MATH 8304: TOPICS IN FUNCTIONAL ANALYSIS
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PROBLEM LIST

ALONSO DELFÍN

This file will be constantly updated during the semester, so that the numbers of problems may change. When writing a solution, make sure to include the complete statement rather than the problem number.

Little proofreading has been done.

1. Review/Preliminaries

Problem 1.1. Let E and F be normed vector spaces and $a\colon E \to F$ a linear map. The operator norm of a was defined by

$$\|a\| := \sup\{\|a\xi\|_F \colon \|\xi\|_E \leq 1\}$$

1.1.1. Show that

$$\|a\| = \sup\{\|a\xi\|_F \colon \|\xi\|_E = 1\} = \sup\left\{\frac{\|a\xi\|_F}{\|\xi\|_E} \colon \xi \neq 0_E\right\} = \inf\{M > 0 \colon \|a\xi\|_F \leq M\|\xi\|_E\}.$$

Conclude that if $a \in \mathcal{B}(E,F)$, then $\|a\xi\|_F \leq \|a\|\|\xi\|_E$ for all $\xi \in E$.

1.1.2. Show that the following statements are equivalent:
1. a is continuous at 0_E,
2. a is continuous on E,
3. $a \in \mathcal{B}(E,F)$.

1.1.3. Show that if F is a Banach space, then so is $\mathcal{B}(E,F)$ when equipped with the operator norm.

> If X is any topological space, we use $C(X)$ to denote the vector space of continuous functions $X \to \mathbb{C}$.

Problem 1.2. Let X be a compact Hausdorff space.

1.2.1. Let A be any Banach algebra. Show that

$$C(X,A) := \{f \colon X \to A \colon \ f \text{ is continuous }\}$$

is a Banach algebra with norm $\|f\| = \sup_{x\in X} \|f(x)\|_A$ and point-wise multiplication.

1.2.2. Show that $C(X,A)$ is unital if A is.

1.2.3. Show that $C(X,A)$ contains a dense subalgebra that is isomorphic to the algebraic tensor product $C(X) \otimes A$. That is, find an injective algebra homomorphism $\varphi\colon C(X)\otimes A \to C(X,A)$ such that $\varphi(C(X)\otimes A)$ is a dense subspace of $C(X,A)$.

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Let X be a locally compact Hausdorff space. The space of continuous functions on X that vanish at infinity is

$$C_0(X) := \{f \in C(X) \colon \{x \in X \colon |f(x)| \geq \varepsilon\} \text{ is compact } \forall \varepsilon > 0\}$$

Problem 1.3. Let X be a locally compact Hausdorff space.

1.3.1. Show that $f \in C_0(X)$ if and only if $f \in C(X)$ and for each $\varepsilon > 0$ there is a compact space $K_\varepsilon \subseteq X$ such that $|f(x)| < \varepsilon$ for all $x \in X \setminus K_\varepsilon$.

1.3.2. Show that $\mathbb{R}$ with the usual topology is locally compact. Then show that $f \in C_0(\mathbb{R})$ if and only if $f \in C(\mathbb{R})$ and

$$\lim_{|x| \to \infty} |f(x)| = 0.$$

1.3.3. Let A be any Banach algebra. Show that

$$C_0(X, A) := \{f \in C(X, A) \colon (x \mapsto \|f(x)\|_A) \in C_0(X)\}$$

is a Banach algebra with norm $\|f\| = \sup_{x \in X} \|f(x)\|_A$ and point-wise multiplication.

1.3.4. Show that $C_0(X, A)$ contains a dense subalgebra that is isomorphic to the algebraic tensor product $C_0(X) \otimes A$.

Recall the classic convolution Banach algebras

$$\ell^1(\mathbb{Z}) := \left\{ a \colon \mathbb{Z} \to \mathbb{C} \colon \|a\|_1 := \sum_{n \in \mathbb{Z}} |a(n)| < \infty \right\}, (a * b)(k) := \sum_{n \in \mathbb{Z}} a(n)b(k-n)$$

and

$$L^1(\mathbb{R}) := \left\{ f \colon \mathbb{R} \to \mathbb{C} \colon \|f\|_1 := \int |f(x)| dx < \infty \right\}, (f * g)(y) := \int_{\mathbb{R}} f(x)g(y-x)dx$$

