Morita Equivalence for C*-algebras.

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Abstract

Morita equivalence was adapted to C*-algebras by Marc Rieffel in the 1970's and it has since become a standard tool for analyzing group C*-algebras, crossed products and representations. Roughly speaking two C*-algebras A and B are Morita equivalent if there is a Hilbert (A, B)-bimodule with some compatibility conditions on the inner products.

The main goal of this talk is to show that two Morita equivalent C*-algebras have equivalent categories of representations. Along the way, I will give many accessible examples. The only two perquisites for following most of the talk are to have some familiarity with Hilbert spaces and with the tensor product of modules.

1 A brief review of Hilbert Modules

Definition 1.1. Let A be a C*-algebra and E a complex vector space which is also a right A-module. An **A-valued right inner product on** E is a map

$$\begin{array}{ccc} E \times E & \to & A \\ (\xi, \eta) & \mapsto & \langle \xi, \eta \rangle_A \end{array}$$

such that for any $\xi, \eta, \eta_1, \eta_2 \in E$, $a \in A$ and $\alpha \in \mathbb{C}$ we have

1. $\langle \xi, \eta_1 + \alpha \eta_2 \rangle_A = \langle \xi, \eta_1 \rangle_A + \alpha \langle \xi, \eta_2 \rangle_A$.
2. $\langle \xi, \eta a \rangle_A = \langle \xi, \eta \rangle_A a$.
3. $\langle \xi, \eta \rangle_A^* = \langle \eta, \xi \rangle_A$.
4. $\langle \xi, \xi \rangle_A \geq 0$ in A.
5. $\langle \xi, \xi \rangle_A = 0 \implies \xi = 0$.

Definition 1.2. Let A be a C*-algebra. A **Hilbert A-module** is a complex vector space E which is a right A-module with an A-valued right inner product and so that E is complete with the norm $\|\xi\| := \|\langle \xi, \xi \rangle_A\|^{1/2}$. We say that E is **full** if $\langle E, E \rangle_A := \text{span}\{\langle \xi, \eta \rangle_A : \xi, \eta \in E\}$ is dense in A.

Example 1.3. Let $\mathcal{H}$ be a Hilbert space with the physicists convention that the inner product is linear in the second variable. Then, $\mathcal{H}$ is clearly a full Hilbert $\mathbb{C}$-module.

Example 1.4. Any C^*-algebra A is clearly a full Hilbert A-module with inner product given by $(a, b) \mapsto a^* b$. More generally, A^n is also a full Hilbert A-module with the obvious "euclidean" inner product.

Example 1.5. The set of continuous sections of a vector bundle over a compact Hausdorff space X equipped with a Riemannian metric g is a Hilbert $C(X)$-module.

A main difference between Hilbert modules and Hilbert spaces is that not every bounded linear map between Hilbert A-modules has an adjoint. We will only be interested in those maps that do have an adjoint.

Definition 1.6. Let E and F be a Hilbert A-modules. A map $t : E \to F$ is said to be **adjointable** if there is a map $t^* : F \to E$ such that for any $\xi \in E$, and $\eta \in F$

$$\langle t(\xi), \eta \rangle = \langle \xi, t^*(\eta) \rangle$$

The space of adjointable maps from E to F is denoted by $\mathcal{L}_A(E, F)$ and $\mathcal{L}_A(E) := \mathcal{L}_A(E, E)$.

It's almost immediate that adjointable maps between Hilbert modules are linear and bounded. A standard proof shows that $\mathcal{L}_A(E)$ is a C*-algebra when equipped with the operator norm. We will have special interest for a particular case of andjointable maps, those of "rank 1":

Definition 1.7. Let E and F be a Hilbert A-modules. For $\xi \in E$ and $\eta \in F$, we define a map $\theta_{\xi,\eta} : F \to E$ by

$$\theta_{\xi,\eta}(\zeta) := \xi \langle \eta, \zeta \rangle_A$$

One easily checks that $\theta_{\xi,\eta} \in \mathcal{L}_A(E, F)$, that $(\theta_{\xi,\eta})^* = \theta_{\eta,\xi} \in \mathcal{L}_A(F, E)$ and that $\|\theta_{\xi,\eta}\| \leq \|\xi\|\|\eta\|$. This gives an analogous of the class of rank-one operators on Hilbert spaces. So, we define an analogous of the compact operators by letting

$$\mathcal{K}_A(E, F) := \overline{\text{span}\{\theta_{\xi,\eta} : \xi \in E, \eta \in F\}}$$

It's also not hard to verify that $\mathcal{K}_A(E) := \mathcal{K}_A(E, E)$ is a closed two sided ideal in $\mathcal{L}_A(E)$, whence $\mathcal{K}(E)$ is also a C^*-algebra.

