L^p-correspondences and their L^p Cuntz-Pimsner algebras.

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Outline

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Spaces of Operators

We denote by $\mathcal{L}(E,F)$ the set of bounded linear maps between Banach spaces E and F. This is a Banach space with norm

$$\|a\| := \sup_{\|\xi\|_E=1} \|a\xi\|_F.$$

If $E = F$, then $\mathcal{L}(E) := \mathcal{L}(E,E)$ is a Banach algebra.

- If $\mathcal{H}_0, \mathcal{H}_1$ are Hilbert spaces, any $a \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ has an adjoint $a^* \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_0)$ characterized by $\langle a\xi, \eta\rangle = \langle \xi, a^*\eta\rangle$.
- If $\mathcal{H}$ is a Hilbert space, a C*-algebra is a norm closed selfadjoint subalgebra of $\mathcal{L}(\mathcal{H})$.
- A right Hilbert module X over a C*-algebra $A \subseteq \mathcal{L}(\mathcal{H}_0)$ is a closed subspace of $\mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ satisfying $xa \in$ X and $x^*y \in A$.
- If (Ω, μ) is a measure space and $p \in [1, \infty)$, an L^p-operator algebra is a norm closed subalgebra of $\mathcal{L}(L^p(\mu))$.

L^p-operator algebras: Examples

Example

Let $(\Omega, \mathfrak{M}, \mu)$ be a measure space and let $p \in [1, \infty)$.

- $\mathcal{L}(L^p(\mu))$ and $\mathcal{K}(L^p(\mu))$ are L^p-operator algebras.
- If Ω is a locally compact and μ is counting measure, then $C_0(\Omega)$ acts via multiplication operators on $L^p(\mu)$ and is therefore an L^p-operator algebra.
- C*-algebras are L^2-operator algebras.
- For $d \in \mathbb{Z}_{\geq 1}$, $j, k \in \{1, \dots, d\}$, let $e_{j,k} \in M_d^p = \mathcal{L}(\ell_d^p)$ be the matrix whose only non-zero entry is the (j,k)-entry, which equals 1. Then, the set of upper triangular matrices

$$T_d^p = \mathrm{span}\{e_{j,k} \colon 1 \leq j < k \leq d\}$$

is an L^p-operator algebra.

L^p Analogues of $\mathcal{O}_d$

Let $d \in \mathbb{Z}_{\geq 2}$. The **Leavitt algebra** L_d is the universal unital complex algebra generated by elements $s_1, s_2, \dots, s_d$, and $t_1, t_2, \dots, t_d$ satisfying

$$t_j s_k = \delta_{j,k} \quad \text{and} \quad \sum_{j=1}^{d} s_j t_j = 1$$

Definition

Let $p \in [1, \infty)$ and let $(\Omega, \mathfrak{M}, \mu)$ be a measure space. A spatial representation of L_d is an algebra homomorphism $\varphi : L_d \to \mathcal{L}(L^p(\mu))$ such that each $\varphi(s_j)$ is a spatial partial isometry with reverse $\varphi(t_j)$.

Given any spatial representation φ of L_d, we define

$$\mathcal{O}_d^p := \overline{\varphi(L_d)} \subseteq \mathcal{L}(L^p(\mu)).$$

L^p-crossed products

Let A be an L^p-operator algebra, G a second countable locally compact group and $\alpha\colon G \to \mathrm{Aut}(A)$ an isometric action.
A covariant representation of (G, A, α) on $L^p(\mu)$ is a pair (u, π) where $u\colon G \to \mathrm{Inv}(L^p(\mu))$ is a strongly continuous group homomorphism and π is a representation of A on $L^p(\mu)$ satisfying

$$\pi(\alpha_g(a)) = u_g \pi(a) u_g^{-1}.$$

$$\sigma_{\ltimes}(x) = \sup_{\substack{(u,\pi) \text{ is a } \sigma\text{-finite, nondegenerate,} \\ \text{contractive, covariant representation of } (G,A,\alpha)}} \|(u \ltimes \pi)x\|$$

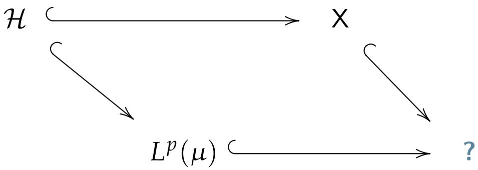
The **full crossed product**, $F^p(G, A, \alpha)$, is the completion of $C_c(G, A, \alpha)/\ker(\sigma_{\ltimes})$ with respect to $\| - \|_{\ltimes}$.

$$\sigma_{\mathrm{r}}(x) = \sup_{\substack{\pi_0 \text{ is a } \sigma\text{-finite, nondegenerate,} \\ \text{contractive representation of } A}} \|(v \ltimes \tilde{\pi}_0)x\|$$

The **reduced crossed product**, $F_{\mathrm{r}}(G, A, \alpha)$, is the completion of $C_c(G, A, \alpha)/\ker(\sigma_{\mathrm{r}})$ with respect to $\| - \|_{\mathrm{r}}$.

