

C^* -Correspondences, Hilbert Bimodules, and their L^p versions

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Outline

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- 2 Project overview
- 3 Review of Hilbert Modules
- 4 L^p -modules
 - L^p -modules: Definition and Examples
 - L^p -modules: Morphisms
 - L^p -modules: New from old
- 5 L^p -correspondences
 - Correspondences
 - Fock Representations
 - Cuntz-Pimsner Algebras
- 6 Future Work

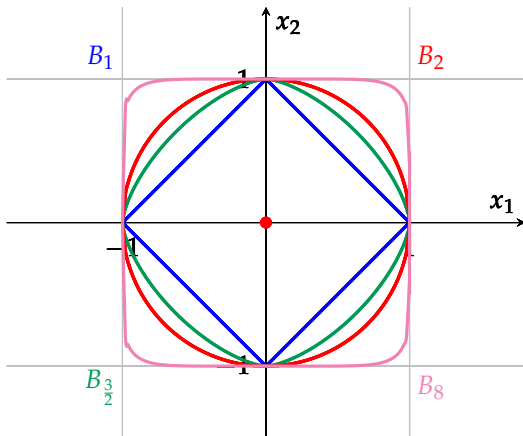
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Unit p -circles in \mathbb{R}^2

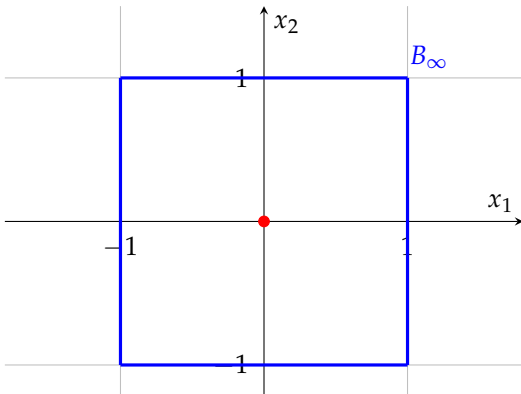
For $p \in [1, \infty)$ let

$$B_p := \{(x_1, x_2) \in \mathbb{R}^2 : |x_1|^p + |x_2|^p = 1\}.$$



Letting $p \rightarrow \infty$

$$\begin{aligned} B_\infty &:= \{(x_1, x_2) \in \mathbb{R}^2: \lim_{p \rightarrow \infty} |x_1|^p + |x_2|^p = 1\} \\ &= \{(x_1, x_2) \in \mathbb{R}^2: \max\{|x_1|, |x_2|\} = 1\}. \end{aligned}$$



p -operator norms in \mathbb{C}^d

Let $d \in \mathbb{Z}_{\geq 1}$ and let $\mathbf{z} = (z_1, \dots, z_d) \in \mathbb{C}^d$. We define

$$\|\mathbf{z}\|_p := \begin{cases} \left(\sum_{j=1}^d |z_j|^p \right)^{1/p} & p \in [1, \infty) \\ \max_{j \in \{1, \dots, d\}} |z_j| & p = \infty \end{cases} .$$

The normed space $(\mathbb{C}^d, \|\cdot\|_p)$ will be denoted by ℓ_d^p . Any linear map $a: \mathbb{C}^d \rightarrow \mathbb{C}^d$ is a $d \times d$ matrix with complex entries and its p -operator norm is defined by

$$\|a\|_p := \max_{\|\mathbf{z}\|_p=1} \|a(\mathbf{z})\|_p.$$

p -operator norms in \mathbb{C}^d

For a linear map $a: \mathbb{C}^d \rightarrow \mathbb{C}^d$ we have defined

$$\|a\|_p := \max_{\|z\|_p=1} \|a(z)\|_p.$$

If $a = (a_{j,k})_{j,k=1}^d$, then

$$\|a\|_1 = \max_{k \in \{1, \dots, d\}} \sum_{j=1}^d |a_{j,k}|,$$

$$\|a\|_2 = \max_{\lambda \in \sigma(a^*a)} \sqrt{|\lambda|},$$

$$\|a\|_\infty = \max_{j \in \{1, \dots, d\}} \sum_{k=1}^d |a_{j,k}|.$$

Otherwise, for a general matrix a , the value $\|a\|_p$ is NP-hard to compute.

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Spaces of Operators

We denote by $\mathcal{L}(E, F)$ the set of bounded linear maps between Banach spaces E and F . This is a Banach space with norm

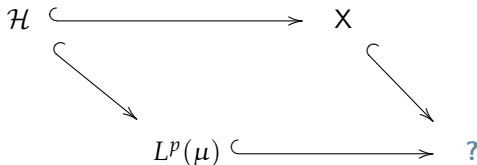
$$\|a\| := \sup_{\|\xi\|_E=1} \|a\xi\|_F.$$

If $E = F$, then $\mathcal{L}(E) := \mathcal{L}(E, E)$ is a Banach algebra.

