Equivalence theorems for Banach algebras of étale groupoids.

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Outline

1. Morita Equivalent Banach Algebras
2. Topological Groupoids
3. Banach algebras of étale groupoids
4. Equivalence Theorems

Outline

Banach Pairs

Fix a Banach algebra B.
A (right) Banach B-module is a Banach space X that is a (right) B-module and such that $\|xb\|_{\mathsf{X}} \leq \|x\|_{\mathsf{X}}\|b\|_B$ for all $x \in \mathsf{X}$, $b \in B$.

Definition

A *Banach B-pair* is a pair (X, Y) such that X is a left Banach B-module, Y is a right Banach B-module, and there is a $\mathbb{C}$-bilinear pairing $\langle - \mid - \rangle_B \colon \mathsf{X} \times \mathsf{Y} \to B$ satisfying

- $\langle bx \mid y \rangle_B = b\langle x \mid y \rangle_B$
- $\langle x \mid yb \rangle_B = \langle x \mid y \rangle_B b$
- $\|\langle x \mid y \rangle_B\|_B \leq \|x\|_{\mathsf{X}}\|y\|_{\mathsf{Y}}$.

We say (X, Y) is *nondegenerate* when $B\mathsf{X} \subseteq \mathsf{X}$ and $\mathsf{Y}B \subseteq \mathsf{Y}$ are both dense subspaces. We say (X, Y) is full when $\langle \mathsf{X} \mid \mathsf{Y} \rangle_B \subseteq B$ is dense.

If A is a C*-algebra and X a right Hilbert A-module, then $(\widetilde{\mathsf{X}}, \mathsf{X})$ is a Banach A-pair.

Banach Correspondences

For a (right) Banach B-module X, we denote $\mathrm{Hom}(\mathsf{X}_B) \subseteq \mathcal{L}(\mathsf{X})$ to the algebra of bounded (right) B-module homomorphisms $\mathsf{X} \to \mathsf{X}$.
For a Banach B-pair (X, Y), we define the Banach algebra $\mathcal{L}_B\big((\mathsf{X}, \mathsf{Y})\big)$ by

$$\mathcal{L}_B\big((\mathsf{X}, \mathsf{Y})\big) = \{(t, s) \colon \langle t(x) \mid y \rangle_B = \langle x, s(y) \rangle_B\} \subseteq \mathrm{Hom}({}_B\mathsf{X}) \times \mathrm{Hom}(\mathsf{Y}_B)^{\mathrm{op}}$$

Definition

Let A and B be Banach algebras. We say $\big((\mathsf{X}, \mathsf{Y}), \varphi_A\big)$ is a Banach (A, B)-correspondence if (X, Y) is a Banach B-pair (X, Y) and $\varphi_A \colon A \to \mathcal{L}_B\big((\mathsf{X}, \mathsf{Y})\big)$ is a contractive algebra homomorphism.

Let $\big((\mathsf{X}, \mathsf{Y}), \varphi_A\big)$ be a Banach (A, B)-correspondence. For each $a \in A$ put $\varphi_A(a) = (t_a, s_a)$ and denote $x \cdot a = t_a(x)$ and $a \cdot y = s_a(y)$.
Then X is a B-A Banach bimodule, Y is an A-B Banach bimodule, and

$$\langle x \cdot a \mid y \rangle_B = \langle x \mid a \cdot y \rangle_B$$

Morita Equivalence

Definition (V. Lafforgue (2002))

Two Banach algebras A and B are *Morita Equivalent* if there are Banach bimodules $\mathsf{X} = {}_B\mathsf{X}_A$, $\mathsf{Y} = {}_A\mathsf{Y}_B$, and bilinear pairings $\langle - \mid - \rangle_B \colon \mathsf{X} \times \mathsf{Y} \to B$ and $(- \mid -)_A \colon \mathsf{Y} \times \mathsf{X} \to A$ such that

- (X, Y) with $\langle - \mid - \rangle_B$ is a Banach (A, B)-correspondence that is full and nondegenerate as a Banach B-pair,
- (Y, X) with $(- \mid -)_A$ is a Banach (B, A)-correspondence that is full and nondegenerate as a Banach A-pair,
- $\langle x_1 \mid y \rangle_B \cdot x_2 = x_1 \cdot (y \mid x_2)_A$ for all $x_1, x_2 \in \mathsf{X}$, $y \in \mathsf{Y}$,
- $y_1 \cdot \langle x \mid y_2 \rangle_B = (y_1 \mid x)_A \cdot y_2$ for all $x \in \mathsf{X}$, $y_1, y_2 \in \mathsf{Y}$.