Goal: Generalizing Hilbert spaces

- Any Hilbert space is an L^2-space.
- Any Hilbert space is a right Hilbert module over $\mathbb{C}$.



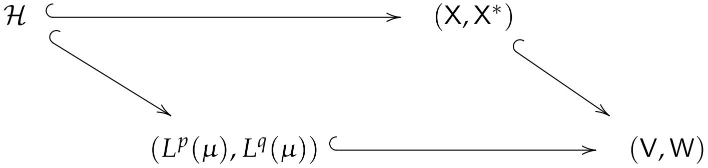
The main goal is to define a proper object to put in place of ? so that the above "diagram of inclusions" commutes.

L^p-modules

For $p \in [1, \infty)$, we denote by q its Hölder conjugate:

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Roughly speaking, an **L^p-module** over an L^p-operator algebra will be a pair (V, W) that makes the following diagram commute.



Outline

L^p-modules: Definition

Let $A \subseteq \mathcal{L}(\mathcal{H}_0)$ be a C*-algebra. Then, any right Hilbert A-module X can be isometrically represented as a closed linear subspace of $\mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ satisfying

1. $xa \in \mathsf{X}$ for all $x \in \mathsf{X}$, $a \in A$,
2. $x^*y \in A$ for all $x, y \in \mathsf{X}$.

Definition

Let $(\Omega_0, \mu_0), (\Omega_1, \mu_1)$ be measure spaces, let $p \in [1, \infty)$, and let $A \subseteq \mathcal{L}(L^p(\mu_0))$ be an L^p operator algebra. An **L^p-module** over A is a pair (X, Y) where $\mathsf{X} \subseteq \mathcal{L}(L^p(\mu_0), L^p(\mu_1))$ and $\mathsf{Y} \subseteq \mathcal{L}(L^p(\mu_1), L^p(\mu_0))$ are closed subspaces satisfying

1. $xa \in \mathsf{X}$ for all $x \in \mathsf{X}$, $a \in A$,
2. $yx \in A$ for all $x \in \mathsf{X}$, $y \in \mathsf{Y}$,
3. $ay \in \mathsf{Y}$ for all $y \in \mathsf{Y}$, $a \in A$.

L^p-modules: Definition

Definition

Let $(\Omega_0, \mu_0), (\Omega_1, \mu_1)$ be measure spaces, let $p \in [1, \infty)$, and let $A \subseteq \mathcal{L}(L^p(\mu_0))$ be an L^p operator algebra. An **L^p-module** over A is a pair (X, Y) where $\mathsf{X} \subseteq \mathcal{L}(L^p(\mu_0), L^p(\mu_1))$ and $\mathsf{Y} \subseteq \mathcal{L}(L^p(\mu_1), L^p(\mu_0))$ are closed subspaces satisfying

1. $xa \in \mathsf{X}$ for all $x \in \mathsf{X}$, $a \in A$,
2. $yx \in A$ for all $x \in \mathsf{X}$, $y \in \mathsf{Y}$,
3. $ay \in \mathsf{Y}$ for all $y \in \mathsf{Y}$, $a \in A$.

An L^p-module comes with a bilinear pairing $\mathsf{Y} \times \mathsf{X} \to A$

$$(y \mid x)_A := yx$$

We say that an L^p-module (X, Y) is **C*-like** if the norms on X and Y can be recovered from the pairing as follows:

$$\|x\| = \sup_{\|y\|=1} \|(y \mid x)_A\| \quad \text{and} \quad \|y\| = \sup_{\|x\|=1} \|(y \mid x)_A\|.$$

L^p-modules: General Examples

Let A be any L^p-operator algebra. Then,

- (A, A) is an L^p-module over A. In general, (A, A) need not to be C*-like. However, if A has a contractive approximate identity (c.a.i), then (A, A) is C*-like.

Let $p \in (1, \infty)$ and let (Ω, μ) be a measure space. Then

- $(L^p(\mu), L^q(\mu))$ is a C*-like L^p-module over $\mathbb{C} = \mathcal{L}(\ell_1^p)$.
- $(L^q(\mu), L^p(\mu))$ is a C*-like L^p-module over $\mathcal{K}(L^p(\mu))$.

Let $p \in [1, \infty)$, and let $d \in \mathbb{Z}_{\geq 1}$. Then

- (ℓ_d^p, ℓ_d^q) is a C*-like L^p-module over $\mathbb{C}$.
- (ℓ_d^q, ℓ_d^p) is a C*-like L^p-module over $\mathcal{K}(\ell_d^p)$.

