# Twisted crossed products of Banach algebras

Alonso Delfín Ares de Parga
(Joint work with Carla Farsi and Judith Packer)

University of Colorado-Boulder

January 7, 2026
JMM 2026: Advances in Operator Algebras

# Outline

1. Standing Assumptions
2. Twisted Crossed Products
3. Representations on $L^p$-spaces

# Outline

1. Standing Assumptions
2. Twisted Crossed Products
3. Representations on $L^p$-spaces

During this talk, $A$ will be a fixed Banach algebra with a contractive approximate identity. That is, there is $(e_\lambda)_{\lambda\in\Lambda}$ in $A$ with $\|e_\lambda\| \leq 1$ and

$$\|e_\lambda a - a\|, \|ae_\lambda - a\| \to 0.$$

Further, we assume $A$ is nondegenerately represented on a Banach space $E$: That is, there is an isometric representation $\pi\colon A \to \mathcal{B}(E)$ such that

$$\overline{\pi(A)E} := \overline{\text{span}\{\pi(a)\xi\colon a \in A, \xi \in E\}} = E.$$

These two assumptions give that $M(A)$, the multiplier algebra of $A$, is nondegenerately represented on $E$ as two sided multipliers:

$$M(A) := \{t \in \mathcal{B}(E)\colon t\pi(a), \pi(a)t \in \pi(A)\ \forall a \in A\}.$$

We also fix a locally compact group $G$ together with $\nu_G$ a left Haar measure. For notational convenience, we let

$$dx := d\nu_G(x).$$

# Outline

# Twisted Actions

A twisted action of $G$ on $A$ is a pair $(\alpha, \sigma)$

$$\begin{aligned} \alpha\colon G &\to \mathrm{Aut}(A) \\ x &\mapsto \alpha_x := \alpha(x) \end{aligned} \qquad \begin{aligned} \sigma\colon G \times G &\to \mathrm{Inv}_1(M(A)) \\ (x, y) &\mapsto \sigma_{x,y} := \sigma(x, y) \end{aligned}$$

such that $\alpha$ is strongly continuous, $\sigma$ is strictly continuous, and

1. $\alpha_{1_G} = \mathrm{id}_A$, $\sigma(1_G, x) = \sigma(x, 1_G) = \mathrm{id}_{M(A)}$,
2. $\alpha_x(\alpha_y(a)) = \sigma_{x,y}\alpha_{xy}(a)\sigma_{x,y}^{-1}$,
3. $\alpha_z(\sigma_{x,y})\sigma_{z,xy} = \sigma_{z,x}\sigma_{zx,y}$.

We call $(G, A, \alpha, \sigma)$ a TBADS:

**T**wisted **B**anach **A**lgebra **D**ynamical **S**ystem.

Let $L^1(G, A, \alpha, \sigma)$ be the Banach space $L^1(G \to A, \nu_G)$ equipped with the twisted multiplication

$$(f *_{\alpha,\sigma} g)(x) := \int_G f(y)\alpha_y(g(y^{-1}x))\sigma_{y,y^{-1}x}dy.$$

# $L^1$-algebra of a TBADS

**Proposition**

*$L^1(G, A, \alpha, \sigma)$ is a Banach algebra with a cai and is nondegenerately represented on itself.*

**Proof.** Consider the Banach bundle $\mathcal{A} = (G \ltimes_{\alpha,\sigma} A, \pi)$ where $G \ltimes_{\alpha,\sigma} A$ is $G \times A$ with multiplication

$$(x, a)(y, b) = (xy, a\alpha_x(b)\sigma_{x,y}),$$

and $\pi\colon G \ltimes_{\alpha,\sigma} A \to A$ is the projection onto the first coordinate.

$$L^1(G \mid \mathcal{A}) \overset{1}{\cong} L^1(G, A, \alpha, \sigma),$$

so the cai of $A$ transfers to a cai for $L^1(G, A, \alpha, \sigma)$ via a result by Fell-Doran (1988) for general Banach bundles. ■

In fact, if $(\psi_U)_{U \subseteq G}$ is the usual cai for $L^1(G)$ then

$$f_{\lambda,U}(x) := \psi_U(x)e_\lambda$$

is the desired cai for $L^1(G, A, \alpha, \sigma)$.

# Covariant Representations

A covariant representation of $(G, A, \alpha, \sigma)$ is a pair $(\pi, u)$ together with a Banach space $E$ where

1. $\pi \colon A \to \mathcal{B}(E)$ is a nondegenerate representation,
2. $u \colon G \to \mathrm{Iso}(E)$ is strongly continuous,
3. $u_x u_y = \pi(\sigma_{x,y}) u_{xy}$,
4. $\pi(\alpha_x(a)) = u_x \pi(a) u_x^{-1}$.