- If $\mathcal{H}_0, \mathcal{H}_1$ are Hilbert spaces, any $a \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ has an adjoint $a^* \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_0)$ characterized by $\langle a\xi, \eta \rangle = \langle \xi, a^*\eta \rangle$.
- If \mathcal{H} is a Hilbert space, a C^* -algebra is a norm closed selfadjoint subalgebra of $\mathcal{L}(\mathcal{H})$.
- A right Hilbert module X over a C^* -algebra $A \subseteq \mathcal{L}(\mathcal{H}_0)$ is a closed subspace of $\mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ satisfying $xa \in X$ and $x^*y \in A$.
- If (Ω, μ) is a measure space and $p \in [1, \infty)$, an L^p -operator algebra is a norm closed subalgebra of $\mathcal{L}(L^p(\mu))$.

Generalizing Hilbert spaces

- Any Hilbert space is an L^2 -space.
- Any Hilbert space is a right Hilbert module over \mathbb{C} .



A main goal of this project was to define a proper object to put in place of $?$ so that the above “diagram of inclusions” commutes.

L^p -modules

For $p \in [1, \infty)$, we denote by q its Hölder conjugate:

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Roughly speaking, an L^p -**module** over an L^p -operator algebra will be a pair (V, W) that makes the following diagram commute.

$$\begin{array}{ccc} \mathcal{H} & \hookrightarrow & (X, X^*) \\ & \searrow & \searrow \\ & & (L^p(\mu), L^q(\mu)) \hookrightarrow (V, W) \end{array}$$

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Hilbert Modules

Let A be a C^* -algebra. Essentially, a Hilbert A -module is a Hilbert space with scalars and values of the scalar product in A instead of in \mathbb{C} .

Definition

A **(right) Hilbert A -module** is a complex vector space X which is a (right) A -module with an A -valued (right) inner product

$$\begin{aligned} X \times X &\rightarrow A \\ (x, y) &\mapsto \langle x, y \rangle_A \end{aligned}$$

such that X is complete with the norm

$$\|x\| := \|\langle x, x \rangle_A\|^{1/2}.$$

Adjointable maps

Definition

Let X and Y be Hilbert A -modules. A map $t : X \rightarrow Y$ is said to be **adjointable** if there is a map $s : Y \rightarrow X$ such that for any $x \in X$, and $y \in Y$

$$\langle t(x), y \rangle_A = \langle x, s(y) \rangle_A.$$

The map s is necessarily unique and henceforth denoted by t^* . We write $\mathcal{L}_A(X, Y)$ for the space of adjointable maps from X to Y .

Fact 1: $\mathcal{L}_A(X, Y) \subseteq \text{Hom}_A(X, Y)$ and $\mathcal{L}_A(X, Y) \subseteq \mathcal{L}(X, Y)$. In general, not every bounded linear map between Hilbert modules is adjointable.

Fact 2: $\mathcal{L}_A(X) := \mathcal{L}_A(X, X)$ is a C^* -algebra when equipped with the operator norm.

Compact-module maps

Definition

Let X and Y be Hilbert A -modules. For $x \in X$ and $y \in Y$, we define a map $\theta_{x,y} : Y \rightarrow X$ by

$$\theta_{x,y}(z) := x\langle y, z \rangle_A \quad \forall z \in Y.$$

We define the set of **compact-module maps** by

$$\mathcal{K}_A(Y, X) := \overline{\text{span}\{\theta_{x,y} : x \in X, y \in Y\}} \subseteq \mathcal{L}_A(Y, X).$$

Fact 1: $\mathcal{K}_A(X) := \mathcal{K}_A(X, X)$ is also a C^* -algebra, for it's a closed two sided ideal in $\mathcal{L}_A(X)$.

Fact 2: In general, elements in $\mathcal{K}_A(Y, X)$ need not to be in $\mathcal{K}(Y, X)$.

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L^p -modules: Definition

Let $A \subseteq \mathcal{L}(\mathcal{H}_0)$ be a C^* -algebra. Then, any right Hilbert A -module X can be isometrically represented as a closed linear subspace of $\mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ satisfying

- 1 $xa \in X$ for all $x \in X, a \in A,$
- 2 $x^*y \in A$ for all $x, y \in X.$

Definition

Let $(\Omega_0, \mu_0), (\Omega_1, \mu_1)$ be measure spaces, let $p \in [1, \infty)$, and let $A \subseteq \mathcal{L}(L^p(\mu_0))$ be an L^p operator algebra. An **L^p -module** over A is a pair (X, Y) where $X \subseteq \mathcal{L}(L^p(\mu_0), L^p(\mu_1))$ and $Y \subseteq \mathcal{L}(L^p(\mu_1), L^p(\mu_0))$ are closed subspaces satisfying

- 1 $xa \in X$ for all $x \in X, a \in A,$
- 2 $yx \in A$ for all $x \in X, y \in Y,$
- 3 $ay \in Y$ for all $y \in Y, a \in A.$

L^p -modules: Definition

Definition

Let $(\Omega_0, \mu_0), (\Omega_1, \mu_1)$ be measure spaces, let $p \in [1, \infty)$, and let $A \subseteq \mathcal{L}(L^p(\mu_0))$ be an L^p operator algebra. An L^p -**module** over A is a pair (X, Y) where $X \subseteq \mathcal{L}(L^p(\mu_0), L^p(\mu_1))$ and $Y \subseteq \mathcal{L}(L^p(\mu_1), L^p(\mu_0))$ are closed subspaces satisfying

- 1 $xa \in X$ for all $x \in X, a \in A$,
- 2 $yx \in A$ for all $x \in X, y \in Y$,
- 3 $ay \in Y$ for all $y \in Y, a \in A$.

