Representations of C*-correspondences on pairs of Hilbert spaces.

Alonso Delfín
CU Boulder.

September 14, 2023

Abstract

In this talk, I will discuss some of the main results in https://arxiv.org/abs/2208.14605. For a pair of C*-algebras (A, B), representing an (A, B) C*-correspondence on a pair of Hilbert spaces $(\mathcal{H}_0, \mathcal{H}_1)$ roughly consists in naturally realizing the correspondence as a closed subspace of $\mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$. This concept is a generalization of R. Exel theory for Hilbert A-B bimodules, originally introduced in 1993. Exel's methods were used as a tool to prove, in its full generality, that any two Morita equivalent C*-algebras have isomorphic K-theory. Extending this theory to C*-correspondences yields necessary and sufficient conditions for an (A,B) C*-correspondence to be a Hilbert A-B bimodule. Another consequence is that, if a right Hilbert A-module X is represented on $(\mathcal{H}_0, \mathcal{H}_1)$, we then get faithful representations of $\mathcal{L}_A(\mathsf{X})$ and $\mathcal{K}_A(\mathsf{X})$, the algebras of adjointable and compact-adjointable maps, on the Hilbert space $\mathcal{H}_1$. This will play a crucial role in my talk next week, where I will talk about the objects we get when the Hilbert spaces are replaced by general L^p spaces for $p \in (1, \infty)$.

1 Motivation

For a C*-algebra A, a right Hilbert A-module X is a right A-module with an A valued right inner product $\langle -, - \rangle_A \colon \mathsf{X} \times \mathsf{X} \to A$ such that $\|x\| := \|\langle x, x\rangle_A\|^{\frac{1}{2}}$ makes X a Banach space. A morphism between two right Hilbert A modules is an adjointable map with respect to their A valued inner products, that is a map $t \colon \mathsf{X} \to \mathsf{Y}$ such that there is a map $t^* \colon \mathsf{X} \to \mathsf{Y}$ satisfying

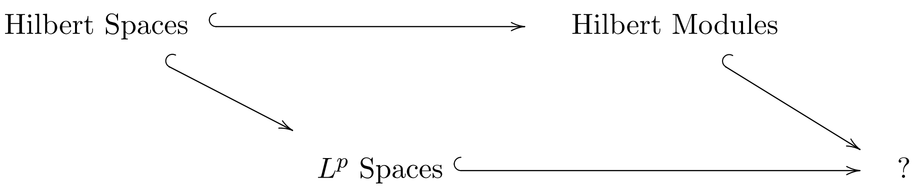
$$(t(x), y)_A = \langle x, t^*(y)\rangle_A$$

for every $x \in \mathsf{X}$, $y \in \mathsf{Y}$. The set of adjointable maps from X to Y is denoted by $\mathcal{L}_A(\mathsf{X}, Y)$ and $\mathcal{L}_A(X) := \mathcal{L}_A(\mathsf{X}, \mathsf{X})$ is a C*-algebra. In particular, if $y \in \mathsf{Y}$ and $x \in \mathsf{X}$, we get a map $\theta_{y,x} \colon \mathsf{X} \to \mathsf{Y}$ by letting $\theta_{y,x}(z) = y\langle x, z\rangle_A$. We define

$$\mathcal{K}_A(\mathsf{X}, \mathsf{Y}) = \overline{\operatorname{span}\{\theta_{y,x} \colon y \in \mathsf{Y}, x \in \mathsf{X}\}} \subseteq \mathcal{L}_A(\mathsf{X}, \mathsf{Y}).$$

The algebra $\mathcal{K}_A(\mathsf{X}) := \mathcal{K}_A(\mathsf{X}, \mathsf{X})$ is a closed two sided ideal in $\mathcal{L}_A(\mathsf{X})$ and therefore a C*-algebra in its own right.

Any Hilbert space $\mathcal{H}$ is a right Hilbert $\mathbb{C}$-module. On the other hand, any Hilbert space is also an L^2-space. Informally, we can represent this as a "diagram of inclusions":



The main motivation for looking at representations of Hilbert modules on pairs of Hilbert spaces was to come up with an object that makes the above "diagram of inclusions" commute.

2 Bimodules vs Correspondences

Any right Hilbert A-module is in particular a Hilbert $\mathcal{K}_A(\mathsf{X})$-$A$-bimodule. For a pair of C*-algebras (A, B), any Hilbert A-B-bimodule is an (A, B) C*-correspondence. It makes sense to study representations of these modules in their most general setting, that is the one of C*-correspondences.

