Fock space construction for Hilbert Modules.

Alonso Delfín
University of Oregon.

February 6, 2020

Abstract

In the mathematical formulation of quantum mechanics, quantum states correspond to vectors in a Hilbert space. The Fock space is a a Hilbert space constructed as a direct sum of tensor products of a given Hilbert space. This construction is important as it allows quantum physicists to make sense of the superposition of states and gives rise to creation and annihilation operators. The generalization of this space using Hilbert Modules gives a way to construct a general class of C^*-algebras that includes the Cuntz algebras and Crossed Products by $\mathbb{Z}$. In this short document, I will go over the Fock space construction and its generalization to Hilbert Modules.

1 Fock Space

In quantum mechanics, the quantum states of a quantum mechanical system are elements of a Hilbert space $\mathcal{H}$. For simplicity, one might think that $\mathcal{H}$ explains a one-particle system. Then, the tensor product $\mathcal{H}^{\otimes 2} := \mathcal{H} \otimes \mathcal{H}$ will represent the states of two non-interacting particles of the same type as the one described by $\mathcal{H}$. More generally, for $n \geq 1$,

$$\mathcal{H}^{\otimes n} := \underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{n\text{times}}$$

represents the states of a collection of n particles of the same type. By convention we set $\mathcal{H}^0 := \mathbb{C}$. Then, we can form the direct sum

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n \geq 0} \mathcal{H}^{\otimes n}$$

The Hilbert space $\mathcal{F}(\mathcal{H})$ is known as the **Fock space** of $\mathcal{H}$. This construction allows quantum physicists to superpose states: if ξ, η and ζ are elements of $\mathcal{H}$, it makes sense to talk about expressions like

$$\xi + \eta \otimes \zeta,$$

This in turn, permits the description of states on which also the number of particles is uncertain and becomes an observable with probabilities and mean values as any other observable.

Now comes the more "mathematical" part of this. For a fixed element $\xi \in \mathcal{H}$, we can define a map c_ξ from $\mathcal{H} \to \mathcal{H}^{\otimes 2}$ by

$$c_\xi(\eta) := \xi \otimes \eta$$

This is a linear map. Using the Physicist's convention of linearity in second coordinate for the inner product, we get,

$$\langle c_\xi(\eta), \zeta_1 \otimes \zeta_2 \rangle = \langle \xi \otimes \eta, \zeta_1 \otimes \zeta_2 \rangle = \langle \xi, \zeta_1 \rangle \langle \eta, \zeta_2 \rangle = \langle \eta, \langle \xi, \zeta_1 \rangle \zeta_2 \rangle$$

Thus, $c_\xi^*(\zeta_1 \otimes \zeta_2) = \langle \xi, \zeta_1 \rangle \zeta_2$. The map c_ξ is a *creation* operator and its adjoint, c_ξ^*, is an *annihilation* operator. This extends to the Folk space $\mathcal{F}(\mathcal{H})$ by letting $c_\xi(\mu) := \xi \otimes \mu$ for any pure tensor $\mu \in \mathcal{H}^{\otimes n}$.

In quantum mechanics, the creation and annihilation operators are useful as they *intertwine* the different $\mathcal{H}^{\otimes n}$'s.

Now, let $d \geq 2$ and let $\mathcal{H}$ be a Hilbert space with dimension d. We will see that, after taking some quotients, the C^* algebra generated by all the creation operators c_ξ for $\xi \in \mathcal{H}$ is in fact $\mathcal{O}_d$, the usual Cuntz algebra.

2 Hilbert Modules

Let A be a C^*-algebra.

Our goal is to generalize the Folk space construction starting with a Hilbert A-module E instead of a Hilbert space $\mathcal{H}$. As before, we will get creation operators c_ξ for each $\xi \in E$. Roughly speaking, we will be interested in the C^* algebra generated by all the c_ξ.

We now briefly recall some definitions about Hilbert Modules and we then explain the interior tensor product construction between two Hilbert Modules.

Definition 2.1. A Hilbert A-module E is a right A-module together with a pairing $\langle \cdot, \cdot \rangle : E \times E \to A$ such that

1. For each $\eta \in E$, the map $\langle \xi, \cdot \rangle : E \to A$ is linear,
2. $\langle \xi, \eta a \rangle = \langle \xi, \eta \rangle a$ for any $\xi, \eta \in E$ and $a \in A$,
3. $\langle \xi, \eta \rangle = \langle \eta, \xi \rangle^*$ for any $\xi, \eta \in E$,
4. $\langle \xi, \xi \rangle \geq 0$ in A for any $\xi \in E$ and if $\langle \xi, \xi \rangle = 0$, then $\xi = 0$.
5. E is complete with the norm $\|\xi\| := \|\langle \xi, \xi \rangle\|^{1/2}$.