An L^p -module comes with a bilinear pairing $Y \times X \rightarrow A$

$$(y \mid x)_A := yx$$

We say that an L^p -module (X, Y) is **C*-like** if the norms on X and Y can be recovered from the pairing as follows:

$$\|x\| = \sup_{\|y\|=1} \|(y \mid x)_A\| \quad \text{and} \quad \|y\| = \sup_{\|x\|=1} \|(y \mid x)_A\|.$$

L^p -modules: General Examples

Let A be any L^p -operator algebra. Then,

- (A, A) is an L^p -module over A . In general, (A, A) need not to be C^* -like. However, if A has a contractive approximate identity (c.a.i), then (A, A) is C^* -like.

Let $p \in (1, \infty)$ and let (Ω, μ) be a measure space. Then

- $(L^p(\mu), L^q(\mu))$ is a C^* -like L^p -module over $\mathbb{C} = \mathcal{L}(\ell_1^p)$.
- $(L^q(\mu), L^p(\mu))$ is a C^* -like L^p -module over $\mathcal{K}(L^p(\mu))$.

Let $p \in [1, \infty)$, and let $d \in \mathbb{Z}_{\geq 1}$. Then

- (ℓ_d^p, ℓ_d^q) is a C^* -like L^p -module over \mathbb{C} .
- (ℓ_d^q, ℓ_d^p) is a C^* -like L^p -module over $\mathcal{K}(\ell_d^p)$.

L^p -modules: “Adjointable maps”

Proposition (D, 2022)

Let A be a C^* -algebra and let X be a right Hilbert A -module isometrically represented on $(\mathcal{H}_0, \mathcal{H}_1)$. Suppose that $\overline{X\mathcal{H}_0} = \mathcal{H}_1$. Then,

$$\mathcal{L}_A(X) \cong \{t \in \mathcal{L}(\mathcal{H}_1) : tx, t^*x \in X \text{ for all } x \in X\}$$

This result motivates the following L^p generalization: If $A \subseteq \mathcal{L}(L^p(\mu_0))$ and (X, Y) is an L^p -module over A with $X \subseteq \mathcal{L}(L^p(\mu_0), L^p(\mu_1))$, we define

$$\mathcal{L}_A((X, Y)) = \{t \in \mathcal{L}(L^p(\mu_1)) : tx \in X \text{ and } yt \in Y \text{ for all } x \in X, y \in Y\}.$$

- If A has a c.a.i and is nondegenerately represented on $L^p(\mu)$, then $\mathcal{L}_A((A, A)) \cong M(A)$,
- $\mathcal{L}_{\mathbb{C}}(L^p(\mu), L^q(\mu)) \cong \mathcal{L}(L^p(\mu))$,
- $\mathcal{L}_{\mathcal{K}(L^p(\mu))}((L^q(\mu), L^p(\mu))) \cong \mathbb{C}$.

L^p -modules: “Compact-module maps”

Proposition (D, 2022)

Let A be a C^* -algebra and let X be a right Hilbert A -module isometrically represented on $(\mathcal{H}_0, \mathcal{H}_1)$. Suppose that $\overline{X\mathcal{H}_0} = \mathcal{H}_1$. Then,

$$\mathcal{K}_A(X) \cong \overline{\text{span} \{xy^* : x, y \in X\}} \subseteq \mathcal{L}(\mathcal{H}_1).$$

We now get the following L^p generalization: If $A \subseteq \mathcal{L}(L^p(\mu_0))$ and (X, Y) is an L^p -module over A with $X \subseteq \mathcal{L}(L^p(\mu_0), L^p(\mu_1))$, we define

$$\mathcal{K}_A((X, Y)) = \overline{\text{span} \{xy : x \in X, y \in Y\}} \subseteq \mathcal{L}(L^p(\mu_1)).$$

$\mathcal{K}_A((X, Y))$ is a closed two sided ideal in $\mathcal{L}_A((X, Y))$.

- If A has a c.a.i, then $\mathcal{K}_A((A, A)) \cong A$,
- $\mathcal{K}_{\mathbb{C}}((L^p(\mu), L^q(\mu))) \cong \mathcal{K}(L^p(\mu))$,
- $\mathcal{K}_{\mathcal{K}(L^p(\mu))}((L^q(\mu), L^p(\mu))) \cong \mathbb{C}$.

L^p -modules: Direct Sum

Let $p \in [1, \infty)$, and let $(X_1, Y_1), \dots, (X_d, Y_d)$ be L^p -modules over a common L^p -operator algebra A , then $(\bigoplus_{j=1}^d X_j, \bigoplus_{j=1}^d Y_j)$ inherits natural p -operator norms that make it an L^p -module over A .