Problem 1.4. For each $a \in \ell^1(\mathbb{Z})$ and $f \in L^1(\mathbb{R})$ define $a^* \colon \mathbb{Z} \to \mathbb{C}$ and $f^* \colon \mathbb{R} \to \mathbb{C}$ by

$$a^*(n) := \overline{a(-n)}, \;\; f^*(x) := \overline{f(-x)}$$

1.4.1. Show that $\ell^1(\mathbb{Z})$ is unital.

1.4.2. Show that $a \mapsto a^*$ makes $\ell^1(\mathbb{Z})$ a Banach $*$-algebra, but that it is not a C*-algebra.

1.4.3. Show that $L^1(\mathbb{R})$ is not unital.

1.4.4. Show that $f \mapsto f^*$ makes $L^1(\mathbb{R})$ a Banach $*$-algebra, but that it is not a C*-algebra.

Problem 1.5. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and let $a \colon \mathcal{H}_1 \to \mathcal{H}_2$, $b \colon \mathcal{H}_2 \to \mathcal{H}_1$ be two functions satisfying

$$\langle a(\xi), \eta \rangle = \langle \xi, b(\eta) \rangle$$

for any $\xi \in \mathcal{H}_1, \eta \in \mathcal{H}_2$. Show that $a \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $b \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$, and that $\|a\| = \|b\|$.

(*Remark/Hint:* Here both a and b are not assumed to be linear to begin with. You will need the closed graph theorem to show boundedness).

Problem 1.6. Let X be a compact Hausdorff space and let μ a regular Borel measure on X such that $\mu(U) > 0$ for every open set $U \subseteq X$. Show that $C(X)$ is faithfully represented on the Hilbert space $L^2(X, \mu)$. That is, find an isometric $*$-homomorphism $\varphi \colon C(X) \to \mathcal{B}(L^2(X, \mu))$.

For a normed space E the weak-$*$ topology on $E' := \mathcal{B}(E, \mathbb{C})$ is the topology generated by the seminorms $\{\rho_\xi : \xi \in E\}$ where $\rho_\xi : E' \to \mathbb{R}_{\geq 0}$ is defined by $\rho_\xi(\varphi) := |\varphi(\xi)|$. In other words, $U \subseteq E'$ is said to be weak-$*$ open if and only if for each $\varphi_0 \in U$ there is $n \in \mathbb{Z}_{\geq 0}$, $\xi_1, \ldots, \xi_n \in E$, $r_1, \ldots, r_n \in [0, \infty)$, and $\psi_1, \ldots, \psi_n \in E'$ such that

$$\varphi_0 \in \{\varphi \in E' : \rho_{\xi_j}(\varphi - \psi_j) < r_j,\ j = 1, \ldots, n\} \subseteq U$$

Problem 1.7. Let E be a normed space.

1.7.1. Show that E' with the weak-$*$ topology is Hausdorff (note that $\varphi = 0_{E'}$ if and only if $\rho_\xi(\varphi) = 0$ for all $\xi \in E$).

1.7.2. Let $(\varphi_\lambda)_{\lambda \in \Lambda}$ be a net in E'. Show that $\varphi_\lambda \to \varphi$ in E' with the weak-$*$ topology if and only if $\varphi_\lambda(\xi) \to \varphi(\xi)$ in $\mathbb{C}$ for each $\xi \in E$.

1.7.3. Let

$$K := \{F \in \mathbb{C}^E : |F(\xi)| \leq \|\xi\| \ \forall \xi \in E\} \subseteq \prod_{\xi \in E} \overline{B_{\|\xi\|}(0)}$$

Show that the toplogy that $\mathrm{Ba}(E') = K \cap E'$ inherits from the product topology in K coincides with the weak-$*$ topology of E'.

1.7.4. Show that $\mathrm{Ba}(E')$ is closed in K and therefore compact.

Problem 1.8. Let E, F be Banach spaces and $u \in \mathcal{B}(E, F')$. Show that u extends uniquely to a weak-$*$ continuous a map $\widetilde{u} : E'' \to F'$ with $\|\widetilde{u}\| = \|u\|$.

Let E be a Banach space and let $F \subseteq E$ be a closed subset. We define $F^\perp \subseteq E'$ by

$$E^\perp = \{\varphi \in E' : \varphi|_F = 0\}.$$

Also, we say $\xi \sim_F \eta$ if and only if $\xi - \eta \in F$ and put $[\xi]_F := \{\eta \in E : \xi \sim_F \eta\}$. The quotient space $E/F := \{[\xi]_F : \xi \in E\}$ is a Banach space with the norm

$$\|[\xi]_F\| := \inf_{\eta \in F} \|\xi - \eta\|$$

Problem 1.9. Let E, F be Banach spaces with $F \subseteq E$.