Example ($p \in (1, \infty)$, $q = \frac{p}{p-1}$, and (Ω, Σ, μ) a measure space)

Then $\mathcal{K}(L^p(\mu))$ and $\mathbb{C}$ are Morita equivalent Banach algebras via the modules $\mathsf{X} = L^q(\mu)$, $\mathsf{Y} = L^p(\mu)$ and the pairings

$$\langle \eta \mid \xi \rangle_{\mathbb{C}} = \int \eta \xi d\mu, \quad (\xi \mid \eta)_{\mathcal{K}(L^p(\mu))} = (\zeta \mapsto \xi \langle \eta \mid \zeta \rangle_{\mathbb{C}})$$

Linking Algebra

Let A and B be Morita Equivalent Banach algebras via the pair $(\mathsf{X},\mathsf{Y}) = ({}_B\mathsf{X}_A, {}_A\mathsf{Y}_B)$.

Definition

The Linking algebra is the Banach space $L = A \oplus \mathsf{X} \oplus \mathsf{Y} \oplus B$ where the multiplication is given by formal matrix multiplication when seeing elements in L as 2×2 matrices in

$$L = \begin{bmatrix} A & \mathsf{Y} \\ \mathsf{X} & B \end{bmatrix}.$$

That is,

$$\begin{bmatrix} a_1 & y_1 \\ x_1 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & y_2 \\ x_2 & b_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 + (y_1 \mid x_2)_A & a_1 \cdot y_2 + y_1 \cdot b_2 \\ x_1 \cdot a_2 + b_1 \cdot x_2 & \langle x_1 \mid y_2 \rangle_B + b_1b_2 \end{bmatrix}.$$

In fact, if $(\mathsf{X},\mathsf{Y}) \oplus B = ({}_B\mathsf{X} \oplus {}_BB, \mathsf{Y}_B \oplus B_B)$, then

$$L = \mathcal{K}_B\big((\mathsf{X},\mathsf{Y}) \oplus B\big) \subseteq \mathcal{L}_B\big((\mathsf{X},\mathsf{Y}) \oplus B\big).$$

Outline

Étale Groupoids

We fix a grouopoid G with set of composable pairs $G^{(2)} \subseteq G \times G$ and unit space $G^{(0)} = \{\gamma \in G \colon \gamma^{-1} = \gamma = \gamma^2\}$. Recall that the range and source maps $r, s \colon G \to G^{(0)}$ are given by $r(\gamma) = \gamma\gamma^{-1}$, $s(\gamma) = \gamma^{-1}\gamma$.

- G is a topological groupoid when G is a topological space such that $\gamma \mapsto \gamma^{-1}$ is a continuous map from G to G, and $(\gamma, \eta) \mapsto \gamma\eta$ is continuous map from $G^{(2)}$ to G.
- G is called étale if G is locally compact, locally Hausdorff, and in addition both s and r are local homeomorphisms.

The condition of being étale implies that both $G_u := s^{-1}(u)$ and $G^u := r^{-1}(u)$ are countable discrete spaces for each $u \in G^{(u)}$.

Thus, for an étale gruopoid G we think of both G_u and G^u as measure spaces equipped with counting measure.

Groupoid Actions

Throughout the talk G will be locally compact, Hausdorff, and étale.

Definition

A left G-space is a locally compact Hausdorff space Z together with a continous open map $r_Z\colon Z \to G^{(0)}$ and a continuous map $(\gamma, z) \mapsto \gamma \cdot z \in Z$ defined on $G * Z = \{(\gamma, z)\colon s(\gamma) = r_Z(z)\}$, such that

1. $r_Z(z) \cdot z = z$ for all $z \in Z$,
2. if $(\gamma', \gamma) \in G^{(2)}$ and $(\gamma, z) \in G * Z$, then $(\gamma', \gamma \cdot z) \in G * Z$ and $(\gamma'\gamma) \cdot z = \gamma' \cdot (\gamma \cdot z)$.

