RESTRICTED ISOMETRIES IN FINITE DIMENSIONAL L^p-SPACES

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CU REU

GOAL

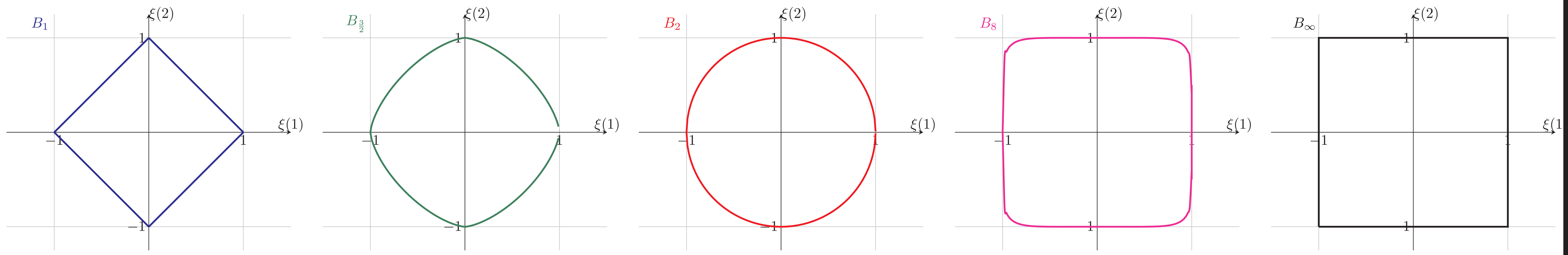
For $p \neq 2$ and $d \in \mathbb{Z}_{\geq 1}$, invertible linear isometries of $\ell_d^p = (\mathbb{C}^d, \| - \|_p)$ have been fully characterized by the Banach-Lamperti theorem. In the same spirit, we characterize, via set transformations, all the restricted isometries from $\ell_d^p = (\mathbb{C}^d, \| - \|_p)$ to $\ell_n^p = (\mathbb{C}^n, \| - \|_p)$ for any $d, n \in \mathbb{Z}_{\geq 1}$. As an application we count all the set transformations by counting all the possible shapes of the restricted isometries.

BACKGROUND

Let $p \in [1, \infty]$, let $d \in \mathbb{Z}_{\geq 2}$, put $X_d = \{1, \ldots, d\}$, and denote by ℓ_d^p to the space of functions $\xi \colon X_d \to \mathbb{C}$ equipped with the usual p-norm:

$$\|\xi\|_p = \begin{cases} \left(\sum_{j=1}^{d} |\xi(j)|^p \right)^{1/p} & \text{if } p \in [1, \infty), \\ \max_{j \in X_d} |\xi(j)| & \text{if } p = \infty \end{cases}$$

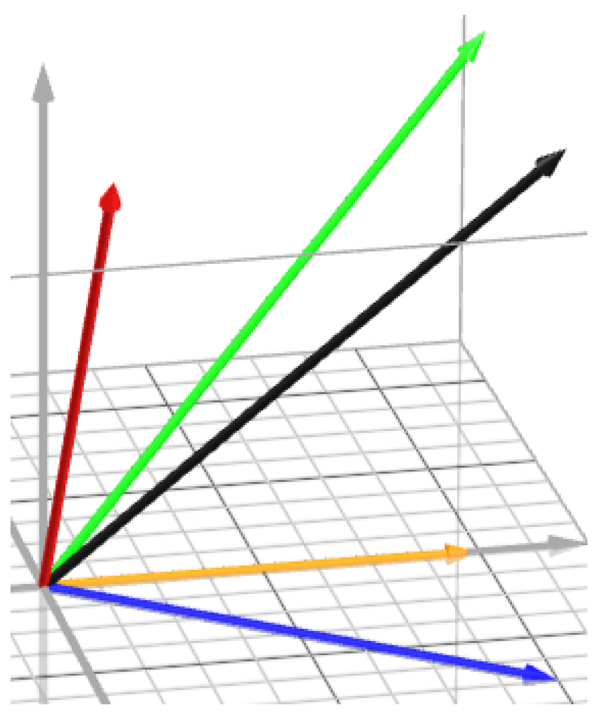
Let $B_p = \{\xi \in \mathbb{R}^2 : \|\xi\|_p = 1\}$, and notice that as $p \to \infty$, B_p becomes more and more like a square.



An *isometry* between two normed spaces V and W is a map $s \in \mathcal{L}(V, W)$ such that $\|s(\xi)\|_W = \|\xi\|_V$ for all $\xi \in V$. We will denote by $\mathrm{Isom}(V, W) \subseteq \mathcal{L}(V, W)$ to the subset consisting of all isometries from V to W.

