C*-modules and p-operator norms

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Outline

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Banach spaces

Definition

A *Banach space* is a complex vector space E with a norm $\| - \|_E$ that makes E a complete metric space.

For a Banach space E, we denote by $\mathrm{Ba}(E)$ its closed unit ball. That is

$$\mathrm{Ba}(E) := \{\xi \in E \colon \|\xi\|_E \leq 1\}.$$

If F is also a Banach space, we let $\mathcal{L}(E, F)$ be the space of bounded linear maps from E to F, which comes equipped with the operator norm

$$\|a\|_{\mathcal{L}(E,F)} := \sup\{\|a\xi\|_F \colon \xi \in \mathrm{Ba}(E)\}.$$

When $E = F$, we set $\mathcal{L}(E) := \mathcal{L}(E, E)$, which becomes a Banach algebra with multiplication given by composition of operators.
The dual of E will be denoted by E', that is

$$E' := \mathcal{L}(E, \mathbb{C}).$$

Finite Dimensional L^p-spaces

In this talk we will be mostly interested in the Banach spaces ℓ_n^p, where $n \in \mathbb{Z}_{\geq 1}$ and $p \in [1, \infty]$. That is, $\ell_n^p = (\mathbb{C}^n, \| - \|_p)$ where

$$\|\xi\|_p := \begin{cases} \left(\sum_{j=1}^{n} |\xi(j)|^p \right)^{1/p} & p \in [1, \infty) \\ \max_{j \in \{1, \dots, n\}} |\xi(j)| & p = \infty \end{cases}.$$

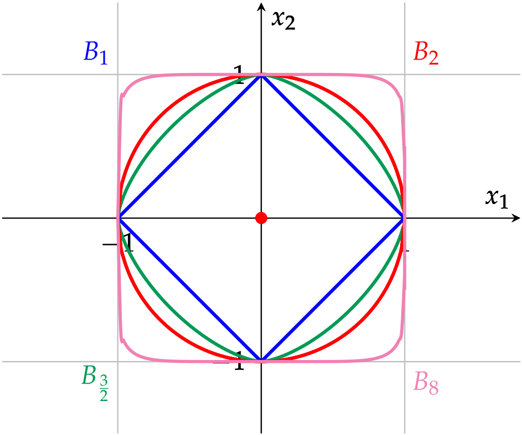
Here for $p, q \in [1, \infty]$ and $d, n \in \mathbb{Z}_{\geq 1}$, the space $\mathcal{L}(\ell_d^p, \ell_n^q)$ is simply the space of $n \times d$ matrices with complex entries equipped with the $p \to q$ operator norm:

$$M_{n,d}^{p \to q}(\mathbb{C}) := \mathcal{L}(\ell_d^p, \ell_n^q), \quad \|a\|_{p \to q} := \max\{\|a\xi\|_q : \xi \in \mathrm{Ba}(\ell_d^p)\}$$

Unit p-circles in $\mathbb{R}^2$

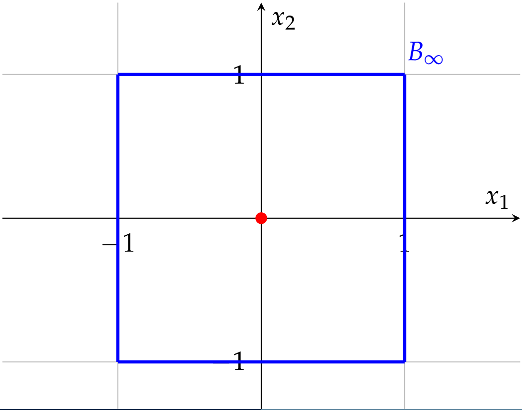
For $p \in [1, \infty)$ let

$$B_p := \{\xi \colon \{1,2\} \to \mathbb{R} \colon |\xi(1)|^p + |\xi(2)|^p = 1\}.$$



