Multiplier Algebras of L^p-Operator Algebras.

Alonso Delfín (Joint work with Ellen Weld)

CU Boulder

February 1, 2024

Outline

1. Notation
2. Multiplier Algebras
3. L^p-operator Algebras
4. Augmentation Ideal

Outline

In what follows A will always be a Banach Algebra.

Definition

A *representation of A on a Banach space E* is a continuous algebra homomorphism $\varphi\colon A \to \mathcal{L}(E)$.

- We say A is *representable on E* if there is an isometric representation on E.
- We say A is *nondegenerately representable on E* if in addition $\varphi(A)E = \operatorname{span}\{\varphi(a)\xi\colon a \in A, \xi \in E\}$ is a dense subspace of E.

Any C*-algebra is nondegenerately representable on a Hilbert space via the universal representation.

Definition

We say that A has a *contractive approximate identity (cai)* if there is a net $(e_\lambda)_{\lambda \in \Lambda}$ such that $\|e_\lambda\| \leq 1$ for all $\lambda \in \Lambda$ and for all $a \in A$,

$$\lim_{\lambda \in \Lambda} \|ae_\lambda - a\| = \lim_{\lambda \in \Lambda} \|e_\lambda a - a\| = 0.$$

Definition

We say that A has an *identity element* if there is an element $1_A \in A$ such that $1_A \cdot a = a = a \cdot 1_A$ for all $a \in A$. In addition, if $\|1_A\| = 1$, we call 1_A a *unit* of A.

Lemma

Let A be a Banach algebra with an identity $1_A \in A$ such that $\|1_A\| \neq 1$. Then, A cannot have a cai.

Proof. On the one hand $\|1_A\| > 1$. On the other hand, if there's a cai

$$\lim_{\lambda} |\|e_\lambda\| - \|1_A\|| \leq \lim_{\lambda} \|e_\lambda - 1_A\| = \lim_{\lambda} \|e_\lambda 1_A - 1_A\| = 0,$$

so $\|1_A\| = \lim_\lambda \|e_\lambda\| \leq 1$. ■

Outline

Double Centralizers

We say that $L \in \mathcal{L}(A)$ is a left centralizer for A if for any $a, b \in A$,

$$L(ab) = L(a)b.$$

The space of left centralizers is denoted by $LC(A)$. Similarly, $R \in \mathcal{L}(A)$ is a right centralizer if for any $a, b \in A$,

$$R(ab) = aR(b).$$

We denote by $RC(A)$ to the space of right centralizers.

Definition

The *multiplier algebra of A* is

$$M(A) = \{(L, R) \in LC(A) \times RC(A) \colon aL(b) = R(a)b\}$$

equipped with the norm $\|(L, R)\| = \max\{\|L\|, \|R\|\}$.

$M(A)$ is a unital Banach subalgebra of $\mathcal{L}(A) \times \mathcal{L}(A)^{\mathrm{op}}$ where the unit is $(\mathrm{id}_A, \mathrm{id}_A)$.

Two Sided Multipliers

It's well known that if A is a C*-algebra which is nondegenerately represented on a Hilbert space $\mathcal{H}$ via $\varphi\colon A \to \mathcal{L}(\mathcal{H})$, then

$$M(A) \cong \{b \in \mathcal{L}(\mathcal{H}) : b\varphi(A) \subseteq \varphi(A), \varphi(A)b \subseteq \varphi(A)\}.$$

In fact, the RHS is an alternative definition for $M(A)$. This alternative definition is therefore independent of the Hilbert space and the representation φ chosen.

Theorem (D, 2023)

Let A be a Banach algebra with a cai and that's nondegenerately represented on a Banach space E via $\varphi\colon A \to \mathcal{L}(E)$. Then $M(A)$ is isometrically isomorphic to

$$\{b \in \mathcal{L}(E) : b\varphi(A) \subseteq \varphi(A), \varphi(A)b \subseteq \varphi(A)\}$$

Question: What happens if we drop the assumptions of nondegeneracy and the existence of a cai?

Algebras with non-unital identities

There is a natural inclusion $\iota\colon A \to M(A)$ be given by $\iota(a) = (L_a, R_a)$, where $L_a(b) = ab$ and $R_a(b) = ba$. If A has a cai, then ι is isometric and

$$\|L_a\| = \|R_a\| = \|a\|.$$

Proposition

Let $1_A \in A$ be an identity for A. Then $M(A)$ is isometrically isomorphic to $(A, \| - \|')$ where

$$\|a\|' = \|\iota(a)\| = \max\{\|L_a\|, \|R_a\|\}$$

In particular $\|a\|' = \|a\|$ when 1_A is a unit.

