# Representations of C*-correspondences on pairs of Hilbert spaces and $L^p$ correspondences

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# Outline

1. Review of Hilbert Modules
2. Representations of C*-correspondences
3. $L^p$-correspondences

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3. $L^p$-correspondences

# Hilbert Modules

Let $A$ be a C$^*$-algebra. Roughly speaking, a Hilbert $A$-module is a Hilbert space with scalars and values of the scalar product in $A$ instead of in $\mathbb{C}$.

## Definition

A **(right) Hilbert $A$-module** is a complex vector space X which is a (right) $A$-module with an $A$-valued (right) inner product

$$\begin{array}{ccc} \mathsf{X} \times \mathsf{X} & \to & A \\ (x, y) & \mapsto & \langle x, y \rangle_A \end{array}$$

such that X is complete with the norm

$$\|x\| := \|\langle x, x \rangle_A\|^{1/2}.$$

# Adjointable maps

**Definition**

Let X and Y be Hilbert $A$-modules. A map $t : \mathsf{X} \to \mathsf{Y}$ is said to be **adjointable** if there is a map $s : \mathsf{Y} \to \mathsf{X}$ such that for any $x \in \mathsf{X}$, and $y \in \mathsf{Y}$

$$\langle t(x), y \rangle_A = \langle x, s(y) \rangle_A.$$

The map $s$ is necessarily unique and henceforth denoted by $t^*$. We write $\mathcal{L}_A(\mathsf{X}, \mathsf{Y})$ for the space of adjointable maps from X to Y and $\mathcal{L}_A(\mathsf{X}) := \mathcal{L}_A(\mathsf{X}, \mathsf{X})$.

**Fact:** $\mathcal{L}_A(\mathsf{X})$ is a C$^*$-algebra when equipped with the operator norm.

**Fact:** Not every bounded linear map between Hilbert modules is adjointable.

# Compact-module maps

**Definition**

Let X and Y be Hilbert $A$-modules. For $x \in \mathsf{X}$ and $y \in \mathsf{Y}$, we define a map $\theta_{x,y} : \mathsf{Y} \to \mathsf{X}$ by

$$\theta_{x,y}(z) := x\langle y, z\rangle_A \ \forall z \in \mathsf{Y}$$

Then, we define the set of **compact-module maps** by

$$\mathcal{K}_A(\mathsf{Y},\mathsf{X}) := \overline{\operatorname{span}\{\theta_{x,y} : x \in \mathsf{X}, y \in \mathsf{Y}\}} \subseteq \mathcal{L}_A(\mathsf{Y},\mathsf{X}).$$

**Fact:** $\mathcal{K}_A(\mathsf{X}) := \mathcal{K}_A(\mathsf{X},\mathsf{X})$ is also a $C^*$-algebra, for it's a closed two sided ideal in $\mathcal{L}_A(\mathsf{X})$.
**Fact:** In general, elements in $\mathcal{K}_A(\mathsf{Y},\mathsf{X})$ need not to be in $\mathcal{K}(\mathsf{Y},\mathsf{X})$.

# Outline

# C*-correspondences

**Definition**

Let $A$ and $B$ be C*-algebras. An $(A, B)$ **C*-correspondence** is a pair $(\mathsf{X}, \varphi)$ where $\mathsf{X}$ is a right Hilbert $B$-module and $\varphi : A \to \mathcal{L}_B(\mathsf{X})$ is a $*$-homomorphism.

$$ {}_A\mathsf{X}_B $$

$$ \langle \varphi(a)x, y\rangle_B = \langle x, \varphi(a^*)y\rangle_B $$

# Hilbert Bimodules

Let $A$ and $B$ be C$^*$-algebras. Let X be a right Hilbert $B$-module that is also a left Hilbert $A$-module satisfying

$$ {}_A\langle x, y\rangle z = x\langle y, z\rangle_B, $$

for all $x, y, z \in$ X. For each $a \in A$ and $x \in$ X put $\lambda(a)x := ax$. Then $\lambda : A \to \mathcal{L}_B(\mathsf{X})$ and $(\mathsf{X}, \lambda)$ is an $(A, B)$ C*-correspondence.

These particular cases of $(A, B)$ C*-correspondences are called **Hilbert $A$-$B$-bimodules**.

# Representations of Hilbert bimodules

Representations of Hilbert bimodules on pairs of Hilbert spaces were introduced by Ruy Exel in 1993.