Now let $(X_j, Y_j)_{j=1}^\infty$ be a sequence of L^p -modules over $A \subseteq \mathcal{L}(L^p(\mu_0))$ with $X_j \subseteq \mathcal{L}(L^p(\mu_0), L^p(\mu_j))$. We define $\bigoplus_{j=1}^\infty (X_j, Y_j)$ as the pair (X, Y) where

$$X := \left\{ (x_j)_{j=1}^\infty \in \mathcal{L}\left(L^p(\mu_0), \bigoplus_{j=1}^\infty L^p(\mu_j)\right) : \lim_{n,m \rightarrow \infty} \sup_{\|\xi\|_p=1} \sum_{j=n}^m \|x_j \xi\|_p^p = 0 \right\},$$

$$Y := \left\{ (y_j)_{j=1}^\infty \in \mathcal{L}\left(\bigoplus_{j=1}^\infty L^p(\mu_j), L^p(\mu_0)\right) : \lim_{n,m \rightarrow \infty} \sup_{\sum_{j=1}^\infty \|\eta_j\|_p^p=1} \left\| \sum_{j=n}^m y_j \eta_j \right\|_p^p = 0 \right\}.$$

L^p -modules: Direct Sum

$$X := \left\{ (x_j)_{j=1}^{\infty} \in \mathcal{L}\left(L^p(\mu_0), \bigoplus_{j=1}^{\infty} L^p(\mu_j)\right) : \lim_{n,m \rightarrow \infty} \sup_{\|\xi\|_p=1} \sum_{j=n}^m \|x_j \xi\|_p^p = 0 \right\},$$

$$Y := \left\{ (y_j)_{j=1}^{\infty} \in \mathcal{L}\left(\bigoplus_{j=1}^{\infty} L^p(\mu_j), L^p(\mu_0)\right) : \lim_{n,m \rightarrow \infty} \sup_{\sum_{j=1}^{\infty} \|\eta_j\|_p^p = 1} \left\| \sum_{j=n}^m y_j \eta_j \right\|_p^p = 0 \right\}.$$

Theorem (D, 2023)

(X, Y) is an L^p -module over A that agrees with the usual definition of direct sum of Hilbert modules when $p = 2$:

$$\bigoplus_{j=1}^{\infty} X_j := \left\{ (x_j)_{j=1}^{\infty} : x_j \in X_j, \sum_{j=1}^{\infty} \langle x_j, x_j \rangle_A \text{ converges in } A \right\}.$$

L^p -modules: External Tensor Product

If (Ω, μ) is a measure space, $p \in [1, \infty)$, and E is a Banach space, there is a well behaved tensor product \otimes_p satisfying

$$L^p(\mu) \otimes_p E \cong L^p(\mu, E) = \left\{ g \in \text{Mble}(\Omega \rightarrow E) : \int_{\Omega} \|g(\omega)\|^p d\mu(\omega) < \infty \right\}$$

In particular, $L^p(\mu) \otimes_p L^p(\nu) \cong L^p(\mu \times \nu)$. This *spatial tensor product* induces a well behaved tensor product on spaces of operators between L^p -spaces, which in turn induces an **external tensor product** on L^p -modules:

Definition

Let (X, Y) be an L^p -module over A and let (V, W) be an L^p -module over B . We define

$$(X, Y) \otimes_p (V, W) = (X \otimes_p V, Y \otimes_p W).$$

$(X \otimes_p V, Y \otimes_p W)$ is an L^p -module over $A \otimes_p B$.

External Tensor Product

In the Hilbert module case, it is well known that

$$\ell^2 \otimes X \cong \bigoplus_{j=1}^{\infty} X,$$

which gives rise to $\ell^2 \otimes A$, the standard Hilbert module of a C^* -algebra A .

L^p -analogue

Let $p \in (1, \infty)$ and let (X, Y) be an L^p -module over A . Then,

$$(\ell^p, \ell^q) \otimes_p (X, Y) = \bigoplus_{j=1}^{\infty} (X, Y).$$

In particular, when $(X, Y) = (A, A)$, we get **the standard L^p -module** $(\ell^p, \ell^q) \otimes_p (A, A) = (\ell^p \otimes_p A, \ell^q \otimes_p A)$ of an L^p -operator algebra A .

External Tensor Product

For a C^* -algebra A , it is well known that

$$\mathcal{K}_A(\ell^2 \otimes A) \cong \mathcal{K}(\ell^2) \otimes A \text{ and } \mathcal{L}_A(\ell^2 \otimes A) \cong M(\mathcal{K}(\ell^2) \otimes A).$$

More generally, it always holds that $\mathcal{L}_A(X) \cong M(\mathcal{K}_A(X))$.

L^p -analogue

Let A be an L^p -operator algebra. If A has a c.a.i., then

$$\mathcal{K}_A((\ell^p \otimes_p A, \ell^q \otimes_p A)) \cong \mathcal{K}(\ell^p) \otimes_p A.$$

If in addition A sits nondegenerately in $\mathcal{L}(L^p(\mu))$, then

$$\mathcal{L}_A((\ell^p \otimes_p A, \ell^q \otimes_p A)) \cong M(\mathcal{K}(\ell^p) \otimes_p A).$$

We have not investigated yet whether $\mathcal{L}_A((X, Y)) \cong M(\mathcal{K}_A((X, Y)))$ for a general L^p -module (X, Y) over an L^p -operator algebra A .

