Filters in Analysis and Topology.

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Abstract

Tychonoff's theorem in point set topology states that an arbitrary product of compact spaces is compact in the product topology. This theorem is of great use to analysts. However, I've found that usually, when taking an analysis course, the proof of this theorem is left to a topology course and vice versa. Hence, it's not a surprise that many students have never seen a proof of this "deep" theorem.

In this talk, I'll introduce the concept of filters and ultrafilters. Then, I'll use basic facts of the latter ones to give a very simple and short proof of Tychonoff's theorem. I'll also sketch how to use filters to construct the Stone-Čech compactification of a set X. In a categorical sense this compactification can be seen as the free compact Hausdorff space on X.

1 Definitions, Examples and Facts

Let X be a set. Informally speaking, a filter on X is a congruent choice of which subsets of X are *large.* More precisely:

Definition 1.1. A **filter** on a set X is a family of subsets, $\mathcal{F} \subseteq \mathcal{P}(X)$, such that

1. $X \in \mathcal{F}$ (that is, the whole set X is large)
2. $\varnothing \notin \mathcal{F}$ (that is, the empty set is not large)
3. If $E \in \mathcal{F}$ and $E \subseteq F$, then $F \in \mathcal{F}$ (any set containing a large set is large)
4. If $E \in \mathcal{F}$ and $F \in F$, then $E \cap F \in \mathcal{F}$ (large sets have large intersections)

We now give several examples.

Example 1.2. Let X be a non-empty set. We describe three filters on X:

i. $\{X\}$ is clearly a filter on X, usually known as the *trivial filter.*

ii. Fix $x \in X$. Define $\mathcal{F}_x := \{A \subseteq X : x \in A\}$. It's easy to check that $\mathcal{F}_x$ is a filter, called *the principal filter generated by x.*

iii. Suppose in addition that X is infinite. Define $\mathcal{F} := \{A \subset X : X \setminus A \text{ is finite }\}$. We now show this is actually a filter by checking 1-4 on definition 1.1:

   1. $X \setminus X$ is clearly a finite set, whence $X \in \mathcal{F}$.
   2. $X \setminus \varnothing$ is infinite and therefore $\varnothing \notin \mathcal{F}$
   3. Assume that $E \in \mathcal{F}$ is such that $E \subseteq F$. Then, $X \setminus F$ is finite because its contained in the finite set $X \setminus E$. Hence, $F \in \mathcal{F}$.

4. Take $E \in \mathcal{F}$ and $F \in \mathcal{F}$. Then, $X \setminus (E \cap F) = (X \setminus E) \cup (X \setminus F)$ is finite, being the union of two finite sets. Therefore, $E \cap F \in \mathcal{F}$.

The filter $\mathcal{F}$ is known as *the cofinite filter on X*.

Definition 1.3. Let X be a set and $\mathcal{C} \subseteq \mathcal{P}(X)$. We say that $\mathcal{C}$ has the **finite intersection property**, henceforth known as FIP, if whenever $E_1, \ldots, E_n \in \mathcal{C}$, then $\bigcap_{k=1}^n E_k \neq \varnothing$.

Remark 1.4. Any filter $\mathcal{F}$ on a set X has the FIP. Indeed, filters are closed under finite intersections and the empty set is never an element of a filter.

Lemma 1.5. *Suppose $\mathcal{C} \subseteq \mathcal{P}(X)$ has the FIP. Then, there is a minimal filter that contains $\mathcal{C}$.*

Proof. We define

$$\mathcal{F} := \left\{ A \in \mathcal{P}(X) : \ \exists\ n \in \mathbb{Z}_{>0}, E_1, \ldots, E_n \in \mathcal{C} \text{ s.t.} \bigcap_{k=1}^{n} E_k \subseteq A \right\}$$

Using the FIP, is not hard to check that $\mathcal{F}$ is indeed a filter. Clearly $\mathcal{C} \subseteq \mathcal{F}$. We only need to check minimality. Let $\mathcal{G}$ be a filter with $\mathcal{C} \subseteq \mathcal{F}$. Take any $A \in \mathcal{F}$, so that there are $E_1, \ldots, E_n \in \mathcal{C}$ whose intersection, call it E, is so that $E \subseteq A$. Since each E_k is in $\mathcal{C} \subseteq \mathcal{G}$ and $\mathcal{G}$ is a filter it follows that $E \in \mathcal{G}$. Hence, $A \in \mathcal{G}$ (again because $\mathcal{G}$ is a filter), whence $\mathcal{F} \subseteq \mathcal{G}$. ■

Definition 1.6. If $\mathcal{C} \subseteq \mathcal{P}(X)$ has the FIP, the filter gotten from the previous lemma is called **the filter generated by** $\mathcal{C}$.

