

Borel Functional Calculus and some applications.

Alonso Delfín
University of Oregon.

November 16, 2017

Abstract.

We all know how to compute the matrix $p(T)$ where p is a polynomial and T a matrix (in fact T can be any linear operator). In general, if a is an element of a Banach algebra A and f is a complex valued function, a functional calculus is roughly a way to define an element $f(a)$ in A .

Tomorrow at 11am in Deady 104, I'll be giving a talk about the Borel Functional Calculus. I'll start from scratch defining what a Functional Calculus is. Then, I'll talk a little bit about the continuous Functional Calculus and finally introduce the Borel Functional Calculus as an extension of this last one. If time allows, I'll give a couple of interesting applications. Throughout my talk I'll assume some familiarity with Banach algebras and C^* -algebras. But, to really get something out of it, it should be enough to know the basics of Lebesgue integration. Thus, my talk should be accesible to anyone who is familiar with the material presented in the first couple of weeks of Math 616.

Functional Calculus.

Q. What's a functional calculus?

Ans. Roughly, if \mathcal{F} is an algebra of complex valued functions and A is a Banach algebra, a functional calculus of an element $a \in A$, is an algebra homomorphism $\Phi : \mathcal{F} \rightarrow A$ such that if $\iota \in \mathcal{F}$ is the function $\iota(z) := z$, then $\Phi(\iota) = a$. For each $f \in \mathcal{F}$, the element $\Phi(f) \in A$ is commonly denoted by $f(a)$. Hence, we have $\iota(a) = a$.

There exist two common methods to construct a functional calculus:

- (1) Suppose there's a dense subalgebra $\mathcal{F}_0 \subset \mathcal{F}$ on which one can define Φ naturally (e.g. the polynomial algebra inside the algebra of continuous functions). Then, we use density to define Φ on all of \mathcal{F} .
- (2) Suppose that for every $f \in \mathcal{F}$ there's a reproduction formula

$$f(z) = \int \kappa(z, \zeta) f(\zeta) d\zeta$$

such that for any fixed ζ , the map $z \mapsto \kappa(z, \zeta)$ is in \mathcal{F} (e.g. the Cauchy formula for the algebra of holomorphic functions). Then, if it makes sense to define the elements $\kappa(a, \zeta) \in A$, we could use vector valued integration to define

$$\Phi(f) = \int \kappa(a, \zeta) f(\zeta) d\zeta$$

for an arbitrary $f \in \mathcal{F}$.

Polynomial Functional Calculus

Here we let \mathcal{F} be the algebra of polynomials in one variable over \mathbb{C} . That is, $\mathcal{F} := \mathbb{C}[z]$. Let A be any unital Banach algebra. Fix $a \in A$. If $p \in \mathcal{F}$ is given by

$$p(z) = \sum_{k=0}^m c_k z^k$$

we define an element $p(a) \in A$ in the obvious way

$$p(a) := \sum_{k=0}^m c_k a^k$$

(by convention we say that $a^0 = 1_A$). Then it's easily seen that the assignment $p \mapsto p(a)$ from \mathcal{F} to A is an algebra homomorphism and that $\iota(a) = a$, where ι is the polynomial given by the identity on \mathbb{C} . Thus, we've defined a functional calculus for the fixed element $a \in A$. This is called the **polynomial functional calculus**.

Entire/Holomorphic Functional Calculus

Here we let \mathcal{F} be the algebra of entire functions on \mathbb{C} . Let A be any unital Banach algebra. Fix $a \in A$. If $f \in \mathcal{F}$ is given by

$$f(z) = \sum_{k=0}^{\infty} c_k z^k$$

we define an element $f(a) \in A$ in the obvious way

$$f(a) := \sum_{k=0}^{\infty} c_k a^k$$

Convergence of the above series is guaranteed since the series for $f(\|a\|)$ converges absolutely and therefore

$$\left\| \sum_{k=0}^{\infty} c_k a^k \right\| \leq \sum_{k=0}^{\infty} |c_k| \|a\|^k < \infty$$

We can go further and extend this homomorphism to all functions f with power series representation around 0 and radius of convergence $R > \|a\|$.

