An introduction to L^p-Operator Algebras and its Multiplier Algebras.

Alonso Delfín Ares de Parga
(joint work with A. Blinov and E. Weld)

CU Boulder

March 20, 2024
Toeplitz Operators Seminar

Outline

1. L^p-Operator Algebras
2. Multiplier Algebras
3. Augmentation Ideal

Outline

1 L^p-Operator Algebras

2 Multiplier Algebras

3 Augmentation Ideal

Representations

For a Banach space E, we will denote the Banach algebra of bounded linear maps $E \to E$ by $\mathcal{L}(E)$.

A **representation of a Banach algebra A on a Banach space E** is a continuous homomorphism $\varphi : A \to \mathcal{L}(E)$. We say φ is **nondegenerate** if

$$\varphi(A)E := \operatorname{span}\{\varphi(a)\xi : a \in A, \xi \in E\}$$

is dense in E.

Let A be a Banach algebra, and let $p \in [1, \infty]$. We say that A is an **L^p-operator algebra** if there is a measure space $(\Omega, \mathfrak{M}, \mu)$ and an isometric representation

$$A \hookrightarrow \mathcal{L}(L^p(\mu))$$

Examples

Example

For any $(\Omega, \mathfrak{M}, \mu)$ and $p \in [1, \infty]$, we trivially have that $\mathcal{L}\big(L^p(\mu)\big)$ is an L^p-operator algebra.

Example

For any $(\Omega, \mathfrak{M}, \mu)$ and $p \in [1, \infty]$, the algebra $\mathcal{K}\big(L^p(\mu)\big)$ of compact operators on $L^p(\mu)$ is an L^p-operator algebra.

Example

Any C^*-algebra is an L^2 operator algebra. However, a general L^2 operator algebra is not necessarily a self-adjoint algebra.

More Examples

Example

Equip M_n, the set of $n \times n$ complex matrices, with the operator norm acting on $(\mathbb{C}^n, \| - \|_p)$ for $p \in [1, \infty]$. Then M_n is equal to $\mathcal{L}(\ell^p(\{1, \dots, n\}))$. To emphasize the dependence on the p-norm, this space is denoted by M_n^p.

Example

For $j, k \in \{1, \dots, n\}$, let $e_{j,k} \in M_n^p$ be the matrix whose only non-zero entry is the entry (j, k) which is equal to 1. Then, the set of upper triangular matrices

$$T_n^p = \operatorname{span}\{e_{j,k} : 1 \leq j \leq k \leq n\}$$

is a subalgebra of M_n^p, which is also an L^p-operator algebra.

Degenerate L^p-Operator Algebras

Let A be the algebra generated by $e_{1,2}$ in T_2^p. That is,

$$A = \left\{ \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} : \lambda \in \mathbb{C} \right\} \subset T_2^p$$

If $\varphi\colon A \to \mathcal{L}(E)$ is a representation on **any** non-zero Banach space E. Then $\varphi(A)E \subset \ker(\varphi(e_{1,2}))$.

(1) $\varphi(e_{1,2}) \neq 0$. There is $\xi \in E$ for which $\varphi(e_{1,2})\xi \neq 0$, whence $\ker(\varphi(e_{1,2}))$ is a proper closed subset of E. Therefore, $\varphi(A)E$ cannot be dense in E.

(2) $\varphi(e_{1,2}) = 0$. Here, $\varphi(A)E = \{0\}$, so again it cannot be dense in E.

Conclusion: A does not admit non-degenerate representations.

A final example

Example

Let $p \in [1, \infty]$ and let Ω be a locally compact topological space. Then $C_0(\Omega)$, with the usual supremum norm, is an L^p-operator algebra.

To see this, let ν be counting measure on Ω and define $\varphi : C_0(\Omega) \to \mathcal{L}\big(\ell^p(\nu)\big)$ by

$$(\varphi(f)\xi)(\omega) := f(\omega)\xi(\omega)$$

One checks that φ is an isometric bijection from $C_0(\Omega)$ to a norm closed subalgebra of $\mathcal{L}(\ell^p(\nu))$.