The pairing $\langle \cdot, \cdot \rangle : E \times E \to A$ satisfying 1-4 above is referred to as an "A-valued inner product".

Notice that Hilbert spaces are precisely Hilbert $\mathbb{C}$-modules, with the Physicist's convention of linearity in second coordinate for the inner product. Thus, when taking an arbitrary C^*-algebra A, the Hilbert A-modules are a good generalization of Hilbert spaces. However, many nice properties of Hilbert spaces, such as complementability of subspaces, are not guaranteed for general Hilbert A-modules. Nevertheless, they provide a good tool to study the C^*-algebra A. For example, one can visualize the multiplier algebra of A using some kind of operators between Hilbert A-modules, the adjointable ones (see the definition below). Also, there is an alternate description of $K_0(A)$ using isomorphism classes of finitely generated projective A-modules.

Definition 2.2. Let E and F be a Hilbert A-modules. A map $t : E \to F$ is said to be adjointable if there is a map $t^* : F \to E$ such that for any $\xi \in E$, and $\eta \in F$

$$\langle t(\xi), \eta \rangle = \langle \xi, t^*(\eta) \rangle$$

The space of adjointable maps from E to F is denoted by $\mathcal{L}(E, F)$ and $\mathcal{L}(E) := \mathcal{L}(E, E)$.

It's easy to check that adjointable maps are bounded linear maps with the usual operator norm and that $\mathcal{L}(E)$ is a C^*-algebra.

Definition 2.3. Let E and F be a Hilbert A-modules. For $\xi \in E$ and $\eta \in F$, we define a map $\theta_{\xi,\eta} : F \to E$ by

$$\theta_{\xi,\eta}(\zeta) := \xi \langle \eta, \zeta \rangle$$

One easily checks that $\theta_{\xi,\eta} \in \mathcal{L}(E,F)$ and that $(\theta_{\xi,\eta})^* = \theta_{\eta,\xi} \in \mathcal{L}(F,E)$. This gives an analogous of the class of rank-one operators on Hilber spaces. So, we define an analogous of the compact operators by letting

$$\mathcal{K}(E,F) := \overline{\operatorname{span}\{\theta_{\xi,\eta} : \xi \in E, \eta \in F\}}$$

It's also not hard to verify that $\mathcal{K}(E) := \mathcal{K}(E,E)$ is a two sided ideal in $\mathcal{L}(E)$. We have to be careful and not call these maps compact operators, in fact they do not have to be compact as maps between the two Banach spaces E and F. For example, if A is an infinite dimensional unital C^* algebra and $E = F = A$ with inner product given by a^*b, then $\mathrm{id}_A = \theta_{1,1} \in \mathcal{K}(A)$ is not a compact operator.

We have the usual direct sum constructions to get new Hilbert A-modules from old. Indeed, if $E_1, \ldots, E_n$ are Hilbert A modules, the direct sum

$$\bigoplus_{n=1}^{k} E_k := \{\xi = (\xi_1, \ldots, \xi_n) : \xi_k \in E_k\}$$

is again a Hilbert A-module with the component-wise right action of A and A-valued inner product

$$\langle \xi, \eta \rangle := \sum_{k=1}^{n} \langle \xi_k, \eta_k \rangle$$

If now $(E_\lambda)_{\lambda\in\Lambda}$ is an arbitrary family of Hilbert A-modules, we can form their direct sum

$$\bigoplus_{\lambda\in\Lambda} E_\lambda := \left\{ \xi = (\xi_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} E_\lambda : \sum_{\lambda\in\Lambda} \langle \xi_\lambda, \xi_\lambda \rangle \text{ converges in } A \right\}$$

which is a right A-module with coordinate-wise action and it becomes a Hilbert A-module when equipped with the well defined A-valued inner product

$$\langle \xi, \eta \rangle := \sum_{\lambda\in\Lambda} \langle \xi_\lambda, \eta_\lambda \rangle$$

We will also need to produce new Hilbert modules via tensor products. There are two different ways to talk about the tensor product of Hilbert modules. The first one is the exterior tensor product, which is a construction that starts with a Hilbert A-module, a Hilbert B-module and produces a Hilbert $A\otimes B$-module, where $A \otimes B$ is the tensor product of C^*-algebras with the spatial norm. This one is not useful to us. The second one, known as the interior tensor product, allows us to start with two Hilbert A-modules together with some extra structure and will produce another A-Hilbert module.

