Cuntz-Pimsner Algebras and a potential L^p version

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Functional Analysis Seminar
University of Oregon

October 29, 2020

Outline

Operator Algebras.

We denote by $\mathcal{L}(E)$ the bounded linear operator on a Banach space E. This is a Banach algebra with norm

$$\|a\| := \sup_{\|\xi\|=1} \|a(\xi)\|$$

- If $\mathcal{H}$ is a Hilbert space, any $a \in \mathcal{L}(\mathcal{H})$ has an adjoint $a^* \in \mathcal{L}(\mathcal{H})$ characterized by $\langle a(\xi), \eta \rangle = \langle \xi, a^*(\eta) \rangle$.
- If $\mathcal{H}$ is a Hilbert space, C*-algebra A is a norm closed selfadjoint subalgebra of $\mathcal{L}(\mathcal{H})$.
- If (X, μ) is a measure space and $p \in [1, \infty)$, an L^p-operator algebra A is a norm closed subalgebra of $\mathcal{L}(L^p(X, \mu))$.

$\mathcal{O}_d$

Let $d \in \mathbb{Z}_{\geq 2}$ and $\mathcal{H}$ an infinite dimensional separable Hilbert space. Then, there are elements $s_1, s_2, \dots, s_d \in \mathcal{L}(\mathcal{H})$ such that

$$s_j^* s_j = 1 \quad \text{and} \quad \sum_{j=1}^{d} s_j s_j^* = 1 \tag{1}$$

For $d = 2$. Let $\{\xi_1, \xi_2, \dots\}$ be an orthonormal basis for $\mathcal{H}$. Define $s_j : \mathcal{H} \to \mathcal{H}$ for $j = 1, 2$ by

$$s_1(\xi_n) = \xi_{2n} \quad \text{and} \quad s_2(\xi_n) = \xi_{2n-1} \quad n \geq 1$$

One quickly checks that

$$s_1^*(\xi_n) = \begin{cases} \xi_{n/2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad s_2^*(\xi_n) = \begin{cases} \xi_{(n+1)/2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$s_1^* s_1 = 1 = s_2^* s_2$

$$\begin{array}{ccccc} & s_1 & & s_1^* & \\ \xi_1 & \mapsto & \xi_2 & \mapsto & \xi_1 \\ \xi_2 & \mapsto & \xi_4 & \mapsto & \xi_2 \\ \xi_3 & \mapsto & \xi_6 & \mapsto & \xi_3 \\ \xi_4 & \mapsto & \xi_8 & \mapsto & \xi_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \quad \text{and} \quad \begin{array}{ccccc} & s_2 & & s_2^* & \\ \xi_1 & \mapsto & \xi_1 & \mapsto & \xi_1 \\ \xi_2 & \mapsto & \xi_3 & \mapsto & \xi_2 \\ \xi_3 & \mapsto & \xi_5 & \mapsto & \xi_3 \\ \xi_4 & \mapsto & \xi_7 & \mapsto & \xi_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

Therefore, $s_1^* s_1 = 1 = s_2^* s_2$.

$s_1s_1^* + s_2s_2^* = 1$

$$
\begin{array}{ccccc}
 & s_1^* & & s_1 & \\
\xi_1 & \mapsto & 0 & \mapsto & 0 \\
\xi_2 & \mapsto & \xi_1 & \mapsto & \xi_2 \\
\xi_3 & \mapsto & 0 & \mapsto & 0 \\
\xi_4 & \mapsto & \xi_2 & \mapsto & \xi_4 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
\quad \text{and} \quad
\begin{array}{ccccc}
 & s_2^* & & s_2 & \\
\xi_1 & \mapsto & \xi_1 & \mapsto & \xi_1 \\
\xi_2 & \mapsto & 0 & \mapsto & 0 \\
\xi_3 & \mapsto & \xi_2 & \mapsto & \xi_3 \\
\xi_4 & \mapsto & 0 & \mapsto & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

Hence, $s_1s_1^* + s_2s_2^* = 1$.

Universal Property for $\mathcal{O}_2$

Definition

We define $\mathcal{O}_d$, the Cuntz algebra of order $d \in \mathbb{Z}_{\geq 2}$, as the sub-C* algebra in $\mathcal{L}(\mathcal{H})$ generated by $s_1, \ldots, s_d$.