Differences between C*-algebras and L^p-operator algebras

1. L^p-operator algebras lack involution,
2. Some L^p-operator algebras can't be nondegenerately represented,
3. Some L^p-operator algebras don't have contractive approximate units,
4. L^p-operator norms are generally hard to compute,
5. L^p-operator norms are not unique.
6. An abstract characterization of L^p-operator algebras, among all Banach algebras, is not known,
   - The class of of L^p-operator algebras is not closed under quotients by two-sided closed ideals,
   - In general, it's hard to show that a given Banach algebra is not an L^p-operator algebra.

Tensor Products

Let $p \in [1, \infty)$, let $(\Omega_0, \mathfrak{M}, \mu)$ and $(\Omega_1, \mathfrak{N}, \nu)$ be measure spaces. There is a tensor product, $\otimes_p$, such that

$$L^p(\mu) \otimes_p L^p(\nu) \cong L^p(\mu \times \nu) \text{ via } (\xi \otimes \eta)(\omega_0, \omega_1) = \xi(\omega_0)\eta(\omega_1)$$

Moreover,

- If $a \in \mathcal{L}(L^p(\mu_1), L^p(\mu_2))$ and $b \in \mathcal{L}(L^p(\nu_1), L^p(\nu_2))$, then there is $a \otimes b \in \mathcal{L}(L^p(\mu_1 \times \nu_1), L^p(\mu_2 \times \nu_2))$, which has the expected properties: bilinearity, $\|a \otimes b\| = \|a\| \|b\|$ and $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$.
- If $A \subseteq \mathcal{L}(L^p(\mu))$ and $B \subseteq \mathcal{L}(L^p(\nu))$ are norm closed subalgebras, we define $A \otimes_p B$ as the closed linear span, in $\mathcal{L}\big(L^p(\mu \times \nu)\big)$, of $a \otimes b$ for $a \in A$ and $b \in B$.

G-L^p-Operator Algebras

Let A be an L^p-operator algebra, let G be a locally compact group with Haar measure ν, and let $\alpha : G \to \mathrm{Aut}(A)$ be an isometric action (i.e. for each $g \in G$ the map $\alpha(g) =: \alpha_g : A \to A$ is isometric and for each $a \in A$, $g \mapsto \alpha_g(a)$ is a continuous map $G \to A$). The triple (G, A, α) is called an **isometric G-L^p-operator algebra**.

We denote by $L^1(G, A, \alpha)$ to the space $L^1(G \to A)$ equipped with twisted convolution:

$$(x * y)(g) = \int_G x(h)\alpha_h(y(h^{-1}g))d\nu(h).$$

Thus, $L^1(G, A, \alpha)$ is a normed algebra.

Crossed Products

Let (G, A, α) be an isometric G-L^p-operator algebra and $(\Omega, \mathfrak{M}, \mu)$ a measure space. A **covariant representation** of (G, A, α) on $L^p(\mu)$ consists of a pair (φ, u) where

- φ is a representation of A on $L^p(\mu)$,
- $u : G \to \mathrm{Inv}(L^p(\mu))$ is a group homomorphism with $g \mapsto u(g)\xi := u_g\xi$ a continuous map for all $\xi \in L^p(\mu)$,
- $\varphi(\alpha_g(a)) = u_g\varphi(a)u_g^{-1}$ for all $g \in G$, $a \in A$.

A covariant representations (φ, u) induces a representation of $L^1(G, A, \alpha)$ on $L^p(\mu)$ via

$$(\varphi \rtimes u)x := \int_G \varphi(x(g))u_g d\nu(g).$$

A contractive representation φ of A on $L^p(\mu)$ induces a covariant representation of (A, G, α) on $L^p(\nu \times \mu)$ denoted $(\widetilde{\varphi}, \lambda)$.