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C^* -correspondences

Definition

Let A and B be C^* -algebras. An (A, B) **C^* -correspondence** is a pair (X, φ) where X is a right Hilbert B -module and $\varphi : A \rightarrow \mathcal{L}_B(X)$ is a $*$ -homomorphism.

$${}_A X_B$$

$$\langle \varphi(a)x, y \rangle_B = \langle x, \varphi(a^*)y \rangle_B$$

Fact 1: Any Hilbert A - B -bimodule X (i.e. a left Hilbert A -module and a right Hilbert B -module with ${}_A \langle x, y \rangle z = x \langle y, z \rangle_B$) is naturally an (A, B) C^* -correspondence.

Fact 2: Any right Hilbert A -module X is naturally a $(\mathcal{K}_A(X), A)$ C^* -correspondence.

Representations of C^* -correspondences

Let (X, φ) be an (A, B) C^* -correspondence and $(\mathcal{H}_0, \mathcal{H}_1)$ a pair of Hilbert spaces. A **representation of (X, φ) on $(\mathcal{H}_0, \mathcal{H}_1)$** consists of a triple (π_A, π_B, π_X) such that

- 1 π_A is a representation of A on \mathcal{H}_1 ,
- 2 π_B is a representation of B on \mathcal{H}_0 ,
- 3 $\pi_X : X \rightarrow \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is a linear map,

satisfying, for any $a \in A$, $x, y \in X$,

$$(i) \pi_X(\varphi(a)x) = \pi_A(a)\pi_X(x), \quad (ii) \pi_X(x)^* \pi_X(y) = \pi_B(\langle x, y \rangle_B).$$

Fact 1: If $x \in X$ and $b \in B$, then (iii) $\pi_X(xb) = \pi_X(x)\pi_B(b)$.

Fact 2: If π_B is isometric, then so is π_X .

Theorem (D, 2022)

Isometric representations of any (A, B) C^ -correspondence on a pair of Hilbert spaces do exist.*

L^p -correspondences

Definition

Let $(\Omega_0, \mu_0), (\Omega_1, \mu_1)$ be measure spaces, let $p \in [1, \infty)$, let A be an L^p -operator algebra, and let $B \subseteq \mathcal{L}(L^p(\mu_0))$ be a concrete L^p operator algebra. An (A, B) L^p -**correspondence** is a pair $((X, Y), \varphi)$ where

- (X, Y) is an L^p module over B with $X \subseteq \mathcal{L}(L^p(\mu_0), L^p(\mu_1))$,
- $\varphi: A \rightarrow \mathcal{L}_B((X, Y))$ is a contractive homomorphism.

When $A = B$ we say that $((X, Y), \varphi)$ is an L^p correspondence over A .

- Let A be an L^p -operator algebra and $\varphi_A: A \rightarrow A$ a contractive automorphism. Then, $((A, A), \varphi_A)$ is an L^p correspondence over A .
- Let (Ω, μ) be a measure space and let $p \in (1, \infty)$. For any $z \in \mathbb{C}$ define $\varphi_{\mathbb{C}}(z) := z \cdot \text{id}_{L^p(\mu)}$. Then, $((L^p(\mu), L^q(\mu)), \varphi_{\mathbb{C}})$ is an L^p correspondence over \mathbb{C} .

A **Fock representation** for (X, φ) is a triple (B, π_A, π_X) where B is a C^* -algebra, $\pi_A: A \rightarrow B$ a $*$ -homomorphism, and $\pi_X: X \rightarrow B$ a linear map s.t.

- 1 $\pi_X(\varphi(a)x) = \pi_A(a)\pi_X(x)$,
- 2 $\pi_A(\langle x, y \rangle_A) = \pi_X(x)^* \pi_X(y)$.

$C^*(B, \pi_A, \pi_X) = C^*$ -algebra generated by $\pi_A(A)$ and $\pi_X(X)$ in B .

L^p -analogue

An **L^p -Fock representation** of $((X, Y), \varphi)$ consist of (B, π_A, π_X, π_Y) where B is an L^p -operator algebra, $\pi_A: A \rightarrow B$ a contractive homomorphism, and both $\pi_X: X \rightarrow B$ and $\pi_Y: Y \rightarrow B$ are contractive linear maps satisfying the following conditions:

- 1 $\pi_X(xa) = \pi_X(x)\pi_A(a)$, and $\pi_X(\varphi(a)x) = \pi_A(a)\pi_X(x)$,
- 2 $\pi_Y(ay) = \pi_A(a)\pi_Y(y)$, and $\pi_Y(y\varphi(a)) = \pi_Y(y)\pi_A(a)$,
- 3 $\pi_A((y | x)_A) = \pi_Y(y)\pi_X(x)$,

$F^p(B, \pi_A, \pi_X, \pi_Y) =$ Banach algebra generated by $\pi_A(A)$, $\pi_X(X)$ and $\pi_Y(Y)$ in B .

Fock Space

Definition

Let A be a C^* -algebra and let (X, φ) be a C^* -correspondence over A . We define $\mathcal{O}(X, \varphi)$, the **Cuntz-Pimsner algebra of (X, φ)** , as the universal C^* -algebra algebra generated by *covariant* Fock representations.

- $\mathcal{O}(\ell_d^2, \varphi_{\mathbb{C}}) \cong \mathcal{O}_d$,
- $\mathcal{O}(A, \varphi_A) \cong C^*(\mathbb{Z}, A, \varphi_A)$.