Corollary

If A is commutative with non unital identity 1_A, then $M(A)$ is isometrically isomorphic to $(A, \| - \|')$ where

$$\|a\|' = \|L_a\| = \sup_{\|b\|=1} \|ab\| \neq \|a\|.$$

Outline

1. Notation
2. Multiplier Algebras
3. L^p-operator Algebras
4. Augmentation Ideal

L^p-operator Algebras.

For $p \in [1, \infty]$, we say that A is an *L^p-operator algebra* if it is representable on $L^p(\Omega, \mu)$ for some measure space (Ω, μ). That is, there is an isometric algebra homomorphism $\varphi \colon A \to \mathcal{L}(L^p(\mu))$.

Example

- C*-algebras are L^2-operator algebras.
- Fix any $p \in [1, \infty]$ and let Ω be a locally compact space equipped with counting measure ν. Then $C_0(\Omega)$ is an L^p-operator algebra via $\varphi \colon C_0(\Omega) \to \mathcal{L}(L^p(\nu))$ given by

$$(\varphi(a)\xi)(\omega) = a(\omega)\xi(\omega).$$

- Fix any $p \in [1, \infty]$. Then T_d^p, the set of strictly upper triangular $d \times d$ matrices acting on ℓ_d^p, is an L^p-operator algebra.

Representable L^p-Multiplier algebras

As a corollary of the work presented for general Banach Algebras we get

Corollary

If A is a nondegenerately representable L^p-operator algebra with a cai, then $M(A)$ is a nondegenerately representable L^p-operator algebra.

Now consider the algebra of 2×2 strictly upper triangular matrices:

$$T_2^p = \left\{ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} : z \in \mathbb{C} \right\} \subset M_2^p = \mathcal{L}(\ell_2^p)$$

It's not hard to see that T_2^p can't have a cai and that it can't be nondegenerately represented on **any** Banach space. However, since $T_2^p \cong \mathbb{C}$ it's clear that $LC(T_2^p) \cong \mathbb{C}$, $RC(T_2^p) \cong \mathbb{C}$, and therefore

$$M(T_2^p) = LC(T_2^p) \times RC(T_2^p) \cong \mathbb{C}^2 \cong C(\{1,2\}) \subset \mathcal{L}(\ell_2^p).$$

That is, $M(T_2^p)$ is a nondegenerately representable L^p-operator algebra.

Outline

L^p-group algebras

Let G be a discrete group and $p \in [1, \infty)$. Then $\ell^1(G)$ acts on $\ell^p(G)$ as a left convolution operator. That is, $\lambda \colon \ell^1(G) \to \mathcal{L}(\ell^p(G))$ is given by

$$(\lambda(a)b)(g) = \sum_{h \in G} a(h)b(h^{-1}g)$$

Definition

For $p \in [1, \infty)$, the *reduced L^p-operator algebra of G* is

$$F^p_r(G) = \overline{\lambda(\ell^1(G))} \subseteq \mathcal{L}(\ell^p(G))$$

When $p = 1$, we have $F^1_r(G) = \ell^1(G)$. Thus, $\ell^1(G)$ is a unital L^1-operator algebra (unit is δ_{1_G}) acting on itself via left multiplication.

The augmentation ideal $\ell_0^1(G)$

Consider the contractive algebra homomorphism $\ell^1(G) \to \mathbb{C}$ given by

$$a \mapsto \sum_{g \in G} a(g)$$

We define the augmentation ideal of $\ell^1(G)$ as the kernel of this map:

Definition

For a discrete group G, *the augmentation ideal of $\ell^1(G)$* is

$$\ell_0^1(G) = \left\{ a \in \ell^1(G) \colon \sum_{g \in G} a(g) = 0 \right\}$$

Since $\ell_0^1(G)$ is a closed proper ideal of $\ell^1(G)$, it follows that $\ell_0^1(G)$ is an L^1-operator algebra degenerately represented on $\ell^1(G)$.

Open Question

Is $\ell_0^1(G)$ nondegenerately representable on some $L^1(\mu)$?

Properties of $\ell^1_0(G)$

For each $g \in G$ we define $\Delta_g \in \ell^1_0(G)$ as

$$\Delta_g := \delta_g - \delta_{1_G}.$$

Proposition

For any discrete group G, the space $\mathrm{span}\{\Delta_g : g \in G\}$ is dense in $\ell^1_0(G)$. If in addition G is finite with $n := \mathrm{card}(G) \geq 2$, then

- $\ell^1_0(G) = \mathrm{span}\{\Delta_g : g \in G\}$
- $\ell^1_0(G)$ *has an identity element $\mathbf{1}_0$. In fact*

$$\mathbf{1}_0 = -\frac{1}{n}\sum_{g \in G} \Delta_g \quad \textit{and} \quad \|\mathbf{1}_0\|_1 = 2 - \frac{2}{n}$$

Since $\mathbf{1}_0$ not a unit when $n > 2$, it follows that $\ell^1_0(G)$ does not have a cai when $n > 2$.

