Operator Spaces.

Alonso Delfín
University of Oregon.

November 19, 2020

Abstract

The field of operator spaces is an important branch of functional analysis, commonly used to generalize techniques from Banach space theory to algebras of operators on Hilbert spaces. A "concrete" operator space E is a closed subspace of $\mathcal{L}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$. Almost 35 years ago, Zhong-Jin Ruan gave an abstract characterization of operator spaces, which allows us to forget about the concrete Hilbert space $\mathcal{H}$. Roughly speaking, an "abstract" operator space consists of a normed space E together with a family of matrix norms on $M_n(E)$ satisfying two axioms. Ruan's Theorem states that any abstract operator space is completely isomorphic to a concrete one.

On this document I will give a basic introduction to operator spaces, providing many examples. I will discuss completely bounded maps (the morphisms in the category of operator spaces) and try to give an idea of why abstract operator spaces are in fact concrete ones. Time permitting, I will explain how operator spaces are used to define a non-selfadjoint version of Hilbert modules and I'll say how this might be useful for my research.

1 Definitions and Examples

Definition 1.1. Let $\mathcal{H}$ be a Hilbert space. An **operator space** E is a closed subspace of $\mathcal{L}(\mathcal{H})$.

Example 1.2.

1. Any C^*-algebra A is an operator space.
2. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Then $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is regarded as an operator space by identifying $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ in $\mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ by
$$\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) \ni a \mapsto \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$$
3. Any Banach space E is an operator space. Indeed, it's well known that B_{E^*} is a compact space when equipped with the weak-$*$ topology and E is identified with a subspace of $C(B_{E^*})$ via the isometric mapping $\xi \mapsto \widehat{\xi}$, where
$$\widehat{\xi}(\varphi) = \varphi(\xi)$$
for any $\varphi \in B_{E^*}$. Since $C(B_{E^*})$ is a C^*-algebra, it follows that E is an operator space.

The main difference between the category of Banach spaces and that of operator spaces is in the morphisms. We will see below that we need to look at linear maps that behave well with respect to some natural matrix norms.

Matrix norms. Let E be an operator space and $n \in \mathbb{Z}_{>0}$. Then, $M_n(E)$, the space of $n \times n$ matrices with entries in E, is a subspace of $M_n(\mathcal{L}(\mathcal{H}))$. Thus, $M_n(E)$ has a natural norm $\|\cdot\|_n$, which comes from the identification of $M_n(\mathcal{L}(\mathcal{H}))$ as $\mathcal{L}(\mathcal{H}^n)$, where $\mathcal{H}^n$ is the ℓ^2 direct sum of $\mathcal{H}$ with it self. More precisely, if $\xi := (\xi_{j,k}) \in M_n(E)$

$$\|\xi\|_n = \|(\xi_{j,k})\|_n := \sup \left\{ \left(\sum_{j=1}^n \left\| \sum_{k=1}^n \xi_{j,k} h_k \right\|^2 \right)^{1/2} : h := (h_1, \ldots, h_n) \in B_{\mathcal{H}^n} \right\}$$

Of course the norm $\|\cdot\|_1$ coincides with the norm of E. The following lemma gives a useful way to compute $\|\xi\|_n$:

Lemma 1.3. *Let E be an operator space and $n \in \mathbb{Z}_{\geq 0}$. Then,*

$$\|\xi\|_n = \|(\xi_{j,k})\|_n = \sup \left\{ \left| \sum_{j=1}^n \sum_{k=1}^n \langle \xi_{j,k} h_k, f_j \rangle \right| : h, f \in B_{\mathcal{H}^n} \right\}$$

Proof. Let $h, f \in B_{\mathcal{H}^n}$. Recall that $\|h\| = \sup_{\|f\|=1} |\langle h, f \rangle|$, from where we get

$$\left| \sum_{j=1}^n \sum_{k=1}^n \langle \xi_{j,k} h_k, f_j \rangle \right|^2 \leq \sum_{j=1}^n \left| \langle \sum_{k=1}^n \xi_{j,k} h_k, f_j \rangle \right| \leq \sum_{j=1}^n \left\| \sum_{k=1}^n \xi_{j,k} h_k \right\|^2 \leq \|\xi\|_n^2$$

