L^p-Spectral Triples: Group and UHF Algebras

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Outline

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Classic Spectral Triple

Definition

A *spectral triple* $(A, \mathcal{H}, D)$ consists of a unital C*-algebra A faithfully represented on a Hilbert space $\mathcal{H}$ via $\varphi\colon A \to \mathcal{B}(\mathcal{H})$, and an unbounded selfadjoint operator $D\colon \mathrm{dom}(D) \subseteq \mathcal{H} \to \mathcal{H}$ such that

1. $(I + D^2)^{-1} \in \mathcal{K}(\mathcal{H})$,
2. $\{a \in A\colon \|[D, \varphi(a)]\|_{\mathcal{B}(\mathcal{H})} < \infty\}$ is a norm dense $*$-subalgebra of A.

Since D is required to be selfadjoint, condition 1 is equivalent to asking for D to have compact resolvent, that is

1' $(D - \lambda I)^{-1} \in \mathcal{K}(\mathcal{H})$ for all $\lambda \in \mathbb{C} \setminus \sigma(D)$.

Let B be a complex normed space. In 1998 M. Rieffel gave a general setting to get compact metrics on certain subspaces of $B' = \mathcal{B}(B, \mathbb{C})$. The ingredients are:

- A subspace $\mathcal{L}$ of B, not necessarily closed.
- A seminorm $L\colon \mathcal{L} \to \mathbb{R}_{\geq 0}$ with $\ker(L) \neq \{0\}$.
- A continuous linear functional, $\varphi\colon \ker(L) \to \mathbb{C}$ with $\|\varphi\| = 1$.
- We ask that $\mathcal{L}$ separate the points of

$$S := \{\omega \in B' : \omega = \varphi \text{ on } \ker(L), \text{ and } \|\omega\| = 1\}.$$

We define an extended pseudo metric $\mathrm{d}_S\colon S \times S \to \mathbb{R}_{\geq 0} \cup \{\infty\}$, by

$$\mathrm{d}_S(\omega, \psi) := \sup\{|\omega(b) - \psi(b)|\colon b \in \mathcal{L},\ L(b) \leq 1\}.$$

Theorem (Rieffel 98)

With the ingredients described above, we put $\mathcal{L}_1 := \{b \in \mathcal{L}\colon L(b) \leq 1\}$. If the image of $\mathcal{L}_1$ in $\mathcal{L}/\ker(L)$ is totally bounded for $\|-\|_{B/\ker(L)}$, then the d_S-topology on S agrees with the weak-$*$ topology.

Length Functions

A length function $\mathbb{L}\colon G \to \mathbb{R}_{\geq 0}$ on a group G is a function satisfying

$$\mathbb{L}(g) = 0 \Leftrightarrow g = 1_G, \quad \mathbb{L}(g^{-1}) = \mathbb{L}(g), \quad L(gh) \leq \mathbb{L}(g) + \mathbb{L}(h).$$

We say $\mathbb{L}$ is *proper* if in addition $B_{\mathbb{L}}(R) := \mathbb{L}^{-1}([0,R]) \subseteq G$ is a finite set for all $R \in \mathbb{R}_{\geq 0}$.

Definition

Let $\mathbb{L}$ be a proper length function on a group G. We say that $\mathbb{L}$ has

- *strong polynomial growth* if there exist constants $C_{\mathbb{L}}, d < \infty$ such that $C_{\mathbb{L}}^{-1} R^d \leq \mathrm{card}(B_{\mathbb{L}}(R)) \leq C_{\mathbb{L}} R^d$ for all $R \geq 1$.
- *bounded doubling* if there exists a constant $C_{\mathbb{L}} < \infty$ such that $\mathrm{card}(B_{\mathbb{L}}(2R)) \leq C_{\mathbb{L}} \mathrm{card}(B_{\mathbb{L}}(R))$ for all $R \geq 1$.
- *polynomial growth* if there exist constants $C_{\mathbb{L}}, d < \infty$ such that $\mathrm{card}(B_{\mathbb{L}}(R)) \leq C_{\mathbb{L}} R^d$ for all $R \geq 1$.

In general SPG $\Rightarrow$ BD $\Rightarrow$ PG; and these are equivalent for finitely generated groups.

Quantum Compact Metrics from Spectral Triples

Let $(A, \mathcal{H}, D)$ be a spectral triple. A. Connes defined an extended pseudometric mk_D on $S(A)$, the state space of A, by

$$\mathrm{mk}_D(\omega, \psi) := \sup\{|\omega(a) - \psi(a)| \colon a \in A,\ \|[D, \varphi(a)]\| \leq 1\}.$$

Definition

We say $(A, \mathcal{H}, D)$ is a *metric spectral triple* when mk_D is a metric whose topology on $S(A)$ agrees with the weak-$*$ topology. This makes $S(A)$ a *quantum compact metric space*.