L^p-modules: "Adjointable maps"

Proposition (D, 2022)

Let A be a C-algebra and let X be a right Hilbert A-module isometrically represented on $(\mathcal{H}_0, \mathcal{H}_1)$. Suppose that $\overline{\mathsf{X}\mathcal{H}_0} = \mathcal{H}_1$. Then,*

$$\mathcal{L}_A(\mathsf{X}) \cong \{t \in \mathcal{L}(\mathcal{H}_1) \colon tx, t^*x \in \mathsf{X} \text{ for all } x \in \mathsf{X}\}$$

This result motivates the following L^p generalization: If $A \subseteq \mathcal{L}(L^p(\mu_0))$ and (X, Y) is an L^p-module over A with $\mathsf{X} \subseteq \mathcal{L}(L^p(\mu_0), L^p(\mu_1))$, we define

$$\mathcal{L}_A\big((\mathsf{X}, \mathsf{Y})\big) = \{t \in \mathcal{L}(L^p(\mu_1)) \colon tx \in \mathsf{X} \text{ and } yt \in \mathsf{Y} \text{ for all } x \in \mathsf{X}, y \in \mathsf{Y}\}.$$

- If A has a c.a.i and is nondegenerately represented on $L^p(\mu)$, then $\mathcal{L}_A\big((A, A)\big) \cong M(A)$,
- $\mathcal{L}_{\mathbb{C}}(L^p(\mu), L^q(\mu)) \cong \mathcal{L}(L^p(\mu))$,
- $\mathcal{L}_{\mathcal{K}(L^p(\mu))}\big((L^q(\mu), L^p(\mu))\big) \cong \mathbb{C}$.

L^p-modules: "Compact-module maps"

Proposition (D, 2022)

Let A be a C-algebra and let X be a right Hilbert A-module isometrically represented on $(\mathcal{H}_0, \mathcal{H}_1)$. Suppose that $\overline{\mathsf{X}\mathcal{H}_0} = \mathcal{H}_1$. Then,*

$$\mathcal{K}_A(\mathsf{X}) \cong \overline{\operatorname{span}\{xy^* : x, y \in \mathsf{X}\}} \subseteq \mathcal{L}(\mathcal{H}_1).$$

We now get the following L^p generalization: If $A \subseteq \mathcal{L}(L^p(\mu_0))$ and (X, Y) is an L^p-module over A with $\mathsf{X} \subseteq \mathcal{L}(L^p(\mu_0), L^p(\mu_1))$, we define

$$\mathcal{K}_A\big((\mathsf{X}, \mathsf{Y})\big) = \overline{\operatorname{span}\{xy : x \in \mathsf{X}, y \in \mathsf{Y}\}} \subseteq \mathcal{L}(L^p(\mu_1)).$$

$\mathcal{K}_A\big((\mathsf{X}, \mathsf{Y})\big)$ is a closed two sided ideal in $\mathcal{L}_A\big((\mathsf{X}, \mathsf{Y})\big)$.

- If A has a c.a.i, then $\mathcal{K}_A\big((A, A)\big) \cong A$,
- $\mathcal{K}_{\mathbb{C}}\big((L^p(\mu), L^q(\mu))\big) \cong \mathcal{K}(L^p(\mu))$,
- $\mathcal{K}_{\mathcal{K}(L^p(\mu))}\big((L^q(\mu), L^p(\mu))\big) \cong \mathbb{C}$.

L^p-modules: Direct Sum

Let $p \in [1, \infty)$, and let $(\mathsf{X}_1, \mathsf{Y}_1), \dots, (\mathsf{X}_d, \mathsf{Y}_d)$ be L^p-modules over a common L^p-operator algebra A, then $(\bigoplus_{j=1}^d \mathsf{X}_j, \bigoplus_{j=1}^d \mathsf{Y}_j)$ inherits natural p-operator norms that make it an L^p-module over A.

Now let $(\mathsf{X}_j, \mathsf{Y}_j)_{j=1}^\infty$ be a sequence of L^p-modules over $A \subseteq \mathcal{L}(L^p(\mu_0))$ with $\mathsf{X}_j \subseteq \mathcal{L}(L^p(\mu_0), L^p(\mu_j))$. We define $\bigoplus_{j=1}^\infty (\mathsf{X}_j, \mathsf{Y}_j)$ as the pair (X, Y) where

$$\mathsf{X} := \left\{ (x_j)_{j=1}^\infty \in \mathcal{L}\left(L^p(\mu_0), \bigoplus_{j=1}^\infty L^p(\mu_j) \right) : \lim_{n,m\to\infty} \sup_{\|\xi\|_p = 1} \sum_{j=n}^m \|x_j \xi\|_p^p = 0 \right\},$$

$$\mathsf{Y} := \left\{ (y_j)_{j=1}^\infty \in \mathcal{L}\left(\bigoplus_{j=1}^\infty L^p(\mu_j), L^p(\mu_0) \right) : \lim_{n,m\to\infty} \sup_{\sum_{j=1}^\infty \|\eta_j\|_p^p = 1} \left\| \sum_{j=n}^m y_j \eta_j \right\|_p = 0 \right\}.$$

