Banach Space Ultraproducts.

Alonso Delfín
University of Oregon.

December 5, 2019

Abstract

Although ultraproducts are probably most associated with logic, the definition is purely set-theoretic. Here, we'll give this definition from scratch (this includes defining what an ultrafilter is) and explain how to modify it to get an ultraproduct construction for Banach spaces.

1 A Brief Review on Filters and Ultrafilters

Definition 1.1. A **filter** on a set X is $\mathcal{F} \subseteq \mathcal{P}(X)$ such that

1. $X \in \mathcal{F}$.
2. $\varnothing \notin \mathcal{F}$.
3. If $A \in \mathcal{F}$ and B is such that $A \subseteq B$, then $B \in \mathcal{F}$.
4. If A and B are both in $\mathcal{F}$, then $A \cap B \in \mathcal{F}$.

Example 1.2. Let X be any set. Below we have three examples of filters on X.

(i) The **trivial filter on** X is X itself.

(ii) Let $x \in X$. The **principal filter generated by** x is $\mathcal{F}_x = \{A \subseteq X : x \in A\}$.

(iii) If X has infinite cardinality, the **cofinite filter on** X is $\mathcal{F} := \{A \subseteq X : X \setminus A \text{ is finite}\}$

Definition 1.3. Let $\mathcal{F}$ be a filter on a set X. We say that $\mathcal{F}$ is an **ultrafilter** if for any $A \subseteq X$, either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.

Example 1.4. Let X be any set and $x \in X$. The filter $\mathcal{F}_x$ defined above is an ultrafilter.

Lemma 1.5. *Every filter is contained in an ultrafilter and maximal filters are ultrafilters.*

Proof. This is a trivial application of Zorn's lemma. ■

2 Ultraproducts

Definition 2.1. Let X be a set and $(E_x)_{x\in X}$ be an arbitrary collection of non-empty sets indexed by X. Let $\mathcal{U}$ be an ultrafilter on X. We define an equivalence relation $\equiv_{\mathcal{U}}$ on $\prod_{x\in X} E_x$ by

$$(\xi_x) \equiv_{\mathcal{U}} (\eta_x) \iff \{x \in X : \xi_x = \eta_x\} \in \mathcal{U}$$

The **set theoretic ultraproduct** of $(E_x)_{x\in X}$ with respect to $\mathcal{U}$ is $\prod_{x\in X} E_x / \equiv_{\mathcal{U}}$. This is sometimes denoted as $\prod_{x\in X} E_x/\mathcal{U}$ or simply by $\prod_{\mathcal{U}} E_x$. When all the E_x are equal to a set E, we get the **set theoretic ultrapower**, usually denoted by $E^X/\mathcal{U}$.

Example 2.2. If $x_0 \in X$, then $\prod_{\mathcal{F}_{x_0}} E_x$ and E_{x_0} are isomorphic as sets.

In the above construction, when each set E_x is a Banach space, we do not always get a Banach space when constructing the set theoretic ultraproduct. We need to modify the construction above to always get a Banach space.

Definition 2.3. Let X be a set and E a topological space. If $\mathcal{U}$ is an ultrafilter on X, we define the ultralimit of $(\xi_x)_{x\in X}$, denoted by $\lim_{\mathcal{U}} \xi_x$, as follows

$$\lim_{\mathcal{U}} \xi_x = \xi \in E \iff \{x \in X : \xi_x \in A\} \in \mathcal{U} \text{ for any open neighborhood } A \text{ of } \xi$$

Theorem 2.4. *If E is a compact Hausdorff space, then for each sequence $(\xi_x)_{x\in X}$, the limit $\lim_{\mathcal{U}} \xi_x$ exists and it's unique.*

Proof. Follows from standard point-set topology arguments. ■

For a set X, let $(E_x)_{x\in X}$ be a family of non-empty Banach spaces. We define a new Banach space

$$\ell^\infty(X, E_x) := \{(\xi_x) \in \prod_{x\in X} E_x : \|(\xi_x)\|_\infty := \sup_{x\in X} \|\xi_x\| < \infty\}$$

with addition and scalar multiplication given componentwise. If $\mathcal{U}$ is an ultrafilter in X, let

$$N_{\mathcal{U}} = \{(\xi_x) \in \ell^\infty(X, E_x) : \lim_{\mathcal{U}} \|\xi_x\| = 0\}$$

One checks that $N_{\mathcal{U}}$ is a closed linear subspace of $\ell^\infty(X, E_x)$

Definition 2.5. For a set X, let $(E_x)_{x\in X}$ be a family of non-empty Banach spaces. Define

$$(E_x)_{\mathcal{U}} := \ell^\infty(X, E_x)/N_{\mathcal{U}}$$

The set $(E_x)_{\mathcal{U}}$ is called **the Banach space ultraproduct of $(E_x)_{x\in X}$ with respect to $\mathcal{U}$**. The equivalence class of an element $(\xi_x) \in \ell^\infty(X, E_x)$ is denoted by $(\xi_x)_{\mathcal{U}}$.

The set $(E_x)_{\mathcal{U}}$ is a Banach space when equipped with the usual quotient norm:

$$\|(\xi_x)_{\mathcal{U}}\| := \inf_{(\eta_x)\in N_{\mathcal{U}}} \|(\xi_x - \eta_x)\|_\infty$$

Moreover, the following result gives a simpler way to compute the norm on $(E_x)_{\mathcal{U}}$:

Lemma 2.6.

$$\|(\xi_x)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|\xi_x\|$$

Proof. Notice that for $(\xi_x) \in \ell^\infty(X, E_x)$, we must have $\sup_{x\in X} \|\xi_x\| < \infty$ and therefore Theorem 2.4 implies that $\lim_{\mathcal{U}} \|\xi_x\|$ exists. The rest is an elementary verification. ■

Example 2.7. If $x_0 \in X$, then $(E_x)_{\mathcal{F}_{x_0}}$ and E_{x_0} are isometrically isomorphic.