What is $M(\ell_0^1(G))$?

For G discrete with $\mathrm{card}(G) > 2$ we have a non unital identity in $\ell_0^1(G)$ and therefore $M(\ell_0^1(G))$ is $(\ell_0^1(G), \|a\|' := \max\{\|L_a\|, \|R_a\|\})$ where

$$\|L_a\| = \sup_{b \in \ell_0^1(G), \|b\|_1 = 1} \|ab\|_1,$$
$$\|R_a\| = \sup_{b \in \ell_0^1(G), \|b\|_1 = 1} \|ba\|_1.$$

If we additionally impose that G is abelian, then $M(\ell_0^1(G))$ is $(\ell_0^1(G), \|a\|' = \|L_a\|)$.

Open Question

Is $M(\ell_0^1(G))$ an L^1-operator algebra?

$G = \mathbb{Z}/3\mathbb{Z}$

For $\mathbb{Z}/3\mathbb{Z} = \{0,1,2\}$, we get $\ell_0^1(\mathbb{Z}/3\mathbb{Z}) = \mathrm{span}\{\Delta_1, \Delta_2\}$, and $\mathbf{1}_0 = \frac{-1}{3}(\Delta_1 + \Delta_2)$. $M(\ell_0^1(\mathbb{Z}/3\mathbb{Z}))$ is the same underlying algebra but now $\mathbf{1}_0$ is the only norm 1 idempotent.

Proposition

$M(\ell_0^1(\mathbb{Z}/3\mathbb{Z}))$ is isometrically isomorphic to $\mathbb{C}^2$ with a norm different from the max norm.

Proof. $\mathrm{Max}(\ell_0^1(\mathbb{Z}/3\mathbb{Z})) = \{\omega_1, \omega_2\}$ where

$$\omega_1(\Delta_1) = \frac{-3}{2} - i\frac{\sqrt{3}}{2} = \omega_2(\Delta_2)$$
$$\omega_1(\Delta_2) = \frac{-3}{2} + i\frac{\sqrt{3}}{2} = \omega_2(\Delta_1)$$

Then $\ell_0^1(\mathbb{Z}/3\mathbb{Z})$ is isomomorphic to $C(\{\omega_1, \omega_2\})$ as algebras, so the norm from $M(\ell_0^1(\mathbb{Z}/3\mathbb{Z}))$ gives a norm on $C(\{\omega_1, \omega_2\})$ different from the max norm. ∎

Known norms that make $\mathbb{C}^2$ an L^1-operator algebra.

Which norms make $\mathbb{C}^2$ an L^1-operator algebra?

- As before, $\mathbb{C}^2$ with the max norm is also an L^1-operator algebra acting on ℓ_2^1 via multiplication operators.
- Let $\mathcal{F}\colon \ell^1(\mathbb{Z}/2\mathbb{Z}) \to C(\mathbb{Z}/2\mathbb{Z})$ the Fourier transform. Then we have algebra isomorphisms

$$\mathbb{C}^2 \cong C(\mathbb{Z}/2\mathbb{Z}) \cong \mathcal{F}^{-1}(C(\mathbb{Z}/2\mathbb{Z})) = \ell^1(\mathbb{Z}/2\mathbb{Z})$$

  which make $\mathbb{C}^2$ an L^1-operator algebra with norm coming from the identification with $\ell^1(\mathbb{Z}/2\mathbb{Z})$.

Open Question

Are these the only two norms that make $\mathbb{C}^2$ an L^1-operator algebra?

We do know that the identification of $M(\ell_0^1(\mathbb{Z}/3\mathbb{Z}))$ with $\mathbb{C}^2$ carries none of these norms.

Thank you!

Questions?

C*-cores

Let A be a unital Banach algebra and let A_{h} denote the set of hermitian elements, that is

$$A_{\mathrm{h}} = \{a \in A \colon \|e^{ita}\| = 1 \forall t \in \mathbb{R}\}.$$

If A_{h} is closed under multiplication, then

$$\mathrm{core}(A) = A_{\mathrm{h}} + iA_{\mathrm{h}}$$

is the largest C*-algebra contained in A.

Theorem (Gardella-Thiel (22))

For $p \neq 2$, G a discrete group, A a unital L^p-operator algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ an action, then

$$\mathrm{core}(A) = \mathrm{core}(F_r^p(G, A, \alpha))$$

We know that $\mathrm{core}(M(\ell_0^1(\mathbb{Z}/3\mathbb{Z}))) = \mathbb{C}$. Is it possible for $M(\ell_0^1(\mathbb{Z}/3\mathbb{Z}))$ to contradict the previous theorem?