Moreover, if f is holomorphic in a neighborhood Ω of $\sigma(a)$, the spectrum of a , then one can use vector valued integration to define

$$f(a) := \frac{1}{2\pi i} \int_{\Gamma} f(\zeta) (\zeta 1_A - a)^{-1} d\zeta$$

where Γ is a collection of closed rectifiable curves such that $\sigma(a) \subset \text{inside}(\Gamma)$ and $\mathbb{C} \setminus \Omega \subset \text{outside}(\Gamma)$.

Continuous Functional Calculus

One of the greatest advantages of the functional calculi described above is that it works for any Banach algebra A and any arbitrary element $a \in A$. However, entire functions and even general holomorphic functions are often not enough for some applications. We want the algebra \mathcal{F} to be “bigger”, say to include the continuous functions. The price to pay is that we’ll have to impose some restrictions on the Banach algebra A and the element a .

Theorem. (*Continuous Functional Calculus*) Let A be a C^* -algebra and $a \in A$ a normal element, that is $aa^* = a^*a$. Then, there exists a unique isometric $*$ -isomorphism $\Phi : \mathcal{C}(\sigma(a)) \rightarrow C^*(\{1, a\})$ such that $\Phi(\iota) = a$, where $\iota : \sigma(a) \hookrightarrow \mathbb{C}$ is the canonical inclusion.

Remark. Below, we clarify the terminology used in the above theorem

- $\mathcal{C}(\sigma(a))$ is the set of complex valued continuous functions on $\sigma(a)$.
- For any $S \subset A$, $C^*(S)$ is the sub C^* algebra generated by S . One has $C^*(S) := \overline{\text{span}(S \cup S^*)}$ where $S^* = \{s^* : s \in S\}$.
- Here, $*$ -isomorphism means a bijective algebra homomorphism such that $\Phi(\bar{f}) = \Phi(f)^*$.
- The isometric part gives that $\|\Phi(f)\| = \|f\|_\infty$.

▼

Sketch of Proof. Define $\mathcal{F}_0 \subset \mathcal{C}(\sigma(a))$ to be the subset of all polynomials in z, \bar{z} with complex coefficients. That is, a typical element p of \mathcal{F}_0 is a function $p : \sigma(a) \rightarrow \mathbb{C}$ given by

$$p(z) = \sum_{j,k=0}^n c_{j,k} z^j \bar{z}^k$$

where $n \in \mathbb{N}_0$ and $c_{j,k} \in \mathbb{C}$. We define a function Φ in \mathcal{F}_0 in the obvious way, that is if p is as above, we let

$$\Phi(p) := \sum_{j,k=0}^n c_{j,k} a^j \bar{a}^{*k}$$

Let $A_0 := \Phi(\mathcal{F}_0) \subset C^*(\{1, a\})$. Then, notice that by definition of the C^* -algebra generated by a set, we have $\overline{A_0} = C^*(\{1, a\})$. Furthermore, the Stone-Weierstrass theorem implies that $\overline{\mathcal{F}_0} = \mathcal{C}(\sigma(a))$. Hence, extending Φ by density yields an algebra homomorphism $\Phi : \mathcal{C}(\sigma(a)) \rightarrow C^*(\{1, a\})$. One still needs to check that this is indeed an isometric $*$ -isomorphism. Let $\text{Max}(\overline{A_0})$ denote the maximal ideal space of $\overline{A_0}$ (which is in turn identified with the non-zero algebra homomorphisms $\omega : \overline{A_0} \rightarrow \mathbb{C}$). The Gelfand transform $\gamma_{\overline{A_0}} : \overline{A_0} \rightarrow \mathcal{C}(\text{Max}(\overline{A_0}))$, is given by