1.9.1. Show that $F^\perp$ is a closed subset of E' with the usual norm topology.

1.9.2. Show that $E'/F^\perp$ is isometrically isomorphic to F'.

1.9.3. Show that $(E/F)'$ is isometrically isomorphic to $F^\perp$.

1.9.4. As subsets of E'', show that the weak-$*$ closure of F is $F^{\perp\perp}$

Let E, F be Banach spaces and $a \in \mathcal{B}(E, F)$. The dual of a is the map $a' : F' \to E'$ defined by

$$a(\varphi)\xi := \varphi(a\xi)$$

for any $\varphi \in F'$ and $\xi \in E$.

Problem 1.10. Let E, F be Banach spaces.

1.10.1. Show that $a' \in \mathcal{B}(F', E')$ and that $\|a\| = \|a'\|$.

1.10.2. If both E and F are Hilbert spaces, find a formula that relates a' with a^*. (*Hint*: The obvious maps $E \to E'$ and $F \to F'$ are bijective)

Problem 1.11. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and let $\mathcal{K} \subseteq \mathcal{H}_1$ be a closed subspace. Show that $\mathcal{K}$, $\mathcal{H}_1/\mathcal{K}$, $\mathcal{H}_1 \oplus_2 \mathcal{H}_2$ are also Hilbert spaces with their natural norms.

2. Tensor Products of Banach spaces

Recall that if V, W, and Z are vector spaces, we put

- $L(V,W)$ = all linear maps $V \to W$,
- $F(V,W) = \{a \in L(V,W) \colon a(V) \text{ is finite-dimensional}\}$,
- $V^\dagger = L(V,\mathbb{C})$, the algebraic dual of V,
- $L(V \times W, Z)$ = all bilinear maps $V \times W \to Z$.

Problem 2.1. Let U, V, W, Z be vector spaces. Show that

2.1.1. $\mathbb{C} \otimes V \cong V \otimes \mathbb{C} \cong V$,
2.1.2. $(V \otimes W) \otimes Z \cong V \otimes (W \otimes Z)$,
2.1.3. $L(V \otimes W, Z) \cong L(V \times W, Z) = L(V, L(W,Z))$,
2.1.4. $F(V,W) \cong V^\dagger \otimes W$.

Here $\cong$ means isomorphism of vector spaces. For **2.1.3.** and **2.1.4.** give a formula for the isomorphism and its inverse.

2.1.5. Show that $L(V,W) \otimes L(U,Z)$ naturally embeds in $L(V \otimes U, W \otimes Z)$. Moreover, if W and Z are finite dimensional, then show the embedding is an isomorphism.

Problem 2.2. Let V be a vector space and $\mathrm{ev} \colon V^\dagger \times V \to \mathbb{C}$ be the evaluation map

$$\mathrm{ev}(\varphi, \xi) := \varphi(\xi)$$

We denote by tr_V the linearization of ev, that is $\mathrm{tr}_V \in (V^\dagger \otimes V)^\dagger$ is the unique map such that the following diagram commutes

$$\begin{tikzcd} V^\dagger \times V \arrow[r, "\mathrm{ev}"] \arrow[d, "\otimes"'] & \mathbb{C} \\ F(V) \cong V^\dagger \otimes V \arrow[ur, dashed, "\mathrm{tr}_V"'] & \end{tikzcd}$$

2.2.1. If $\dim(V) = n < \infty$, then $F(V) \cong V^\dagger \otimes V \cong M_n(\mathbb{C})$. Show that under these idendifications, tr_V agrees with the usual trace of a matrix.

2.2.2. (Trace Duality) Let W be another vector space. For each $z = \sum_{j=1}^n \xi_j \otimes \eta_j \in V \otimes W$ define a map $S_z \colon W^\dagger \to V$ by

$$S_z(\psi) := \sum_{j=1}^n \psi(\eta_j)\xi_j.$$

Show that $z \to S_z$ is a well defined injection $V \otimes W \to F(W^\dagger, V)$ and that

$$\varphi(z) = \mathrm{tr}_V(S_z \circ L_\varphi) = \mathrm{tr}_{W^\dagger}(L_\varphi \circ S_z),$$

for any $\varphi \in (V \otimes W)^\dagger$ represented by $L_\varphi \in L(V, W^\dagger)$ via the map from **2.1.3.** with $Z = \mathbb{C}$.