2 Morita Equivalence

Given a Hilbert A-module E, there is a close connection between the C^*-algebras A and $\mathcal{K}(E)$. Observe that E is a left $\mathcal{K}(E)$-module when equipped with the obvious left action $v \cdot \xi := v(\xi)$. Further, there is a $\mathcal{K}(E)$-valued left inner product on E defined by

$$(\xi, \eta) := \theta_{\xi,\eta}$$

for any $\xi, \eta \in E$. Indeed:

- $(\xi_1 + \alpha\xi_2, \eta) = \theta_{\xi_1+\alpha\xi_2,\eta} = \theta_{\xi_1,\eta} + \alpha\theta_{\xi_2,\eta}$.
- $(v\xi, \eta) = \theta_{v\xi,\eta} = v\theta_{\xi,\eta} = v(\xi, \eta)$.
- $(\xi, \eta)^* = \theta^*_{\xi,\eta} = \theta_{\eta,\xi} = (\eta, \xi)$.
- $\langle (\xi, \xi)\eta, \eta \rangle = \langle \xi\langle \xi, \eta \rangle, \eta \rangle = \langle \xi, \eta \rangle^* \langle \xi, \eta \rangle \geq 0$, whence $(\xi, \xi) \geq 0$.
- If $(\xi, \xi) = 0$, then $\langle \xi, \xi \rangle = 0$ and therefore $\xi = 0$.
- $\|(\xi, \xi)\| = \|\langle \xi, \xi \rangle\|$ ($\leq$ is immediate and $\geq$ requires some play with functional calculus). Form this, it follows that E is complete with the norm induced by $(\cdot, \cdot)$.

Hence E is also a left Hilbert $\mathcal{K}(E)$-module. Even better, the right action of A on E is compatible with the left action of $\mathcal{K}(E)$ on E. Indeed, for $v \in \mathcal{K}(E)$, $\xi \in E$ and $a \in A$

$$(v \cdot \xi)a = v(\xi)a = v(\xi a) = v \cdot (\xi a)$$

The correct terminology is to say that E is a Hilbert $(\mathcal{K}(E), A)$-bimodule.

Definition 2.1. Two C^*-algebras A and B are said to be **Morita equivalent** if there is a Hilbert (A, B)-bimodule E (we use ${}_A(\cdot,\cdot)$ for A-valued inner product and $\langle\cdot,\cdot\rangle_B$ for the B-valued one) such that

1. E is a full left Hilbert A-module, E is a full right Hilbert B-module.
2. For all $\xi, \eta, \zeta \in E$, $a \in A$ and $b \in B$

   (2.1) $\langle a\xi, \eta\rangle_B = \langle \xi, a^*\eta\rangle_B$.

   (2.2) ${}_A(\xi b, \eta) = {}_A(\xi, \eta b^*)$.

   (2.3) ${}_A(\xi, \eta) \cdot \zeta = \xi \cdot \langle \eta, \zeta\rangle_B$.

If A and B are Morita equivalent C^*-algebras, then the module E implementing the equivalence is called an A-B **imprimitivity bimodule**.

Example 2.2. We already saw that any full Hilbert A-module implements a Morita equivalence between the C^*-algebras A and $\mathcal{K}_A(E)$. In particular, if $\mathcal{H}$ is an infinite dimensional Hilbert space, then $\mathbb{C}$ and $\mathcal{K}(\mathcal{H})$ are Morita equivalent C^*-algebras via the $\mathcal{K}(\mathcal{H})$-$\mathbb{C}$ imprimitivity bimodule $\mathcal{H}$.

Example 2.3. Morita equivalence is weaker than isomorphism. Indeed, given $\varphi : A \to B$, an isomorphism of C*-algebras, we can construct an imprimitive bimodule whose underlying space is B, right action of A is $a \cdot b := \varphi(a)b$, left action is left multiplication on B, and inner products are given by

$$ {}_A(b_1, b_2) := \varphi^{-1}(b_1 b_2^*), \qquad \langle b_1, b_2\rangle_B := b_1^* b_2 $$

Example 2.4. Let X be a locally compact Hausdorff space and $\mathcal{H}$ a Hilbert space. The C^*-algebras $A := C_0(X, \mathcal{K}(\mathcal{H}))$ and $B := C_0(X)$ are Morita equivalent. To see this we construct an (A, B) imprimitive bimodule whose underlying space is$C_0(X, \mathcal{H})$ and operations as follows