Crossed Products

Given an isometric G-L^p-operator algebra we define a norm on $L^1(G,A,\alpha)$ by

$$\|x\|_{\rtimes} = \sup_{\substack{(\varphi,u) \text{ is a } \sigma\text{-finite, nondegenerate,} \\ \text{contractive, covariant representation of } (G,A,\alpha)}} \|(\varphi \rtimes u)x\|$$

The **full** crossed product is $F^p(G,A,\alpha) := \overline{L^1(G,A,\alpha)}^{\|-\|_{\rtimes}}$. Similarly, if

$$\|x\|_{\mathrm{r}} = \sup_{\substack{\varphi \text{ is a } \sigma\text{-finite, nondegenerate,} \\ \text{contractive representation of } A}} \|(\widetilde{\varphi} \rtimes \lambda))x\|$$

Then, $F_{\mathrm{r}}(G,A,\alpha) := \overline{L^1(G,A,\alpha)}^{\|-\|_{\mathrm{r}}}$ is the **reduced** crossed product. These two coincide when G is amenable.

L^p-group algebras

Let G be a discrete group and $p \in [1, \infty)$. Then $\ell^1(G)$ acts on $\ell^p(G)$ as a left convolution operator. That is, $\lambda \colon \ell^1(G) \to \mathcal{L}(\ell^p(G))$ is given by

$$(\lambda(a)b)(g) = \sum_{h \in G} a(h)b(h^{-1}g)$$

Definition

For $p \in [1, \infty)$, the *reduced L^p-operator algebra of G* is

$$F_r^p(G) = \overline{\lambda(\ell^1(G))} \subseteq \mathcal{L}(\ell^p(G))$$

When $p = 1$, we have $F_r^1(G) = \ell^1(G)$. Thus, $\ell^1(G)$ is a unital L^1-operator algebra (unit is δ_{1_G}) acting on itself via left multiplication.

Outline

1. L^p-Operator Algebras
2. Multiplier Algebras
3. Augmentation Ideal

Double Centralizers of a Banach Algebra A

We say that $L \in \mathcal{L}(A)$ is a left centralizer for A if for any $a, b \in A$,

$$L(ab) = L(a)b.$$

The space of left centralizers is denoted by $LC(A)$. Similarly, $R \in \mathcal{L}(A)$ is a right centralizer if for any $a, b \in A$,

$$R(ab) = aR(b).$$

We denote by $RC(A)$ to the space of right centralizers.

Definition

The *multiplier algebra of A* is

$$M(A) = \{(L, R) \in LC(A) \times RC(A) \colon aL(b) = R(a)b\}$$

equipped with the norm $\|(L, R)\| = \max\{\|L\|, \|R\|\}$.

$M(A)$ is a unital Banach subalgebra of $\mathcal{L}(A) \times \mathcal{L}(A)^{\mathrm{op}}$ where the unit is $(\mathrm{id}_A, \mathrm{id}_A)$.

Two Sided Multipliers

It's well known that if A is a C*-algebra which is nondegenerately represented on a Hilbert space $\mathcal{H}$ via $\varphi\colon A \to \mathcal{L}(\mathcal{H})$, then

$$M(A) \cong \{b \in \mathcal{L}(\mathcal{H})\colon b\varphi(A) \subseteq \varphi(A), \varphi(A)b \subseteq \varphi(A)\}.$$

In fact, the RHS is an alternative definition for $M(A)$. This alternative definition is therefore independent of the Hilbert space and the representation φ chosen.

Theorem (D, 2023)

Let A be a Banach algebra with a cai and that's nondegenerately represented on a Banach space E via $\varphi\colon A \to \mathcal{L}(E)$. Then $M(A)$ is isometrically isomorphic to

$$\{b \in \mathcal{L}(E)\colon b\varphi(A) \subseteq \varphi(A), \varphi(A)b \subseteq \varphi(A)\}$$

Question: What happens if we drop the assumptions of nondegeneracy and the existence of a cai?

