Operator Spaces

Alonso Delfín

Functional Analysis Seminar
University of Oregon

November 19, 2020

Outline

Operator Algebras.

We denote by $\mathcal{L}(E,F)$ the bounded linear operators between Banach spaces E and F. This is a Banach space (algebra if $E=F$) with norm

$$\|a\| := \sup_{\|\xi\|_E=1} \|a(\xi)\|_F$$

- If $\mathcal{H}$ is a Hilbert space, any $a \in \mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H},\mathcal{H})$ has an adjoint $a^* \in \mathcal{L}(\mathcal{H})$ characterized by $\langle a(\xi),\eta\rangle = \langle \xi, a^*(\eta)\rangle$.
- If $\mathcal{H}$ is a Hilbert space, C*-algebra A is a norm closed selfadjoint subalgebra of $\mathcal{L}(\mathcal{H})$.
- If (X,μ) is a measure space and $p \in [1,\infty)$, an L^p-operator algebra A is a norm closed subalgebra of $\mathcal{L}(L^p(X,\mu))$.

Definition and Examples

Definition

Let $\mathcal{H}$ be a Hilbert space. An **operator space** E is a closed subspace of $\mathcal{L}(\mathcal{H})$.

Example

Any C^*-algebra A is an operator space.

Definition and Examples

Definition

Let $\mathcal{H}$ be a Hilbert space. An **operator space** E is a closed subspace of $\mathcal{L}(\mathcal{H})$.

Example

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Then $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is regarded as an operator space by identifying $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ in $\mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ by

$$\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) \ni a \mapsto \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$$

Definition and Examples

Definition

Let $\mathcal{H}$ be a Hilbert space. An **operator space** E is a closed subspace of $\mathcal{L}(\mathcal{H})$.

Example

Any Banach space E is an operator space. Indeed, it's well known that B_{E^*} is a compact space when equipped with the weak-$*$ topology and E is identified with a subspace of $C(B_{E^*})$ via the isometric mapping $\xi \mapsto \widehat{\xi}$, where

$$\widehat{\xi}(\varphi) = \varphi(\xi)$$

for any $\varphi \in B_{E^*}$. Since $C(B_{E^*})$ is a C^*-algebra, it follows that E is an operator space.

Matrix Norms

Let E be an operator space and $n \in \mathbb{Z}_{>0}$. Then, $M_n(E)$, the space of $n \times n$ matrices with entries in E, is a subspace of $M_n(\mathcal{L}(\mathcal{H}))$. Thus, $M_n(E)$ has a natural norm $\|\cdot\|_n$, which comes from the identification of $M_n(\mathcal{L}(\mathcal{H}))$ as $\mathcal{L}(\mathcal{H}^n)$, where $\mathcal{H}^n$ is the ℓ^2 direct sum of $\mathcal{H}$ with it self.

More precisely, if $\xi := (\xi_{j,k}) \in M_n(E)$

$$\|(\xi_{j,k})\|_n := \sup\left\{ \left(\sum_{j=1}^{n} \left\| \sum_{k=1}^{n} \xi_{j,k} h_k \right\|^2 \right)^{1/2} : h := (h_k) \in B_{\mathcal{H}^n} \right\}$$

Completely bounded maps.

The main difference between the category of Banach spaces and that of operator spaces is in the morphisms.

Let E and F be operator spaces and $u : E \to F$ a linear map. For each $n \in \mathbb{Z}_{>0}$, u induces a linear map $u_n : M_n(E) \to M_n(F)$ in the obvious way

$$u_n((\xi_{j,k})) := (u(\xi_{j,k})).$$

Further, we set $\|u_n\| := \sup\{\|u_n(\xi)\|_n : \xi \in M_n(E),\ \|\xi\|_n = 1\}$. We say that u is **completely bounded (c.b.)** if

$$\|u\|_{\mathrm{cb}} := \sup_{n \in \mathbb{Z}_{>0}} \|u_n\| < \infty.$$

We denote by $\mathrm{CB}(E,F) \subset \mathcal{L}(E,F)$ to the set of all c.b maps from E to F.

Definition

Let E, F be operator spaces and $u \in \mathrm{CB}(E,F)$.

1. If $\|u\|_{\mathrm{cb}} \leq 1$ we say u is **completely contractive.**
2. If each u_n is an isometry, we say u is a **complete isometry.**
3. We say E and F are **completely isomorphic** if u is an isomorphism with $u^{-1} \in \mathrm{CB}(F,E)$.
4. We say E and F are **completely isometrically isomorphic** if u is a complete isomorphism that's also a complete isometry.