L^p-modules: Direct Sum

$$\mathsf{X} := \left\{ (x_j)_{j=1}^{\infty} \in \mathcal{L}\left(L^p(\mu_0), \bigoplus_{j=1}^{\infty} L^p(\mu_j) \right) : \lim_{n,m\to\infty} \sup_{\|\xi\|_p=1} \sum_{j=n}^{m} \|x_j \xi\|_p^p = 0 \right\},$$

$$\mathsf{Y} := \left\{ (y_j)_{j=1}^{\infty} \in \mathcal{L}\left(\bigoplus_{j=1}^{\infty} L^p(\mu_j), L^p(\mu_0) \right) : \lim_{n,m\to\infty} \sup_{\sum_{j=1}^{\infty} \|\eta_j\|_p^p=1} \left\| \sum_{j=n}^{m} y_j \eta_j \right\|_p = 0 \right\}.$$

Theorem (D, 2023)

(X, Y) *is an* L^p*-module over* A *that agrees with the usual definition of direct sum of Hilbert modules when* $p = 2$*:*

$$\bigoplus_{j=1}^{\infty} \mathsf{X}_j := \left\{ (x_j)_{j=1}^{\infty} : x_j \in \mathsf{X}_j, \sum_{j=1}^{\infty} \langle x_j, x_j \rangle_A \ \text{ converges in } A \right\}.$$

L^p-modules: External Tensor Product

If (Ω,μ) is a measure space, $p\in[1,\infty)$, and E is a Banach space, there is a well behaved tensor product $\otimes_p$ satisfying

$$L^p(\mu)\otimes_p E\cong L^p(\mu,E)=\left\{g\in \mathrm{Mble}(\Omega\to E)\colon \int_\Omega \|g(\omega)\|^p d\mu(\omega)<\infty\right\}$$

In particular, $L^p(\mu)\otimes_p L^p(\nu)\cong L^p(\mu\times\nu)$. This *spatial tensor product* induces a well behaved tensor product on spaces of operators between L^p-spaces, which in turn induces an **external tensor product** on L^p-modules:

Definition

Let (X,Y) be an L^p-module over A and let (V,W) be an L^p-module over B. We define

$$(\mathsf{X},\mathsf{Y})\otimes_p(\mathsf{V},\mathsf{W})=(\mathsf{X}\otimes_p\mathsf{V},\mathsf{V}\otimes_p\mathsf{W}).$$

$(\mathsf{X}\otimes_p\mathsf{V},\mathsf{V}\otimes_p\mathsf{W})$ is an L^p-module over $A\otimes_p B$.

External Tensor Product

In the Hilbert module case, it is well known that

$$\ell^2 \otimes \mathsf{X} \cong \bigoplus_{j=1}^{\infty} \mathsf{X},$$

which gives rise to $\ell^2 \otimes A$, the standard Hilbert module of a C*-algebra A.

L^p-analogue

Let $p \in (1, \infty)$ and let (X, Y) be an L^p-module over A. Then,

$$(\ell^p, \ell^q) \otimes_p (\mathsf{X}, \mathsf{Y}) = \bigoplus_{j=1}^{\infty} (\mathsf{X}, \mathsf{Y}).$$

In particular, when $(\mathsf{X}, \mathsf{Y}) = (A, A)$, we get **the standard L^p-module** $(\ell^p, \ell^q) \otimes_p (A, A) = (\ell^p \otimes_p A, \ell^q \otimes_p A)$ of an L^p-operator algebra A.

External Tensor Product

For a C*-algebra A, it is well known that

$$\mathcal{K}_A(\ell^2 \otimes A) \cong \mathcal{K}(\ell^2) \otimes A \text{ and } \mathcal{L}_A(\ell^2 \otimes A) \cong M(\mathcal{K}(\ell^2) \otimes A).$$

More generally, it always holds that $\mathcal{L}_A(\mathsf{X}) \cong M(\mathcal{K}_A(\mathsf{X}))$.

L^p-analogue

Let A be an L^p-operator algebra. If A has a c.a.i., then

$$\mathcal{K}_A\big((\ell^p \otimes_p A, \ell^q \otimes_p A)\big) \cong \mathcal{K}(\ell^p) \otimes_p A.$$

If in addition A sits nondegenerately in $\mathcal{L}(L^p(\mu))$, then

$$\mathcal{L}_A\big((\ell^p \otimes_p A, \ell^q \otimes_p A)\big) \cong M(\mathcal{K}(\ell^p) \otimes_p A).$$

We have not investigated yet whether $\mathcal{L}_A\big((\mathsf{X},\mathsf{Y})\big) \cong M\big(\mathcal{K}_A\big((\mathsf{X},\mathsf{Y})\big)\big)$ for a general L^p-module (X,Y) over an L^p-operator algebra A.