Below we present precise definitions.

Definition 2.1. A *Hilbert A-B-bimodule* X is at the same time a right Hilbert B-module and a left Hilbert A-module such that

$$_A\langle x, y\rangle z = x\langle y, z\rangle_B$$

Definition 2.2. An (A, B) *C*-correspondence* is a pair (X, φ) where X is a right Hilbert B-module and $\varphi\colon A \to \mathcal{L}_A(\mathsf{X})$ is a $*$-homomorphism. We say A acts nondegenerately on X whenever $\varphi(A)\mathsf{X}$ is dense in X.

Remark 2.3. Any Hilbert A-B-bimodule X is in fact an (A, B) C*-correspondence with A acting nondegenerately on X. Indeed, it's standard to check that both A and B act nongeneretaly on X. Now define $\varphi(a)$ to be the left action of the module X. That is, $\varphi(a)x = ax$. Then,

$$z\langle ax, y\rangle_B = {}_A\langle z, ax\rangle y = {}_A\langle ax, z\rangle^* y = {}_A\langle x, z\rangle^* a^* y = {}_A\langle z, x\rangle a^* y = z\langle x, a^* y\rangle_B.$$

Now, if $b \in \overline{\langle X, X\rangle_B}$ and $zb = 0$ for all $z \in \mathsf{X}$, then $b = 0$. Thus, we conclude that $\langle ax, y\rangle_B = \langle x, a^* y\rangle_B$, whence A acts via $\langle -, -\rangle_B$-adjoitable maps. Therefore (X, φ) is an (A, B) C*-correspondence.

3 Representations on pairs of Hilbert Spaces.

In 1993 Ruy Exel defined representation of Hilbert bimodules on pairs of Hilbert spaces to show that if A and B are C*-algebras and there is a Hilbert A-B-bimodule X with $\overline{_A\langle X, X\rangle} = A$ and $\overline{\langle X, X\rangle_B} = B$, then there is an explicit isomorphism $K_i(A) \cong K_i(B)$ for $i = 0, 1$. This fact was already proved by Brown-Green-Rieffel but only for A and B separable and in their proof the isomorphism was not given explicitly. Exel's definition is given below

Definition 3.1. Let X be a Hilbert A-B-bimodule and let $(\mathcal{H}_0, \mathcal{H}_1)$ be a pair of Hilbert spaces. A *representation of* X *on* $(\mathcal{H}_0, \mathcal{H}_1)$ is a triple $(\lambda_A, \rho_B, \pi_{\mathsf{X}})$ such that λ_A is a representation of A on $\mathcal{H}_1$, ρ_B is a representation of B on $\mathcal{H}_0$, and $\pi_{\mathsf{X}}\colon \mathsf{X} \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is a linear map, such that for all $a \in A, b \in B$, and $x, y \in \mathsf{X}$, the following compatibility conditions are satisfied.

1. $\pi_{\mathsf{X}}(ax) = \lambda_A(a)\pi_{\mathsf{X}}(x)$,
2. $\pi_{\mathsf{X}}(xb) = \pi_{\mathsf{X}}(x)\rho_B(b)$,
3. $\lambda_A({}_A\langle x, y\rangle) = \pi_{\mathsf{X}}(x)\pi_{\mathsf{X}}(y)^*$,
4. $\rho_B(\langle x, y\rangle_B) = \pi_{\mathsf{X}}(x)^*\pi_{\mathsf{X}}(y)$.

If π_{X} is an isometry, we say the representation $(\lambda_A, \rho_B, \pi_{\mathsf{X}})$ is *isometric.*

Remark 3.2.

- Conditions 1 and 2 above are actually redundant, for they follow from conditions 3 and 3 respectively.
- The map π_{X} is automatically bounded and in fact isometric when either λ_A or ρ_B are faithful. Indeed, for instance
$$\|\pi_{\mathsf{X}}(x)\|^2 = \|\pi_{\mathsf{X}}(x)^*\pi(x)\| = \|\rho_B(\langle x, x\rangle_B)\| \leq \|\langle x, x\rangle_B\| = \|x\|^2$$