Let $\mathsf{X}$ be a Hilbert $A$-$B$-bimodule. Exel showed that there are two Hilbert spaces, say $\mathcal{H}_0$ and $\mathcal{H}_1$, such that

$$A \hookrightarrow \mathcal{L}(\mathcal{H}_1),$$
$$B \hookrightarrow \mathcal{L}(\mathcal{H}_0),$$
$$\mathsf{X} \hookrightarrow \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1),$$

where both module actions and inner products are the obvious ones inherited from the operators acting on the Hilbert spaces.

# Representations of C*-correspondences

Let $(\mathsf{X}, \varphi)$ be an $(A, B)$ C*-correspondence and $(\mathcal{H}_0, \mathcal{H}_1)$ a pair of Hilbert spaces. A **representation of $(\mathsf{X}, \varphi)$ on $(\mathcal{H}_0, \mathcal{H}_1)$** consists of a triple $(\pi_A, \pi_B, \pi_{\mathsf{X}})$ such that

1. $\pi_A$ is a representation of $A$ on $\mathcal{H}_1$,
2. $\pi_B$ is a representation of $B$ on $\mathcal{H}_0$,
3. $\pi_{\mathsf{X}} \colon \mathsf{X} \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is a linear map,

satisfying, for any $a \in A$, $b \in B$, $x, y \in \mathsf{X}$,

(i) $\pi_{\mathsf{X}}(\varphi(a)x) = \pi_A(a)\pi_{\mathsf{X}}(x)$

(ii) $\pi_{\mathsf{X}}(xb) = \pi_{\mathsf{X}}(x)\pi_B(b)$

(iii) $\pi_{\mathsf{X}}(x)^*\pi_{\mathsf{X}}(y) = \pi_B(\langle x, y\rangle_B)$

**Fact:** if either $\pi_A$ or $\pi_B$ are isometric, then so is $\pi_{\mathsf{X}}$.

**Theorem (D, 2022)**

*Isometric representations of any $(A, B)$ C*-correspondence on a pair of Hilbert spaces do exist.*

# Application # 1

**Theorem (D, 2022)**

*Let $(\mathsf{X}, \varphi)$ be an $(A, B)$ C*-correspondence such that $A$ acts nondegenerately on $\mathsf{X}$. Then, there is an $A$-valued left inner product on $\mathsf{X}$ making it a Hilbert $A$-$B$ bimodule if and only if $\mathcal{K}_B(\mathsf{X}) \subseteq \varphi(A)$.*

**Proof.** $(\Rightarrow)$ Easy: $\theta_{x,y}(z) = x\langle y, z\rangle_B = \varphi({}_A\langle x, y\rangle)z$.
$(\Leftarrow)$ Assume that $\mathcal{K}_B(\mathsf{X}) \subseteq \varphi(A)$ and let $(\pi_A, \pi_B, \pi_{\mathsf{X}})$ be an isometric representation of $(\mathsf{X}, \varphi)$. Then,

$$ {}_A\langle x, y\rangle := \pi_A^{-1}\big(\pi_{\mathsf{X}}(x)\pi_{\mathsf{X}}(y)^*\big). $$

is a well defined left $A$ valued inner product making $\mathsf{X}$ a Hilbert $A$-$B$-bimodule. ■

# Application # 2: Representations of $\mathcal{K}_B(\mathsf{X})$ and $\mathcal{L}_B(\mathsf{X})$

**Theorem (D, 2022)**

*Let $(\mathsf{X}, \varphi)$ be an $(A, B)$ C*-correspondence and let $(\pi_A, \pi_B, \pi_{\mathsf{X}})$ be a representation of $(\mathsf{X}, \varphi)$ on $(\mathcal{H}_0, \mathcal{H}_1)$ with $\pi_B$ injective. Suppose that $\pi_{\mathsf{X}}(\mathsf{X})\mathcal{H}_0$ is dense in $\mathcal{H}_1$. Then,*

- $\mathcal{K}_B(\mathsf{X})$ *is $*$-isomorphic to the closure of* $\operatorname{span}\{\pi_{\mathsf{X}}(x)\pi_{\mathsf{X}}(y)^* \colon x, y \in \mathsf{X}\}$ *in* $\mathcal{L}(\mathcal{H}_1)$,
- $\mathcal{L}_B(\mathsf{X})$ *is $*$-isomorphic to* $\{t \in \mathcal{L}(\mathcal{H}_1) \colon t\pi_{\mathsf{X}}(x), t^*\pi_{\mathsf{X}}(x) \in \pi_{\mathsf{X}}(\mathsf{X}) \text{ for all } x \in \mathsf{X}\}$.

**Remark:** We can always find a representation with $\pi_B$ injective and $\pi_{\mathsf{X}}(\mathsf{X})\mathcal{H}_0$ dense in $\mathcal{H}_1$.

