Multiplier Algebras of L^p-Operator Algebras

Alonso Delfín
(Joint work with Andrey Blinov and Ellen Weld)

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Outline

1. L^p-operator Algebras
2. Multiplier Algebras
3. Representability of $M(A)$ on L^q-spaces

Outline

L^p-operator Algebras.

For $p \in [1, \infty)$, a Banach algebra A is an *L^p-operator algebra* if there is a measure space (Ω, μ) and an isometric representation

$$\varphi\colon A \to \mathcal{L}(L^p(\mu)).$$

Example

- C*-algebras are L^2-operator algebras.
- $\mathcal{L}(L^p(\mu))$, $\mathcal{K}(L^p(\mu))$.
- $\mathcal{L}(\ell^p_d) = M^p_d(\mathbb{C})$.
- Fix any $p \in [1, \infty)$ and let Ω be a locally compact space equipped with counting measure ν. Then $C_0(\Omega)$ is an L^p-operator algebra via $\varphi\colon C_0(\Omega) \to \mathcal{L}(L^p(\nu))$ given by

$$(\varphi(a)\xi)(\omega) = a(\omega)\xi(\omega).$$

- Fix any $p \in [1, \infty)$. Then T^p_d, the set of strictly upper triangular $d \times d$ matrices acting on ℓ^p_d, is an L^p-operator algebra.

Differences between C*-algebras and L^p-operator algebras

1. L^p-operator algebras lack involution,
2. Some L^p-operator algebras can't be nondegenerately represented,
3. Some L^p-operator algebras don't have cai's,
4. L^p-operator norms are generally hard to compute,
5. L^p-operator norms are not unique,
6. An abstract characterization of L^p-operator algebras, among all Banach algebras, is not known,
    - In general, it's hard to show whether a given Banach algebra is an L^p-operator algebra,
    - The class of of L^p-operator algebras is not closed under quotients by two-sided closed ideals (Gardella-Thiel, 2016; Blecher-Phillips, 2020).

Theorem (Bernau-Lacey (1977))

Let $p \in [1, \infty)$ and let $e \in \mathcal{L}(L^p(\mu))$ be a bicontractive idempotent (i.e. $e^2 = e$, $\|e\| \leq 1$, and $\|1 - e\| \leq 1$). Then $\|1 - 2e\| = 1$.

Outline

Double Centralizers

Let A be a Banach algebra. We say that $L \in \mathcal{L}(A)$ is a left centralizer for A if for any $a, b \in A$,

$$L(ab) = L(a)b.$$

The space of left centralizers is denoted by $LC(A)$. Similarly, $R \in \mathcal{L}(A)$ is a right centralizer if for any $a, b \in A$,

$$R(ab) = aR(b).$$

We denote by $RC(A)$ to the space of right centralizers.

Definition

The *multiplier algebra of A* is

$$M(A) = \{(L, R) \in L(A) \times RC(A) \colon aL(b) = R(a)b\}$$

equipped with the norm $\|(L, R)\| = \max\{\|L\|, \|R\|\}$.

$M(A)$ is a unital Banach subalgebra of $\mathcal{L}(A) \times \mathcal{L}(A)^{\mathrm{op}}$ where the unit is $(\mathrm{id}_A, \mathrm{id}_A)$.

Two Sided Multipliers

It's well known that if A is a C*-algebra which is nondegenerately represented on a Hilbert space $\mathcal{H}$ via $\varphi\colon A \to \mathcal{L}(\mathcal{H})$, then

$$M(A) \cong \{b \in \mathcal{L}(\mathcal{H}) \colon b\varphi(A) \subseteq \varphi(A), \varphi(A)b \subseteq \varphi(A)\}.$$

In fact, the RHS is an alternative definition for $M(A)$.

Theorem (B. E. Johnson, 1964)

Let A be a Banach algebra with a cai and that's nondegenerately represented on a Banach space E via $\varphi\colon A \to \mathcal{L}(E)$. Then φ extends to a nondegenerate representation of $M(A)$ on E.