Let $\mathcal{F}(X) = \bigoplus_{n=0}^{\infty} X^{\otimes \varphi^n}$, let $c: X \rightarrow \mathcal{L}_A(\mathcal{F}(X))$ be given by $c(x)\kappa = x \otimes \kappa$, and let $\mathcal{Q}_A(X) = \mathcal{L}_A(\mathcal{F}(X))/\mathcal{K}_{J_X}(\mathcal{F}(X))_{J_X}$.

Theorem (Pimsner-Katsura)

$\mathcal{O}(X, \varphi)$ is the C^* -subalgebra in $\mathcal{Q}_A(X)$ generated by the images of $c(X)$ and by $\varphi^{\infty}(A)$ under the quotient map.

L^p -Cuntz Algebras

For the L^p -correspondence $((\ell_d^p, \ell_d^q), \varphi_{\mathbb{C}})$, consider $\mathcal{F}^p := \bigoplus_{n=0}^{\infty} \ell_{d^n}^p$, $\mathcal{F}^q := \bigoplus_{n=0}^{\infty} \ell_{d^n}^q$, the creation operator $c: \ell_d^p \rightarrow \mathcal{L}(\mathcal{F}^p)$, and the annihilation operator $v: \ell_d^q \rightarrow \mathcal{L}(\mathcal{F}^p)$.

Theorem (D, 2023)

Let $\mathcal{Q} := \mathcal{L}(\mathcal{F}^p)/\mathcal{K}(\mathcal{F}^q)$ with quotient map $q_0: \mathcal{L}(\mathcal{F}^p) \rightarrow \mathcal{Q}$. Then $(\mathcal{Q}, q_0(\varphi_{\mathbb{C}}), q_0(c), q_0(v))$ is a covariant L^p -Fock representation for $((\ell_d^p, \ell_d^q), \varphi_{\mathbb{C}})$ and $F^p(\mathcal{Q}, q_0(\varphi_{\mathbb{C}}), q_0(c), q_0(v))$ is isometrically isomorphic to \mathcal{O}_d^p , the L^p analogue of Cuntz algebra.

Question

Is $(\mathcal{Q}, q_0(\varphi_{\mathbb{C}}), q_0(c), q_0(v))$ the universal L^p -Fock covariant representation for $((\ell_d^p, \ell_d^q), \varphi_{\mathbb{C}})$?

L^p -Crossed Products

Assume A is an L^p -operator algebra that has a bicontractive approximate identity and that sits nongenerately in $\mathcal{L}(L^p(\mu))$. For the L^p -correspondence $((A, A), \varphi_A)$, with φ_A isometric, consider $\mathcal{F}^p(A) := \ell^p \otimes_p A$, $\mathcal{F}^q := \ell^q \otimes_p A$, and $\mathcal{Q}_A := \mathcal{L}_A((\mathcal{F}^p(A), \mathcal{F}^q(A))) / \mathcal{K}_A((\mathcal{F}^p(A), \mathcal{F}^q(A)))$

Theorem (D, 2023)

\mathcal{Q}_A is an L^p -operator algebra and there are isometric maps φ_A^∞, c_A , and v_A from A to $\mathcal{L}_A((\mathcal{F}^p(A), \mathcal{F}^q(A)))$ such that $(\mathcal{Q}_A, q_0(\varphi_A^\infty), q_0(c_A), q_0(v_A))$ is a covariant L^p -Fock representation for $((A, A), \varphi_A)$.

Question

Is $F^p(\mathcal{Q}_A, q_0(\varphi_A^\infty), q_0(c_A), q_0(v_A))$ isometrically isomorphic to $F^p(\mathbb{Z}, A, \varphi_A)$, the L^p -crossed product of A by \mathbb{Z} ?

$\frac{1}{2}$ **Ans:** There is $\gamma: F^p(\mathbb{Z}, A, \varphi_A) \rightarrow F^p(\mathcal{Q}_A, q_0(\varphi_A^\infty), q_0(c_A), q_0(v_A))$, a contractive algebra homomorphism.

Outline

- 1 Warm up: p -norms
- 2 Project overview
- 3 Review of Hilbert Modules
- 4 L^p -modules
 - L^p -modules: Definition and Examples
 - L^p -modules: Morphisms
 - L^p -modules: New from old
- 5 L^p -correspondences
 - Correspondences
 - Fock Representations
 - Cuntz-Pimsner Algebras
- 6 Future Work

Future Lines of work

- Independence of the representation,
- Algebras generated by the Fock spaces for other L^p -modules,
- Universality L^p -Fock representations coming from Fock spaces,
- C^* -likeness of direct sums of L^p -modules,
- Bimodules and Morita equivalent L^p -operator algebras.

Thank you!
Questions?