$$[\gamma_{\overline{A_0}}(b)](\omega) := \omega(b)$$

for any $b \in \overline{A_0}$ and any $\omega \in \text{Max}(\overline{A_0})$. Notice that since a is a normal element, then $\overline{A_0} = C^*(\{1, a\})$ is commutative and this gives that $\gamma_{\overline{A_0}}$ is an isometric $*$ -isomorphism. Furthermore, the map from $\text{Max}(\overline{A_0})$ to $\sigma_{\overline{A_0}}(a)$ sending ω to $\omega(a)$ is a homeomorphism. Now, since $\overline{A_0}$ is a sub C^* -algebra of A and $1 \in \overline{A_0}$, the spectral permanence theorem gives that $\sigma_{\overline{A_0}}(a) = \sigma(a)$. Putting altogether, we have that $\text{Max}(\overline{A_0}) \cong \sigma(a)$, which in turns gives that $\mathcal{C}(\text{Max}(\overline{A_0})) = \mathcal{C}(\sigma(a))$. Thus, the Gelfand transform actually gives rise to an isometric $*$ -isomorphism from $C^*(\{1, a\}) \rightarrow \mathcal{C}(\sigma(a))$. This turns out to be the inverse of our map Φ . By definition of Φ , it follows that $\Phi(\iota) = a$, which in turn gives the uniqueness of such isomorphism. \square

Borel Functional Calculus

In this section we restrict our attention to $A = \mathcal{B}(\mathcal{H})$, the C^* -algebra of bounded operators on a Hilbert space \mathcal{H} . The unital element is the identity map $I \in \mathcal{B}(\mathcal{H})$, which is given, as usual, by $I(\xi) = \xi$ for any $\xi \in \mathcal{H}$.

Definition. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be **positive** if

$$\langle T\xi, \xi \rangle \geq 0 \quad \forall \xi \in \mathcal{H}$$

We will write $T \geq 0$ when T is positive. \blacktriangle

Definition. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a **projection** if it is self-adjoint and idempotent, that is

$$T = T^* = T^2$$

Definition. Let X be a compact Hausdorff space. We think of X as a measurable space with the Borel σ -algebra, which we will denote by \mathcal{S} . A **spectral measure** in (X, \mathcal{H}) is a function $\mathcal{P} : \mathcal{S} \rightarrow \mathcal{B}(\mathcal{H})$, such that

- (i) $\mathcal{P}(E)$ is a projection for any $E \in \mathcal{S}$.
- (ii) $\mathcal{P}(\emptyset) = 0$ and $\mathcal{P}(X) = I$.
- (iii) If $(E_n)_{n=1}^\infty$ is a sequence of mutually disjoint Borel sets, then

$$\mathcal{P}\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mathcal{P}(E_n)$$

where the convergence of the series is interpreted in SOT sense.

(iv) For any $\xi, \eta \in \mathcal{H}$, the functions $\mathcal{P}_{\xi, \eta} : \mathcal{S} \rightarrow \mathbb{C}$, defined by

$$\mathcal{P}_{\xi, \eta}(E) := \langle \mathcal{P}(E)\xi, \eta \rangle$$

are complex valued regular measures in (X, \mathcal{S}) .

▲

Proposition. *If \mathcal{P} is a spectral measure in (X, \mathcal{H}) , then*

- (1) $E \subset F$ implies that $\mathcal{P}(E) \leq \mathcal{P}(F)$, that is $\mathcal{P}(F) - \mathcal{P}(E)$ is positive.
- (2) $E \cap F = \emptyset$ implies $\mathcal{P}(E) \perp \mathcal{P}(F)$, that is $\mathcal{P}(E)\mathcal{P}(F) = \mathcal{P}(F)\mathcal{P}(E) = 0$.
- (3) $\mathcal{P}(E \cap F) = \mathcal{P}(E)\mathcal{P}(F)$.

Proof. We only show (1) to get a feeling on how to work with spectral measures. Well, $\mathcal{P}(F \setminus E)$ is a projection and therefore,

$$\langle \mathcal{P}(F \setminus E)\xi, \xi \rangle = \langle \mathcal{P}(F \setminus E)\xi, \mathcal{P}(F \setminus E)\xi \rangle = \|\mathcal{P}(F \setminus E)\xi\|^2 \geq 0$$

for any $\xi \in \mathcal{H}$. Hence, $\mathcal{P}(F \setminus E)$ is positive. Since $F = E \sqcup (F \setminus E)$, it follows that $\mathcal{P}(F) = \mathcal{P}(E) + \mathcal{P}(F \setminus E)$. Thus, $\mathcal{P}(F) - \mathcal{P}(E) = \mathcal{P}(F \setminus E)$ is positive and we are done. ■

Our goal is to define a functional calculus on $\mathcal{B}(\mathcal{H})$ on a class which is bigger than the continuous functions.