The algebra $\mathcal{O}_d$ is a simple, unital C*-algebra and has the following universal property: If A is a unital C*-algebra containing elements $a_1, \ldots, a_d$ such that

$$a_j^* a_j = 1 \quad \text{and} \quad \sum_{j=1}^{d} a_j a_j^* = 1,$$

then there is a unique $*$-homomorphism $\varphi : \mathcal{O}_d \to A$ such that $\varphi(s_j) = a_j$.

$\mathcal{E}_d$

Definition

For $d \in \mathbb{Z}_{\geq 2}$, look at the generating isometries $s_1, \dots, s_{d+1} \in \mathcal{O}_{d+1}$ and let $\mathcal{E}_d$ be the sub-C* algebra in $\mathcal{L}(\mathcal{H})$ generated by $s_1, \dots, s_d$. That is, $\mathcal{E}_d$ is the universal unital C*-algebra generated by d isometries, whose orthogonal ranges do not add up to 1.

The Cuntz algebra $\mathcal{O}_d$ has elements satisfying the relations of $\mathcal{E}_d$, so by universality there is a surjective map $\mathcal{E}_d \to \mathcal{O}_s$. The kernel of this map is the ideal in $\mathcal{E}_d$ generated by $s_{d+1}s_{d+1}^* = 1 - \sum_j^d s_j s_j^*$, which we denote by $\mathcal{J}_d$. Then $\mathcal{E}_d / \mathcal{J}_d \cong \mathcal{O}_d$.

A-valued right inner product.

Definition

Let A be a C*-algebra and E a complex vector space which is also a right A-module. An A-**valued right inner product on** E is a map

$$\begin{array}{ccc} E \times E & \to & A \\ (\xi, \eta) & \mapsto & \langle \xi, \eta \rangle_A \end{array}$$

such that for any $\xi, \eta, \eta_1, \eta_2 \in E$, $a \in A$ and $\alpha \in \mathbb{C}$ we have

1. $\langle \xi, \eta_1 + \alpha\eta_2 \rangle_A = \langle \xi, \eta_1 \rangle_A + \alpha \langle \xi, \eta_2 \rangle_A$.
2. $\langle \xi, \eta a \rangle_A = \langle \xi, \eta \rangle_A a$.
3. $\langle \xi, \eta \rangle_A^* = \langle \eta, \xi \rangle_A$.
4. $\langle \xi, \xi \rangle_A \geq 0$ in A.
5. $\langle \xi, \xi \rangle_A = 0 \implies \xi = 0$.

Hilbert Modules

Definition

Let A be a C*-algebra. A **Hilbert A-module** is a complex vector space E which is a right A-module with an A-valued right inner product and so that E is complete with the norm

$$\|\xi\| := \|\langle \xi, \xi \rangle_A\|^{1/2}.$$

We say that E is **full** if

$$\mathrm{span}\langle E, E\rangle_A := \mathrm{span}\{\langle \xi, \eta \rangle_A : \xi, \eta \in E\}$$

is dense in A.

Hilbert Modules: Examples

Example

Let $\mathcal{H}$ be a Hilbert space with the physicists convention that the inner product is linear in the second variable. Then, $\mathcal{H}$ is clearly a full Hilbert $\mathbb{C}$-module.

Example

Any C^*-algebra A is clearly a full Hilbert A-module with inner product given by $(a, b) \mapsto a^*b$. More generally, A^n is also a full Hilbert A-module with the obvious "euclidean" inner product.

Example

The set of continuous sections of a vector bundle over a compact Hausdorff space X equipped with a Riemannian metric g is a Hilbert $C(X)$-module.

Direct Sum of Hilbert Modules

Example

If $(E_\lambda)_{\lambda\in\Lambda}$ is an arbitrary family of Hilbert A-modules, we can form their direct sum

$$\bigoplus_{\lambda\in\Lambda} E_\lambda := \left\{ \xi = (\xi_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} E_\lambda : \sum_{\lambda\in\Lambda} \langle \xi_\lambda, \xi_\lambda \rangle \text{ converges in } A \right\}$$

which is a right A-module with coordinate-wise action and it becomes a Hilbert A-module when equipped with the well defined A-valued inner product

$$\langle \xi, \eta \rangle := \sum_{\lambda\in\Lambda} \langle \xi_\lambda, \eta_\lambda \rangle$$

Adjointable maps

A main difference between Hilbert modules and Hilbert spaces is that not every bounded linear map between Hilbert A-modules has an adjoint.