- Left action $A \curvearrowright C_0(X, \mathcal{H})$ is $(a \cdot f) \in C_0(X, \mathcal{H})$ given by

  $$ (a \cdot f)(x) := a(x)\big(f(x)\big) $$

  for any $a \in C_0(X, \mathcal{K}(\mathcal{H}))$ and $f \in C_0(X, \mathcal{H})$.
- Right action $C_0(X, \mathcal{H}) \curvearrowleft B$ is $(f \cdot b) \in C_0(X, \mathcal{H})$ given by

  $$ (f \cdot b)(x) := f(x)b(x) $$

  for any $b \in C_0(X)$ and $f \in C_0(X, \mathcal{H})$.
- Left A-valued inner product is ${}_A(f, g) \in C_0(X, \mathcal{K}(\mathcal{H}))$ given by

  $$ {}_A(f, g)(x) := \theta_{f(x), g(x)} $$

  for $f, g \in C_0(X, \mathcal{H})$.
- Right B-valued inner product is $\langle f, g\rangle_B \in C_0(X, \mathcal{K}(\mathcal{H}))$ given by

  $$ \langle f, g\rangle_B(x) := \langle f(x), g(x)\rangle_{\mathbb{C}} $$

  for $f, g \in C_0(X, \mathcal{H})$.

That $C_0(X, \mathcal{H})$ is indeed a (A, B)-bimodule follows working pointwise and using that $\mathcal{H}$ is a $(\mathcal{K}(\mathcal{H}), \mathbb{C})$-bimodule. Some analysis is needed to actually check the fullness of the modules but we omit this.

If A and B are Morita equivalent, there is an equivalence between the categories of representations of A and representations of B. To see this, we need to discuss first inner tensor products of Hilbert modules.

3 Inner Tensor product and the Induced representation

Let A and B be C^*-algebras. Suppose E is a Hilbert B-module, that F is a Hilbert A-module and that there is a $*$-homomorphism $\varphi : B \to \mathcal{L}(F)$. This naturally makes F a left B-module with the action induced by φ. We can then form the algebraic tensor product of E and F over B, denoted by $E \odot_B F$. To do so, we start with the algebraic tensor product $E \odot F$ and take the quotient by the subspace generated by

$$\{\xi b \otimes \eta - \xi \otimes \varphi(b)\eta : \xi \in E, \eta \in F, b \in B\}$$

This quotient is $E \odot_B F$. We abuse notation and call the image of $\xi \otimes \eta$ in $E \odot_B F$ also by $\xi \otimes \eta$. Then, $E \odot_B F$ is a right A-module with an action defined by

$$(\xi \otimes \eta)a = \xi \otimes (\eta a)$$

We now define an A-valued inner product on $E \odot_B F$. First we put

$$\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle := \langle \eta, \varphi(\langle \xi, \xi' \rangle)\eta' \rangle$$

for any $\xi, \xi' \in E$ and $\eta, \eta' \in F$. One checks that this is indeed a well defined A-valued inner product on $E \odot_B F$, so to get a Hilbert A-module we complete $E \odot_B F$ with respect to the norm induced by this inner product. We denote the completion $E \otimes_\varphi F$ and we call it the interior tensor product of E and F by φ.

Theorem 3.1. *If A and B are Morita equivalent C^*-algebras, then the category of representations of A is equivalent to the one on B.*

Sketch of Proof. Let E be the A-B imprimitivity bimodule implementing the equivalence and $\pi : B \to \mathcal{L}(\mathcal{H}_\pi)$ be a representation of B. Write $\langle \cdot, \cdot \rangle_B$ for the B-valued right inner product on E. Then, regarding $\mathcal{H}_\pi$ as a right $\mathbb{C}$-module, we can form the Hilbert space $E \otimes_\pi \mathcal{H}_\pi$ whose inner product on elementary tensors looks like

$$\langle \xi_1 \otimes \upsilon_1, \xi_2 \otimes \upsilon_2 \rangle = \langle \upsilon_1, \pi(\langle \xi_1, \xi_2 \rangle_B)\upsilon_2 \rangle$$

for $\xi_k \in E$ and $\upsilon_k \in \mathcal{H}_B$. We define $\mathrm{Ind}\pi : A \to \mathcal{L}(E \otimes_\pi \mathcal{H}_\pi)$ by first letting

$$[\mathrm{Ind}\pi(a)](\xi \otimes \upsilon) = (a\xi) \otimes \upsilon$$

and then extending to all $E \otimes_\pi \mathcal{H}_\pi$. Using that A is Morita equivalent to B, this gives a $*$-homomorphism and therefore $\mathrm{Ind}\pi$ is a representation of A. One checks that π is irreducible if and only if $\mathrm{Ind}\pi$ is irreducible and every irreducible representation of A is of this form. The Functor Ind from the category of representations of A to the one of representations of B is the one implementing the equivalence. "□"

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