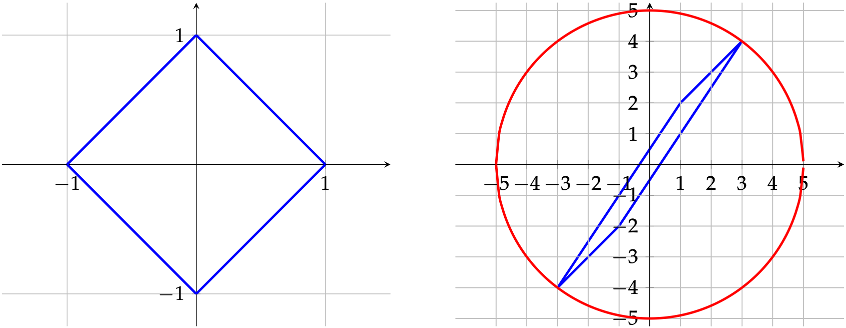
Letting $p \to \infty$

$$B_\infty := \{\xi\colon \{1,2\} \to \mathbb{R}\colon : \lim_{p\to\infty} |\xi(1)|^p + |\xi(2)|^p = 1\}$$
$$= \{\xi\colon \{1,2\} \to \mathbb{R}\colon \max\{|\xi(1)|, |\xi(2)|\} = 1\}.$$



$(1 \to 2)$-Operator Norm: Example

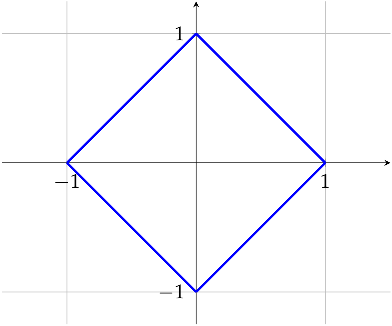
Let $a : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $a = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$. How to find $\|a\|_{1\to 2}$?

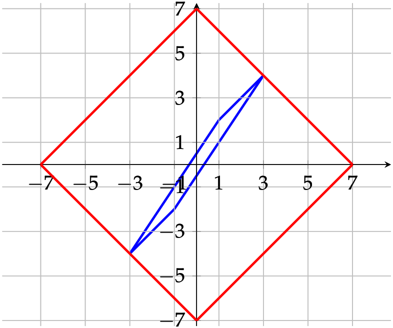


$$\|a\|_{1\to 2} = 5$$

1-Operator Norm: Example

Let $a : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $a = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$. How to find $\|a\|_{1\to 1}$?





$$\|a\|_{1\to 1} = 7$$

p-operator norms in $\mathbb{C}^n$: Known Cases

$M_n^p(\mathbb{C})$ is the algebra of $d \times d$ complex valued matrices equipped with the p-operator norm:

$$M_n^p(\mathbb{C}) = \mathcal{L}(\ell_n^p)$$

For $a \in M_n^p(\mathbb{C})$ we defined $\|a\|_{p\to p} := \max_{\|\xi\|_p \leq 1} \|a\xi\|_p$.

If $a = (a_{j,k})_{j,k=1}^n$, then

$$\|a\|_{1\to 1} = \max_{k\in\{1,\dots,n\}} \sum_{j=1}^n |a_{j,k}|,$$

$$\|a\|_{2\to 2} = \max_{\lambda\in\sigma(\overline{a}^T a)} \sqrt{|\lambda|},$$

$$\|a\|_{\infty\to\infty} = \max_{j\in\{1,\dots,n\}} \sum_{k=1}^n |a_{j,k}|.$$

Otherwise, for a general matrix a, the value $\|a\|_{p\to p}$ is NP-hard to compute.

Outline

1. Preliminaries
2. Hölder Duality
3. C*-like L^p-modules
4. Main results

Classic Hölder Duality

For $p \in [1, \infty]$ we denote p' its Hölder conjugate. That is,

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Let (Ω, μ) be a measure space and consider the dual pairing $\langle -, - \rangle_p \colon L^{p'}(\mu) \times L^p(\mu) \to \mathbb{C}$ given by

$$\langle \eta, \xi \rangle_p = \int_\Omega \eta \xi d\mu.$$

This defines $\Phi \colon L^{p'}(\mu) \to L^p(\mu)'$ by $[\Phi(\eta)](\xi) = \langle \eta, \xi \rangle_p$.