Proposition

Let E, F and G be operator spaces and $u \in \mathrm{CB}(E, F)$, $v \in \mathrm{CB}(F, G)$, then $vu \in \mathrm{CB}(E, G)$ and

$$\|vu\|_{\mathrm{cb}} \leq \|v\|_{\mathrm{cb}} \|u\|_{\mathrm{cb}}.$$

Proposition

Let E, F be operator spaces and $u \in \mathrm{CB}(E, F)$ a rank one operator. That is, $u(\xi) = \varphi(\xi)\eta$ for $\varphi \in E^*$ and $\eta \in F$. Then, $\|u\|_{\mathrm{cb}} = \|u\|$.

Ruan's Axioms

Proposition

Let $E \subset \mathcal{L}(\mathcal{H})$ *be an operator space and* $\|\cdot\|_n$ *the norms on* $M_n(E)$ *defined above. For* $\xi := (\xi_{j,k}) \in M_n(E)$ *we have*

(R1) *If* $\alpha := (\alpha_{j,k}), \beta := (\beta_{j,k}) \in M_n := M_n(\mathbb{C})$

$$\|\alpha\xi\beta\|_n \leq \|\alpha\|\|\xi\|_n\|\beta\|.$$

(R2) *If* $\eta := (\eta_{j,k}) \in M_m(E)$, *then*

$$\left\|\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}\right\|_{n+m} = \text{máx}\{\|\xi\|_n, \|\eta\|_m\}$$

Ruan's Theorem

Theorem

(Ruan 1987) Suppose that E *is a vector space, and that for each* $n \in \mathbb{Z}_{>0}$ *we are given a norm* $\| \cdot \|_n$ *on* $M_n(E)$ *satisfying conditions (R1) and (R2) above. Then* E *is completely isometrically isomorphic to a subspace of* $\mathcal{L}(\mathcal{H})$*, for some Hilbert space* $\mathcal{H}$*.*

Proof. Let $\mathcal{U}$ be the collection of all completely contractive maps $u : E \to M_n$ for some $n \in \mathbb{Z}_{\geq 0}$. For each $u \in \mathcal{U}$, we define $n_u \in \mathbb{Z}_{\geq 0}$ to be so that M_{n_u} is the co-domain of u. Then,

$$M := \bigoplus_{u \in \mathcal{U}}^{\sup} M_{n_u}$$

is a C^*-algebra and hence an operator space. We define $v : E \to M$ by $v(\xi) = (u(\xi))_{u \in \mathcal{U}}$. One checks v is a complete contraction. Furthermore, if $\xi \in M_n(E)$, by Hahn-Banach there is a complete contraction $u : E \to M_n(E)$ such that $\|u_n(\xi)\| = \|\xi_n\|$. Now consider the projection $p_u : M \to M_{n_u}$ and notice that

$$\|v_n(\xi)\| = \|(v(\xi_{j,k}))\| \geq \|(p_u(u(\xi_{j,k})))\| = \|u_n(\xi)\| = \|\xi_n\|.$$

Thus, v_n is an isometry and therefore v is a complete isometry. ■

Column space

Let $\mathcal{H}$ be any Hilbert space. There are several (completely isometrically isomorphic) ways of giving $\mathcal{H}$ a canonical operator space structure which we call the column Hilbert space and denote by $\mathcal{H}^c$. Informally, one should think of $\mathcal{H}^c$ as a "column in $\mathcal{L}(\mathcal{H})$". We now give 3 equivalent descriptions of $\mathcal{H}^c$ for a general Hilbert space $\mathcal{H}$.

Definition 1

Identify $\mathcal{H}$ with $\mathcal{L}(\mathbb{C}, \mathcal{H})$ by regarding each $h \in \mathcal{H}$ as a map $t_h : \mathbb{C} \to \mathcal{H}$ defined by $t_h(\lambda) := \lambda h$. Notice that the operator $t_h^* : \mathcal{H} \to \mathbb{C}$ is such that $t_h^*(f) = \langle f, h \rangle$ and therefore

$$(t_h^* t_f)(1) = \langle f, h \rangle$$

Using this we notice that the induced norm in $M_n(\mathcal{H}^c)$ takes a nice form:

$$\|(t_{h_{j,k}})\| = \left\| \left(\sum_{i=1}^{n} t_{h_{i,j}}^* t_{h_{i,k}} \right)_{j,k} \right\|^{1/2} = \left\| \left(\sum_{i=1}^{n} \langle h_{i,k}, h_{i,j} \rangle \right)_{j,k} \right\|^{1/2}$$

where we've used the C^*-identity.