For the reverse inequality, ... ■

Completely bounded linear maps. Let E and F be operator spaces and $u : E \to F$ a linear map. For each $n \in \mathbb{Z}_{>0}$, u induces a linear map $u_n : M_n(E) \to M_n(F)$ in the obvious way

$$u_n((\xi_{j,k})) := (u(\xi_{j,k})).$$

Further, we set $\|u_n\| := \sup\{\|u_n(\xi)\|_n : \xi \in M_n(E),\ \|\xi\|_n = 1\}$. We say that u is **completely bounded (c.b.)** if

$$\|u\|_{\text{cb}} := \sup_{n \in \mathbb{Z}_{>0}} \|u_n\| < \infty.$$

We will denote by $\text{CB}(E, F) \subset \mathcal{L}(E, F)$ to the set of all c.b maps from E to F. Notice that when equipped with the norm $\|\cdot\|_{\text{cb}}$, $\text{CB}(E, F)$ is a Banach space and therefore also an operator space.

Definition 1.4. Let E, F be operator spaces and $u \in \text{CB}(E, F)$.

1. If $\|u\|_{\text{cb}} \leq 1$ we say u is **completely contractive.**
2. If each u_n is an isometry, we say u is a **complete isometry.**
3. We say E and F are **completely isomorphic** if u is an isomorphism with $u^{-1} \in \text{CB}(F, E)$.
4. We say E and F are **completely isometrically isomorphic** if u is a complete isomorphism that's also a complete isometry.

Proposition 1.5. *Let E, F and G be operator spaces and $u \in \text{CB}(E, F)$, $v \in \text{CB}(F, G)$, then $vu \in \text{CB}(E, G)$ and $\|vu\|_{\text{cb}} \leq \|v\|_{\text{cb}} \|u\|_{\text{cb}}$.*

Proof. Since

$$\|(vu)_n\| = \sup_{\|\xi\|_n = 1} \|v_n(u_n(\xi))\|_n \leq \|v_n\| \|u_n\|,$$

it follows that $\|vu\| = \sup_{n \in \mathbb{Z}_{>0}} \|(vu)_n\| \leq \sup_{n \in \mathbb{Z}_{>0}} \|v_n\| \|u_n\| = \|v\|_{\text{cb}} \|u\|_{\text{cb}}$. ■

Proposition 1.6. *Let E, F be operator spaces and $u \in \mathrm{CB}(E,F)$ a rank one operator. That is, $u(\xi) = \varphi(\xi)\eta$ for $\varphi \in E^*$ and $\eta \in F$. Then, $\|u\|_{\mathrm{cb}} = \|u\|$.*

Proof. We always have $\|u\| \leq \|u\|_{cb}$. For the reverse inequality, we recall first that $\|u\| = \|\varphi\|\|\eta\|$. Then, notice that for any $\xi = (\xi_{j,k}) \in M_n(E)$ we have using the lemma above

$$\|\varphi_n(\xi)\|_n = \|(\varphi(\xi_{j,k}))\| = \sup_{x,y\in B_{\ell^2_n}} \left|\sum_{j,k} \varphi(\xi_{j,k}) x_k \overline{y_j}\right| \leq \|\varphi\| \sup_{x,y\in B_{\ell^2_n}} \left\|\sum_{j,k} \xi_{j,k} x_k \overline{y_j}\right\| \leq \|\varphi\|\|\xi\|_n$$

Hence,

$$\|u_n(\xi)\|_n = \|\eta\|\|\varphi_n(\xi)\|_n \leq \|\eta\|\|\varphi\|\|\xi\|_n = \|u\|\|\xi\|_n$$

from where it follows that $\|u\|_{\mathrm{cb}} \leq \|u\|$. ■

Definition 1.7. Let $n \in \mathbb{Z}_{>0}$. We write M_n instead of $M_n(\mathbb{C})$. We use the natural norm on M_n, which comes from the identification of M_n with $\mathcal{L}(\ell^2(\{1,\dots,n\}))$. If E is an operator space, the multiplication of elements in $M_n(E)$ with M_n is done in the obvious way.