Definition

Let E and F be a Hilbert A-modules. A map $t : E \to F$ is said to be **adjointable** if there is a map $t^* : F \to E$ such that for any $\xi \in E$, and $\eta \in F$

$$\langle t(\xi), \eta \rangle = \langle \xi, t^*(\eta) \rangle$$

The space of adjointable maps from E to F is denoted by $\mathcal{L}_A(E, F)$ and $\mathcal{L}_A(E) := \mathcal{L}_A(E, E)$.

It's almost immediate that adjointable maps between Hilbert modules are linear and bounded. A standard argument shows that $\mathcal{L}_A(E)$ is a C*-algebra when equipped with the operator norm.

Generalized Compact Operators

We will have special interest for a particular case of adjointable maps, those of "rank 1":

Definition

Let E and F be a Hilbert A-modules. For $\xi \in E$ and $\eta \in F$, we define a map $\theta_{\xi,\eta} : F \to E$ by

$$\theta_{\xi,\eta}(\zeta) := \xi\langle \eta, \zeta \rangle_A$$

One easily checks that

- $\theta_{\xi,\eta} \in \mathcal{L}_A(E,F)$
- $(\theta_{\xi,\eta})^* = \theta_{\eta,\xi} \in \mathcal{L}_A(F,E)$
- $\|\theta_{\xi,\eta}\| \leq \|\xi\|\|\eta\|$

Generalized Compact Operators

The maps $\theta_{\xi,\eta}$ give an analogous of the of rank-one operators on Hilbert spaces. So, we define an analogous of the compact operators by letting

$$\mathcal{K}_A(E,F) := \overline{\mathrm{span}\{\theta_{\xi,\eta} : \xi \in E, \eta \in F\}}$$

It's also not hard to verify that $\mathcal{K}_A(E) := \mathcal{K}_A(E,E)$ is a closed two sided ideal in $\mathcal{L}_A(E)$, whence $\mathcal{K}(E)$ is also a C^*-algebra.

C*-correspondences

Definition

Let A be a C*-algebra. A C*-correspondence over A is a pair (E, φ) where E is a Hilbert right A-module and a $\varphi : A \to \mathcal{L}_A(E)$ is a $*$-homomorphism.

Remark

Even though it's not strictly necessary, from now on we will assume that the map φ of a C*-correspondence over A is injective and therefore isometric. This is done for simplicity.

Inner Tensor Product

Given (E, φ_E) and (F, φ_F) two C*-correspondence over A, we use the inner tensor product construction to produce $(E \otimes_{\varphi_F} F, \widetilde{\varphi_E})$, a new C*-correspondence over A.

More precisely, $E \otimes_{\varphi_F} F$ is a Hilbert module such that the middle action is glued:

$$\xi a \otimes \eta - \xi \otimes \varphi_F(a)\eta = 0,$$

it's an A-module with right action satisfying

$$(\xi \otimes \eta)a = \xi \otimes (\eta a),$$

it has an A-valued right inner product such that

$$\langle \xi_1 \otimes \eta_1, \xi_1 \otimes \eta_2 \rangle = \langle \eta_1, \varphi_F(\langle \xi_1, \xi_2 \rangle)\eta_2 \rangle.$$

Inner Tensor Product

In fact, $E \otimes_{\varphi_F} F$ is the completion of the algebraic tensor product $E \odot_A F$ with respect to the norm induced by the above inner product.

Finally, $\widetilde{\varphi_E} : A \to \mathcal{L}_A(E \otimes_{\varphi_F} F)$ is defined on elementary tensors by

$$\widetilde{\varphi_E}(a)(\xi \otimes \eta) = (\varphi_E(a)(\xi)) \otimes \eta$$

From now on we'll do an abuse of notation and drop the $\widetilde{}$ on top of φ_F. In fact, any adjointable map acting on a Hilbert A module, also acts on $E \otimes -$ by only acting on the E portion.