Outline

Representations of C*-correspondences

Let (X, φ) be an (A, B) C*-correspondence and $(\mathcal{H}_0, \mathcal{H}_1)$ a pair of Hilbert spaces. A **representation of (X, φ) on $(\mathcal{H}_0, \mathcal{H}_1)$** consists of a triple $(\lambda_A, \rho_B, \pi_{\mathsf{X}})$ such that

1. λ_A is a representation of A on $\mathcal{H}_1$,
2. ρ_B is a representation of B on $\mathcal{H}_0$,
3. $\pi_{\mathsf{X}} : \mathsf{X} \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is a linear map,

satisfying, for any $a \in A$, $x, y \in \mathsf{X}$,

$$(i)\ \pi_{\mathsf{X}}(\varphi(a)x) = \lambda_A(a)\pi_{\mathsf{X}}(x), \qquad (ii)\ \pi_{\mathsf{X}}(x)^*\pi_{\mathsf{X}}(y) = \rho_B(\langle x, y\rangle_B).$$

Fact 1: If $x \in \mathsf{X}$ and $b \in B$, then (iii) $\pi_{\mathsf{X}}(xb) = \pi_{\mathsf{X}}(x)\rho_B(b)$.
Fact 2: If π_B is isometric, then so is π_{X}.

Theorem (D, 2022)

Given any (A, B) C-correspondence (X, φ) and ρ_B, a faithful representation of B on $\mathcal{H}_0$, there is a Hilbert space $\mathcal{H}_1$ and maps λ_A, π_{X} such that $(\lambda_A, \rho_B, \pi_{\mathsf{X}})$ is an isometric representations of (X, φ) on $(\mathcal{H}_0, \mathcal{H}_1)$.*

L^p-correspondences

Definition

Let $(\Omega_0, \mu_0), (\Omega_1, \mu_1)$ be measure spaces, let $p \in [1, \infty)$, let A be an L^p-operator algebra, and let $B \subseteq \mathcal{L}(L^p(\mu_0))$ be a concrete L^p operator algebra. An (A, B) **L^p-correspondence** is a pair $((\mathsf{X}, \mathsf{Y}), \varphi)$ where

- (X, Y) is an L^p module over B with $\mathsf{X} \subseteq \mathcal{L}(L^p(\mu_0), L^p(\mu_1))$,
- $\varphi \colon A \to \mathcal{L}_B((\mathsf{X}, \mathsf{Y}))$ is a contractive homomorphism.

When $A = B$ we say that $((\mathsf{X}, \mathsf{Y}), \varphi)$ is an L^p correspondence over A.

- Let A be an L^p-operator algebra and $\varphi_A \colon A \to A$ a contractive automorphism. Then, $\big((A, A), \varphi_A\big)$ is an L^p correspondence over A.
- Let (Ω, μ) be a measure space and let $p \in (1, \infty)$. For any $z \in \mathbb{C}$ define $\varphi_{\mathbb{C}}(z) := z \cdot \mathrm{id}_{L^p(\mu)}$. Then, $\big((L^p(\mu), L^q(\mu)), \varphi_{\mathbb{C}}\big)$ is an L^p correspondence over $\mathbb{C}$.

A **Fock representation** for (X, φ) is a triple $(B, \sigma_A, \pi_{\mathsf{X}})$ where B is a C*-algebra, $\sigma_A \colon A \to B$ a *-homomorphism, and $\pi_{\mathsf{X}} \colon \mathsf{X} \to B$ a linear map s.t.

1. $\pi_X(\varphi(a)x) = \sigma_A(a)\pi_{\mathsf{X}}(x)$,
2. $\sigma_A(\langle x, y\rangle_A) = \pi_X(x)^*\pi_X(y)$.

$C^*(B, \sigma_A, \pi_{\mathsf{X}})$ = C*-algebra generated by $\sigma_A(A)$ and $\pi_{\mathsf{X}}(\mathsf{X})$ in B.

L^p-analogue

An **L^p-Fock representation** *of* $((\mathsf{X}, \mathsf{Y}), \varphi)$ *consist of* $(B, \sigma_A, \pi_{\mathsf{X}}, \pi_{\mathsf{Y}})$ *where* B *is an* L^p*-operator algebra,* $\sigma_A \colon A \to B$ *a contractive homomorphism, and both* $\pi_{\mathsf{X}} \colon \mathsf{X} \to B$ *and* $\pi_{\mathsf{Y}} \colon \mathsf{Y} \to B$ *are contractive linear maps satisfying the following conditions:*

1. $\pi_{\mathsf{X}}(xa) = \pi_{\mathsf{X}}(x)\sigma_A(a)$, *and* $\pi_{\mathsf{X}}(\varphi(a)x) = \sigma_A(a)\pi_{\mathsf{X}}(x)$,
2. $\pi_Y(ay) = \sigma_A(a)\pi_{\mathsf{Y}}(y)$, *and* $\pi_Y(y\varphi(a)) = \pi_{\mathsf{Y}}(y)\sigma_A(a)$,
3. $\sigma_A((y \mid x)_A) = \pi_{\mathsf{Y}}(y)\pi_{\mathsf{X}}(x)$,

$F^p(B, \sigma_A, \pi_{\mathsf{X}}, \pi_{\mathsf{Y}})$ = *Banach algebra generated by* $\sigma_A(A)$, $\pi_{\mathsf{X}}(\mathsf{X})$ *and* $\pi_{\mathsf{Y}}(\mathsf{Y})$ *in* B.