Each $(\pi, u)$ induces a representation $\pi \rtimes u \colon L^1(G, A, \alpha, \sigma) \to \mathcal{B}(E)$ via

$$(\pi \rtimes u)(f) := \int_G \pi(f(x)) u_x dx.$$

Fact: The map $(\pi, u) \mapsto \pi \rtimes u$ is a bijection between covariant representations of $(G, A, \alpha, \sigma)$ and nondegenerate representations of $L^1(G, A, \alpha, \sigma)$.

# Twisted Crossed Products

We fix a class $\mathcal{R}$ consisting of covariant representations of $(G, A, \alpha, \sigma)$. satisfying $\|\pi\| \leq C_{\mathcal{R}}$ for all $(\pi, u) \in \mathcal{R}$. On $L^1(G, A, \alpha, \sigma)$ define a seminorm by

$$\|f\|_{\mathcal{R}} := \sup\{\|(\pi \rtimes u)(f)\| \colon (\pi, u) \in \mathcal{R}\}.$$

**Definition**

The **twisted crossed product of** $(G, A, \alpha, \sigma)$ **with respect to** $\mathcal{R}$ is the Hausdorff completion of $L^1(G, A, \alpha, \sigma)/\ker(\| - \|_{\mathcal{R}})$. It will be denoted by $F_{\mathcal{R}}(G, A, \alpha, \sigma)$.

Notice that $F_{\mathcal{R}}(G, A, \alpha, \sigma)$ has a $C_{\mathcal{R}}$-approximate identity. Indeed,

$$\|(\pi \rtimes u) f_{U,\lambda}\| = \left\| \int_G \pi(\psi_U(x) e_\lambda) u_x dx \right\| \leq \|\pi\| \int_G \psi_U(x) dx \leq C_{\mathcal{R}}.$$

Since there is an $\mathcal{R}$-isometric map $\tau \colon L^1(G, A, \alpha, \sigma) \to F_{\mathcal{R}}(G, A, \alpha, \sigma)$, the net $(\tau(f_{U,\lambda}))_{U,\lambda}$ is the desired bai.

# Universal Property

If $C_{\mathcal{R}} \leq 1$, then $F_{\mathcal{R}}(G,A,\alpha,\sigma)$ is the isometric universal Banach algebra generated by $\mathcal{R}$-continuous covariant representations.
The formulas

$$(\lambda_A(a)f)(x) := af(x) \qquad (\rho_A(a)f)(x) := f(x)\alpha_x(a),$$
$$(\lambda_G(y)f)(x) := \alpha_y(f(y^{-1}x))\sigma_{y,y^{-1}x} \qquad (\rho_G(y)f)(x) := f(xy^{-1})\sigma_{xy^{-1},y}\Delta(y^{-1}),$$

extend to well defined maps $(\lambda_A,\rho_A)\colon A \to M(F_{\mathcal{R}}(G,A,\alpha,\sigma))$ and $(\lambda_G,\rho_G)\colon G \to M(F_{\mathcal{R}}(G,A,\alpha,\sigma))$ such that $((\lambda_A,\rho_A),(\lambda_G,\rho_G))$ is a covariant representation of $(G,A,\alpha,\sigma)$ on $F_{\mathcal{R}}(G,A,\alpha,\sigma)$.

**Theorem (D., Farsi, Packer: 2025)**

*Let $B$ be a Banach algebra and let $(k_A,k_G)$ be a covariant representation of $(G,A,\alpha,\sigma)$ on $B$. If $(k_A \rtimes k_G)(L^1(G,A,\alpha,\sigma))$ is dense in $M(B)$ and $\|(k_A \rtimes k_G)(f)\| = \|f\|_{\mathcal{R}}$ for all $f \in L^1(G,A,\alpha,\sigma)$, then*

$$B \overset{1}{\cong} F_R(G,A,\alpha,\sigma).$$

# Equivalent twisted actions

Two twisted actions $(\alpha, \sigma)$ and $(\beta, \omega)$ of $G$ on $A$ are *exterior equivalent* if there is $\theta \colon G \to \mathrm{Inv}_1(M(A))$ strictly continuous such that

1. $\beta_x(a) = \theta_x \alpha_x(a) \theta_x^{-1}$,
2. $\omega_{x,y}\theta_{xy} = \theta_x \alpha_x(\theta_y)\sigma_{x,y}$.

In such case we write $(\alpha, \sigma) \overset{\theta}{\sim} (\beta, \omega)$, and for each $(\pi, u) \in \mathcal{R}$ we define the map $v = v_{\pi,u} \colon G \to \mathrm{Iso}(E)$ by

$$v_x := \widehat{\pi}(\theta_x) u_x.$$

Let $\mathcal{R}_\theta := \{(\pi, v_{\pi,u}) \colon (\pi, u) \in \mathcal{R}\}$.

