Review of Hilbert Modules

Def 1 Let A be a C^*-alg and X a $\mathbb{C}$-vector space which is also a right A-module. An A-valued inner product on X is a map

$$X \times X \longrightarrow A$$
$$(x, y) \longmapsto \langle x, y \rangle_A$$

st
- $\forall x$, $y \mapsto \langle x, y \rangle_A$ is a linear module map $X \to A$.
- $\langle x, y \rangle_A^* = \langle y, x \rangle_A$
- $\langle x, x \rangle_A \geq 0$ in A
- $\langle x, x \rangle_A = 0 \Rightarrow x = 0$

Def 2 A right Hilbert A-mod is a $\mathbb{C}$-vector space X with an A valued inner prod s.t X is complete with the norm

$$\|x\| := \|\langle x, x \rangle_A\|^{1/2}$$

We say X is <u>full</u> if

$$\langle X, X \rangle_A := \operatorname{span}\{\langle x, y \rangle_A : x, y \in X\}$$

is a dense subspace of A.

Def 3 Let X, Y be right Hilbert A-modules. A map $t : X \to Y$ is adjointable if there is a map $t^* : Y \to X$ s.t.

$$\langle tx, y\rangle_A = \langle x, t^*y\rangle_A$$

We denote by $\mathcal{L}_A(X,Y)$ the set of adjointable maps.

Prop

- $\mathcal{L}_A(X,Y) \subset \mathcal{L}(X,Y)$
- $\mathcal{L}_A(X) := \mathcal{L}_A(X,X)$ is a C^*-algebra

Def 4 Let X, Y be right Hilbert A-modules. We define the rank 1 operator $\Theta_{x,y} : Y \longrightarrow X$ by

$$\Theta_{x,y}(y_0) = x\langle y, y_0\rangle_A$$

and

$$K_A(Y,X) = \overline{\text{span}}\{\Theta_{x,y} : x \in X, y \in Y\}$$

Prop

- $\Theta_{x,y}^* = \Theta_{y,x}$
- $K_A(Y,X) \subset \mathcal{L}_A(Y,X)$
- $K_A(X) := K_A(X)$ is a closed two sided ideal in $\mathcal{L}_A(X)$.

Def 5 Let A and B be C^*-algebras. Then, an A-B imprimitivity bimodule is a ~~a~~ vector space X such that

- X is a full left Hilbert A-module, X is a full right Hilbert B-mod
- $\forall x, y \in X$, $a \in A$, $b \in B$

$$\langle ax, y\rangle_B = \langle x, a^* y\rangle_B \quad \text{and} \quad {}_A\langle xb, y\rangle = {}_A\langle x, yb^*\rangle$$

- $\forall x, y, z \in X$

$${}_A\langle x, y\rangle z = x\langle y, z\rangle_B$$

Rmk 2^{nd} condition implies X is an A-B bimodule in the algebraic sense: $a(xb) = (ax)b$.

Ex If X is a full right Hilbert A-module. Then X is a full left Hilbert $K_A(X)$-module where the $K_A(X)$-valued inner product is

$$_{K_A(X)}\langle x, y \rangle = \Theta_{x,y}$$

It is easy to check that X is in fact a $K_A(X)$-A imprimitivity bimodule.

Modules over Operator Algebras

	Concrete		Abstract
C^*-alg	$A \subseteq \mathcal{L}(\mathcal{H})$ closed selfadj subalg.	$\Longleftrightarrow$	A - A Banach $*$-algebra $\|a^*a\| = \|a\|^2$
Operator Spaces.	$E \subseteq \mathcal{L}(\mathcal{H})$ closed subspace.	$\Longleftrightarrow$	E - Banach space $M_n(E) \leftarrow$ matrix norms
Hilbert C^*-modules	$X \subseteq \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ closed subspace.		Abs def (soon)

$A \to C^*$- alg

A rep of A on a Hilbert space $\mathcal{H}$, is a map

$$\pi : A \longrightarrow \mathcal{L}(\mathcal{H})$$

$$a \longmapsto \pi(a)$$

$$\underset{\mathcal{H}}{\overset{\xi}{\cap}} \longmapsto \pi(a)\xi \in \mathcal{H}$$

$$\left.\begin{array}{l}\text{s.t} \quad \pi \text{ is an algebra homomorphism} \\ \pi(a^*) = \pi(a)^*\end{array}\right\} \|\pi\| \leq 1$$

THE concrete example of a Hilbert A-module.

Let $A \subset \mathcal{L}(\mathcal{H}_0)$ be a concrete C^*-alg.