Definition. Let $B_\infty(X)$ denote all the bounded measurable complex valued functions on X . $B_\infty(X)$ is a C^* -algebra with involution given by $f \mapsto \bar{f}$. Of course, since X is compact, we have $\mathcal{C}(X) \subset B_\infty(X)$ ▲

Lemma. *If \mathcal{P} is a spectral measure on (X, \mathcal{H}) and $f \in B_\infty(X)$, then the map $\varphi_f : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$, given by*

$$\varphi_f(\xi, \eta) := \int_X f d\mathcal{P}_{\xi, \eta}$$

is a bounded bilinear form with $\|\varphi_f\| \leq \|f\|_\infty$.

Corollary. *Let \mathcal{P} be a spectral measure on (X, \mathcal{H}) and $f \in B_\infty(X)$. There is a unique $T_f \in \mathcal{B}(\mathcal{H})$ such that*

$$\langle T_f \xi, \eta \rangle = \int_X f d\mathcal{P}_{\xi, \eta}$$

for all $\xi, \eta \in \mathcal{H}$.

Definition. Let \mathcal{P} be a spectral measure on (X, \mathcal{H}) and $f \in B_\infty(X)$. The unique operator T_f obtained in the previous corollary is called **the integral of f with respect to \mathcal{P}** . We use the following convenient notation

$$\int_X f d\mathcal{P} := T_f$$

▲

Remark. For any $E \in \mathcal{S}$ we have $\langle \mathcal{P}(E)\xi, \eta \rangle = \mathcal{P}_{\xi, \eta}(E) = \int_X \chi_E d\mathcal{P}_{\xi, \eta}$. Thus, we actually have

$$\mathcal{P}(E) = \int_X \chi_E d\mathcal{P}$$

This shows that spectral measures behave as common measures. ▼

The following theorem is key to obtain the Borel functional calculus as an extension of the continuous one:

Theorem. Suppose $\theta : \mathcal{C}(X) \rightarrow \mathcal{B}(\mathcal{H})$ is a $*$ -homomorphism, with $\theta(\mathbf{1}) = I$ (here $\mathbf{1} : X \rightarrow \mathbb{C}$ is the constant function $\mathbf{1}(x) := 1$ for all $x \in X$). Then there is a unique spectral measure \mathcal{P} in (X, \mathcal{H}) such that

$$\theta(f) = \int_X f d\mathcal{P}$$

Sketch of Proof. For each $\xi, \eta \in \mathcal{H}$ define $\tau_{\xi, \eta} : \mathcal{C}(X) \rightarrow \mathbb{C}$ by

$$\tau_{\xi, \eta}(f) := \langle \theta(f)\xi, \eta \rangle$$

One checks that $\tau_{\xi, \eta}$ is a bounded linear functional and therefore, by the Riesz-Markov theorem, there is a regular Borel measure $\mu_{\xi, \eta}$ such that

$$\tau_{\xi, \eta}(f) = \int_X f d\mu_{\xi, \eta},$$

for all $f \in \mathcal{C}(X)$. Now, for each $f \in \mathcal{C}(X)$, we define $\varphi_f : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ by

$$\varphi_f(\xi, \eta) := \int_X f d\mu_{\xi, \eta}$$

Then, φ_f is seen to be a bounded bilinear form and therefore there is a unique $\Psi(f) \in \mathcal{B}(\mathcal{H})$ such that

$$\varphi_f(\xi, \eta) = \langle \Psi(f)\xi, \eta \rangle$$

for all $\xi, \eta \in \mathcal{H}$. We have then that $\Psi = \theta$ on $\mathcal{C}(X)$. Therefore, for any $f \in \mathcal{C}(X)$ and any $\xi, \eta \in \mathcal{H}$ we have

$$\langle \Psi(f)\xi, \eta \rangle = \int_X f d\mu_{\xi, \eta}$$

Since the right hand side is well defined when $f \in B_\infty(X)$, we actually get a map $\Psi : B_\infty(X) \rightarrow \mathcal{B}(\mathcal{H})$. We define $\mathcal{P} : \mathcal{S} \rightarrow \mathcal{B}(\mathcal{H})$ by

$$\mathcal{P}(E) := \Psi(\chi_E)$$

for any $E \in \mathcal{S}$. One checks that \mathcal{P} is actually a spectral measure. Furthermore, $\mathcal{P}_{\xi, \eta} = \mu_{\xi, \eta}$ for any $\xi, \eta \in \mathcal{H}$. Therefore, for any $f \in \mathcal{C}(X)$ and any $\xi, \eta \in \mathcal{H}$, we get

$$\left\langle \left(\int_X f d\mathcal{P} \right) \xi, \eta \right\rangle = \int_X f d\mathcal{P}_{\xi, \eta} = \int_X f d\mu_{\xi, \eta} = \langle \theta(f)\xi, \eta \rangle$$