Definition 1.7. A filter $\mathcal{F}$ on X is said to be an **ultrafilter** if for all $A \subset X$, either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.

Informally speaking, an ultrafilter on X says that any subset is either large or colarge. Equilalently, ultrafilters are characterized by the fact that large sets are not the finite union of small sets. This equivalence is made more precise in the following remark:

Remark 1.8. A filter $\mathcal{F}$ on X is an ultrafilter if and only if whenever $E_1, \ldots, E_n \subset X$ are such that $\bigcup_{k=1}^n E_k \in \mathcal{F}$, it follows that $E_k \in \mathcal{F}$ for at least one k.

Proof. Suppose first that $\mathcal{F}$ is an ultrafilter and that $\bigcup_{k=1}^n E_k \in \mathcal{F}$. If $E_k \notin \mathcal{F}$ for all k, then $X \setminus E_k \in \mathcal{F}$ for all k and therefore $X \setminus (\bigcup_{k=1}^n E_k) \in \mathcal{F}$, a contradiction. Conversely, assume that $\mathcal{F}$ is a filter on which $\bigcup_{k=1}^n E_k \in \mathcal{F}$ implies that $E_k \in \mathcal{F}$ for at least one k. If $A \subset X$, then $A \cup (X \setminus A) = X \in \mathcal{F}$ and therefore either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$. This finishes the proof. ■

Example 1.9. The principal filter generated by a fixed $x \in X$ is an ultrafilter.

Lemma 1.10. *Every filter $\mathcal{F}$ on a set X is contained in an ultrafilter.*

Proof. Denote by $\mathfrak{F}$ to the family of filters containing $\mathcal{F}$ and order it by inclusion. A routine verification gives that

$$\bigcup_{\mathcal{G} \in \mathfrak{F}} \mathcal{G}$$

is a filter containing $\mathcal{F}$. Suffices to apply Zorn's lemma to $\mathfrak{F}$ and notice that the maximal filter is in fact an ultrafilter. ■

2 Generalized Limits

Let's briefly recall how convergence of a sequence works on a topological space M.

Example 2.1. A sequence $(a_n)_{n\in\mathbb{Z}_{>0}}$ in M is actually function $f : \mathbb{Z}_{>0} \to M$ where

$$a_n := f(n)$$

We say that $a_n \to a \in M$ (i.e. $\lim_{n\to\infty} f(n) = a$) if for every open set $U \subset M$ with $a \in U$, there is $N \in \mathbb{Z}_{>0}$ such that $f(n) \in U$ for all $n \geq N$. This means that all but finitely many natural numbers are mapped to U by f, that is $\mathbb{Z}_{>0} \setminus f^{-1}(U)$ is a finite set. In other words, if $\mathcal{F}$ denotes the cofinite filter on $\mathbb{Z}_{>0}$, then $f(n) \to a$ means that $f^{-1}(U) \in \mathcal{F}$ for any open set U containing a.

Given a filter $\mathcal{F}$ we will generalize the concept of convergence, keeping in mind the usual definition for the convergence of a sequence. First we see an example that shows why ultrafilters are important for this generalization

Example 2.2. Let M be a topological space and $a \neq b$. Suppose further that a and b are separated by open sets, that is there are open sets U and V with $a \in U$, $b \in V$ such that $U \cap V = \varnothing$. Consider the sequence $(a, b, a, b, a, b, a, b, \ldots)$. This sequence does not converge in the usual sense, which is the one that comes from the cofinite filter on $\mathbb{Z}_{>0}$. This is because $f^{-1}(U)$ and $f^{-1}(V)$ are both infinite and coinfinite subsets of $\mathbb{Z}_{>0}$. However, since $f^{-1}(U) \cap f^{-1}(V) = \varnothing$, both $^{-1}(U)$ and $^{-1}(V)$ can't be in the same filter, moreover an ultrafilter on $\mathbb{Z}_{>0}$ will give preference to either $f^{-1}(U)$ or $f^{-1}(V)$, therefore deciding whether a or b should be the limit of the sequence.

Convergence of filters will not be limited to sequences. Given a(n) (ultra)filter $\mathcal{F}$ on a set X and a map of sets $f : X \to Y$, we need to push forward the (ultra)filter $\mathcal{F}$ to get a(n) (ultra)filter on Y. This is done in the obvious way by letting

$$f_*\mathcal{F} := \{B \subseteq Y : f^{-1}(B) \in \mathcal{F}\}$$

A routine verification shows that $f_*\mathcal{F}$ is a(n) (ultra)filter on Y.

Definition 2.3. Let M be a topological space and $\mathcal{G}$ a filter on M. We say that $\mathcal{G}$ **converges to** $a \in M$ if for every open set $U \subseteq M$ with $a \in U$, we have $U \in \mathcal{G}$.