As before, when all the Banach spaces E_x are equal to E, we get the **Banach space ultrapower**, which we still denote by $E^X/\mathcal{U}$. There is an isometric embedding $\Phi : E \to E^X/\mathcal{U}$ defined by

$$\Phi(\xi) = (\xi_x)_{\mathcal{U}}$$

where each $\xi_x := \xi$. Banach space ultrapowers are only interesting in the infinite-dimensional case. Indeed, if E is finite dimensional, the map $\Phi : E \to E^X/\mathcal{U}$ is an isometric isomorphism. To check this, it suffices to show that Φ is surjective, so let's take any $(\xi_x)_{\mathcal{U}}$ in $E^X/\mathcal{U}$. Since $M := \sup_{x\in X} \|\xi_x\| < \infty$, then (ξ_x) is in $\{\xi \in E : \|\xi\| \leq M\}$, which is a compact set because E is finite dimensional. Thus, by Theorem 2.4, the limit $\lim_{\mathcal{U}} \xi_x$ exists and its unique. Call the limit ξ. Clearly $\lim_{\mathcal{U}}(\xi_x - \xi) = 0$ and therefore $(\xi_x)_{\mathcal{U}} = \Phi(\xi)$.

Theorem 2.8. *The following classes of Banach spaces are closed under Banach space ultraproducts:*

(1) Banach algebras

(2) C^*-algebras

(3) $C(K)$ spaces where K is a compact Hausdorff space.

(4) $L^p(\mu)$ spaces for $1 \leq p < \infty$

Proof. To prove (1) one checks that the natural multiplication

$$(\xi_x)_{\mathcal{U}}(\eta_x)_{\mathcal{U}} := (\xi_x\eta_x)_{\mathcal{U}}$$

is well defined in $(E_x)_{\mathcal{U}}$. Then, it follows that

$$\|(\xi_x)_{\mathcal{U}}(\eta_x)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|\xi_x\eta_x\| \leq \lim_{\mathcal{U}} \|\xi_x\|\|\eta_x\| = \|(\xi_x)_{\mathcal{U}}\|\|(\eta_x)_{\mathcal{U}}\|$$

For (2), one checks that the natural involution

$$(\xi_x)_{\mathcal{U}}^* := (\xi_x^*)_{\mathcal{U}}$$

is well defined, and then notice that

$$\|(\xi_x)_{\mathcal{U}}(\xi_x)_{\mathcal{U}}^*\| = \lim_{\mathcal{U}} \|\xi_x\xi_x^*\| = \lim_{\mathcal{U}} \|\xi_x\|^2 = \|(\xi_x)_{\mathcal{U}}\|^2.$$

Now, (3) follows because $C(K)$-spaces are commutative C^*-algebras and therefore the ultraproduct will also be a commutative C^*-algebra, so Gelfand-Naimark gives that the ultraproduct is also a $C(K)$ space. Finally, (4) requires the representation theorem for L^p spaces, which says that a Banach lattice with the property that $\|x+y\|^p = \|x\|^p + \|y\|^p$ whenever $x \wedge y = 0$ is an $L^p(\nu)$ space for some measure ν. ■

Having introduced the ultraproduct of Banach spaces, it makes sense to to talk about the ultraproduct of operators between Banach spaces. Let $(E_x)_{x\in X}$ and $(F_x)_{x\in X}$ be families of Banach spaces indexed by the same set X. Suppose that for each $x \in X$ we have $a_x \in \mathcal{L}(E_x, F_x)$ and that

$$\sup_{x\in X} \|a_x\| < \infty$$

Then, if $\mathcal{U}$ is an ultrafilter, we define a map $(a_x)_{\mathcal{U}} : (E_x)_{\mathcal{U}} \to (F_x)_{\mathcal{U}}$ by

$$(a_x)_{\mathcal{U}}(\xi_x)_{\mathcal{U}} = (a_x\xi_x)_{\mathcal{U}}$$

One checks that if $\lim_{\mathcal{U}} \|\xi_x\| = 0$, then $\lim_{\mathcal{U}} \|a_x\xi_x\| = 0$, so the definition above is well defined. Furthermore, one has that $(a_x)_{\mathcal{U}} \in \mathcal{L}\big((E_x)_{\mathcal{U}}, (F_x)_{\mathcal{U}}\big)$.

Corollary 2.9. *For $1 \leq p < \infty$, we have that L^p-operator algebras are closed under Banach space ultraproducts.*

Sketch of Proof. If $(L^p(\mu_x))_{x\in X}$ is a family of L^p spaces indexed by X and $(A_x)_{x\in X}$ is a family such that each A_x is a norm-closed subalgebra of $\mathcal{L}(L^p(\mu_x))$, then one checks that for any ultrafilter $\mathcal{U}$, the algebra $(A_x)_{\mathcal{U}}$ is a norm-closed subalgebra of $\Big(\mathcal{L}(L^p(\mu_x))\Big)_{\mathcal{U}}$. Having the definition of ultraproduct of operators, we can check that

$$\Big(\mathcal{L}(L^p(\mu_x))\Big)_{\mathcal{U}} = \mathcal{L}\Big(\big(L^p(\mu_x)\big)_{\mathcal{U}}\Big)$$

So the result follows from Theorem 2.8, which assures that $\big(L^p(\mu_x)\big)_{\mathcal{U}}$ is an L^p space. "□"

Department of Mathematics, University of Oregon, Eugene OR 97403-1222, USA.
E-mail address: alonsod@uoregon.edu