L^p -modules: Direct Sum

If $((X_j, Y_j))_{j=1}^{\infty}$ is a sequence of L^p -modules over A , a naive generalization of the finite case does not work. Say that for each $j \geq 1$, $X_j \subseteq \mathcal{L}(L^p(\mu_0), L^p(\mu_j))$. Define

$$X_w = \left\{ (x_j)_{j=1}^{\infty} : x_j \in X_j, \sup_{\|\xi\|_p=1} \sum_{j=1}^{\infty} \|x_j \xi\|_p^p < \infty \right\},$$
$$Y_w = \left\{ (y_j)_{j=1}^{\infty} : y_j \in Y_j, \sup_{\sum_{j=1}^{\infty} \|\eta_j\|_p^p=1} \left\| \sum_{j=1}^{\infty} y_j \eta_j \right\|_p < \infty \right\}.$$

Then, in general (X_w, Y_w) is not an L^p -module over A .

L^p -modules: Direct Sum

$$X_w = \left\{ (x_j)_{j=1}^{\infty} : x_j \in X_j, \sup_{\|\xi\|_p=1} \sum_{j=1}^{\infty} \|x_j \xi\|_p^p < \infty \right\},$$
$$Y_w = \left\{ (y_j)_{j=1}^{\infty} : y_j \in Y_j, \sup_{\sum_{j=1}^{\infty} \|\eta_j\|_p^p=1} \left\| \sum_{j=1}^{\infty} y_j \eta_j \right\|_p < \infty \right\}.$$

Counterexample:

For any $p \in [1, \infty)$, let $(X_j, Y_j) = (\ell^q, \ell^p)$ for all $j \in \mathbb{Z}_{\geq 1}$. Define $x_j: \ell^p \rightarrow \mathbb{C}$ by $x_j \xi = \xi(j)$ and $y_j: \mathbb{C} \rightarrow \ell^p$ by $y_j \zeta = \zeta \delta_j$. Then,

- $(y_j | x_j)_{\mathcal{K}(\ell^p)} = y_j x_j = \theta_{\delta_j, \delta_j} \in \mathcal{K}(\ell^p)$,
- $(x_j)_{j=1}^{\infty} \in X_w$, $(y_j)_{j=1}^{\infty} \in Y_w$,
- $\sum_{j=1}^{\infty} \theta_{\delta_j, \delta_j}$ is not convergent in $\mathcal{K}(\ell^p)$.

L^p -modules: Direct Sum

Theorem (D, 2023)

$\bigoplus_{j=1}^{\infty} (X_j, Y_j)$ is an L^p -module over A that agrees with the usual definition of direct sum of Hilbert modules when $p = 2$. That is, if A is a C^* -algebra and X_j are right Hilbert A -modules, with

$$\bigoplus_{j=1}^{\infty} X_j := \left\{ (x_j)_{j=1}^{\infty} : x_j \in X_j, \sum_{j=1}^{\infty} \langle x_j, x_j \rangle_A \text{ converges in } A \right\},$$

then

$$\bigoplus_{j=1}^{\infty} (X_j, \widetilde{X}_j) \cong \left(\bigoplus_{j=1}^{\infty} X_j, \bigoplus_{j=1}^{\infty} \widetilde{X}_j \right).$$

Application # 1

Theorem (D, 2022)

Let (X, φ) be an (A, B) C^* -correspondence such that A acts nondegenerately on X . Then, there is an A -valued left inner product on X making it a Hilbert A - B bimodule if and only if $\mathcal{K}_B(X) \subseteq \varphi(A)$.

Proof. (\Rightarrow) Easy: $\theta_{x,y}(z) = x\langle y, z \rangle_B = \varphi({}_A\langle x, y \rangle)z$.

(\Leftarrow) Assume that $\mathcal{K}_B(X) \subseteq \varphi(A)$ and let (π_A, π_B, π_X) be an isometric representation of (X, φ) . Then,

$${}_A\langle x, y \rangle := \pi_A^{-1}(\pi_X(x)\pi_X(y)^*).$$

is a well defined left A valued inner product making X a Hilbert A - B -bimodule. ■

Application # 2: Tensor Products of C^* -correspondences

Theorem (D, 2022)

Let (X, φ_X) be an (A, B) C^* -correspondence, let (Y, φ_Y) be a (B, C) C^* -correspondence such that B acts nondegenerately on Y , and let $\pi_C: C \rightarrow \mathcal{L}(\mathcal{H}_0)$ be any nondegenerate representation of C on a Hilbert space \mathcal{H}_0 . Then:

- 1 There are Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ and maps such that (π_A, π_B, π_X) is a representation of (X, φ_X) on $(\mathcal{H}_1, \mathcal{H}_2)$, and (π_B, π_C, π_Y) is a representation of (Y, φ_Y) on $(\mathcal{H}_0, \mathcal{H}_1)$.
- 2 There is a map $\pi: X \otimes_{\varphi_Y} Y \rightarrow \mathcal{L}(\mathcal{H}_0, \mathcal{H}_2)$ satisfying $\pi(x \otimes y) = \pi_X(x)\pi_Y(y)$ and such that (π_A, π_C, π) is a representation of $(X \otimes_{\varphi_Y} Y, \widetilde{\varphi_X})$ on $(\mathcal{H}_0, \mathcal{H}_2)$.
- 3 $X \otimes_{\varphi_Y} Y \cong \overline{\pi_X(X)\pi_Y(Y)}$ via π .