- We say Z is free if $\gamma \cdot z = z$ implies $\gamma = r_Z(z)$;
- We say Z is proper if the map $\Theta : G * Z \to Z \times Z$ given by $\Theta(\gamma, z) = (\gamma \cdot z, z)$ is a proper map of $G * Z$ into $Z \times Z$, i.e., Θ is a closed map such that the inverse image of compact sets are compact.

Right G-spaces are defined similarly except that the structure map is denoted by s_Z instead of r_Z.

Groupoid Equivalences

Definition

Let G and H be groupoids. A (G,H)-equivalence is a space Z such that

1. Z is a free and proper left G-space;
2. Z is a free and proper right H-space;
3. the actions of G and H on Z commute;
4. r_Z induces a homeomorphism of Z/H onto $G^{(0)}$;
5. s_Z induces a homeomorphism of $G\backslash Z$ onto $H^{(0)}$.

Let Z be a (G,H)-equivalence.

- There is a continuous map $Z *_s Z \to G$, $(z_1,z_2) \mapsto {}_G[z_1,z_2]$,

$$ {}_G[z_1,z_2] \cdot z_2 = z_1 \ \forall \ (z_1,z_2) \in Z *_s Z. $$

- There is a continuous map $Z *_r Z \to G$, $(z_1,z_2) \mapsto [z_1,z_2]_H$,

$$ z_1 \cdot [z_1,z_2]_H = z_2 \ \forall \ (z_1,z_2) \in Z *_r Z. $$

The linking groupoid

Define the opposite space of a (G, H)-equivalence Z to be a copy $Z^{\text{op}} := \{\bar{z} : z \in Z\}$ of Z with the structure of a (H, G)-equivalence determined by

$$r(\bar{z}) = s(z), s(\bar{z}) = r(z), \eta \cdot \bar{z} = \overline{z \cdot \eta^{-1}}, \bar{z} \cdot \gamma = \overline{\gamma^{-1} \cdot z}.$$

Definition

The linking groupoid K of of a (G, H)-equivalence Z is defined as the disjoint union $K = G \sqcup Z^{\text{op}} \sqcup Z \sqcup H$.

K is a locally compact Hausdorff groupoid, and the groupoid operations are the ones inherited from G, Z^{op}, Z, and H:

- $K^{(0)} = G^{(0)} \sqcup H^{(0)}$,
- source and range are the inherited ones,
- multiplication and inversion restrict to the ones on G and H, and obey

$$z_1\overline{z_2} = {}_G[z_1, z_2],\ z_1\overline{z_2} = [z_1, z_2]_H,\ z^{-1} = \bar{z},\ \text{ and } \bar{z}^{-1} = z.$$

Outline

Banach groupoid algebras

Recall G will be locally compact, Hausdorff, and étale.

$$C_c(G) := \{f \colon G \to \mathbb{C} \colon \; f \text{ is continous and has compact support}\}$$

We get a convolution product that makes $C_c(G)$ into an algebra:

$$(f * g)(\gamma) = \sum_{\{\eta \in G \colon r(\eta) = r(\gamma)\}} f(\eta) g(\eta^{-1}\gamma).$$

There's three submultiplicative norms on $C_c(G)$:

$$\|f\|_{I,s} = \sup_{u \in G^{(0)}} \sum_{\gamma \in G_u} |f(\gamma)|, \quad \|f\|_{I,r} = \sup_{u \in G^{(0)}} \sum_{\gamma \in G^u} |f(\gamma)|,$$

and $\|f\|_I = \max\{\|f\|_{I,s}, \|f\|_{I,r}\}$. The completions of $C_c(G)$ with respect to these norms are respectively denoted by