Proposition 1.8. *Let $E \subset \mathcal{L}(\mathcal{H})$ be an operator space and $\|\cdot\|_n$ the norms on $M_n(E)$ defined above. For $\xi := (\xi_{j,k}) \in M_n(E)$ and $\alpha := (\alpha_{j,k}), \beta := (\beta_{j,k}) \in M_n$, we have*

(R1) $\|\alpha\xi\beta\|_n \leq \|\alpha\|\|\xi\|_n\|\beta\|$.

(R2) If $\eta := (\eta_{j,k}) \in M_m(E)$, then

$$\left\|\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}\right\|_{n+m} = \max\{\|\xi\|_n, \|\eta\|_m\}$$

Proof. To see *(R1)* we define for any $\alpha \in M_n$ an element $\widetilde{\alpha} := (\alpha_{j,k}\mathrm{id}_{\mathcal{H}}) \in M_n(\mathcal{L}(\mathcal{H}))$. Notice that $\|\alpha\| = \|\widetilde{\alpha}\|_n$. Furthermore, $\alpha\xi\beta = \widetilde{\alpha}\xi\widetilde{\beta} \in M_n(E)$. Therefore

$$\|\alpha\xi\beta\|_n = \|\widetilde{\alpha}\xi\widetilde{\beta}\|_n \leq \|\widetilde{\alpha}\|_n\|\xi\|_n\|\widetilde{\beta}\|_n = \|\alpha\|\|\xi\|_n\|\beta\|$$

For *(R2)*, notice that

$$\|\xi\| = \left\|\begin{pmatrix} \xi \\ 0 \end{pmatrix}\right\| = \left\|\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}\begin{pmatrix} \mathrm{id}_{\mathcal{H}^n} \\ 0 \end{pmatrix}\right\| \leq \left\|\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}\right\|_{n+m} \|\mathrm{id}_{\mathcal{H}^n}\| = \left\|\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}\right\|_{n+m}$$

and similarly

$$\|\eta\| \leq \left\|\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}\right\|_{n+m}$$

This gives one inequality. For the reverse one, we take $h \in \mathcal{H}^n$ and $f \in \mathcal{H}^m$ and notice that

$$\left\|\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}\begin{pmatrix} \xi \\ 0 \end{pmatrix}\right\|^2 = \left\|\begin{pmatrix} \xi h \\ \eta f \end{pmatrix}\right\|^2 = \|\xi h\|^2 + \|\eta f\|^2 \leq \max\{\|\xi\|_n, \|\eta\|_m\}(\|h\|^2 + \|f\|^2).$$

This finishes the proof. ■

Theorem 1.9. *(Ruan 1987) Suppose that E is a vector space, and that for each $n \in \mathbb{Z}_{>0}$ we are given a norm $\|\cdot\|_n$ on $M_n(E)$ satisfying conditions (R1) and (R2) above. Then E is completely isometrically isomorphic to a subspace of $\mathcal{L}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$.*