Covariant Fock Representations

Lemma

*Let $(B, \sigma_A, \pi_{\mathsf{X}})$ be any Fock representation for (X, φ). Then, there is a *-homomorphism $\lambda_{\mathcal{K}} \colon \mathcal{K}_A(\mathsf{X}) \to B$ that satisfies $\lambda_{\mathcal{K}}(\theta_{x,y}) = \pi_{\mathsf{X}}(x)\pi_{\mathsf{X}}(y)^*$ for all $x, y \in \mathsf{X}$.*

For a C* correspondence (X, φ) over A, T. Katsura defined ideal J_{X} by

$$\begin{aligned} J_{\mathsf{X}} &= \varphi^{-1}(\mathcal{K}_A(\mathsf{X})) \cap (\ker(\varphi))^{\perp} \\ &= \{a \in A : \varphi(a) \in \mathcal{K}_A(\mathsf{X}) \text{ and } ab = 0 \text{ for all } b \in \ker(\varphi)\}. \end{aligned}$$

Definition

A Fock representation $(B, \sigma_A, \pi_{\mathsf{X}})$ for (X, φ) is said to be **covariant** if

$$\lambda_{\mathcal{K}}(\varphi(a)) = \sigma_A(a) \ \forall a \in J_{\mathsf{X}}.$$

Covariant L^p-Fock Representations

Let $((\mathsf{X},\mathsf{Y}),\varphi)$ be an L^p-correspondence over an L^p-operator algebra A. We define

$$J_{(\mathsf{X},\mathsf{Y})} = \{a \in A \colon \varphi(a) \in \mathcal{K}_A((\mathsf{X},\mathsf{Y})) \text{ and } ab = 0 \text{ for all } b \in \ker(\varphi)\}$$

Definition

Let $p \in [1,\infty)$, let A be an L^p-operator algebra, and let $(B,\sigma_A,\pi_{\mathsf{X}},\pi_{\mathsf{Y}})$ be an L^p-Fock representation for a L^p-correspondence $((\mathsf{X},\mathsf{Y}),\varphi)$ over A. We say $(B,\sigma_A,\pi_{\mathsf{X}},\pi_{\mathsf{Y}})$ is **covariant** if there is a contractive homomorphism $\lambda_{\mathcal{K}} \colon \mathcal{K}_A((\mathsf{X},\mathsf{Y})) \to B$ satisfying

$$\lambda_{\mathcal{K}}(\theta_{x,y}) = \pi_{\mathsf{X}}(x)\pi_{\mathsf{Y}}(y) \text{ for all } x \in \mathsf{X}, y \in \mathsf{Y},$$
$$\lambda_{\mathcal{K}}(\varphi(a)) = \sigma_A(a) \text{ for all } a \in J_{(\mathsf{X},\mathsf{Y})}.$$

Cuntz-Pimsner Algebras come from the Fock Space

Definition

Let A be a C*-algebra and let (X, φ) be a C^*-correspondence over A. We define $\mathcal{O}(\mathsf{X}, \varphi)$, the **Cuntz-Pimsner algebra of (X, φ)**, as the universal C*-algebra algebra generated by *covariant* Fock representations.

- $\mathcal{O}(\ell_d^2, \varphi_{\mathbb{C}}) \cong \mathcal{O}_d$,
- $\mathcal{O}(A, \varphi_A) \cong C^*(\mathbb{Z}, A, \varphi_A)$.

Let $\mathcal{F}(\mathsf{X}) = \bigoplus_{n=0}^{\infty} \mathsf{X}^{\otimes_\varphi n}$, let $c \colon \mathsf{X} \to \mathcal{L}_A(\mathcal{F}(\mathsf{X}))$ be given by $c(x)\kappa = x \otimes \kappa$, and let $\mathcal{Q}_A(\mathsf{X}) = \mathcal{L}_A(\mathcal{F}(\mathsf{X}))/\mathcal{K}_{J_\mathsf{X}}(\mathcal{F}(\mathsf{X})J_\mathsf{X})$.

Theorem (Pimsner-Katsura)

$\mathcal{O}(\mathsf{X}, \varphi)$ is the C^*-subalgebra in $\mathcal{Q}_A(\mathsf{X})$ generated by the images of $c(\mathsf{X})$ and by $\varphi^\infty(A)$ under the quotient map.

L^p-Cuntz Algebras

For the L^p-correspondence $\big((\ell_d^p, \ell_d^q), \varphi_{\mathbb{C}}\big)$, consider $\mathcal{F}^p := \bigoplus_{n=0}^{\infty} \ell_{d^n}^p$, $\mathcal{F}^q := \bigoplus_{n=0}^{\infty} \ell_{d^n}^q$, the creation operator $c\colon \ell_d^p \to \mathcal{L}(\mathcal{F}^p)$, and the annihilation operator $v\colon \ell_d^q \to \mathcal{L}(\mathcal{F}^p)$.