Problem 2.3. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. Show that the formula

$$\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle := \langle \xi_1, \eta_1 \rangle \langle \xi_2, \eta_2 \rangle$$

for $\xi_1, \eta_1 \in \mathcal{H}_1$, $\xi_2, \eta_2 \in \mathcal{H}_2$ extends to an actual inner product on the algebraic tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$. Conclude that the completion of $\mathcal{H}_1 \otimes \mathcal{H}_2$ under the norm induced by this inner product is a Hilbert space.

Problem 2.4. Let E, F be normed vector spaces and let G be a Banach space. Show that every $\Phi \in \mathcal{B}(E \times F, G)$ has a unique extension $\widetilde{\Phi} \in \mathcal{B}(\widetilde{E} \times \widetilde{F}, G)$ satisfying $\|\Phi\| = \|\widetilde{\Phi}\|$, where $\widetilde{E}$ and $\widetilde{F}$ are the completions of E and F.

Problem 2.5. Show that the projective tensor product does not respect subspaces. That is, give an example of Banach spaces E, F, a subspace $G \subseteq E$, and $x \in G \otimes F$ such that

$$\|x\|_{\pi,E,F} < \|x\|_{\pi,G,F}.$$

Problem 2.6. Let E and F be Banach spaces and isometrically identify $(E \otimes^\pi F)'$ with both $\mathcal{B}(E, F')$ and $\mathcal{B}(F, E')$ as done in class. Show that

$$\|x\|_\pi = \sup\{|\varphi(x)| \colon \varphi \in \mathrm{Ba}(\mathcal{B}(E, F'))\} = \sup\{|\psi(x)| \colon \psi \in \mathrm{Ba}(\mathcal{B}(F, E'))\}$$

Problem 2.7. Let c_0 be the Banach space of complex valued sequences converging to 0 equipped with the sup norm. Show that

$$\|(\delta_1 \otimes \delta_1) + (\delta_2 \otimes \delta_2)\|_{\pi,c_0,c_0} = 1$$

where $\delta_1 = (1, 0, 0, \ldots)$ and $\delta_2 = (0, 1, 0, \ldots)$.

Problem 2.8. Let E and F be Banach spaces. Show that even though E is usually not complemented in E'', we do have that $E \otimes^\pi F$ is an isometric subspace of $E'' \otimes^\pi F''$. That is show that the algebraic inclusion $\iota \colon E \otimes F \hookrightarrow E'' \otimes F''$ is isometric with respect to the projective norm.

Problem 2.9. Set $E = \ell^2(\mathbb{Z}_{\geq 1}) \otimes^\pi \ell^2(\mathbb{Z}_{\geq 1})$ and let $\{\delta_n \colon n \in \mathbb{Z}_{\geq 1}\}$ be the standard unit basis for $\ell^2(\mathbb{Z}_{\geq 1})$ (i.e. $\delta_n(k) = \delta_{n,k}$).

2.9.1. Show that $\ell^1(\mathbb{Z}_{\geq 1})$ is isometrically isomorphic to the subspace of E generated by $\{\delta_n \otimes \delta_n \colon n \in \mathbb{Z}_{\geq 1}\}$.
2.9.2. Show that the isometric copy of $\ell^1(\mathbb{Z}_{\geq 1})$ in E is complemented.
2.9.3. Show that $\ell^2(\mathbb{Z}_{\geq 1}) \otimes^\pi \ell^2(\mathbb{Z}_{\geq 1})$ is not reflexive even though $\ell^2(\mathbb{Z}_{\geq 1})$ is.
2.9.4. How much of the above is still true for $\ell^p(\mathbb{Z}_{\geq 1})$ when $p \in [1, \infty) \setminus \{2\}$?

Problem 2.10. Let E, F be Banach spaces and $a \in \mathcal{B}(E, F)$. Show that the following statements are equivalent

(1) $E/\ker(a) \overset{1}{\cong} F$ via a,
(2) a is surjective and $\|\eta\|_F = \inf\{\|\xi\|_E \colon a(\xi) = \eta\}$,
(3) $a(B_1^E(0)) = B_1^F(0)$.