Thus,

$$\theta(f) = \int_X f d\mathcal{P}$$

as we wanted to show. Uniqueness of \mathcal{P} follows from the fact $\mathcal{P}_{\xi, \eta} = \mu_{\xi, \eta}$, which was already proven. “□”

As a corollary, we get the **Borel functional calculus** for normal operators. Indeed, let $T \in \mathcal{B}(\mathcal{H})$ be a normal operator and put $X := \sigma(T)$, which is Hausdorff. Let $\theta : \mathcal{C}(\sigma(T)) \rightarrow \mathcal{B}(\mathcal{H})$ be the continuous functional calculus for T . Then, since θ is known to be a $*$ -homomorphism, the previous theorem implies the existence of a unique spectral measure \mathcal{P} such that for any $f \in \mathcal{C}(\sigma(T))$,

$$\theta(f) = \int_{\sigma(T)} f d\mathcal{P}$$

In particular, if $\iota : \sigma(T) \hookrightarrow \mathbb{C}$ is the canonical inclusion, we have

$$T = \int_{\sigma(T)} \iota d\mathcal{P}$$

Such spectral measure \mathcal{P} is called the **resolution of the identity** for T . Moreover, we extend the continuous functional calculus for any $f \in B_\infty(\sigma(T))$ by setting

$$f(T) := \int_{\sigma(T)} f d\mathcal{P}$$

The assignment $f \mapsto f(T)$ gives a $*$ -homomorphism from $B_\infty(\sigma(T))$ to $\mathcal{B}(\mathcal{H})$ which extends the continuous functional calculus θ . Such assignment is known as the Borel functional calculus for T .

Applications

Theorem. *Suppose $T \in \mathcal{B}(\mathcal{H})$ is normal with resolution of the identity given by \mathcal{P} . Then, $\lambda \in \sigma(T)$ is an eigenvalue of T if and only if $\mathcal{P}(\{\lambda\}) \neq 0$.*

Proof. Suppose first that $\lambda \in \sigma(T)$ is an eigenvalue of T . Let $\xi \in \mathcal{H}$ be an eigenvalue for λ . That is, $\xi \neq 0$ and $T\xi = \lambda\xi$. For each $n \in \mathbb{N}$ we define

$$f_n(z) := \begin{cases} \frac{1}{z - \lambda} & \text{if } |z - \lambda| > \frac{1}{n} \\ 0 & \text{if } |z - \lambda| \leq \frac{1}{n} \end{cases}$$

Then, $f_n \in B_\infty(\sigma(T))$ for each $n \in \mathbb{N}$. Next, we define sets $E_n \in \mathcal{S}$ by

$$E_n := \left\{ z \in \sigma(T) : |z - \lambda| > \frac{1}{n} \right\}$$

Hence, $f_n(z)(z - \lambda) = \chi_{E_n}(z)$ and therefore $f_n(T)(T - \lambda) = \chi_{E_n}(T)$. Thus,

$$\mathcal{P}(E_n)\xi = \chi_{E_n}(T)\xi = f_n(T)(T - \lambda)\xi = f_n(T)(T\xi - T\xi) = 0$$

Furthermore, notice that $E_n \subset E_{n+1}$ and that $E := \bigcup_{n=1}^{\infty} E_n = \{z \in \sigma(T) : z \neq \lambda\}$. Thus,

$$\mathcal{P}(E)\xi = \lim_{n \rightarrow \infty} \mathcal{P}(E_n)\xi = 0$$

But then,

$$\mathcal{P}(\{\lambda\})\xi = \mathcal{P}(\sigma(T) \setminus E)\xi = \mathcal{P}(\sigma(T))\xi - \mathcal{P}(E)\xi = I\xi - 0 = \xi$$

Since $\xi \neq 0$, this shows indeed that $\mathcal{P}(\{\lambda\}) \neq 0$.