Example (Christ-Rieffel 2017)

Let G be a countable discrete group, let $\mathbb{L}$ be a proper length function on G, and let $D_{\mathbb{L}} \colon C_c(G) \to \ell^2(G)$ be given by

$$(D_{\mathbb{L}}\xi)(g) := \mathbb{L}(g)\xi(g).$$

Then $(C^*_{\mathrm{r}}(G), \ell^2(G), D_{\mathbb{L}})$ is a metric spectral triple when $\mathbb{L}$ has bounded doubling.

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Definition

Let $p \in [1, \infty)$ and let A be a Banach algebra. We say A is an *L^p-operator algebra* if there is a measure space (X, μ) and an isometric homomorphism $\varphi\colon A \to \mathcal{B}(L^p(\mu))$.

Minor modifications to the classic definition give the following:

Definition

Let $p \in [1, \infty)$ and let (X, μ) be a measure space. An *L^p-spectral triple* $(A, L^p(\mu), D)$ consists of a unital L^p-operator algebra A, a unital isometric homomorphism $\varphi\colon A \to \mathcal{B}(L^p(\mu))$, and an unbounded operator $D\colon \mathrm{dom}(D) \subseteq L^p(\mu) \to L^p(\mu)$, satisfying

1. $(I + D^2)^{-1}, (D - \lambda I)^{-1} \in \mathcal{K}(L^p(\mu))$ for all $\lambda \in \mathbb{C} \setminus \sigma(D)$,
2. $\{a \in A\colon \|[D, \varphi(a)]\| < \infty\}$ is a norm dense subalgebra of A.

$S(A) := \{\omega \in A'\colon \|\omega\| = \omega(1_A) = 1\}$ is the state space of a unital Banach algebra A. Thus, we will say $(A, L^p(\mu), D)$ is a *metric L^p-spectral triple* when the mk_D-metric topology on $S(A)$ agrees with the weak-$*$ topology.

Outline

Let $p \in [1,\infty)$ and let G be a discrete group. Then $\ell^1(G)$ acts on $\ell^p(G)$ as left convolution operators. That is, $\lambda_p \colon \ell^1(G) \to \mathcal{B}(\ell^p(G))$ is given by

$$(\lambda_p(a)b)(g) = \sum_{h\in G} a(h)b(h^{-1}g)$$

Definition

For $p \in [1,\infty)$, the *reduced L^p-operator algebra of G* is

$$F^p_{\mathrm{r}}(G) = \overline{\lambda_p(\ell^1(G))} \subseteq \mathcal{B}(\ell^p(G))$$

Fact: $F^2_{\mathrm{r}}(G) = C^*_{\mathrm{r}}(G)$.

Theorem (D., Farsi, Packer (2025))

Let $p \in [1,\infty)$, let G be a countable discrete group, and let $\mathbb{L}$ be a proper length function on G. Then, $(F^p_{\mathrm{r}}(G), \ell^p(G), D_{\mathbb{L}})$ is an L^p-spectral triple

Quantum Compact Metrics on $S(F_{\mathrm{r}}^p(\mathbb{Z}))$

For $\beta \in (0,1]$ define $\mathbb{L}_\beta \colon \mathbb{Z} \to \mathbb{R}_{\geq 0}$ by $\mathbb{L}_\beta(n) := |n|^\beta$.

Theorem (Rieffel (2002))

Let ω be a translation-bounded function on $\mathbb{Z}$ such that $\omega(0) = 0$. If $\mathbb{L}_\beta/\omega$ is a bounded function (ignoring $n = 0$) for some β with $1/2 < \beta \leq 1$, then mk_{D_ω} is a quantum compact metric on the state space of $C_{\mathrm{r}}^*(\mathbb{Z}) = C(\mathbb{T})$.

Theorem (D., Farsi, Packer (2025))

Let $\frac{1}{p} + \frac{1}{q} = 1$, and let ω be a translation-bounded function on $\mathbb{Z}$ such that $\omega(0) = 0$. If $\mathbb{L}_\beta/\omega$ is a bounded function (ignoring $n = 0$) for some β with $1/q < \beta \leq 1$, then mk_{D_ω} is a quantum compact metric on the state space of $F_{\mathrm{r}}^p(\mathbb{Z})$.

Outline

Christensen-Ivan AF Metric Spectral Triples

Let $A = \lim_{\rightarrow} A_n \subseteq \mathcal{B}(\mathcal{H})$ an infinite dimensional unital C* AF-algebra with $A_n \subseteq \mathcal{B}(\mathcal{H}_n)$ and $\mathcal{H}_n \subseteq \mathcal{H}$. Let $P_n \colon \mathcal{H} \to \mathcal{H}_n$ and $\alpha = (\alpha_n)_{n=0}^{\infty}$ a sequence in $\mathbb{R}_{\geq 0}$ with $\alpha_0 = 0$. Put

$$D_\alpha = \sum_{n=1}^{\infty} \alpha_n (P_n - P_{n-1}).$$

Theorem (Christensen, Ivan (2006))

There exists $\alpha = (\alpha_n)_{n=0}^{\infty}$, a sequence in $\mathbb{R}_{\geq 0}$, with $\alpha_0 = 0$ such that $(A, \mathcal{H}, D_\alpha)$ is a metric spectral triple.