Theorem 3.3 (Exel, 1993)**.** *Let* X *be a Hilbert* A*-*B*-bimodule and* ρ_B *a nondegenerate representation of* B *on a Hilbert space* $\mathcal{H}_0$*. Then there is a Hilbert space* $\mathcal{H}_1$*, a non degenerate representation* λ_A *of* A *on* $\mathcal{H}_1$*, and a linear map* $\pi_{\mathsf{X}}\colon \mathsf{X} \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ *such that* $(\lambda_A, \rho_B, \pi_{\mathsf{X}})$ *is a representation of* X *on* $(\mathcal{H}_0, \mathcal{H}_1)$*.*

Sketch of Proof. Let L_{X} be the Linking algebra of the bimodule X. That is,

$$L_{\mathsf{X}} := \begin{pmatrix} A & \mathsf{X} \\ \widetilde{\mathsf{X}} & B \end{pmatrix} := \left\{ \begin{pmatrix} a & x \\ y & b \end{pmatrix} : a \in A, x \in \mathsf{X}, y \in \widetilde{\mathsf{X}}, b \in B \right\}$$

which is a C*-algebra with multiplication given by the matrix algebra structure inherited by the actions and inner products of the bimodule. The given representaion ρ_B can be extended (via states and GNS construction) to a representation π of L_{X} on a Hilbert space $\mathcal{H}$ that contains a copy of $\mathcal{H}_0$ and such that

$$\pi \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \Big|_{\mathcal{H}_0} = \rho_B(b).$$

We now define

$$\mathcal{H}_1 := \overline{\pi \begin{pmatrix} 0 & \mathsf{X} \\ 0 & 0 \end{pmatrix} \mathcal{H}_0}, \ \lambda_A(a) := \pi \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \Big|_{\mathcal{H}_1}, \text{ and } \pi_{\mathsf{X}}(x) = \pi \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \Big|_{\mathcal{H}_0}.$$

Conditions 1 and 2 in Definition 3.1 are now immediately checked. "□"

Exel's result depends on X having a left valued A-inner product, so it can't be adapted to a general C*-correspondence. However, we can use different methods to provide an analogous result. To that end, we first need to have a definition for representations of C*-correspondences on pairs of Hilbert spaces.

Definition 3.4. Let (X, φ) be an (A, B) C*-correspondence and let $(\mathcal{H}_0, \mathcal{H}_1)$ be a pair of Hilbert spaces. A *representation of* (X, φ) *on* $(\mathcal{H}_0, \mathcal{H}_1)$ is a triple $(\lambda_A, \rho_B, \pi_{\mathsf{X}})$ such that λ_A is a representation of A on $\mathcal{H}_1$, ρ_B is a representation of B on $\mathcal{H}_0$, and $\pi_{\mathsf{X}}\colon \mathsf{X} \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is a linear map, such that for all $a \in A, b \in B$, and $x, y \in \mathsf{X}$, the following compatibility conditions are satisfied.

1. $\pi_{\mathsf{X}}(\varphi(a)x) = \lambda_A(a)\pi_{\mathsf{X}}(x)$,
2. $\rho_B(\langle x, y\rangle_B) = \pi_{\mathsf{X}}(x)^*\pi_{\mathsf{X}}(y)$.

If π_{X} is an isometry, we say the representation $(\lambda_A, \rho_B, \pi_{\mathsf{X}})$ is *isometric.*

As before, condition 2 automatically implies that $\pi_{\mathsf{X}}(xb) = \pi_{\mathsf{X}}(x)\rho_B(b)$. Similarly, boundedness of π_{X} is automatic and the isometric condition is implied by faithfulness of ρ_B.

Theorem 3.5 (D, 2022)**.** *Let* (X, φ) *be an* (A, B) *C*-correspondence and* ρ_B *a nondegenerate representation of* B *on a Hilbert space* $\mathcal{H}_0$*. Then there is a Hilbert space* $\mathcal{H}_1$*, a non degenerate representation* λ_A *of* A *on* $\mathcal{H}_1$*, and a linear map* $\pi_{\mathsf{X}}\colon \mathsf{X} \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ *such that* $(\lambda_A, \rho_B, \pi_{\mathsf{X}})$ *is a representation of* (X, φ) *on* $(\mathcal{H}_0, \mathcal{H}_1)$*.*