Conversely, suppose $\mathcal{P}(\{\lambda\}) \neq 0$. Let $P := \mathcal{P}(\{\lambda\})$ and find $\eta \in P(\mathcal{H})$ so that $\eta \neq 0$. Since P is a projection, we have $P\eta = \eta$. Then,

$$T\eta = TP\eta = \left(\int_{\sigma(T)} \iota d\mathcal{P} \right) \left(\int_{\sigma(T)} \chi_{\{\lambda\}} d\mathcal{P} \right) \eta = \left(\int_{\{\lambda\}} \iota d\mathcal{P} \right) \eta = (\lambda P)\eta$$

But $(\lambda P)\eta = \lambda\eta$. Thus, λ is indeed an eigenvalue. ■

Definition. An operator $U \in \mathcal{B}(\mathcal{H})$ is said to be **unitary** if $UU^* = I = U^*U$. ▲

Theorem. A normal operator $T \in \mathcal{B}(\mathcal{H})$ is unitary if and only if

$$\sigma(T) \subset \partial\mathbb{D} := \{z \in \mathbb{C} : |z| = 1\}$$

Proof. Suppose first that T is unitary. Then, $\|T\| = \|T^*\| = 1$. Thus, $\sigma(T) \subset \{z \in \mathbb{C} : |z| \leq 1\}$ and $\sigma(T^*) \subset \{z \in \mathbb{C} : |z| \leq 1\}$. However, since $(T^*)^{-1} = T$, we also have

$$\sigma(T) = \sigma((T^*)^{-1}) = \left\{ \frac{1}{\lambda} : \lambda \in \sigma(T^*) \right\} \subset \left\{ \frac{1}{\lambda} : |\lambda| \leq 1 \right\} = \{\lambda : |\lambda| \geq 1\}$$

So it follows that $\sigma(T) \subset \{z \in \mathbb{C} : |z| = 1\}$ as wanted.

Conversely, suppose that $\sigma(T) \subset \partial\mathbb{D}$. Then, since the functional calculus θ is a $*$ homomorphism and $\iota\bar{\iota} = \mathbb{1}$ on $\partial\mathbb{D}$, it follows that

$$TT^* = \theta(\iota)\theta(\iota)^* = \theta(\iota\bar{\iota}) = \theta(\mathbb{1}) = I$$

An analogous computation yields $T^*T = I$, so T is unitary. ■

Theorem. An operator $U \in \mathcal{B}(\mathcal{H})$ is unitary if and only if $U = e^{iT}$ for a self adjoint operator $T \in \mathcal{B}(\mathcal{H})$.

Proof. Suppose first that U is unitary. Let $g : [0, 2\pi) \rightarrow \partial\mathbb{D}$ be given by $g(t) = e^{it}$. It's well known that g is a continuous bijection whose inverse $f : \partial\mathbb{D} \rightarrow [0, 2\pi)$ is not continuous. However, $f \in B_\infty(\partial\mathbb{D})$. Thus, since U is unitary, the above theorem gives $\sigma(U) \subset \partial\mathbb{D}$. Moreover, U is normal so we can use the Borel functional calculus of U to define an operator $T := f(U)$. Let \mathcal{P} be the resolution of the identity for U . Since f is real valued, we have

$$T^* = f(U)^* = \int_{\sigma(U)} \bar{f} d\mathcal{P} = \int_{\sigma(U)} f d\mathcal{P} = f(U) = T$$

Thus, T is self adjoint. Moreover, notice that $g \circ f = \iota$ on $\sigma(U)$. Hence, it follows that $U = (g \circ f)(U) = g(f(U)) = e^{iT}$ as wanted.

Conversely, suppose T is self adjoint. Then T has real spectrum and it's normal. Let \mathcal{R} be the resolution of the identity for T . Then,

$$e^{iT}(e^{iT})^* = \left(\int_{\sigma(T)} e^{it} d\mathcal{R}(t) \right) \left(\int_{\sigma(T)} e^{-it} d\mathcal{R}(t) \right) = \int_{\sigma(T)} \mathbb{1} d\mathcal{R}(t) = I$$

An analogous computation yields $(e^{iT})^*e^{iT} = I$, so e^{iT} is unitary. ■