Theorem (Classic Hölder Duality)

Φ is an isometric isomorphism from $L^{p'}(\mu)$ to $L^p(\mu)'$ whenever

1. $p \in (1, \infty)$.
2. $p = 1$ *and* μ *is sigma-finite,*
3. $p = \infty$ *and* μ *is the counting measure on* $\{1, \ldots, n\}$ *for* $n \in \mathbb{Z}_{\geq 1}$.

Hölder Duality in finite dimension

$\ell_n^{p'}$ is isometrically isomorphic to $(\ell_n^p)'$ via the map $\Phi\colon \ell_n^{p'} \to (\ell_n^p)'$

$$[\Phi(\eta)](\xi) = \sum_{j=1}^{n} \eta(j)\xi(j) \in \mathbb{C}.$$

This can be reinterpreted through a 'p-operator space' perspective:

$$\ell_n^p \cong \mathcal{L}(\ell_1^p, \ell_n^p) = M_{n,1}^p(\mathbb{C}) \quad \ell_n^{p'} \cong \mathcal{L}(\ell_n^p, \ell_1^p) = M_{1,n}^p(\mathbb{C}).$$

The pairing is now multiplication of a $1 \times n$-matrix by a $n \times 1$-matrix:

$$(\eta \mid \xi) = \begin{bmatrix} \eta(1) & \dots & \eta(n) \end{bmatrix} \begin{bmatrix} \xi(1) \\ \vdots \\ \xi(n) \end{bmatrix} = \sum_{j=1}^{d} \eta(j)\xi(j).$$

Furthermore, Hölder duality gives

$$\|\xi\|_p = \max\{|(\eta \mid \xi)| \colon \eta \in \mathrm{Ba}(\ell_n^{p'})\}, \quad \|\eta\|_{p'} = \max\{|(\eta \mid \xi)| \colon \xi \in \mathrm{Ba}(\ell_n^p)\}.$$

Outline

Definition

Let (Ω_0, μ_0) and (Ω_1, μ_1) be measure spaces, let $p \in [1, \infty]$, and let $A \subseteq \mathcal{L}(L^p(\mu_0))$ be a closed subalgebra. An L^p-module over A is a pair (X, Y) such that $\mathsf{X} \subseteq \mathcal{L}(L^p(\mu_0), L^p(\mu_1))$ and $\mathsf{Y} \subseteq \mathcal{L}(L^p(\mu_1), L^p(\mu_0))$ are closed subspaces s.t.

1. $xa \in \mathsf{X}$ for all $x \in \mathsf{X}$, $a \in A$,
2. $ay \in \mathsf{Y}$ for all $y \in \mathsf{Y}$, $a \in A$.
3. $yx \in A$ for all $x \in \mathsf{X}$, $y \in \mathsf{Y}$,

Condition 3 gives a pairing $(- \mid -)_A \colon \mathsf{Y} \times \mathsf{X} \to A$ defined by

$$(y \mid x)_A = yx \in A.$$

Definition

We say that the module (X, Y) is *C*-like* if both the norms in X and Y are recovered by the pairing, that is for every $x \in \mathsf{X}$ and $y \in \mathsf{Y}$ we have

4. $\|x\| = \sup\{\|(y \mid x)_A\| \colon y \in \mathrm{Ba}(\mathsf{Y})\}$,
5. $\|y\| = \sup\{\|(y \mid x)_A\| \colon x \in \mathrm{Ba}(\mathsf{X})\}$.