Algebras with non-unital identities

There is a natural inclusion $\iota\colon A \to M(A)$ be given by $\iota(a) = (L_a, R_a)$, where $L_a(b) = ab$ and $R_a(b) = ba$. If A has a cai, then ι is isometric and

$$\|L_a\| = \|R_a\| = \|a\|.$$

Proposition

Let $1_A \in A$ be an identity for A, potentially not a unit (i.e. $\|1_A\| \neq 1$). Then $M(A)$ is isometrically isomorphic to $(A, \| - \|')$ where

$$\|a\|' = \|\iota(a)\| = \max\{\|L_a\|, \|R_a\|\}$$

In particular $\|a\|' = \|a\|$ when 1_A is a unit.

Corollary

If A is commutative with non unital identity 1_A, then $M(A)$ is isometrically isomorphic to $(A, \| - \|')$ where

$$\|a\|' = \|L_a\| = \sup_{\|b\|=1} \|ab\| \neq \|a\|.$$

Representable L^p-Multiplier algebras

As a corollary of the work presented for general Banach Algebras we get

Corollary

If A is a nondegenerately representable L^p-operator algebra with a cai, then $M(A)$ is a nondegenerately representable L^p-operator algebra.

Now consider the algebra of 2×2 strictly upper triangular matrices:

$$ST_2^p = \left\{ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} : z \in \mathbb{C} \right\} \subset M_2^p = \mathcal{L}(\ell_2^p)$$

ST_2^p does not have a cai and that it can't be nondegenerately represented on **any** Banach space. However, since $ST_2^p \cong \mathbb{C}$ it's clear that $LC(ST_2^p) \cong \mathbb{C}$, $RC(ST_2^p) \cong \mathbb{C}$, and therefore

$$M(ST_2^p) = LC(ST_2^p) \times RC(ST_2^p) \cong \mathbb{C}^2 \cong C(\{1,2\}) \subset \mathcal{L}(\ell_2^p).$$

That is, $M(ST_2^p)$ is a nondegenerately representable L^p-operator algebra.

Outline

1 L^p-Operator Algebras

2 Multiplier Algebras

3 Augmentation Ideal

The augmentation ideal $\ell_0^1(G)$

Consider the contractive algebra homomorphism $\ell^1(G) \to \mathbb{C}$ given by

$$a \mapsto \sum_{g \in G} a(g)$$

We define the augmentation ideal of $\ell^1(G)$ as the kernel of this map:

Definition

For a discrete group G, *the augmentation ideal of* $\ell^1(G)$ is

$$\ell_0^1(G) = \left\{ a \in \ell^1(G) \colon \sum_{g \in G} a(g) = 0 \right\}$$

$\ell_0^1(G)$ is an L^1-operator algebra degenerately represented on $\ell^1(G)$.

Open Question

Is $\ell_0^1(G)$ nondegenerately representable on some $L^1(\mu)$?

Partial Answer: It's impossible to nondegenertely represent $\ell_0^1(G)$ on any Banach space when G is finite.

Properties of $\ell_0^1(G)$

For each $g \in G$ we define $\Delta_g \in \ell_0^1(G)$ as

$$\Delta_g := \delta_g - \delta_{1_G}.$$

Proposition

For any discrete group G, the space $\operatorname{span}\{\Delta_g : g \in G\}$ is dense in $\ell_0^1(G)$. If in addition G is finite with $n := \operatorname{card}(G) \geq 2$, then

- $\ell_0^1(G) = \operatorname{span}\{\Delta_g : g \in G\}$
- $\ell_0^1(G)$ *has an identity element $\mathbf{1}_0$. In fact*

$$\mathbf{1}_0 = -\frac{1}{n}\sum_{g\in G}\Delta_g \quad \textit{and} \quad \|\mathbf{1}_0\|_1 = 2 - \frac{2}{n}$$

Since $\mathbf{1}_0$ not a unit when $n > 2$, it follows that $\ell_0^1(G)$ does not have a cai when $n > 2$.