Theorem (D, 2023)

Let $\mathcal{Q} := \mathcal{L}(\mathcal{F}^p)/\mathcal{K}(\mathcal{F}^q)$ with quotient map $q_0\colon \mathcal{L}(\mathcal{F}^p) \to \mathcal{Q}$. Then $(\mathcal{Q}, q_0(\varphi_{\mathbb{C}}), q_0(c), q_0(v))$ is a covariant L^p-Fock representation for $\big((\ell_d^p, \ell_d^q), \varphi_{\mathbb{C}}\big)$ and $F^p(\mathcal{Q}, q_0(\varphi_{\mathbb{C}}), q_0(c), q_0(v))$ is isometrically isomorphic to $\mathcal{O}_d^p$, the L^p analogue of Cuntz algebra.

Question

Is $(\mathcal{Q}, q_0(\varphi_{\mathbb{C}}), q_0(c), q_0(v))$ the universal L^p-Fock covariant representation for $\big((\ell_d^p, \ell_d^q), \varphi_{\mathbb{C}}\big)$?

L^p-Crossed Products

Assume A is an L^p-operator algebra that has a bicontractive approximate identity and that sits nongenerately in $\mathcal{L}(L^p(\mu))$. For the L^p-correspondence $\big((A,A),\varphi_A\big)$, with φ_A isometric, consider $\mathcal{F}^p(A) := \ell^p \otimes_p A$, $\mathcal{F}^q := \ell^q \otimes_p A$, and $\mathcal{Q}_A := \mathcal{L}_A\big((\mathcal{F}^p(A),\mathcal{F}^q(A))\big)/\mathcal{K}_A\big((\mathcal{F}^p(A),\mathcal{F}^q(A))\big)$

Theorem (D, 2023)

$\mathcal{Q}_A$ is an L^p-operator algebra and there are isometric maps φ_A^∞, c_A, and v_A from A to $\mathcal{L}_A\big((\mathcal{F}^p(A),\mathcal{F}^q(A))\big)$ such that $(\mathcal{Q}_A, q_0(\varphi_A^\infty), q_0(c_A), q_0(v_A))$ is a covariant L^p-Fock representation for $\big((A,A),\varphi_A\big)$.

Question

Is $F^p(\mathcal{Q}_A, q_0(\varphi_A^\infty), q_0(c_A), q_0(v_A))$ isometrically isomorphic to $F^p(\mathbb{Z}, A, \varphi_A)$, the L^p-crossed product of A by $\mathbb{Z}$?

$\frac{1}{2}$**Ans:** There is $\gamma\colon F^p(\mathbb{Z}, A, \varphi_A) \to F^p(\mathcal{Q}_A, q_0(\varphi_A^\infty), q_0(c_A), q_0(v_A))$, a contractive algebra homomorphism.

Outline

Future Lines of work

- Independence of the representation,
- Algebras generated by the Fock spaces for other L^p-modules,
- Universality L^p-Fock representations coming from Fock spaces,
- C*-likeness of direct sums of L^p-modules,
- Bimodules and Morita equivalent L^p-operator algebras.

Thank you!

Questions?

L^p-modules:Direct Sum

If $\big((\mathsf{X}_j, \mathsf{Y}_j)\big)_{j=1}^{\infty}$ is a sequence of L^p-modules over A, a naive generalization of the finite case does not work. Say that for each $j \geq 1$, $\mathsf{X}_j \subseteq \mathcal{L}(L^p(\mu_0), L^p(\mu_j))$. Define

$$\mathsf{X}_{\mathrm{w}} = \left\{ (x_j)_{j=1}^{\infty} : x_j \in \mathsf{X}_j, \sup_{\|\xi\|_p = 1} \sum_{j=1}^{\infty} \|x_j \xi\|_p^p < \infty \right\},$$

$$\mathsf{Y}_{\mathrm{w}} = \left\{ (y_j)_{j=1}^{\infty} : y_j \in \mathsf{Y}_j, \sup_{\sum_{j=1}^{\infty} \|\eta_j\|_p^p = 1} \Big\| \sum_{j=1}^{\infty} y_j \eta_j \Big\|_p < \infty \right\}.$$

Then, in general $(\mathsf{X}_{\mathrm{w}}, \mathsf{Y}_{\mathrm{w}})$ is not an L^p-module over A.

L^p-modules:Direct Sum

$$\mathsf{X}_{\mathrm{w}} = \left\{ (x_j)_{j=1}^{\infty} \colon x_j \in \mathsf{X}_j,\ \sup_{\|\xi\|_p = 1} \sum_{j=1}^{\infty} \|x_j \xi\|_p^p < \infty \right\},$$

$$\mathsf{Y}_{\mathrm{w}} = \left\{ (y_j)_{j=1}^{\infty} \colon y_j \in \mathsf{Y}_j,\ \sup_{\sum_{j=1}^{\infty} \|\eta_j\|_p^p = 1} \Big\| \sum_{j=1}^{\infty} y_j \eta_j \Big\|_p < \infty \right\}.$$

Counterexample:

For any $p \in [1, \infty)$, let $(\mathsf{X}_j, \mathsf{Y}_j) = (\ell^q, \ell^p)$ for all $j \in \mathbb{Z}_{\geq 1}$. Define $x_j \colon \ell^p \to \mathbb{C}$ by $x_j \xi = \xi(j)$ and $y_j \colon \mathbb{C} \to \ell^p$ by $y_j \zeta = \zeta \delta_j$. Then,

- $(y_j \mid x_j)_{\mathcal{K}(\ell^p)} = y_j x_j = \theta_{\delta_j, \delta_j} \in \mathcal{K}(\ell^p)$,
- $(x_j)_{j=1}^{\infty} \in \mathsf{X}_{\mathrm{w}}$, $(y_j)_{j=1}^{\infty} \in \mathsf{Y}_{\mathrm{w}}$,
- $\sum_{j=1}^{\infty} \theta_{\delta_j, \delta_j}$ is not convergent in $\mathcal{K}(\ell^p)$.

L^p-modules:Direct Sum

Theorem (D, 2023)

$\bigoplus_{j=1}^{\infty}(\mathsf{X}_j, \mathsf{Y}_j)$ is an L^p-module over A that agrees with the usual definition of direct sum of Hilbert modules when $p = 2$. That is, if A is a C-algebra and X_j are right Hilbert A-modules, with*

$$\bigoplus_{j=1}^{\infty} \mathsf{X}_j := \left\{ (x_j)_{j=1}^{\infty} : x_j \in \mathsf{X}_j, \sum_{j=1}^{\infty} \langle x_j, x_j \rangle_A \ \text{ converges in } A \right\},$$

then

$$\bigoplus_{j=1}^{\infty} (\mathsf{X}_j, \widetilde{\mathsf{X}_j}) \cong \left(\bigoplus_{j=1}^{\infty} \mathsf{X}_j, \bigoplus_{j=1}^{\infty} \widetilde{\mathsf{X}_j} \right).$$

Spatial Partial Isometries

Let $p \in [1, \infty)$, let $(\Omega_0, \mathfrak{M}, \mu)$ and $(\Omega_1, \mathfrak{N}, \nu)$ be σ-finite measure spaces, and let $s : L^p(\mu) \to L^p(\nu)$ be a linear map.

We say s is a **spatial partial isometry** if there are $E \in \mathfrak{M}$, $F \in \mathfrak{N}$, $S : \mathfrak{M}|_E \to \mathfrak{N}|_F$ a bijective measurable set transformation with $\nu|_{\mathfrak{N}|_F}$ σ-finite, and $g : F \to \mathbb{C}$ a measurable function such that $|g| = 1$ a.e. $[\nu|_F]$ such that

$$s(\xi) = \begin{cases} \left(\frac{dS_*(\mu|_E)}{d\nu|_{\mathfrak{N}|_F}} \right)^{1/p} S_*(\xi|_E)g & \text{on } F \\ 0 & \text{on } \Omega_1 \setminus F \end{cases}$$

for all $\xi \in L^p(\mu)$. Turns out that

$$\|s(\xi)\|_p = \|\xi|_E\|_p.$$

The partial isometry implemented by $(F, E, S^{-1}, (S^{-1})_*(g)^{-1})$ is called the **reverse of s**.

L^p Analogues of $\mathcal{O}_d$

Let $d \in \mathbb{Z}_{\geq 2}$. The **Leavitt algebra** L_d is the universal unital complex algebra generated by elements $s_1, s_2, \ldots, s_d$, and $t_1, t_2, \ldots, t_d$ satisfying

$$t_j s_k = \delta_{j,k} \quad \text{and} \quad \sum_{j=1}^{d} s_j t_j = 1$$

Definition

Let $p \in [1, \infty)$ and let $(\Omega, \mathfrak{M}, \mu)$ be a measure space. A spatial representation of L_d is an algebra homomorphism $\varphi : L_d \to \mathcal{L}(L^p(\mu))$ such that each $\varphi(s_j)$ is a spatial partial isometry with reverse $\varphi(t_j)$.

Given any spatial representation φ of L_d, we define

$$\mathcal{O}_d^p := \overline{\varphi(L_d)} \subseteq \mathcal{L}(L^p(\mu)).$$

Known Facts about $\mathcal{O}_d^p$

Let $d \geq 2$ be an integer and let $p \in [1, \infty)$. Then

1. $\mathcal{O}_d^p$ is simple.
2. $\mathcal{O}_d^p$ is purely infinite
3. $\mathcal{O}_d^2 = \mathcal{O}_d$.
4. $\mathcal{O}_2^p \otimes_p \mathcal{O}_2^p \cong \mathcal{O}_2^p$ if and only if $p = 2$.
5. $\mathcal{O}_d^p$ has the same K-theory as when $p = 2$:

$$K_0(\mathcal{O}_d^p) \cong \mathbb{Z}/(d-1)\mathbb{Z} \text{ and } K_1(\mathcal{O}_d^p) \cong \{0\}.$$

6. $K_*^{\text{top}}(\mathcal{O}_d^p) \cong K_*^{\text{alg}}(\mathcal{O}_d^p)$