In this case $M(A)$ is again isometrically isomorphic to

$$\{b \in \mathcal{L}(E) \colon b\varphi(A) \subseteq \varphi(A), \varphi(A)b \subseteq \varphi(A)\}$$

Question: What happens if we drop the assumptions of nondegeneracy and the existence of a cai?

Outline

Representable L^p-Multiplier algebras

Now consider the algebra of strictly upper triangular 2×2 matrices:

$$T_2^p = \left\{ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} : z \in \mathbb{C} \right\} \subset M_2^p(\mathbb{C}) = \mathcal{L}(\ell_2^p)$$

It's not hard to see that T_2^p can't have a cai and that it can't be nondegenerately represented on **any** Banach space. However, since $T_2^p \cong \mathbb{C}$, then $LC(T_2^p) \cong \mathbb{C}$ and $RC(T_2^p) \cong \mathbb{C}$. Therefore, for any $q \in [1, \infty)$

$$M(T_2^p) = LC(T_2^p) \times RC(T_2^p) \cong \mathbb{C}^2 \cong C(\{1, 2\}) \subset \mathcal{L}(\ell_2^q).$$

That is, $M(T_2^p)$ is a nondegenerately representable L^q-operator algebra for any $p, q \in [1, \infty)$.

L^p-group algebras

Let G be a discrete group and $p \in [1, \infty)$. Then $\ell^1(G)$ acts on $\ell^p(G)$ as left convolution operators. That is, $\lambda_p \colon \ell^1(G) \to \mathcal{L}(\ell^p(G))$ is given by

$$(\lambda_p(a)b)(g) = \sum_{h \in G} a(h)b(h^{-1}g)$$

Definition

For $p \in [1, \infty)$, the *reduced L^p-operator algebra of G* is

$$F^p_{\mathrm{r}}(G) = \overline{\lambda_p(\ell^1(G))} \subseteq \mathcal{L}(\ell^p(G))$$

We define $\ell^1_0(G)$ as the kernel of the map $\ell^1(G) \ni a \mapsto \sum_{g \in G} a(g) \in \mathbb{C}$. That is,

$$\ell^1_0(G) = \left\{ a \in \ell^1(G) \colon \sum_{g \in G} a(g) = 0 \right\}$$

Our main object of study will be $F^p_{\mathrm{r},0}(G) = \overline{\lambda_p(\ell^1_0(G))} \subseteq \mathcal{L}(\ell^p(G))$.

Properties of the augmentation ideal

Recall $\ell_0^1(G) = \left\{a \in \ell^1(G) \colon \sum_{g\in G} a(g) = 0\right\}$; $F^p_{\mathrm{r},0}(G) = \overline{\lambda_p(\ell_0^1(G))}$.
For each $g \in G$ we define $\Delta_g \in \ell_0^1(G)$ as

$$\Delta_g := \delta_g - \delta_{1_G}.$$

Proposition

For any $p \in [1, \infty)$, the set $\mathrm{span}\{\Delta_g \colon g \in G\}$ is dense in $F^p_{\mathrm{r},0}(G)$. If in addition G is finite with $n := \mathrm{card}(G) \geq 2$, then

$$\mathbf{1}_0 = -\frac{1}{n} \sum_{g\in G} \Delta_g$$

is an algebraic identity for $F^p_{\mathrm{r},0}(G)$.

When $p = 1$ we have $F^p_{\mathrm{r},0}(G) = \ell_0^1(G)$ and $\|\mathbf{1}_0\|_1 = 2 - \frac{2}{n}$.

$F^p_{\mathrm{r},0}(G)$ is a nonunital L^p-operator Algebra

Recall $\ell^1_0(G) = \left\{a \in \ell^1(G) \colon \sum_{g\in G} a(g) = 0\right\}$; $F^p_{\mathrm{r},0}(G) = \overline{\lambda_p(\ell^1_0(G))}$.