Definition 2.4. Let X be a set, $\mathcal{F}$ be a filter on X, M be a topological space and $f : X \to M$ a map of sets. We say that $a \in M$ is an **$\mathcal{F}$-limit point of** f if the filter $f_*\mathcal{F}$ converges to a.

Remark 2.5. Suppose $f : \mathbb{Z}_{>0} \to M$ is a sequence and $\mathcal{F}$ is the cofinite filter on $\mathbb{Z}_{>0}$. Then, the sequence converges to a if and only if a is an $\mathcal{F}$-limit point of f.

We are now ready to characterize compact and Hausdorff spaces using ultrafilters.

Theorem 2.6. *A topological space M is compact if and only if every ultrafilter on M converges to at least one point.*

Proof. Suppose first that M is compact but that there is an ultrafilter $\mathcal{F}$ that does not converge to any point. For each $a \in M$ there is an open set $U_a \subseteq M$ with $a \in U_a$ and $U_a \notin \mathcal{F}$. Clearly $\{U_a\}_{a\in M}$ is an open cover of M and therefore, by compactness, there is a finite subcover $\{U_{a_k}\}_{k=1}^n$. Thus, since $\bigcup_k U_{a_k} = M$ is in the ultra filter $\mathcal{F}$, it follows from Remark 1.8 that $U_{a_k} \in \mathcal{F}$ for at least one k, a contradiction.

For the converse, assume that M is not compact. We show that there is an ultrafilter that doesn't converge. There is an open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of M that doesn't have a finite subcover. We claim that the family $\mathcal{C} := \{M \setminus U_\lambda\}_{\lambda\in\Lambda}$ has the FIP. Indeed, otherwise we could find a finite subset $L \subset \Lambda$ such that

$$\varnothing = \bigcap_{\lambda\in L} (M \setminus U_\lambda) = M \setminus \bigcup_{\lambda\in L} U_\lambda,$$

which in turn means that $\{U_\lambda\}_{\lambda\in L}$ is a finite subcover of M, this is a contradiction and the claim is proved. Hence, by Lemmas 1.5 and 1.10 there is an ultrafilter $\mathcal{F}$ with $\mathcal{C} \subseteq \mathcal{F}$. To show that $\mathcal{F}$ does not converge, assume on the contrary that it converges to a point $a \in M$. There is $\ell \in \Lambda$ such that $a \in U_\ell$ and therefore $U_\lambda \in \mathcal{F}$. Since $\mathcal{F}$ is an ultrafilter, then $M \setminus U_\ell \notin \mathcal{F}$, but $M \setminus U_\ell \in \mathcal{C} \subseteq \mathcal{F}$, a contradiction. ∎

Theorem 2.7. *A topological space M is Hausdorff if and only if every ultrafilter on M converges to at most one point.*

Proof. First suppose that M is Hausdorff but that there is an ultrafilter $\mathcal{F}$ that converges to both $a \in M$ and $b \in M$ with $a \neq b$. Then, there are open sets $U, V \subset M$ such that $a \in U$, $b \in V$ and $U \cap V = \varnothing$. Convergence of $\mathcal{F}$ to both a and b implies $U \in \mathcal{F}$ and $V \in \mathcal{F}$ and therefore $\varnothing = U \cap V \in \mathcal{F}$, a contradiction.

Now assume that M is not Hausdorff. Suffices to exhibit an ultrafilter that converges to two different points in M. Indeed, since M is not Hausdorff there are $a, b \in M$ with $a \neq b$ but such that any open neighborhood of a intersected with any open neighborhood of b. Let $\mathcal{C}_a$ denote all the open neighborhoods of a and define $\mathcal{C}_b$ analogously. Then, $\mathcal{C} := \mathcal{C}_a \cap \mathcal{C}_b$ has the FIP by construction. Thus, by Lemmas 1.5 and 1.10 there is an ultrafilter $\mathcal{F}$ with $\mathcal{C} \subseteq \mathcal{F}$. The ultrafilter $\mathcal{F}$ clearly converges to both a and b, we are done. ∎

Corollary 2.8. *Let X be a set, M a compact Hausdorff space and $f : X \to M$ a map of sets. Fix $x \in X$ and consider $\mathcal{F}_x$ the principal filter generated by x. Then f has a unique $\mathcal{F}$-limit point given by $f(x)$.*

Proof. Since $\mathcal{F}_x$ is an ultrafilter on X, it follows that $f_*\mathcal{F}_x$ is an ultrafilter on the compact Hausdorff space M and therefore it converges to exactly one point. Pick any open set $U \subset M$ such that $f(x) \in U$. We only need to show that $U \in f_*\mathcal{F}_x$. Well, since $x \in f^{-1}(U)$ it follows that $f^{-1}(U) \in \mathcal{F}_x$ and we are done. ∎