Covariant Fock Representations

Lemma

Let (B, π_A, π_X) be any Fock representation for (X, φ) . Then, there is a $*$ -homomorphism $\pi_{\mathcal{K}}: \mathcal{K}_A(X) \rightarrow B$ that satisfies $\pi_{\mathcal{K}}(\theta_{x,y}) = \pi_X(x)\pi_X(y)^*$ for all $x, y \in X$.

For a C^* correspondence (X, φ) over A , T. Katsura defined ideal J_X by

$$\begin{aligned} J_X &= \varphi^{-1}(\mathcal{K}_A(X)) \cap (\ker(\varphi))^\perp \\ &= \{a \in A : \varphi(a) \in \mathcal{K}_A(X) \text{ and } ab = 0 \text{ for all } b \in \ker(\varphi)\}. \end{aligned}$$

Definition

A Fock representation (B, π_A, π_X) for (X, φ) is said to be **covariant** if

$$\pi_{\mathcal{K}}(\varphi(a)) = \pi_A(a) \quad \forall a \in J_X.$$

Covariant L^p -Fock Representations

Let $((X, Y), \varphi)$ be an L^p -correspondence over an L^p -operator algebra A . We define

$$J_{(X, Y)} = \{a \in A : \varphi(a) \in \mathcal{K}_A((X, Y)) \text{ and } ab = 0 \text{ for all } b \in \ker(\varphi)\}$$

Definition

Let $p \in [1, \infty)$, let A be an L^p -operator algebra, and let (B, π_A, π_X, π_Y) be an L^p -Fock representation for a L^p -correspondence $((X, Y), \varphi)$ over A . We say (B, π_A, π_X, π_Y) is **covariant** if there is a contractive homomorphism $\pi_{\mathcal{K}} : \mathcal{K}_A((X, Y)) \rightarrow B$ satisfying

$$\begin{aligned}\pi_{\mathcal{K}}(\theta_{x, y}) &= \pi_X(x)\pi_Y(y) \text{ for all } x \in X, y \in Y, \\ \pi_{\mathcal{K}}(\varphi(a)) &= \pi_A(a) \text{ for all } a \in J_{(X, Y)}.\end{aligned}$$

Spatial Partial Isometries

Let $p \in [1, \infty)$, let $(\Omega_0, \mathfrak{M}, \mu)$ and $(\Omega_1, \mathfrak{N}, \nu)$ be σ -finite measure spaces, and let $s : L^p(\mu) \rightarrow L^p(\nu)$ be a linear map.

We say s is a **spatial partial isometry** if there are $E \in \mathfrak{M}$, $F \in \mathfrak{N}$, $S : \mathfrak{M}|_E \rightarrow \mathfrak{N}|_F$ a bijective measurable set transformation with $\nu|_{\mathfrak{N}|_F}$ σ -finite, and $g : F \rightarrow \mathbb{C}$ a measurable function such that $|g| = 1$ a.e. $[\nu|_F]$ such that

$$s(\xi) = \begin{cases} \left(\frac{dS_*(\mu|_E)}{d\nu|_{\mathfrak{N}|_F}} \right)^{1/p} S_*(\xi|_E)g & \text{on } F \\ 0 & \text{on } \Omega_1 \setminus F \end{cases}$$

for all $\xi \in L^p(\mu)$. Turns out that

$$\|s(\xi)\|_p = \|\xi|_E\|_p.$$

The partial isometry implemented by $(F, E, S^{-1}, (S^{-1})_*(g)^{-1})$ is called the **reverse of s** .

L^p Analogues of \mathcal{O}_d

Let $d \geq 2$ be an integer. The **Leavitt algebra** L_d is the universal unital complex algebra generated by elements s_1, s_2, \dots, s_d , and t_1, t_2, \dots, t_d satisfying

$$t_j s_k = \delta_{j,k} \quad \text{and} \quad \sum_{j=1}^d s_j t_j = 1$$

Definition

Let $p \in [1, \infty)$ and let $(\Omega, \mathfrak{M}, \mu)$ be a measure space. A spatial representation of L_d is an algebra homomorphism $\varphi : L_d \rightarrow \mathcal{L}(L^p(\mu))$ such that each $\varphi(s_j)$ is a spatial partial isometry with reverse $\varphi(t_j)$.

Given any spatial representation φ of L_d , we define

$$\mathcal{O}_d^p := \overline{\varphi(L_d)} \subseteq \mathcal{L}(L^p(\mu)).$$

Known Facts about \mathcal{O}_d^p

Let $d \geq 2$ be an integer and let $p \in [1, \infty)$. Then

- 1 \mathcal{O}_d^p is simple.
- 2 \mathcal{O}_d^p is purely infinite
- 3 $\mathcal{O}_d^2 = \mathcal{O}_n$.
- 4 $\mathcal{O}_2^p \otimes_p \mathcal{O}_2^p \cong \mathcal{O}_2^p$ if and only if $p = 2$.
- 5 \mathcal{O}_d^p has the same K -theory as when $p = 2$:

$$K_0(\mathcal{O}_d^p) \cong \mathbb{Z}/(d-1)\mathbb{Z} \quad \text{and} \quad K_1(\mathcal{O}_d^p) \cong \{0\}.$$

- 6 $K_*^{\text{top}}(\mathcal{O}_d^p) \cong K_*^{\text{alg}}(\mathcal{O}_d^p)$