$$F_{I,s}(G), \; F_{I,r}(G), \text{ and } F_I(G).$$

Reduced L^p groupoid algebras

Fix $p \in [1, \infty]$. Each $u \in G^{(0)}$ induces a representation of $C_c(G)$ on $\ell^p(G_u)$, denoted $\mathrm{Ind}_u \colon C_c(G) \to \mathcal{L}(\ell^p(G_u))$, and defined by

$$[(\mathrm{Ind}_u f)\xi](\gamma) = \sum_{\{\eta \in G \colon r(\eta) = r(\gamma)\}} f(\eta)\xi(\eta^{-1}\gamma)$$

for every $f \in C_c(G), \xi \in \ell^p(G_u)$, and $\gamma \in G_u$. This gives the *p-reduced* norm on $C_c(G)$:

$$\|f\|_{p,\mathrm{red}} = \sup_{u \in G^{(0)}} \|\mathrm{Ind}_u f\|_{\mathcal{L}(\ell^p(G_u))}.$$

We denote by $F^p_{\mathrm{red}}(G)$ to the completion of $C_c(G)$ w.r.t. $\|f\|_{p,\mathrm{red}}$.

Proposition

If $p \in (1, \infty)$ and p' is its Hölder conjugate, then

$$\|f\|_{p,\mathrm{red}} \leq \|f\|_{I,s}^{1/p} \|f\|_{I,r}^{1/p'} \leq \|f\|_I$$

Proof. Apply the Riesz-Thorin interpolation theorem. ∎

Full L^p groupoid algebras and amenability

Let $\mathcal{R}_p$ be the class of all contractive representations of $(C_c(G), \| - \|_I)$ on L^p spaces. Put

$$\|f\|_{L^p} = \sup_{\varphi \in \mathcal{R}_p} \|\varphi(f)\|.$$

$F^p(G)$ is defined as the completion of $C_c(G)$ w.r.t. $\|f\|_{L^p}$.

Proposition

$F^1_{\mathrm{red}}(G) = F^1(G) = F_{I,s}(G)$ and $F^\infty_{\mathrm{red}}(G) = F^\infty(G) = F_{I,r}(G)$.

Theorem (Gardella-Lupini (2017))

If G is an amenable groupoid, then $F^p_{\mathrm{red}}(G)$ and $F^p(G)$ are p-completely isometrically isomorphic.

Outline

Known results

Theorem (P. Mulhy, J. Renault, D.P. Williams (1987))

If G and H are equivalent groupoids, then $F^2(G)$ and $F^2(G)$ are Morita equivalent C-algebras.*

Theorem (Known since 1983, proved A. Sims, D.P. Williams (2012))

If G and H are equivalent groupoids, then $F^2_{\mathrm{red}}(G)$ and $F^2_{\mathrm{red}}(G)$ are Morita equivalent C-algebras.*

Theorem (W. Paravicini (2008))

If G and H are equivalent groupoids, then $F_I(G)$ and $F_I(H)$ are Morita equivalent Banach algebras.

Unconditional Completions

An unconditional completion $\mathcal{A}(G)$ of $C_c(G)$ is a Banach algebra containing $C_c(G)$ as a dense subalgebra and having the property

$$|f(\gamma)| \leq |g(\gamma)| \ \forall \gamma \in G \implies \|f\|_{\mathcal{A}(G)} \leq \|g\|_{\mathcal{A}(G)}$$

for all $f, g \in C_c(G)$.

Theorem (W. Paravicini (2008))

If G and H are equivalent groupoids, then $\mathcal{A}(G)$ and $\mathcal{A}(H)$ are Morita equivalent Banach algebras.

Unfortunately, the algebras $F^p_{\mathrm{red}}(G)$ and $F_p(G)$ are not generally unconditional completions of $C_c(G)$. They are when $p = 1, \infty$, but we have counterexamples for all other $p \in (1, \infty)$.

Equivalence Theorem for $F^p_{\mathrm{red}}(G)$

Theorem (Chung, D., Wang (2025))

Let $p \in (1, \infty)$. If G and H are equivalent groupoids, then $F^p_{\mathrm{red}}(G)$ and $F^p_{\mathrm{red}}(G)$ are Morita equivalent L^p-operator algebras.