The proof relies on the GNS construction and the existence of a separating and cyclic vector on $\mathcal{H}$.

L^p UHF-algebras

General (spatial) L^p AF-Algebras were recently defined and classified by N. C. Phillips and M. G. Viola, but we need more structure on the acting L^p-spaces. A concrete and more manageable case is the UHF one:

Definition

Let $d\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 2}$ be a sequence. An L^p UHF-algebra of infinite tensor product type is the direct limit of $((A_n)_{n=0}^{\infty}, (\varphi_{m,n})_{0\leq n\leq m})$ where each A_n is an L^p-operator algebra that acts on a probability space and such that each A_n algebraically isomorphic to M_n where

$$M_n := \mathbb{M}_{d(0)\cdots d(n)}(\mathbb{C}) \cong \bigotimes_{j=0}^{n} \mathbb{M}_{d(j)}(\mathbb{C}).$$

Fix $p \in [1, \infty)$ and $d \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 2}$. Let $(X_n, \mu_n)_{n=0}^{\infty}$ be probability spaces and let $(\rho_j)_{j=0}^{\infty}$ be unital algebra homomorphism

$$\rho_j \colon \mathbb{M}_{d(j)}(\mathbb{C}) \to \mathcal{B}(L^p(X_j, \mu_j)).$$

In addition, we assume that for each $j \in \mathbb{Z}_{\geq 1}$

1. X_j consists of finitely many points and is a compact metric space with metric $\mathrm{dist}_j \colon X_j \times X_j \to \mathbb{R}_{\geq 0}$ bounded by 1,

and in particular for $j = 0$

2. $L^p(X_0, \mu_0) = \ell_1^p = \mathbb{C}$ and $\rho_0(z)\zeta = z\zeta$ for any $z, \zeta \in \mathbb{C}$.

Now, for each $m \in \mathbb{Z}_{\geq 0}$, we define

$$(X_{\leq m}, \mu_{\leq m}) := \prod_{j=0}^{m} (X_j, \mu_j), \quad (X_{\geq m}, \mu_{\geq m}) := \prod_{j=m}^{\infty} (X_j, \mu_j).$$

On M_n we define $\|a_0 \otimes \ldots \otimes a_n\| := \|\rho_0(a_0) \otimes_p \ldots \otimes_p \rho_n(a_n)\|$, which makes M_n an L^p-operator algebra acting on $L^p(\mu_{\leq n})$. Filling in tensor identities, we get isometric embeddings $\rho_{n,\infty} \colon M_n \to \mathcal{B}(L^p(\mu_{\geq 0}))$.

$$A(\rho, d) := \overline{\bigcup_{j=0}^{\infty} \rho_{n,\infty}(M_n)} \subseteq \mathcal{B}(L^p(\mu_{\geq 0})).$$

Metric L^p UHF triple

With $(X_n, \mu_n)_{n=0}^{\infty}$ as before, we define

- $\iota_n : L^p(\mu_{\leq n}) \to L^p(\mu_{\geq 0})$ by $(\iota_n \xi)(x_0, \ldots, x_n, \mathbf{x}) := \xi(x_0, \ldots, x_n)$,
- $\pi_n : L^p(\mu_{\geq 0}) \to L^p(\mu_{\leq n})$ by

$$(\pi_n \eta)(x_0, \ldots, x_n) := \int_{X_{\geq n+1}} \eta(x_0, \ldots, x_n, \mathbf{x}) d\mu_{\geq n+1}(\mathbf{x}),$$

- $P_n : L^p(\mu_{\geq 0}) \to L^p(\mu_{\geq 0})$ by $P_n := \iota_n \circ \pi_n$.

Facts: $P_n P_m = P_m P_n = P_n$ for $m \in \mathbb{Z}_{\geq n}$, $\|P_n\| = 1$, $P_n \xrightarrow{\text{SOT}} \mathrm{Id}_{\geq 0}$, and for any $a \in \rho_{n,\infty}(M_n)$, if $m \in \mathbb{Z}_{\geq n}$, $aP_m = P_m a$.

Theorem (D., Farsi, Packer (2025))

There exists $\alpha = (\alpha_n)_{n=0}^{\infty}$, a sequence in $\mathbb{R}_{\geq 0}$, with $\alpha_0 = 0$ such that $(A(d, \rho), L^p(\mu_{\geq 0}), D_\alpha)$ is a metric L^p-spectral triple, where

$$D_\alpha = \sum_{n=1}^{\infty} \alpha_n (P_n - P_{n-1}).$$

Thank you!

Questions?