Definition 2

Fix a unit vector $f \in \mathcal{H}$. Look at the rank one operators $\theta_{h,f}(y) := \langle y, f\rangle h$ and identify $\mathcal{H}^c$ with $\{\theta_{h,f} : h \in \mathcal{H}\} \subset \mathcal{L}(\mathcal{H})$. This gives an operator structure which is independent (up to complete isometry) of the unit vector f chosen to begin with. Further, such structure coincides with the above one:

$$\|(\theta_{h_{j,k},f})_{j,k}\| = \left\| \left(\sum_{i=1}^{n} \theta_{f,h_{h,j}} \theta_{h_{j,k},f} \right)_{j,k} \right\|^{1/2} = \left\| \left(\sum_{i=1}^{n} \langle h_{i,k}, h_{i,j} \rangle \theta_{f,f} \right)_{j,k} \right\|^{1/2} =$$

where we've used again the C^*-identity and that f has norm 1.

Definition 3

Fix an orthonormal basis for $\mathcal{H}$ and regard elements in $\mathcal{L}(\mathcal{H})$ as infinite matrices with respect to this basis. Then let $\mathcal{H}^{c}$ consist of all the matrices in $\mathcal{L}(\mathcal{H})$ that are zero except on the first column. That way, if $\mathcal{H} = \ell^2$ we can describe the column space as $\mathcal{H}^{c} = \overline{\text{span}}\{\delta_{j,1} : j \in \mathbb{Z}_{>0}\} \subset \mathcal{L}(\ell^2)$ which is of course isometric to ℓ^2. For a general Hilbert space $\mathcal{H}$ this description gives the same operator structure as the one described above.

Theorem

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be vector spaces. Then $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is completely isometrically isomophic to $\mathrm{CB}(\mathcal{H}_1^c, \mathcal{H}_2^c)$.

The row space

We define the Hilbert row space $\mathcal{H}^{\mathrm{r}}$ similarly. This will end up being the operator set whose underlying space is $\mathcal{H}$ and whose matrix norms are given by

$$\|(h_{j,k})\| = \left\| \left(\sum_{i=1}^{n} \langle h_{k,i}, h_{j,i} \rangle \right)_{j,k} \right\|^{1/2}$$

Even though $\mathcal{H}^{\mathrm{c}}$ and $\mathcal{H}^{\mathrm{r}}$ are the same Hilbert space, their operator space structure is not the same. Indeed, if $(h_{j,k})$ is an $n \times n$ orthonormal matrix with elements in $\mathcal{H}$, its norm induced by $\mathcal{H}^{\mathrm{c}}$ is 1, whereas its norm induced by $\mathcal{H}^{\mathrm{r}}$ is $\sqrt{n}$.

notation

- A (concrete) operator algebra is a subalgebra A of $\mathcal{L}(\mathcal{H})$. A concrete right A-operator module E is a subspace E of $\mathcal{L}(\mathcal{H})$, which is right invariant under multiplication by the algebra $A \subset \mathcal{L}(\mathcal{H})$.
- If $t : E \to E$ is a module map, we get a dual map $t^* : \mathrm{Hom}_A(E, A) \to \mathrm{Hom}_A(E, A)$ given by

$$t^*(\varphi) = \varphi \circ t$$

  Turns out that if $t \in \mathrm{CB}(E, E)$ is completely contractive, then t^* is completely contractive on $\mathrm{CB}_A(E, A)$.
- If $\xi \in E$ and $\varphi \in \mathrm{CB}_A(E, A)$ we get a rank one map $\theta_{\xi,\varphi} \in \mathrm{CB}(E, E)$ given by

$$\theta_{\xi,\varphi}(\eta) = \xi\varphi(\eta) \in E$$

  It's easy to check that $\|\theta_{\xi,\varphi}\| \leq \|\xi\|\|\varphi\|$.

Rigged Modules

For any operator space E, we write $C_n(E) := M_{n,1}(E)$.

Definition

Suppose A is an operator algebra, E a right A-operator module and that there is a net of positive integers $(n_\lambda)_{\lambda \in \Lambda}$ together with A-module maps $u_\lambda : E \to C_{n_\lambda}(A)$, $v_\lambda : C_{n_\lambda}(A) \to E$ such that

- u_λ and v_λ are completely contractive.
- $t_\lambda := v_\lambda u_\lambda \to \mathrm{id}_E$ strongly.
- The maps v_λ are right A-essential.
- For all $\gamma \in \Lambda$, $u_\gamma v_\lambda u_\lambda \to u_\gamma$ uniformly.

Then, we say E is a **right A-rigged module**.

If E is a right A-rigged module we define

$$\widetilde{E} := \{\varphi \in \mathrm{CB}_A(E, A) : t_\lambda^* \varphi \to \varphi \text{ uniformly}\}$$

and we let $\mathcal{K}(E)$ be the closure in $\mathrm{CB}_A(E, E)$ of $\{\theta_{\xi,\varphi} : \xi \in E, \varphi \in \widetilde{E}\}$.

Theorem

If A is a C^*-algebra, then Hilbert A-modules are right A-rigged module.

Questions?