- If either $p = 1$ or G is amenable, then $F^p_{\mathrm{r},0}(G)$ acts degenerately on $\ell^p(G)$.
- For any finite group G with $n := \mathrm{card}(G) > 2$, if $p \neq 2$, then $\mathbf{1}_0$ is not unital in $F^p_{\mathrm{r},0}(G)$. In fact,

$$\|\mathbf{1}_0\|_{F^p_{\mathrm{r}}(G)} > 1 \iff p \neq 2$$

Let $a_0 := (n-1, -1, \ldots, -1) \in \ell^1_0(G)$ and for each $\varepsilon \in (-1,1) \setminus \{0\}$ put $b_\varepsilon := a_0 + \varepsilon(1, \ldots, 1) \in \ell^p(G)$. Then $\mathbf{1}_0 b_\varepsilon = a_0$. We only need to find ε_p such that

$$\|a_0\|_p > \|b_{\varepsilon_p}\|_p.$$

Here's a *proof by picture* using the function $\varepsilon \mapsto \|a_0\|_p - \|b_\varepsilon\|_p$.

- For any finite group G with $\mathrm{card}(G) > 2$, if $p \neq 2$, then $F^p_{\mathrm{r},0}(G)$ has no cai and can't be nondegenerately represented on **any** Banach space.

Nonrepresentable L^p-Multiplier algebras

For any finite group G with $\mathrm{card}(G) > 2$, if $p \neq 2$, then $F^p_{\mathrm{r},0}(G)$ has no cai and can't be nondegenerately represented on **any** Banach space.
If in addition G is abelian

$$M^p_0(G) := M(F^p_{\mathrm{r},0}(G)) \cong (\ell^1_0(G), |||-|||_p)$$

where

$$|||a|||_p = \sup\{\|ab\|_{F^p_{\mathrm{r}}(G)} : b \in \ell^1_0(G), \|b\|_{F^p_{\mathrm{r}}(G)} = 1\}.$$

Theorem (Blinov, D., Weld (2024))

Let $p_0 = 1.606$ and let $p_0' = \frac{p_0}{p_0 - 1}$. Take any $p \in [1, p_0] \cup [p_0', \infty)$. Then $M^p_0(\mathbb{Z}/3\mathbb{Z})$ is not an L^q-operator algebra for any $q \in [1, \infty)$.

Proof. Set $\gamma := \frac{1}{3}\exp(\frac{2\pi i}{3})$. The element

$$e := \gamma\Delta_{[1]} + \overline{\gamma}\Delta_{[2]} \in M^p_0(\mathbb{Z}/3\mathbb{Z})$$

is a bicontractive idempotent and $|||\mathbf{1}_0 - 2e|||_p > 1$
$(|||\mathbf{1}_0 - 2e|||_1 = \frac{2}{\sqrt{3}})$. ■

Thank you!

Questions?

Known norms that make $\mathbb{C}^n$ an L^1-operator algebra.

Which norms make $\mathbb{C}^n$ an L^1-operator algebra?

- As before, $\mathbb{C}^n$ with the max norm is also an L^1-operator algebra acting on ℓ^1_n via multiplication operators.
- Let $\mathcal{F}\colon \ell^1(\mathbb{Z}/n\mathbb{Z}) \to C(\mathbb{Z}/n\mathbb{Z})$ the Fourier transform. Then we have algebra isomorphisms

$$\mathbb{C}^n \cong C(\mathbb{Z}/n\mathbb{Z}) \cong \mathcal{F}^{-1}(C(\mathbb{Z}/n\mathbb{Z})) = \ell^1(\mathbb{Z}/n\mathbb{Z})$$

  which make $\mathbb{C}^n$ an L^1-operator algebra with norm coming from the identification with $\ell^1(\mathbb{Z}/n\mathbb{Z})$.

Open Question

Are these the only two norms that make $\mathbb{C}^n$ an L^1-operator algebra?

For $n = 2$, the identification of $M^1_0(\mathbb{Z}/3\mathbb{Z})$ with $\mathbb{C}^2$ carries none of these norms.