Optimizing quadratic functions of two variables

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Let $f(x,y)$ be a quadratic function such that

- $f(0,0) = 0$
- $f(x,y)$ has a critical point at $(0,0)$.

In general, we can write such a function as

$$f(x,y) = ax^2 + bxy + cy^2$$

for constants a, b, and c. This activity will use the geogebra app

`https://www.geogebra.org/m/zu8subte`

to visualize $f(x,y)$. Vary a, b, and c with the sliders to answer the following questions.

1. Set the sliders to show the graph of $f(x,y) = 3x^2 - xy + y^2$. i.e. $a = 3$, $b = -1$, and $c = 1$.

   (a) What type of critical point does f have?

   (b) Compute f_{xx} and f_{yy} at $(0,0)$.

   (c) Using the fact that f_{xx} and f_{yy} describe concavity in the x and y directions, describe how your computation in (b) agree with your answer to (a).

2. Set the sliders to show the graph of $f(x,y) = x^2 + 3xy + y^2$.

   (a) What type of critical point does f have?

   (b) Compute f_{xx} and f_{yy} at $(0,0)$.

   (c) Does your computation in (b) contradict your answer to (a)? Why or why not?

3. Set the sliders to $a = -1$ and $c = -0.5$. That is, we are considering functions of the form

$$f(x, y) = -x^2 + bxy - 0.5y^2.$$

   (a) Compute f_{xx}, f_{xy}, and f_{yy}.

   (b) Find a value of b such that $f(x, y)$ has a maximum.

   (c) Find a value of b such that $f(x, y)$ has a pringle point[1].

   (d) What are the ranges of values for b such that $f(x, y)$ has a maximum?

   (e) Is it possible to find a b such that $f(x, y)$ has a minimum? Explain your answer using concavity and your computations from part (a).

4. Set the sliders to show the graph of $f(x, y) = 2x^2 - xy - y^2$. Toggle the checkbox to display the $z = 0$ trace. You should see that the trace is two lines.

   (a) By visual inspection, estimate equations of the two lines that form the $z = 0$ trace.

[1] Or if you prefer, a "saddle point".

(b) Factor $f(x, y)$ into the product of two linear terms.

(c) Use your answer to (b) to find the equations of the two lines that form the $z = 0$ trace.

5. Vary a, b, and c. Observe the types of curves that the $z = 0$ trace is made of. It might help to unclick the second checkbox to hide the surface.

   (a) What are all the possible types of traces that show up?

   (b) Describe the relation between when $f(x, y)$ has a pringle point and the $z = 0$ trace of $f(x, y)$.

6. Recall that factoring a quadratic equation is the same process as finding its roots. Thus, use the quadratic formula to describe when $f(x, y) = ax^2 + bxy + cy^2$ factors into the product of two linear terms.

7. Writing $f(x, y) = ax^2 + bxy + cy^2$, compute and express f_{xx}, f_{xy}, and f_{yy} in terms of a, b, and c (using the general form of $f(x, y) = ax^2 + bxy + cy^2$). What is the determinant of the Hessian of $f(x, y)$ expressed in terms of a, b, and c.

8. The discriminant of the quadratic polynomial $f(x, y) = ax^2 + bxy + cy^2$ is $b^2 - 4ac$. Compare this to the determinant of the Hessian you computed in the previous part. What is the relationship between the two?

9. Combine your answers from problems 5, 6, and 7 to describe when $f(x, y)$ has a maximum/minimum, or pringle point in terms of f_{xx}, f_{xy}, and f_{yx}.

10. Circle the correct options.
    Let $D = f_{xx}f_{yy} - f_{xy}^2$. Then
    - If $D > 0$ and $f_{xx} > 0$, then $f(x, y)$ has a **max / min / saddle point**.
    - If $D > 0$ and $f_{xx} < 0$, then $f(x, y)$ has a **max / min / saddle point**.
    - If $D < 0$ then $f(x, y)$ has a **max / min / saddle point**.