We now explain the construction of the interior tensor product on its full generality. Let A and B be C^*-algebras. Suppose E is a Hilbert B-module, that F is a Hilbert A-module and that there is a $*$-homomorphism $\phi : B \to \mathcal{L}(F)$. This naturally makes F a left B-module with the action induced by ϕ. We can then form the algebraic tensor product of E and F over B, denoted by $E \odot_B F$. To do so, we start with the algebraic tensor product $E \odot F$ and take the quotient by the subspace generated by

$$\{\xi b \otimes \eta - \xi \otimes \phi(b)\eta : \xi \in E, \eta \in F, b \in B\}$$

This quotient is $E \odot_B F$. We abuse notation and call the image of $\xi \otimes \eta$ in $E \odot_B F$ also by $\xi \otimes \eta$. Then, $E \odot_B F$ is a right A-module with an action defined by

$$(\xi \otimes \eta)a = \xi \otimes (\eta a)$$

We now define an A-valued inner product on $E \odot_B F$. First we put

$$\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle := \langle \eta, \phi(\langle \xi, \xi' \rangle)\eta' \rangle$$

for any $\xi, \xi' \in E$ and $\eta, \eta' \in F$. One checks that this is indeed a well defined A-valued inner product on $E \odot_B F$, so to get a Hilbert A-module we complete $E \odot_B F$ with respect to the norm induced by this inner product. We denote the completion $E \otimes_\phi F$ and we call it the interior tensor product of E and F by ϕ.

3 Fock space using Hilbert Modules

To generalize the Fock space construction to Hilbert modules instead of Hilbert spaces, we are interested in using the interior tensor product construction described above in the case where $A = B$ so that with two Hilbert A-modules E and F together with a $*$-homomorphism $\phi : A \to \mathcal{L}(F)$, we can form a new Hilbert A-module $E \otimes_\phi F$.

Let E be a Hilbert A-module and $\phi : A \to \mathcal{L}(E)$ a $*$-homomorphism, we can form a new Hilbert A-module

$$E^{\otimes 2} := E \otimes_\phi E$$

Next, to talk about $E^{\otimes 3}$, one should except associativity of the interior tensor product. This is indeed the case, as one can check that

$$(E \otimes_\phi E) \otimes_\phi E = E \otimes_\phi (E \otimes_\phi E)$$

where for the right hand side construction we have that ϕ induces a $*$-homomorphism, also labeled as ϕ, from A to $\mathcal{L}(E \otimes_\phi E)$ given by

$$\phi(a)(\xi \otimes \eta) := (\phi(a)\xi) \otimes \eta$$

Therefore, for any $n \geq 1$, it makes sense to define a Hilbert A-module

$$E^{\otimes n} = \underbrace{E \otimes_\phi \cdots \otimes_\phi E}_{n \text{ times}}$$

By convention we put $E^{\otimes 0} = A$. We can now construct the Fock space type by letting

$$\mathcal{F}(E) := \bigoplus_{n \geq 0} E^{\otimes n}$$

We are taking the direct sum of Hilbert modules above, so $\mathcal{F}(E)$ is again a Hilbert A-module. Exactly as we did with the Hilbert space case, for each $\xi \in E$ we have a creation operator $c_\xi \in \mathcal{L}(\mathcal{F}(E))$ so that if $\mu \in E^{\otimes n}$ is an elementary tensor, then

$$c_\xi(\mu) := \xi \otimes \mu \in E^{\otimes(n+1)}$$

A routine computation gives that the adjoint c_ξ^* acts on an elementary tensor $\nu = \nu_1 \otimes \cdots \otimes \nu_n \in E^{\otimes n}$ as

$$c_\xi^*(\nu) = (\phi(\langle \xi, \nu_1 \rangle)\nu_2) \otimes \nu_3 \otimes \ldots \otimes \nu_n \in E^{\otimes(n-1)}$$

It's now easy to check that if we regard each c_ξ as a map $E^{\otimes n} \to E^{\otimes(n+1)}$, then

$$c_\xi^* c_\eta = \phi(\langle \xi, \eta \rangle) \in \mathcal{L}(E^{\otimes n}), \tag{1}$$

and

$$c_\eta c_\xi^* = \theta_{\eta,\xi} \in \mathcal{L}(E^{\otimes(n+1)}), \tag{2}$$

where, as we did with $\phi(a)$ before, $\theta_{\eta,\xi}$ only acts on the first element of a tensor. Similarly, we also have

$$c_{\eta_1} \cdots c_{\eta_n} c_{\xi_1}^* \cdots c_{\xi_n}^* = \theta_{\eta_1 \otimes \cdots \otimes \eta_n, \xi_1 \otimes \cdots \otimes \xi_n} \in \mathcal{L}(E^{\otimes n}). \tag{3}$$

Now, for each $m \geq 0$, we can also define a finite version of the Fock sapce by

$$\mathcal{F}^m(E) := \bigoplus_{n=0}^{m} E^{\otimes n}$$

This is again a Hilbert A-module and we denote by $J(E)$ to the C^*-algebra in $\mathcal{L}(\mathcal{F}(E))$ generated by

$$\{\mathcal{L}(\mathcal{F}^m(E)) : m \geq 0\}$$

If we set

$$M(E) := \{t \in \mathcal{L}(\mathcal{F}(E)) : tJ(E) \subset J(E), J(E)t \subset J(E)\}$$

one checks that $M(E)$ is the multiplier algebra of $J(E)$. Notice that for each $\xi \in E$, we have $c_\xi \in M(E)$. Finally, we denote by s_ξ to the class of c_ξ in the quotient algebra $M(E)/J(E)$. This gives the setting for the following definition.

