by Ionescu and Russo ([2]), for which one can characterize rationality by looking at the families of rational curves through a point of the variety.
The idea is that X is rational if through a point $x \in X_{reg}$, there exists a family of rational curves parametrized by a proper subscheme $\mathcal{C} \subset \mathrm{Chow}(X)$, such that:
- All C, $[C] \in \mathcal{C}$, are smooth at x;
- The family covers X;
- The induced morphism $\rho : \mathcal{C} \to PT_x$ is generically one to one.

The idea is that the family should play the role of the lines through a point in $\mathbb{P}^n$. Building on Ionescu-Russo work, we make this precise. Namely, we show that $\mathcal{C}$ determines a birational map $\phi : X \dashrightarrow \mathbb{P}^n$, that restricts to an isomorphism in an open neighborhood of x. Even more, we prove that the universal family $\pi : \mathcal{U} \to \mathcal{C}$ gives a resolution of the birational map. The family of rational curves carries information about the geometry of X.

THEOREM 1 There exists a birational map $\eta : \mathcal{U} \to \mathbb{P}^n$, such that the induced map $\phi = \eta \circ \tau^{-1} : X \dashrightarrow \mathbb{P}^n$ restricts to an isomorphism in an open neighborhood of x.



Using this result, we gain control on the geometry of rational varieties. We define a new intrinsic invariant for rational varieties, called rational degree, and we study rational varieties of small rational degree. We investigate how rationality deforms in families.

Varieties of small rational degree

The map ϕ, as in the Theorem 1, is induced by a linear system $|\Gamma| \subset |D|$, where
$$D = \tau_*(\pi^*(\rho^*(H)))$$
with $H \in |\mathcal{O}(1)|$. We can give the following definition.

DEFINITION 1 We define the rational degree of X as
$$\mathrm{ratdeg}(X) = \min\{\mathrm{ratdeg}_{\mathcal{C}}(X)\},$$
where if $\mathcal{C}$ is as in Theorem 1 and $C \in [\mathcal{C}]$, we define $\mathrm{ratdeg}_{\mathcal{C}}(X) = D \cdot C$.

EXAMPLE:

1) When $\dim X = 2$, then $\mathrm{ratdeg}(X) = \min\{C^2 | [C] \in \mathcal{C}\}$.
Moreover, one can easily see that:
$\mathrm{ratdeg}(\mathbb{P}^2) = 1$
$\mathrm{ratdeg}(\mathbb{P}^1 \times \mathbb{P}^1) = 2$
$\mathrm{ratdeg}(\mathbb{F}_n) = n$ for $n \geq 1$.

2) If $X \to Y$ is a birational morphism, then $\mathrm{ratdeg}(X) \leq \mathrm{ratdeg}(Y)$.

Using Theorem 1, for small values of the rational degree we have the following results.

THEOREM 2 Let X be a smooth rational variety, with $\dim X = n$ and $\mathrm{ratdeg}(X) = 1$. Then there exists a birational morphism $X \to \mathbb{P}^n$.

THEOREM 3 Let X be a rational variety with $\dim X = 3$ and $\mathrm{ratdeg}(X) = 2$. Then there exists a birational morphism $X \to Y$, where Y is a variety of minimal degree. In particular Y is one of the following:
i) $Q \subset \mathbb{P}^4$
ii) $S(0, 0, 2) \subset \mathbb{P}^4$
iii) $S(0, 1, 1) \subset \mathbb{P}^4$
iv) $S(0, 1, 2) \subset \mathbb{P}^5$
v) $S(1, 1, 1) \subset \mathbb{P}^5$
vi) $C(v_2(\mathbb{P}^2))$
where $S(a_0, a_1, a_2)$ is the image in $\mathbb{P}^{\sum a_i + 2}$ of $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(a_0) \oplus \mathcal{O}(a_1) \oplus \mathcal{O}(a_2))$ under the tautological map. All the cases are effective.

COROLLARY 1 Let X be a smooth rational variety, with $\dim X = n$, $\rho(X) = 1$ and $\mathrm{ratdeg}(X) = 2$. Then $X \cong Q \subset \mathbb{P}^{n+1}$.

Rationality in families

Using this point of view of families of rational curves on varieties, one proposes a possible approach towards the following long-standing conjecture.

CONJECTURE 1 Let $X \to Y$ be a smooth morphism. Then the set of rational fibers is countable union of closed subset of Y.

Although this remains open at the moment, we prove the following property regarding the deformation of families of rational curves.

THEOREM 4 Let $F : X \to Y$ be a morphism of proper projective varieties. Then the set of rational fibers is countable union of locally closed subset of Y . In the particular case $\dim Y = 1$, the locus of rational fibers is either an open set, or an Iitaka set.

While this statement can also be proved by general arguments, we hope that the methods used here may show some light toward the conjecture.

Using the same approach, an analogous result for unirationality can be proved.

Centers of rationality

DEFINITION 2 A point x on X is said to be a *center of rationality* if there exists a birational map $\phi : X \dashrightarrow \mathbb{P}^n$, that is an isomorphism in an open neighborhood U of x.

The notion of center of rationality was inspired by the following question which was raised by Pandharipande: if a variety X of dimension n is rational, then is it the case that for every point $p \in X$ there exists an open neighborhood U of p isomorphic to an open set of the projective space $\mathbb{P}^n$?
The general expectation is that the answer should be negative. The expected counter example was the blown up of $\mathbb{P}^3$ along a curve of positive genus. Howerver, as one can see in what follows, this can not be the case.

PROPOSITION 1 Suppose that $x \in X$ is a center of rationality. Let $f : Y \to X$ be the blow up along a smooth center Z such that $x \in Z$. Then every point $y \in f^{-1}(x)$ is a center of rationality.
At this point, it seems reasonable to focus on the following problem:

QUESTION 1 Let $f : Y \to X$ be the blow down of a smooth divisor $Z \subset Y$, with $f(Z)$ smooth. Fix a point $x \in f(Z)$, and suppose that every point $y \in f^{-1}(x)$ is a center of rationality. Does this implies that the point x is center of rationality on X?

Note that if the answer to the above question is positive, then the weak factorization theorem would imply that every point of a smooth rational variety is a center of rationality.

References

- D. Fusi. *Geometry of rational varieties of small rational degree* in preparation.
- P. Ionescu, F. Russo. *Conic-connected manifolds* to appear in J. Reine Angew. Math..