The Fock space

Definition

Given (E, φ) a C*-correspondence over A, the Fock space of E is the Hilbert A-module given by

$$\mathcal{F}(E) := \bigoplus_{n \geq 0} E^{\otimes n},$$

where $E^{\otimes 0} := A$ and $E^{\otimes n} := \underbrace{E \otimes_\varphi \ldots \otimes_\varphi E}_{n \text{ times}}$.

An arbitrary element of $\mathcal{F}(E)$ is a tuple $(\kappa_n)_{n \geq 0}$ where each κ_n is an element of the n^{th} degree tensor product of E.

Creation and Annihilation Operators

For a fixed $\xi \in E$ and any $n \in \mathbb{Z}_{\geq 1}$ we have a *creation operator* $c_\xi : E^{\otimes n} \to E^{\otimes(n+1)}$ given by

$$c_\xi(\eta) := \xi \otimes \eta, \quad \forall\ \eta \in E^{\otimes n}$$

If $n = 0$ we set $c_\xi : A \to E$

$$c_\xi(a) := \xi a, \quad \forall\ \in A$$

Each c_ξ is an adjointable map where, if $n \in \mathbb{Z}_{\geq 1}$, $c_\xi^* : E^{\otimes(n+1)} \to: E^{\otimes n}$ is an *annihilation operator*, satisfying

$$c_\xi^*(\zeta \otimes \eta) = \varphi(\langle \xi, \zeta \rangle)\eta, \quad \forall\ \zeta \in E, \eta \in E^{\otimes n}$$

and $c_\xi^* : E \to A$ is simply

$$c_\xi^*(\zeta) = \langle \xi, \zeta \rangle, \quad \forall\ \zeta \in E,$$

Creation and Annihilation Operators

Notice that c_ξ increases the degree by one, whereas c_ξ^* decreases the degree by one. Let $\xi, \zeta \in E$ and $n \geq 0$. The map $c_\xi^* c_\zeta : E^{\otimes n} \to E^{\otimes n}$ satisfies:

$$c_\xi^* c_\zeta = \varphi(\langle \xi, \zeta \rangle) \in \mathcal{L}_A(E^{\otimes n}),$$

Also the map $c_\xi c_\zeta^* : E^{\otimes(n+1)} \to E^{\otimes(n+1)}$, satisfies

$$c_\xi c_\zeta^* = \theta_{\xi,\zeta} \in \mathcal{L}_A(E^{\otimes(n+1)}).$$

We abuse notation and consider the elements c_ξ as elements of $\mathcal{L}_A(\mathcal{F}(E))$ acting coordinate-wise:

$$c_\xi\big((\kappa_n)_{n\geq 0}\big) := \big(c_\xi(\kappa_n)\big)_{n\geq 0} \quad \text{and} \quad c_\xi^*\big((\kappa_n)_{n\geq 0}\big) := \big(c_\xi^*(\kappa_n)\big)_{n\geq 1}$$

$\mathcal{T}_E$

Definition

Let (E, φ) be a C^*-correspondence over A. We define $\mathcal{T}_E$, the Toeplitz algebra of E, as the C^*-subalgebra in $\mathcal{L}_A(\mathcal{F}(E))$ generated by the creation operators $\{c_\xi : \xi \in E\}$.

Universal property: Suppose B is another C^*-algebra and that there is a $*$-homomoprhism $\pi : A \to B$, a linear map $t : E \to B$ such that

1. $t(\xi)^* t(\zeta) = \pi(\langle \xi, \zeta \rangle)$ for $\xi, \zeta \in E$,
2. $\pi(a)t(\xi) = t(\varphi(a)\xi)$

Then π has is a unique extension $\widehat{\pi} : \mathcal{T}_E \to B$ that sends c_ξ to $t(\xi)$.

Theorem

Let $d \in \mathbb{Z}_{\geq 2}$ and regard $\mathbb{C}^d$ as a Hilbert $\mathbb{C}$-module. Let $\varphi : \mathbb{C} \to \mathcal{L}_{\mathbb{C}}(\mathbb{C}^d)$ by given by

$$\varphi(z)(\zeta_1, \ldots, \zeta_d) := (z\zeta_1, \ldots, z\zeta_d)$$

Then $(\mathbb{C}^d, \varphi)$ is a C^* correspondence and $\mathcal{T}_{\mathbb{C}^d} \cong \mathcal{E}_d$.