Definition 3.1. The **Cuntz-Krieger algebra** of a pair (E, ϕ) (where E is a Hilbert A-module and the map $\phi : A \to \mathcal{L}(E)$ is an isometric $*$-homomorphism), denoted by $\mathcal{O}_E$, is given by the C^*-algebra generated in $M(E)/J(E)$ by all the operators s_ξ, with $\xi \in E$. Similarly, the Toeplitz algebra of (E, ϕ), denoted by $\mathcal{T}_E$, is the C^*-algebra generated in $\mathcal{L}(\mathcal{F}(E))$ by all the operators c_ξ, with $\xi \in E$.

Even though requiring ϕ to be isometric is not used at any step in the constructions we gave above, this is done for simplicity. In fact, this gives

$$\sup_{\|\eta\|=1} \|\xi \otimes \eta\|^2 = \sup_{\|\eta\|=1} \|\langle \xi \otimes \eta, \xi \otimes \eta \rangle\| = \sup_{\|\eta\|=1} \|\langle \eta, \phi(\langle \xi, \xi \rangle)\eta \rangle\| = \|\phi(\langle \xi, \xi \rangle)\| = \|\langle \xi, \xi \rangle\| = \|\xi\|^2$$

which forces the norm of c_ξ to be equal to $\|\xi\|$.

To look at some properties of the algebras $\mathcal{O}_E$ and $\mathcal{T}_E$ we just defined, is useful to identify both A and $\mathcal{L}(E)$ with their images in $M(E)/J(E)$, given by

$$a(\xi_1 \otimes \ldots \otimes \xi_n) = (\phi(a)\xi_1) \otimes \ldots \otimes \xi_n$$

for $a \in A$ and

$$t(\xi_1 \otimes \ldots \otimes \xi_n) = t(\xi_1) \otimes \ldots \otimes \xi_n$$

for $t \in \mathcal{L}(E)$. More generally, we identify $\mathcal{L}(E^{\otimes k})$ for $k \geq 1$ with its image in $M(E)/J(E)$ given by

$$t(\xi_1 \otimes \ldots \otimes \xi_k \otimes \ldots \otimes \xi_n) = t(\xi_1 \otimes \ldots \otimes \xi_k) \otimes \ldots \otimes \xi_n$$

for each $t \in \mathcal{L}(E^{\otimes k})$. With this in mind and using (1), (2) and (3) above, it's easy to check the following relations for the elements in $\mathcal{O}_E$:

- $s_\xi^* s_\eta = \langle \xi, \eta \rangle \in A$,
- $s_\eta s_\xi^* = \theta_{\eta,\xi} \in \mathcal{K}(E) \subset \mathcal{L}(E)$,
- $s_{\eta_1} \cdots s_{\eta_n} s_{\xi_1}^* \cdots s_{\xi_n}^* = \theta_{\eta_1 \otimes \cdots \otimes \eta_n, \xi_1 \otimes \cdots \otimes \xi_n} \in \mathcal{K}(E^{\otimes n}) \subset \mathcal{L}(E^{\otimes n})$,
- $s_\xi a = s_{\xi a}$ and $a s_\xi = s_{\phi(a)\xi}$ for any $a \in A$,
- $t s_\xi = s_{t(\xi)}$ for any $t \in \mathcal{L}(E)$.

Example 3.2. Let $d \geq 2$ and regard $E = \mathbb{C}^d$ as a Hilbert space, i.e. a Hilbert $\mathbb{C}$-module. If $(\xi_k)_{k=1}^d$ is an orthonormal basis for E, we have that $\mathcal{O}_E$ is generated by $(s_{\xi_k})_{k=1}^d$. We clearly have that $s_{\xi_j}^* s_{\xi_k} = \delta_{j,k} \in \mathbb{C}$ and moreover, one checks that

$$\sum_{k=1}^d s_{\xi_k} s_{\xi_k}^* = \mathrm{id}_E$$

This is a good indicator that $\mathcal{O}_E$ is in fact the usual Cuntz algebra $\mathcal{O}_d$. Indeed, by the universal property of $\mathcal{O}_d$, we get a surjective $*$-homomorphism $\Phi : \mathcal{O}_d \to \mathcal{O}_E$. Since $\mathcal{O}_d$ is simple, Φ is also injective.

Department of Mathematics, University of Oregon, Eugene OR 97403-1222, USA.
E-mail address: `alonsod@uoregon.edu`