Examples

- Let X be a right Hilbert module over a C*-algebra A. Then $(\mathsf{X}, \widetilde{\mathsf{X}})$ is a C*-like L^2-module over A.
- $(L^p(\mu), L^{p'}(\mu))$ is a C*-like L^p-module over $\mathbb{C}$.
- $(L^{p'}(\mu), L^p(\mu))$ is a C*-like L^p-module over $\mathcal{K}(L^p(\mu))$.
- $A \subseteq \mathcal{L}(L^p(\mu))$ a closed subalgebra. Then (A, A) is an L^p-module over A, it is C*-like as long as A has a contractive approximate unit.

Definition (Column-Row modules)

Let (X, Y) be an L^p-module over A, let $n \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$, and define

$$M^p_{n,1}(\mathsf{X}) := \ell^p_n \otimes_p \mathsf{X}, \quad M^p_{1,n}(\mathsf{Y}) = \ell^{p'}_n \otimes_p \mathsf{Y}.$$

- $(M^p_{n,1}(\mathsf{X}), M^p_{1,n}(\mathsf{Y}))$ is an L^p-module over A.

Question: When is $(M^p_{n,1}(\mathsf{X}), M^p_{1,n}(\mathsf{Y}))$ C*-like?

Outline

Positive Results

Theorem (REU(G)-2025)

Let $p \in [1, \infty)$, let $d, n \in \mathbb{Z}_{\geq 1}$, and ley $A \subseteq M_d^p(\mathbb{C})$ a subalegbra. Then $(M_{n,1}^p(A), M_{1,n}^p(A))$ is C-like whenever*

- $A = M_d^p(\mathbb{C})$.
- A *is any block diagonal subalgebra of* $M_d^p(\mathbb{C})$.

Comments:

1. Both instances are likely to hold as well when $n = \infty$.
2. What happens for $(L^p(\mu) \otimes_p A, L^{p'}(\mu) \otimes_p A)$ when μ is a more general measure?
3. Observe that in both cases, $\mathrm{id}_d \in A$. Is this a sufficient condition?

Counter Examples

Consider $A \subseteq M_2^p(\mathbb{C})$ defined by $A = \left\{ \begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix} : z \in \mathbb{C} \right\}$. A is simply $\mathbb{C}$ with trivial multiplication, so $(M_{n,1}^p(A), M_{1,n}^p(A))$ has 0 pairing. Hence, $(M_{n,1}^p(A), M_{1,n}^p(A))$ cannot be C*-like. Note that $\mathrm{id}_2 \notin A$.

The example above is likely generailzable to any nilpotent subalgebra of $M_d^p(\mathbb{C})$.

Let $A = \left\{ u \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} u \colon \lambda_1, \lambda_2 \in \mathbb{C} \right\}$, where $u = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Then $(M_{n,1}^1(A), M_{1,n}^1(A))$ is not C*-like, even though $\mathrm{id}_2 \in A$.

We still don't know whether this last example is C*-like for $p \neq 1, 2$.

Thank you!
Questions?

$(p \to q)$-operator norms in $\mathbb{C}^d$: Known cases

$M_d^{p\to q}(\mathbb{C})$ is the algebra of $d \times d$ complex valued matrices equipped with the $(p \to q)$-operator norm:

$$M_d^{p\to q}(\mathbb{C}) = \mathcal{L}(\ell_d^p, \ell_d^q)$$

For $a \in M_d^{p\to q}(\mathbb{C})$ we defined $\|a\|_{p\to q} := \max_{\|\xi\|_{p_1} \leq 1} \|a\xi\|_q$.

For $a = (a_{j,k})_{j,k=1}^d$:

$$\|a\|_{1\to 2} = \max_k \|(a_{1,k}, \dots, a_{d,k})\|_2,$$

$$\|a\|_{1\to\infty} = \max_k \|(a_{1,k}, \dots, a_{d,k})\|_\infty,$$

$$\|a\|_{2\to\infty} = \max_j \|(a_{j,1}, \dots, a_{j,d})\|_\infty.$$

However, the computability of $\|a\|_{2\to 1}$, $\|a\|_{\infty\to 1}$, and $\|a\|_{\infty\to 2}$ is NP-hard.