$X \subset \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ be a closed subspace st

- $xa \in X \quad \forall\, x \in X,\ a \in A \quad \left(\begin{array}{l} a : \mathcal{H}_0 \to \mathcal{H}_0 \\ x : \mathcal{H}_0 \to \mathcal{H}_1 \end{array} \right)$
- $x^* y \in A \quad \forall\, x, y \in X \quad \left(\begin{array}{l} x^* : \mathcal{H}_1 \to \mathcal{H}_0 \\ y : \mathcal{H}_0 \to \mathcal{H}_1 \end{array} \right)$

$$\langle x, y \rangle_A = x^* y \in A, \quad \|x\|_{HM} = \|x^* x\|^{1/2}_{A \subset \mathcal{L}(\mathcal{H}_0)}$$

$$\|x\|^2_{HM} = \|x^* x\|_{\mathcal{L}(\mathcal{H}_0)} = \sup_{\|\xi\|=1} |\langle x^* x \xi, \xi \rangle| = \sup_{\|\xi\|} \langle x\xi, x\xi \rangle = \|x\|^2_{\mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)}$$

Prop 1 Let X be as above $(X \subset \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1))$ and suppose that $X \cdot \mathcal{H}_0 := \operatorname{span}\{x\xi : x \in X, \xi \in \mathcal{H}_0\}$ is dense in $\mathcal{H}_1$. Then

$$K_A(X) = \overline{\operatorname{span}}\{xy^* : x, y \in X\} \subset \mathcal{L}(\mathcal{H}_1)$$

$$\Theta_{x,y}(z) = x\langle y, z\rangle = xy^*z = (xy^*)z$$

"Prop 2" Same hyp as Prop 1

$$\mathcal{L}_A(X) \overset{\cancel{\subseteq}}{\underset{\supseteq}{=}} \{b \in \mathcal{L}(\mathcal{H}_1) : bx, b^*x \in X\}$$

Linking algebra: Let X be an A-B imprimitivity bimodule.

$$\begin{pmatrix} A & X \\ \tilde{X} & B \end{pmatrix} = \left\{ \begin{pmatrix} a & x \\ \tilde{y} & b \end{pmatrix} : \begin{array}{l} a \in A, \ x \in X \\ \tilde{y} \in \tilde{X}, \ b \in B \end{array} \right\}$$

$$\tilde{X} = \{ \tilde{x} : x \in X \} \qquad \lambda \tilde{x} = \widetilde{\bar{\lambda} \cdot x}$$

$\tilde{X}$ is a B-A imprimitivity bimodule.

$$\begin{pmatrix} a_1 & x_1 \\ \tilde{y}_1 & b_1 \end{pmatrix}\begin{pmatrix} a_2 & x_2 \\ \tilde{y}_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + {}_A\langle x_1, y_2\rangle & a_1x_2 + x_1b_2 \\ \tilde{y}_1 a + b_1\tilde{y}_2 & \langle y_1, x_2\rangle_B + b_1b_2 \end{pmatrix}$$

In fact The linking algebra is a C^*-alg.

$$\mathbb{L} = K_B(X \oplus B)$$

Thm Let X be any Hilbert A-module.
Then $X \hookrightarrow \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ $\quad A \hookrightarrow \mathcal{L}(\mathcal{H}_0)$
s.t X looks like "THE example above"

Idea proof look at the linking algebra of X as an $\mathcal{K}_A(X) - A$ imprimitivity bimodule. Then take any faithful repn of A

$\pi_A : A \hookrightarrow \mathcal{L}(\mathcal{H}_0)$ $\qquad \mathcal{H}_1 = X \otimes_A \mathcal{H}_0$

$\pi_K : \mathcal{K}_A(X) \hookrightarrow \mathcal{L}(\mathcal{H}_1)$ $\qquad \pi_K(k)(x \otimes \xi) = kx \otimes \xi$

Finally represent on $\underline{\mathcal{H}_1} \oplus \underline{\mathcal{H}_0}$

$$\left[\Pi\begin{pmatrix} K & x \\ \tilde{y} & a \end{pmatrix}\right]\begin{pmatrix} \eta_1 \\ \xi_0 \end{pmatrix} = \begin{pmatrix} \pi_K(k)\eta_1 + C_x(\xi_0) \\ C_y^*(\eta_1) + \pi_A(a)\xi_0 \end{pmatrix}$$

where $x \longmapsto \underline{C_x} : \mathcal{H}_0 \to \mathcal{H}_1$ $\qquad \|C_x\| = \|x\|$

↳ Creation op by x