**Theorem (D., Farsi, Packer: 2025)**

*If* $(\alpha, \sigma) \overset{\theta}{\sim} (\beta, \omega)$, *then* $\mathcal{R}_\theta$ *is a class of covariant representations of* $(G, A, \beta, \omega)$ *and*

$$F_{\mathcal{R}}(G, A, \alpha, \sigma) \overset{1}{\cong} F_{\mathcal{R}_\theta}(G, A, \beta, \omega).$$

# Outline

# $L^p$-operator algebras

From now on, we fix $p \in [1, \infty)$ and assume that $A$ acts nondegenerately on a separable $L^p$-space. That is, there is a measure space $(\Omega_A, \mu_A)$ such that

$$A \subseteq \mathcal{B}(L^p(\Omega_A, \mu_A)) \quad \text{and} \quad \overline{AL^p(\Omega_A, \mu_A)} = L^p(\Omega_A, \mu_A).$$

Let $\mathcal{R}^p = \mathcal{R}^p(G, A, \alpha, \sigma)$ be the class of **all** contractive covariant representations of $(G, A, \alpha, \sigma)$ on $L^p$-spaces. We define the **$L^p$-twisted crossed product** as

$$F^p(G, A, \alpha, \sigma) := F_{\mathcal{R}^p}(G, A, \alpha, \sigma)$$

### Corollary

*If* $(\alpha, \sigma) \overset{\theta}{\sim} (\beta, \omega)$, *then* $F^p(G, A, \alpha, \sigma) \overset{1}{\cong} F^p(G, A, \beta, \omega)$.

**Proof.** $\mathcal{R}^p(G, A, \alpha, \sigma)_\theta = \mathcal{R}^p(G, A, \beta, \omega)$. ■

# A $p$-version of the Packer-Raeburn untwisting trick

Let $p \in (1,\infty)$ and consider

$$\mathrm{St}_p(A) := \mathcal{K}(L^p(G)) \otimes_p A \subseteq \mathcal{B}(L^p(G \times \Omega_A, \nu_G \times \mu_A)).$$

Let $p' \in (1,\infty)$ be the Hölder conjugate of $p$ (i.e, $\frac{1}{p} + \frac{1}{p'} = 1$). The map $L^p(G \to L^{p'}(G \to A)) \to \mathrm{St}_p(A)$ given by $\psi \mapsto K_\psi$, where

$$(K_\psi \xi)(x,w) := \int_G \psi(x,y)\xi(y,w)dy,$$

has dense range and is such that $\|K_\psi\| \leq \|\psi\|$.

**Theorem ( D., Farsi, Packer: 2025)**

*There is a genuine action $\beta$ of $G$ on $\mathrm{St}_p(A)$ such*

$$\mathcal{K}(L^p(G)) \otimes_p F^p(G,A,\alpha,\sigma) \overset{1}{\cong} F^p(G,\mathrm{St}_p(A),\beta).$$

# Reduced twisted crossed product

For any nondegenerate representation $\pi_0\colon A \to \mathcal{B}(L^p(\Omega,\mu))$, let $E := L^p(G \to L^p(\Omega,\mu))$ and define $\pi\colon A \to \mathcal{B}(E)$ by

$$(\pi(a)\xi)(x) := \pi_0(\alpha_x^{-1}(a))(\xi(x)).$$

Define also $u\colon G \to \mathrm{Iso}(E)$ by

$$(u_y\xi)(x) := \widehat{\pi_0}(\alpha_x^{-1}(\sigma_{y,y^{-1}x}))(\xi(y^{-1}x)).$$

Fact: $\mathrm{reg}(\pi_0) := (\pi, u)$ is a covariant representation of $(G, A, \alpha, \sigma)$ on $E$. Set $\mathcal{R}_{\mathrm{r}}^p := \{\mathrm{reg}(\pi_0)\colon \pi_0 \in \mathrm{Rep}^p(A)\}$. We define the **reduced $L^p$-twisted crossed product** by $F_{\mathrm{r}}^p(G, A, \alpha, \sigma) := F_{\mathcal{R}_{\mathrm{r}}^p}(G, A, \alpha, \sigma)$.

## Conjecture

If $G$ is amenable then $F_{\mathrm{r}}^p(G, A, \alpha, \sigma) \overset{1}{\cong} F^p(G, A, \alpha, \sigma)$

## Conjecture (Rigidity for $p \neq 2$)

$$F_{\mathrm{r}}^p(G, A, \alpha, \sigma) \overset{1}{\cong} F_{\mathrm{r}}^p(G, A, \beta, \omega) \iff (\alpha, \sigma) \sim (\beta, \omega).$$

# Thank you!

# Questions?

References, details:
arXiv:2509.24106 [math.FA]