Proof idea: Given a (G, H) equivalence Z, the space $C_c(Z)$ has a natural structure of a $C_c(G)$-$C_c(H)$-bimodule (actions being left and right translation). Define $\langle - \mid - \rangle_{C_c(H)} \colon C_c(Z^{\mathrm{op}}) \times C_c(Z) \to C_c(H)$ by

$$\langle \phi \mid \psi \rangle_{C_c(H)}(\eta) = \sum_{r(\gamma) = r_Z(z)} \phi(\overline{\gamma^{-1} \cdot z})\psi(\gamma^{-1} \cdot z \cdot \eta),$$

for any $s_Z(z) = r(\eta)$. Also $(- \mid -)_{C_c(G)} \colon C_c(Z) \times C_c(Z^{\mathrm{op}}) \to C_c(G)$ by

$$(\psi \mid \phi)_{C_c(G)}(\gamma) = \sum_{r(\eta) = s_Z(z)} \psi(z \cdot \eta)\phi(\overline{\gamma^{-1} \cdot z \cdot \eta}),$$

for any $r_Z(z) = s(\gamma)$.

Next, we realize both $C_c(Z^{\mathrm{op}})$ and $C_c(Z)$as p-operator spaces, and complete them in the p-operator norm. The respective completions X_Z and Y_Z are now our candidate to implement the Morita equivalence between $F^p_{\mathrm{red}}(G)$-$F^p_{\mathrm{red}}(H)$.
Indeed, we have

$$\left\| \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix} \right\|_{F^p_{\mathrm{red}}(K)} = \|f\|_{F^p_{\mathrm{red}}(G)}, \ \left\| \begin{bmatrix} 0 & 0 \\ 0 & g \end{bmatrix} \right\|_{F^p_{\mathrm{red}}(K)} = \|g\|_{F^p_{\mathrm{red}}(H)},$$

and $C_c(G)$ contains an approximate identity $(e_\lambda)_{\lambda\in\Lambda}$ with respect to the inductive limit topology for both the convolution product on $C_c(G)$ and the actions of $C_c(G)$ on $C_c(Z)$ and $C_c(Z^{\mathrm{op}})$, where each e_λ is a finite sum of elements of the form $(\psi \mid \phi)_{C_c(G)}$.

Consequences

Suppose that G and H are countable discrete groups acting freely and properly on the left and right, respectively, of a compact Hausdorff space Z, such that the actions commute.
Then the transformation groupoids $G \ltimes Z/H$ and $H \ltimes G\backslash Z$ are equivalent and therefore

$$F^p_{\mathrm{red}}(G \ltimes Z/H) \cong_{\mathrm{M}} F^p_{\mathrm{red}}(H \ltimes G\backslash Z)$$

A result by Y. Choi, E. Gardella, and H. Thiel shows that for any $p \in [1, \infty)$, the algebra $F^p_{\mathrm{red}}(G \ltimes X)$ is isometrically isomorphic to the crossed product $F^p_{\mathrm{red}}(G, C(X))$. Thus,

$$F^p_{\mathrm{red}}(G, C(Z/H)) \cong_{\mathrm{M}} F^p_{\mathrm{red}}(H, C(G\backslash Z)),$$

which is a p-version of the Green-Rieffel imprimitivity theorem.

What's Next?

- The main result is likely to hold also when G and H are locally Hausdorff. This is more technical to prove.
- Is $F^p_{\mathrm{red}}(K)$ isometrically isomorphic to the Linking algebra L of the equivalence bimodule $(\mathsf{X}_Z, \mathsf{Y}_Z)$ between $F^p_{\mathrm{red}}(G)$ and $F^p_{\mathrm{red}}(H)$?
- For the full algebra case, so far we only know

$$\left\| \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix} \right\|_{F^p(K)} \leq \|f\|_{F^p(G)}, \quad \left\| \begin{bmatrix} 0 & 0 \\ 0 & g \end{bmatrix} \right\|_{F^p_{\mathrm{red}}(K)} \leq \|g\|_{F^p(H)}.$$

When $p = 2$, the reverse inequality comes Renault's disintegration theorem. We don't have a version of this for $p \neq 2$.

Thank you!

Questions?