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# Swapping Hilbert spaces by $L^p$ spaces

For concrete C*-algebras $A \subseteq \mathcal{L}(\mathcal{H}_0)$, $B \subseteq \mathcal{L}(\mathcal{H}_1)$, we think of an $(A, B)$ C*-correspondence $(\mathsf{X}, \varphi)$ concretely as $\mathsf{X}$ being a closed subspace of $\mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ satisfying

- $xb \in \mathsf{X}$ for all $x \in \mathsf{X}$, $b \in B$,
- $x^*y \in B$ for all $x, y \in \mathsf{X}$,

and $\varphi \colon A \to \{t \in \mathcal{L}(\mathcal{H}_1) \colon tx, t^*x \in \mathsf{X} \text{ for all } x \in \mathsf{X}\}$ being a $*$-homomorphism.

Furthermore, notice that $\mathsf{X}^* = \{x^* \colon x \in \mathsf{X}\}$ is a closed subset of $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_0)$ that satisfies

- $by \in \mathsf{X}^*$ for all $b \in B$, $y \in \mathsf{X}^*$,
- $y\varphi(a) \in \mathsf{X}^*$ for all $y \in \mathsf{X}^*$, $a \in A$.

# $L^p$-correspondences

If $A$ is a Banach algebra, $p \in [1, \infty)$, and there is a measure space $(\Omega, \mu)$ and an isometric homomorphism $A \to \mathcal{L}(L^p(\Omega, \mu))$, we say $A$ is an $L^p$-operator algebra.

## Definition

Let $(\Omega_0, \mu_0), (\Omega_1, \mu_1)$ be measure spaces, let $p \in [1, \infty)$, and let $A \subseteq \mathcal{L}(L^p(\mu_1))$ and $B \subseteq \mathcal{L}(L^p(\mu_0))$ be $L^p$ operator algebras. An $(A, B)$ $L^p$**-correspondence** is a triple $(\mathsf{X}, \mathsf{Y}, \varphi)$ where

1. $\mathsf{X} \subseteq \mathcal{L}(L^p(\mu_0), L^p(\mu_1))$ is a closed subspace,
2. $\mathsf{Y} \subseteq \mathcal{L}(L^p(\mu_1), L^p(\mu_0))$ is a closed subspace,
3. $xb \in \mathsf{X}$ for all $x \in \mathsf{X}$, $b \in B$,
4. $yx \in B$ for all $x \in \mathsf{X}$, $y \in \mathsf{Y}$,
5. $by \in \mathsf{Y}$ for all $y \in \mathsf{Y}$, $b \in B$,
6. $\varphi \colon A \to \{t \in \mathcal{L}(L^p(\mu_1)) \colon tx \in \mathsf{X}, yt \in \mathsf{Y} \ \forall \ x \in \mathsf{X}, \ y \in \mathsf{Y}\}$ is a contractive homomorphism.

# Examples

1. Let $(\Omega, \mu)$ be a measure space, let $p \in (1, \infty)$ and let $q$ be its Hölder conjugate (i.e. $\frac{1}{p} + \frac{1}{q} = 1$). For any $z \in \mathbb{C}$ define $\varphi_{\mathbb{C}}(z) := z \cdot \mathrm{id}_{L^p(\mu)}$. Then, $(L^p(\mu), L^q(\mu), \varphi_{\mathbb{C}})$ is a $(\mathbb{C}, \mathbb{C})$ $L^p$-correspondence
2. Let $(\Omega, \mu)$ be a measure space, let $p \in (1, \infty)$, let $A \subseteq \mathcal{L}(L^p(\mu))$ be an $L^p$-operator algebra, and let $\varphi \colon A \to A$ be a contractive automorphism. Observe that $A$ acts on $A$ via $\varphi$ acting as left multiplication and that $(A, A, \varphi)$ is an $(A, A)$ $L^p$ correspondence.

# Applications

To any $(A, A)$ C*-correspondence $(\mathsf{X}, \varphi)$ we can associate two important universal algebras: its Toeplitz algebra $\mathcal{T}(\mathsf{X}, \varphi)$ and its Cuntz-Pimsner algebra $\mathcal{O}(\mathsf{X}, \varphi)$. These algebras include important examples of C*-algebras such as Cuntz algebras, Cuntz Krieger algebras, and crossed products by $\mathbb{Z}$.

**Question**

Can we get $L^p$-operator algebras from $L^p$-correspondences in a similar way?

**Answer:** YES!

$$\mathcal{O}(\ell_d^p, \ell_d^q, \varphi_{\mathbb{C}}) \cong \mathcal{O}_d^p$$

$$\mathcal{O}(A, A, \varphi) \overset{?}{\cong} F^p(\mathbb{Z}, A, \varphi)$$

# Questions?