Let Ω be any set and E a Banach space. We say a function $a \colon \Omega \to E$ is *summable* if there is $\xi \in E$ such that for all $\varepsilon > 0$ there is a finite set $\Lambda_\varepsilon \subseteq \Omega$ such that

$$\Big\|\xi - \sum_{\lambda \in \Lambda} a(\lambda)\Big\|_E < \varepsilon$$

for all finite subsets $\Lambda \subseteq \Omega$ with $\Lambda \subseteq \Lambda_\varepsilon$. In such case we say that $\sum_{\omega \in \Omega} a(\omega)$ converges to ξ and write

$$\xi = \sum_{\omega \in \Omega} a(\omega).$$

Problem 2.11. Prove the following statements about summable functions:

2.11.1. Show that $a\colon \mathbb{Z}_{\geq 0} \to E$ converges to $\xi \in E$ if and only if for any bijection $\sigma\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ we have

$$\xi = \sum_{n=1}^{\infty} a(\sigma(n)).$$

2.11.2. Show that if $n \mapsto \|a(n)\|_E$ is summable then so is $a\colon \mathbb{Z}_{>0} \to E$. Give an example that the converse is false.

2.11.3. Show that $a\colon \Omega \to \mathbb{R}_{\geq 0}$ is summable if and only if $\{\sum_{\lambda\in\Lambda} a(\lambda)\colon \Lambda \subseteq \Omega \text{ is finite}\}$ is a bounded subset of $\mathbb{R}$, and in this case

$$\sum_{\omega\in\Omega} a(\omega) = \sup\Big\{\sum_{\lambda\in\Lambda} a(\lambda)\colon \Lambda \subseteq \Omega \text{ is finite}\Big\}.$$

2.11.4. Suppose that $E = \mathbb{C}$ and let ν be counting measure on Ω. Show that $a\colon \Omega \to \mathbb{C}$ is summable if and only if $\sum_{\omega\in\Omega} a(\omega) = \int_\Omega a d\nu$.

2.11.5. Let $a\colon \Omega \to E$ be summable. Show that there is a countable set $N \subseteq \Omega$ such that $a(\omega) = 0$ for all $\omega \in \Omega \setminus N$.

For any $p \in [1,\infty)$, we say a is *absolutely p-summable* if the function $\omega \mapsto \|a(\omega)\|_E^p$ is summable. We define

$$\ell^p(\Omega; E) := \{\text{absolutely } p\text{-summable functions } \Omega \to E\},$$

which is a Banach space with norm

$$\|a\|_p := \Big(\sum_{\omega\in\Omega} \|a(\omega)\|_E^p\Big)^{1/p}.$$

Problem 2.12. Let E be a Banach space.

2.12.1. Modify the proof given in class of $\ell^1(\mathbb{Z}_{>0}) \otimes^\pi E \overset{1}{\cong} \ell^1(\mathbb{Z}_{>0}; E)$ to show that $\ell^1(\Omega) \otimes^\pi E \overset{1}{\cong} \ell^1(\Omega; E)$ for any set Ω.

2.12.2. Show that $\ell^p(\Omega) \otimes^\pi E$ is not isometrically isomorphic to $\ell^p(\Omega; E)$ when $p \neq 1$.

Problem 2.13. In the definitions introduced in Problems 1.2 and 1.3 replace the Banach algebra A by a Banach space E to get the Banach spaces (potentially not algebras now) $C(X, E)$ and $C_0(X, E)$.

2.13.1. Show that $C(X) \otimes^I E \overset{1}{\cong} C(X, E)$,

2.13.2. Show that $C_0(X) \otimes^I E \overset{1}{\cong} C_0(X, E)$.

Problem 2.14. Let c_0 be as defined in Problem 2.7 and let $\ell^p := \ell^p(\mathbb{Z}_{<0})$ for $p \in [1,\infty)$

2.14.1. Show that $c_0 \otimes^I c_0 \overset{1}{\cong} c_0$,

2.14.2. Show that $\ell^2 \otimes^I \ell^2$ is not a Hilbert space,

2.14.3. Show that if $q\colon \ell^1 \to c_0$ is a quotient map, then $\mathrm{id}_{\ell^1} \otimes^I q$ is not a quotient map.

Problem 2.15. Let $p \in (1,\infty)$, let $p' = \frac{p}{p-1}$ be its Hölder conjugate, and let (X, Σ, μ) be a measure space. Show that $\mathcal{K}(L^p(X,\mu))$ (the algebra of compact operators on $L^p(\mu)$) is isometrically isomorphic to $L^p(X,\mu) \otimes^I L^{p'}(X,\mu)$. Conclude that $\mathcal{K}(L^p(X,\mu))$ and $\mathcal{K}(L^{p'}(X,\mu))$ are isometrically isomorphic.