Proof. Consider $v_1 := c_{(1,0)}$ and $v_2 := c_{(0,1)}$.

$$(v_1^* v_1) = c_{(1,0)}^* c_{(1,0)} = \varphi(\langle (1,0), (1,0) \rangle) = \varphi(1) = \mathrm{id}$$

and similarly $v_2^* v_2 = \mathrm{id}$. This gives a surjective $*$-homomorphism $\psi : \mathcal{E}_2 \to \mathcal{T}_{\mathbb{C}^2}$.

Proof. Let $\pi : \mathbb{C} \to \mathcal{E}_2$ be given by $\pi(z) := z1$ and $t : \mathbb{C}^2 \to \mathcal{E}_2$ be given by

$$t(\zeta_1, \zeta_2) = \zeta_1 s_1 + \zeta_2 s_2$$

It's obvious that π is a $*$-homomorphism and that t is a linear map. Since $s_j^* s_j = 1$, we get

$$t(\zeta_1, \zeta_2)^* t(\eta_1, \eta_2) = (\overline{\zeta_1}\eta_1 + \overline{\zeta_2}\eta_2)1 = \pi(\langle (\zeta_1, \zeta_2), (\eta_1, \eta_2) \rangle)$$

Finally,

$$\pi(z)t(\zeta_1, \zeta_2) = z1(\zeta_1 s_1 + \zeta_2 s_2) = z\zeta_1 s_1 + z\zeta_2 s_2 = t(\varphi(z)(\zeta_1, \zeta_2))$$

Hence, universality gives the $*$-homomorphism $\widehat{\pi} : \mathcal{T}_{\mathbb{C}^2} \to \mathcal{E}_2$, the inverse of ψ. ■

Remark

Notice that $\mathcal{T}_{\mathbb{C}^2} \neq \mathcal{O}_2$. Indeed, if $v_1 := c_{(1,0)}$ and $v_2 := c_{(0,1)}$ are asin the proof, then $v_1 v_1^* + v_2 v_2^*$ does not quite act as the identity on all degrees of $\mathcal{F}(\mathbb{C}^2)$. In fact, it only fails to do so at degree 0, because the adjoint kills everything at $n = 0$. Indeed,

$$(v_1 v_1^* + v_2 v_2^*)\big((\kappa_n)_{n\geq 0}\big) = \big((\kappa_n)_{n\geq 1}\big)$$

Thus, $1 - (v_1 v_1^* + v_2 v_2^*)$ is in fact a rank one operator who extracts the 0 coefficient, that is

$$1 - (v_1 v_1^* + v_2 v_2^*) = \theta_{(1,0,0,\ldots),(1,0,0,\ldots)}$$

This problem will no longer occur when we define the Cuntz-Pimsner algebra $\mathcal{O}_{\mathbb{C}^2}$, as we will be taking a quotient by a set that contains $1 - (v_1 v_1^* + v_2 v_2^*)$.

Killing some finite rank operators

Definition

For a C* correspondence (E, φ) over A, we define the ideal $J_E := \varphi^{-1}(\mathcal{K}_A(E))$.

Lemma

Let (E, φ) be a C correspondence over A. Then $\mathcal{F}(E)J_E$ is a Hilbert J_E-module and*

$$\mathcal{K}_{J_E}(\mathcal{F}(E)J_E) = \overline{\text{span}}\{\theta_{\kappa a,\tau} : \kappa, \tau \in \mathcal{F}(E), a \in J_X\} \trianglelefteq \mathcal{L}_A(\mathcal{F}(E))$$

We will now consider the quotient C*-algebra $\mathcal{Q}_A(E) := \mathcal{L}_A(\mathcal{F}(E))/\mathcal{K}_{J_E}(\mathcal{F}(E)J_E)$ together with the quotient map $q : \mathcal{L}_A(\mathcal{F}(E)) \to \mathcal{Q}_A(E)$.