Sketch of Proof. Let $\mathcal{U}$ be the collection of all completely contractive maps $u : E \to M_n$ for some $n \in \mathbb{Z}_{\geq 0}$. For each $u \in \mathcal{U}$, we define $n_u \in \mathbb{Z}_{\geq 0}$ to be so that M_{n_u} is the co-domain of u. Then,

$$M := \bigoplus_{u\in\mathcal{U}}^{\sup} M_{n_u}$$

is a C^*-algebra and hence an operator space. We define $v : E \to M$ by $v(\xi) = (u(\xi))_{u\in\mathcal{U}}$. One checks v is a complete contraction. Furthermore, if $\xi \in M_n(E)$, by Hahn-Banach there is a complete contraction $u : E \to M_n(E)$ such that $\|u_n(\xi)\| = \|\xi_n\|$. Now consider the projection $p_u : M \to M_{n_u}$ and notice that

$$\|v_n(\xi)\| = \|(v(\xi_{j,k}))\| \geq \|(p_u(u(\xi_{j,k})))\| = \|u_n(\xi)\| = \|\xi_n\|.$$

Thus, v_n is an isometry and therefore v is a complete isometry. "□"

2 Column and Row Hilbert space.

Let $\mathcal{H}$ be any Hilbert space. There are several (completely isometrically isomorphic) ways of giving $\mathcal{H}$ a canonical operator space structure which we call the column Hilbert space and denote by $\mathcal{H}^{\mathrm{c}}$. Informally, one should think of $\mathcal{H}^{\mathrm{c}}$ as a "column in $\mathcal{L}(\mathcal{H})$". We now give 3 equivalent descriptions of $\mathcal{H}^{\mathrm{c}}$ for a general Hilbert space $\mathcal{H}$.

1. Identify $\mathcal{H}$ with $\mathcal{L}(\mathbb{C}, \mathcal{H})$ by regarding each $h \in \mathcal{H}$ as a map $t_h : \mathbb{C} \to \mathcal{H}$ defined by $t_h(\lambda) := \lambda h$. Notice that the operator $t_h^* : \mathcal{H} \to \mathbb{C}$ is such that $t_h^*(f) = \langle f, h\rangle$ and therefore

$$(t_h^* t_f)(1) = \langle f, h\rangle$$

Using this we notice that the induced norm in $M_n(\mathcal{H}^{\mathrm{c}})$ takes a nice form:

$$\|(t_{h_{j,k}})\| = \left\| \left(\sum_{i=1}^{n} t^*_{h_{i,j}} t_{h_{i,k}} \right)_{j,k} \right\|^{1/2} = \left\| \left(\sum_{i=1}^{n} \langle h_{i,k}, h_{i,j}\rangle \right)_{j,k} \right\|^{1/2}$$

where we've used the C^*-identity.

2. Fix a unit vector $f \in \mathcal{H}$. Look at the rank one operators $\theta_{h,f}(y) := \langle y, f\rangle h$ and identify $\mathcal{H}^{\mathrm{c}}$ with $\{\theta_{h,f} : h \in \mathcal{H}\} \subset \mathcal{L}(\mathcal{H})$. This gives an operator structure which is independent (up to complete isometry) of the unit vector f chosen to begin with. Further, such structure coincides with the above one:

$$\|(\theta_{h_{j,k},f})_{j,k}\| = \left\| \left(\sum_{i=1}^{n} \theta_{f,h_{h,j}} \theta_{h_{j,k},f} \right)_{j,k} \right\|^{1/2} = \left\| \left(\sum_{i=1}^{n} \langle h_{i,k}, h_{i,j}\rangle \theta_{f,f} \right)_{j,k} \right\|^{1/2} = \left\| \left(\sum_{i=1}^{n} \langle h_{i,k}, h_{i,j}\rangle \right)_{j,k} \right\|^{1/2}$$

where we've used again the C^*-identity and that f has norm 1.

3. Fix an orthonormal basis for $\mathcal{H}$ and regard elements in $\mathcal{L}(\mathcal{H})$ as infinite matrices with respect to this basis. Then let $\mathcal{H}^{\mathrm{c}}$ consist of all the matrices in $\mathcal{L}(\mathcal{H})$ that are zero except on the first column. That way, if $\mathcal{H} = \ell^2$ we can describe the column space as $\mathcal{H}^{\mathrm{c}} = \overline{\mathrm{span}}\{\delta_{j,1} : j \in \mathbb{Z}_{>0}\} \subset \mathcal{L}(\ell^2)$ which is of course isometric to ℓ^2. For a general Hilbert space $\mathcal{H}$ this description gives the same operator structure as the one described above.