What is $M(\ell^1_0(G))$?

For G discrete with $\mathrm{card}(G) > 2$ we have a non unital identity in $\ell^1_0(G)$ and therefore $M(\ell^1_0(G))$ is $(\ell^1_0(G), \|a\|' := \max\{\|L_a\|, \|R_a\|\})$ where

$$\|L_a\| = \sup_{b\in\ell^1_0(G), \|b\|_1=1} \|ab\|_1,$$
$$\|R_a\| = \sup_{b\in\ell^1_0(G), \|b\|_1=1} \|ba\|_1.$$

If G is abelian, then $M(\ell^1_0(G))$ is $(\ell^1_0(G), \|a\|' = \|L_a\|)$.

Theorem (Blinov-D-Weld (2024))

For G finite ($n = \mathrm{card}(G)$) and abelian $M(\ell^1_0(G))$ is isometrically isomorphic to $\mathbb{C}^{n-1}$ with a norm different from the max norm.

Open Question

Is $M(\ell^1_0(G))$ an L^1-operator algebra?

Partial Answer: It is not when $G = \mathbb{Z}/3\mathbb{Z}$.

Known norms that make $\mathbb{C}^2$ an L^1-operator algebra.

Which norms make $\mathbb{C}^2$ an L^1-operator algebra?

- As before, $\mathbb{C}^2$ with the max norm is also an L^1-operator algebra acting on ℓ_2^1 via multiplication operators.
- Let $\mathcal{F}\colon \ell^1(\mathbb{Z}/2\mathbb{Z}) \to C(\mathbb{Z}/2\mathbb{Z})$ the Fourier transform. Then we have algebra isomorphisms

$$\mathbb{C}^2 \cong C(\mathbb{Z}/2\mathbb{Z}) \cong \mathcal{F}^{-1}(C(\mathbb{Z}/2\mathbb{Z})) = \ell^1(\mathbb{Z}/2\mathbb{Z})$$

  which make $\mathbb{C}^2$ an L^1-operator algebra with norm coming from the identification with $\ell^1(\mathbb{Z}/2\mathbb{Z})$.

Open Question

Are these the only two norms that make $\mathbb{C}^2$ an L^1-operator algebra?

We do know that the identification of $M(\ell_0^1(\mathbb{Z}/3\mathbb{Z}))$ with $\mathbb{C}^2$ carries none of these norms.

$G = \mathbb{Z}/3\mathbb{Z}$

Theorem (Bernau-Lacey (1977))

Let $p \in [1, \infty)$ and let $e \in \mathcal{L}(L^p(\mu))$ be a bicontractive idempotent (i.e. $e^2 = e$, $\|e\| \leq 1$, and $\|1 - e\| \leq 1$). Then $\|1 - 2e\| = 1$.

For $\mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$, we get $\ell_0^1(\mathbb{Z}/3\mathbb{Z}) = \operatorname{span}\{\Delta_1, \Delta_2\}$, and $\mathbf{1}_0 = \frac{-1}{3}(\Delta_1 + \Delta_2)$.

Theorem (Blinov-D-Weld (2024))

$M(\ell_0^1(\mathbb{Z}/3\mathbb{Z}))$ cannot be isometrically represented on any L^p-space.

Proof. The element

$$\frac{e^{2\pi i/3}}{3}\Delta_1 + \frac{e^{-2\pi i/3}}{3}\Delta_2$$

is a bicontractive idempotent with $\|L_{\mathbf{1}_0 - 2e}\| = \frac{2}{\sqrt{3}} > 1$. ■

Unfortunately, this argument doesn't seem to work for higher order groups.

Thank you!

Questions?