$\mathcal{O}_E$

Definition

Let (E, φ) be a C^*-correspondence over A. We define $\mathcal{O}_E$, the Cuntz-Pimsner algebra of E, as the C^*-subalgebra in $\mathcal{Q}_A(E)$ generated by the image of the creation operators $\{q(c_\xi) : \xi \in E\}$.

Universal property. Suppose B is another C^*-algebra and that there is a $*$-homomoprhism $\pi : A \to B$, a linear map $t : E \to B$ s.t.

1. $t(\xi)^* t(\zeta) = \pi(\langle \xi, \zeta \rangle)$ for $\xi, \zeta \in E$,
2. $\pi(a)t(\xi) = t(\varphi(a)\xi)$

Further, if $\Theta_t : \mathcal{K}_A(E) \to B$ is s.t. $\Theta_t(\theta_{\xi,\zeta}) = t(\xi)t(\zeta)^*$ for $\xi, \zeta \in E$, and suppose also that

3. $\pi(a) = \Theta_t(\varphi(a))$ for $a \in J_E$

Then there π has is a unique extension $\widehat{\pi} : \mathcal{O}_E \to B$ that sends $q(c_\xi)$ to $t(\xi)$.

Theorem

Let $d \in \mathbb{Z}_{\geq 2}$ and regard $\mathbb{C}^d$ as a Hilbert $\mathbb{C}$-module. Let $\varphi : \mathbb{C} \to \mathcal{L}_{\mathbb{C}}(\mathbb{C}^d)$ by given by

$$\varphi(z)(\zeta_1, \ldots, \zeta_d) := (z\zeta_1, \ldots, z\zeta_d)$$

Then $(\mathbb{C}^d, \varphi)$ is a C^* correspondence and $\mathcal{O}_{\mathbb{C}^d} \cong \mathcal{O}_d$.

Proof. We will show that $\mathcal{O}_{\mathbb{C}^2}$ fits the universal property for $\mathcal{O}_2$. Well, if $v_1 := q(c_{(1,0)})$ and $v_2 := q(c_{(0,1)})$. Using our computations from the Toeplitz case we have

$$v_1^* v_1 = v_2^* v_2 = q(\mathrm{id}_{\mathcal{L}_A(\mathcal{F}(\mathbb{C}^2))}) = 1$$

Proof. Notice that $\theta_{(1,0,0,\ldots),(1,0,0,\ldots)} \in \mathcal{K}_{J_E}(\mathcal{F}(E)J_E)$ because in this case $\mathcal{L}_C(\mathbb{C}^2) = \mathcal{K}_{\mathbb{C}}(\mathbb{C}^2)$ and therefore $J_{\mathbb{C}^2} = \mathbb{C}$. Thus, following the computations from a previous Remark we have

$$v_1v_1^* + v_2v_2^* = q(\mathrm{id} - \theta_{(1,0,0,\ldots),(1,0,0,\ldots)}) = q(\mathrm{id}) = 1$$

Hence, universality gives a surjective $*$-homomorphism $\mathcal{O}_2 \to \mathcal{O}_{\mathbb{C}^2}$ sending s_j to v_j, for $j = 1, 2$. Since $\mathcal{O}_2$ is simple,such homomorphism has to be injective and we are done. ■

The L^p-Cuntz algebra

Definition

Let $d \geq 2$ be an integer. We define the **Leavitt algebra** L_d to be the universal complex unital algebra generated by elements $s_1, s_2, \ldots, s_d, t_1, t_2, \ldots, t_d$ satisfying

$$t_j s_k = \delta_{j,k} \quad \text{and} \quad \sum_{j=1}^{d} s_j t_j = 1$$

There is a well defined norm on L_d that comes from a particular kind of algebraic representations of L_d on σ-finite L^p spaces. The completion of L_d with respect to this norm is the L^p-Cuntz algebra $\mathcal{O}_d^p$.

I have been working on a Fock space-type construction for an L^p operator algebra A that yields a class of L^p-operator algebras that contains the L^p-Cuntz algebras. Looks like looking at the notion of Rigged Hilbert modules introduced by D.P. Blecher gives a reasonable starting point.

One needs to give an operator space type definition for L^p spaces. Some issues that might appear along the road will require to fix a representation $\pi : A \to \mathcal{L}(L^p(X, \mu))$, choose a particular tensor product and a particular type of completion.

Questions?