Problem 2.16. Let E and F be Banach spaces with at least one of them finite dimensional. Show that $(E \otimes^\pi F)' \overset{1}{\cong} E' \otimes^I F'$ and that $(E \otimes^I F)' \overset{1}{\cong} E' \otimes^\pi F'$.

Let $p \in [1, \infty)$, let (X, Σ, μ) be a measure space, and let E be a Banach space. We define $L^p(\mu) \otimes_p E$ as the completion of $L^p(\mu) \otimes E$ with the norm

$$\Big\| \sum_{j=1}^{n} f_j \otimes \xi \Big\|_p := \Big(\int_X \Big\| \sum_{j=1}^{n} f_j(x)\xi_j \Big\|_E^p d\mu(x) \Big)^{1/p}.$$

Problem 2.17. Show that $L^1(\mu) \otimes_1 E \overset{1}{\cong} L^1(\mu) \otimes^\pi E$.

3. Operator Spaces

Problem 3.1. Show that the tensor product of Hilbert spaces from Problem 2.3 coincides with the tensor product $\otimes_2$. Then show that if $a \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $b \in \mathcal{B}(\mathcal{G}_1, \mathcal{G}_2)$, then $a \otimes b$ extends uniquely to $a \otimes_2 b \in \mathcal{B}(\mathcal{H}_1 \otimes_2 \mathcal{G}_1, \mathcal{H}_2 \otimes_2 \mathcal{G}_2)$ with $\|a \otimes_2 b\| = \|a\|\|b\|$.

Problem 3.2. Let $\alpha \in M_n$ and let $x \in \mathcal{B}(\mathcal{H})$. Show that the norm of $\alpha \otimes x$ seen, via Problem 3.1, as an element in $\mathcal{B}(\mathbb{C}^n \otimes_2 \mathcal{H})$ agrees with its norm as an element in $\mathcal{B}(\mathcal{H}^n)$ via the action defined, for $\vec{\xi} = (\xi_j)_{j=1}^n \in \mathcal{H}^n$ and $j \in \{1, \dots, n\}$, by

$$((\alpha \otimes x)\vec{\xi}\,)_j := \sum_{k=1}^{n} \alpha_{j,k} x \xi_k \in \mathcal{H}.$$

Problem 3.3. Let $\alpha \in M_n$. Show that $\|\alpha\|$ is the largest singular value of α. That is, show that if $\sigma(\alpha^*\alpha) = \{\text{eigenvalues of } \alpha^*\alpha\}$, then

$$\|\alpha\| = \max_{\lambda \in \sigma(\alpha^*\alpha)} \sqrt{|\lambda|}.$$

Let $\mathcal{H}$ be a Hilbert space and for each $\xi, \eta \in \mathcal{H}$ define the rank-1 operator $\theta_{\xi,\eta} \in \mathcal{B}(\mathcal{H})$ by

$$\theta_{\xi,\eta}(\zeta) := \langle \zeta, \eta \rangle \xi.$$

The compact operators on $\mathcal{H}$, $\mathcal{K}(\mathcal{H}) \trianglelefteq \mathcal{B}(\mathcal{H})$, coincide with the closure of finite rank operators, that is

$$\mathcal{K}(\mathcal{H}) = \overline{\operatorname{span}\{\theta_{\xi,\eta} \colon \xi, \eta \in \mathcal{H}\}}.$$

Problem 3.4. Let $\mathcal{H}$ be a Hilbert space and let $u \colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\overline{\mathcal{H}})$ be the isometry given by $u(a) = a'$ (see Problem 1.10).

3.4.1. Show that $u(\theta_{\xi,\eta}) = \theta_{\overline{\eta},\overline{\xi}}$ for all $\xi, \eta \in \mathcal{H}$,

3.4.2. show that $u(\mathcal{K}(\mathcal{H})) = \mathcal{K}(\overline{\mathcal{H}})$,

3.4.3. is u in $CB(\mathcal{K}(\mathcal{H}), \mathcal{K}(\overline{\mathcal{H}}))$?

Problem 3.5. Let $\mathcal{H}$ be a Hilbert space and fix $\eta_0 \in \mathcal{H}$ with $\|\eta_0\| = 1$. Show that $\{\theta_{\overline{\eta}_0,\overline{\xi}} \colon \xi \in \mathcal{H}\} \subset \mathcal{B}(\overline{\mathcal{H}})$ is completely isometrically isomorphic to $\mathcal{H}_{\text{row}}$.