Theorem 2.1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be vector spaces. Then $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is completely isometrically isomophic to* $\mathrm{CB}(\mathcal{H}_1^{\mathrm{c}}, \mathcal{H}_2^{\mathrm{c}})$.

We define the Hilbert row space $\mathcal{H}^{\mathrm{r}}$ similarly. This will end up being the operator set whose underlying space is $\mathcal{H}$ and whose matrix norms are given by

$$\|(h_{j,k})\| = \left\| \left(\sum_{i=1}^{n} \langle h_{k,i}, h_{j,i} \rangle \right)_{j,k} \right\|^{1/2}$$

Even though $\mathcal{H}^{\mathrm{c}}$ and $\mathcal{H}^{\mathrm{r}}$ are the same Hilbert space, their operator space structure is not the same. Indeed, if $(h_{j,k})$ is an $n \times n$ orthonormal matrix with elements in $\mathcal{H}$, its norm induced by $\mathcal{H}^{\mathrm{c}}$ is 1, whereas its norm induced by $\mathcal{H}^{\mathrm{r}}$ is $\sqrt{n}$.

3 Generalization of Hilbert Modules

Notation. A (concrete) operator algebra is a subalgebra A of $\mathcal{L}(\mathcal{H})$. A concrete right A-operator module E is a subspace E of $\mathcal{L}(\mathcal{H})$, which is right invariant under multiplication by the algebra $A \subset \mathcal{L}(\mathcal{H})$.

If $t : E \to E$ is a module map, we get a dual map $t^* : \mathrm{Hom}_A(E, A) \to \mathrm{Hom}_A(E, A)$ given by

$$t^*(\varphi) = \varphi \circ t$$

Turns out that if $t \in \mathrm{CB}(E, E)$ is completely contractive, then t^* is completely contractive on $\mathrm{CB}_A(E, A)$.

If $\xi \in E$ and $\varphi \in \mathrm{CB}_A(E, A)$ we get a rank one map $\theta_{\xi,\varphi} \in \mathrm{CB}(E, E)$ given by

$$\theta_{\xi,\varphi}(\eta) = \xi\varphi(\eta) \in E$$

It's easy to check that $\|\theta_{\xi,\varphi}\| \leq \|\xi\|\|\varphi\|$.

For any operator space E, we write $C_n(E) := M_{n,1}(E)$.

Definition 3.1. Suppose A is an operator algebra, E a right A-operator module and that there is a net of positive integers $(n_\lambda)_{\lambda \in \Lambda}$ together with A-module maps $u_\lambda : E \to C_{n_\lambda}(A)$, $v_\lambda : C_{n_\lambda}(A) \to E$ such that

- u_λ and v_λ are completely contractive.
- $t_\lambda := v_\lambda u_\lambda \to \mathrm{id}_E$ strongly.
- The maps v_λ are right A-essential.
- For all $\gamma \in \Lambda$, $u_\gamma v_\lambda u_\lambda \to u_\gamma$ uniformly.

Then, we say E is a **right A-rigged module**.

If E is a right A-rigged module we define

$$\widetilde{E} := \{\varphi \in \mathrm{CB}_A(E, A) : t_\lambda^* \varphi \to \varphi \text{ uniformly}\}$$

and we let $\mathcal{K}(E)$ be the closure in $\mathrm{CB}_A(E, E)$ of $\{\theta_{\xi,\varphi} : \xi \in E, \varphi \in \widetilde{E}\}$.

Theorem 3.2. *If A is a C^*-algebra, then Hilbert A-modules are right A-rigged module.*

Department of Mathematics, University of Oregon, Eugene OR 97403-1222, USA.
E-mail address: `alonsod@uoregon.edu`