3 Tychonoff's Theorem

We start by recalling the product topology. Let $(M_\lambda)_{\lambda\in\Lambda}$ be an arbitrary family of topological spaces , let $M := \prod_{\lambda\in\Lambda} M_\lambda$ and let $\pi_\lambda : M \to M_\lambda$ be the canonical projection. We topologize M as follows: $U \subseteq M$ is declared open if and only if it can be written as union of elements that look like $\prod_{\lambda\in\Lambda} U_\lambda$ where each $U_\lambda \subseteq M_\lambda$ is open and $U_\lambda \neq M_\lambda$ for finitely many λ's. This gives a topology on M known as **the product topology**. In fact, the product topology on M is generated by $\{\pi^{-1}(U_\lambda) : \lambda \in \Lambda, U_\lambda \subseteq M_\lambda \text{ open }\}$.

Theorem 3.1. *(Tychonoff) An arbitrary product of compact spaces is compact in the product topology*

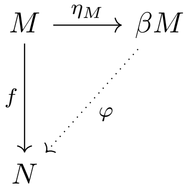
Proof. Say $(M_\lambda)_{\lambda\in\Lambda}$ is an arbitrary family of compact spaces. According to Theorem 2.6, to show that $M := \prod_{\lambda\in\Lambda} M_\lambda$ is compact it suffices to show that any ultrafilter $\mathcal{F}$ on M converges to at least one point. Since each $(\pi_\lambda)_*\mathcal{F}$ is an ultrafilter on M_λ, it converges to at least one point say a_λ. Thus, any open set $U_\lambda \subseteq M_\lambda$ with $a_\lambda \in U_\lambda$ is such that $\pi_\lambda^{-1}(U_\lambda) \in \mathcal{F}$. We claim that the ultrafilter $\mathcal{F}$ converges to $a := (a_\lambda)_{\lambda\in\Lambda} \in M$. Indeed, if $U \subseteq M$ is open with $a \in U$ we can find $U_{\lambda_1}, \ldots, U_{\lambda_n}$ such that

$$a \in \bigcap_{k=1}^{n} \pi_{\lambda_k}^{-1}(U_{\lambda_k}) \subseteq U.$$

This gives that $a_{\lambda_k} \in U_{\lambda_k}$, so it follows that $\pi_\lambda^{-1}(U_{\lambda_k}) \in \mathcal{F}$ and therefore (by part 3 on Definition 1.1) that $U \in \mathcal{F}$. ∎

4 Stone-Čech Compactification

Categorically speaking the Stone-Čech Compactification of M, denoted by βM, is the free compact Hausdorff space on M. More precisely, let **Top** be the category of toplogical spaces with continuous maps and **CH** the subcategory of compact Hausdorff space. Consider the forgetful functor **CH** $\to$ **Top**, then the Stone-Čech Compactification comes from a reflection of the forgetful functor denoted by (β, η). That is, for each $M \in \mathrm{Ob}(\mathbf{Top})$ there is $\beta M \in \mathrm{Ob}(\mathbf{CH})$ and $\eta_M : M \to \beta M$ a continuous map such that for any compact Hausdorff space N and continuous map $f : M \to N$ there is a unique continuous $\varphi : \beta M \to N$ such that $\varphi \circ \eta_M = f$. In other words, we have the following commutative diagram



As usual with category theory, the object βM is unique up to isomorphism, provided that we can construct one. There are many known constructions for βM. Here we sketch a simple one using ultrafilters:

$$\beta M := \{\mathcal{F} : \mathcal{F} \text{ is an ultrafilter on } M\}$$

To topologize βM, we define for each $A \subseteq M$

$$U_A := \{\mathcal{F} \in \beta X : A \in \mathcal{F}\}$$

It's not hard to check that $\{U_A\}_{A \subseteq M}$ gives a basis for a topology on βM and that this makes βM into a compact Hausdorff space. Finally, we define $\eta_M : M \to \beta M$ by $\eta_M(a) := \mathcal{F}_a$. Now we need to check that η_X is indeed continuous, this is left as a exercise for the audience. Finally, if N is a compact Hausdorff space and $f : M \to N$ a continuous map we define $\varphi : \beta M \to N$ by

$$\varphi(\mathcal{F}) := \text{the unique } \mathcal{F}\text{-limit point of } f$$

Then, for any $a \in M$ by Corollary 2.8 it follows that $(\varphi \circ \eta_M)(a) = \varphi(\mathcal{F}_a) = f(a)$, whence $\varphi \circ \eta_M = f$ as wanted.

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