Sketch of Proof. We define $\mathcal{H}_1 = \mathsf{X} \otimes_{\rho_B} \mathcal{H}_0$ and get the induced representation $\lambda_A\colon A \to \mathcal{L}(\mathcal{H}_1)$ by letting $\lambda_A(a)(x \otimes \xi) := \varphi(a)x \otimes \xi$. We also get creation operators via $\pi_{\mathsf{X}}\colon \mathsf{X} \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ defined by $\pi_{\mathsf{X}}(x)\xi := x \otimes \xi$. it is easily checked that $\pi_{\mathsf{X}}(x)^*\colon \mathcal{H}_1 \to \mathcal{H}_0$ satisfies

$$\pi_{\mathsf{X}}(x)^*(y \otimes \xi) = \rho_B(\langle x, y\rangle_B)\xi.$$

From here condition 2 in Definition 3.1 follows immediately. Similarly, $\lambda_A(a)\pi_{\mathsf{X}}(x)\xi = \varphi(a)x \otimes \xi = \pi_{\mathsf{X}}(\varphi(a)x)\xi$, so condition 1 also holds. "□"

Remark 3.6. If the correspondence from Theorem 3.5 is actually a bimodule, then the maps λ_A and π_{X} constructed in the proof also satisfy condition 3 in Definition 3.1. This shows that the proof of Theorem 3.5 is also an alternative proof for Theorem 3.3.

4 Applications

The first application of this theory is that we have necessary and sufficient conditions for a general (A, B) C*-correspondence to be a Hilbert A-B-bimodule.

Theorem 4.1 (D, 2022). *Let (X, φ) be an (A, B) C*-correspondence such that A acts nondegenerately on X. Then there is an A-valued left inner product on X making it an A-B-bimodule if and only if $\mathcal{K}_B(\mathsf{X}) \subseteq \varphi(A)$.*

Sketch of Proof. If the correspondence is a bimodule, then $\varphi({}_A\langle x, y\rangle) = \theta_{x,y}$ and we are done.

Conversely, assume that $\mathcal{K}_B(\mathsf{X}) \subseteq \varphi(A)$ and let $(\lambda_A, \rho_B, \pi_{\mathsf{X}})$ be the isometric representation of (X, φ) on $(\mathcal{H}_0, \mathcal{H}_1)$ obtained in Theorem 3.5 (start with any nondegenerate and faithful ρ_B, for instance the universal representation of B). Then, we can check that our hypothesis implies that $\pi_{\mathsf{X}}(x)\pi_{\mathsf{X}}(y)^* \in \lambda_A(A)$. Thus, we can define

$$
{}_A\langle x, y\rangle = \lambda_A^{-1}\big(\pi_{\mathsf{X}}(x)\pi_{\mathsf{X}}(y)^*\big).
$$

It's an immediate computation to check that

$$
\pi_{\mathsf{X}}(\varphi({}_A\langle x, y\rangle)z) := \pi_{\mathsf{X}}(x\langle y, z\rangle_B),
$$

whence the fact that π_{X} is isometric implies that X is indeed a bimodule. "□"

The second application, which is a result we will use in the next talk for the L^p-case, gives a nice way to represent the C*-algebras $\mathcal{L}_A(\mathsf{X})$ and $\mathcal{K}_A(\mathsf{X})$ of a right Hilbert A-module X given a representation of X on $(\mathcal{H}_0, \mathcal{H}_1)$.

Proposition 4.2 (D, 2022). *Let X be a right Hilbert A-module and let $(\rho_A, \pi_{\mathsf{X}})$ be a representation of X on $(\mathcal{H}_0, \mathcal{H}_1)$, that is $\rho_A(\langle x, y\rangle_A) = \pi_{\mathsf{X}}(x)^*\pi_{\mathsf{X}}(y)$. Assume that $\pi_X(\mathsf{X})\mathcal{H}_0$ is dense in $\mathcal{H}_1$. Then,*

- $\mathcal{K}_A(\mathsf{X}) \cong \overline{\operatorname{span}\{\pi_{\mathsf{X}}(x)\pi_{\mathsf{X}}(y)^* \colon x, y \in \mathsf{X}\}} \subseteq \mathcal{L}(\mathcal{H}_1)$
- $\mathcal{L}_A(\mathsf{X}) \cong \{b \in \mathcal{L}(\mathcal{H}_1) \colon b\pi_{\mathsf{X}}(x), b^*\pi_{\mathsf{X}}(x) \in \pi_{\mathsf{X}}(\mathsf{X}) \text{ for all } x \in \mathsf{X}\}.$

Department of Mathematics, University of Colorado, Boulder CO 80309-0395, USA.
E-mail address: alonso.delfin@colorado.edu
Website.