If $\xi \in \ell_d^p$ and $E \subseteq X_d$, we get $\xi|_E$, the restriction of ξ to E, by defining the function $\xi|_E \colon X_d \to \mathbb{C}$ by

$$\xi|_E(j) = \begin{cases} \xi(j) & \text{if } j \in E \\ 0 & \text{if } j \notin E \end{cases}.$$



To the left is a visual example for a vector and possible corresponding restricted vectors. The unrestricted vector $\xi = \begin{bmatrix} 3 & 4 & 5 \end{bmatrix}^T$ is colored black. A restriction to the second and third entries (the "y" and "z" coordinates) results in the green vector $\xi|_{\{2,3\}} = \begin{bmatrix} 0 & 4 & 5 \end{bmatrix}^T$. Restricting to $\{1, 3\}$ and $\{1, 2\}$ entries gives the red $\xi|_{\{1,3\}} = \begin{bmatrix} 3 & 0 & 5 \end{bmatrix}^T$ and blue $\xi|_{\{1,2\}} = \begin{bmatrix} 3 & 4 & 0 \end{bmatrix}^T$ vectors respectively. It is possible to restrict to one entry. The orange vector $\xi|_{\{2\}} = \begin{bmatrix} 0 & 4 & 0 \end{bmatrix}^T$ shows this.

Let $d, n \in \mathbb{Z}_{\geq 1}$ and let $p \in [1, \infty)$. We say $s \in \mathcal{L}(\ell_d^p, \ell_n^p)$ is a *restricted isometry* if there is a subset $E \subseteq X_d$ such that

$$\|s(\xi)\|_p = \|\xi|_E\|_p$$

The space of restricted isometries from ℓ_d^p to ℓ_n^p is denoted as $\mathrm{RI}(\ell_d^p, \ell_n^p)$.

EXAMPLES: RESTRICTED ISOMETRIES (RI)

$$s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{RI}(\ell_3^p, \ell_2^p)$$

for any $p \in [1, \infty]$ (here $E = \{1, 3\}$)

$$s = \begin{bmatrix} 0 & (1/3)^{1/p} & 0 \\ 0 & (2/3)^{1/p} & 0 \end{bmatrix} \in \mathrm{RI}(\ell_3^p, \ell_2^p)$$

for a fixed $p \in [1, \infty)$ (here $E = \{2\}$)

SET TRANSFORMATIONS GIVE RESTRICTED ISOMETRIES

Let $d, n \in \mathbb{Z}_{\geq 1}$. A *set transformation* from X_d to X_n is a triple (E, F, S) where $E \subseteq X_d$, $F \subseteq X_n$, and $S \colon \mathcal{P}(E) \to \mathcal{P}(F)$ is a function satisfying the following conditions

1. $S(E_1) = \varnothing$ if and only if $E_1 = \varnothing$,
2. $S(E_1 \cup E_2) = S(E_1) \cup S(E_2)$ for any disjoint $E_1, E_2 \in \mathcal{P}(E)$,
3. $S(E \setminus E_1) = F \setminus S(E_1)$ for any $E_1 \in \mathcal{P}(E)$.

Let $E = \{1, 2\} \subset X_3$, $F = \{1, 2, 3\} \subset X_4$ and define $S \colon \mathcal{P}(E) \to \mathcal{P}(F)$ so that

$$S(\{1\}) = \{1, 2\}, \quad S(\{2\}) = \{3\}$$

We denote by $\mathrm{ST}(d, n)$ the set of all set transformations from X_d to X_n. For each $S \in \mathrm{ST}(d, n)$, define $S_* \colon \mathbb{C}^d \to \mathbb{C}^n$ for $j \in X_d$ by

$$S_*(\delta_j) = \begin{cases} \sum_{k \in S(\{j\})} \delta_k & \text{if } j \in E \\ \mathbf{0}_n & \text{if } j \notin E \end{cases}$$

$$S_* = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $p \in [1, \infty)$. Suppose that $\mathbf{c} \colon F \to \mathbb{R}_{>0}$ satisfies $\|\mathbf{c}|_{S(E_{S,k})}\|_p = 1$ for every $k \in F$, where

$$E_{S,k} := \bigcap_{\{K \in S(\mathcal{P}(E)) \colon k \in K\}} S^{-1}(K)$$

$$E_{S,1} = E_{S,2} = \{1\}, E_{S,3} = \{2\}$$

and that $g \colon F \to \mathbb{T}$ is a given function. We define $s(E, F, S, \mathbf{c}, g) \colon \ell_d^p \to \ell_n^p$ by

$$(s(E, F, S, \mathbf{c}, g)\xi)(k) = \begin{cases} \mathbf{c}(k)(S_*(\xi))(k)g(k) & \text{if } k \in F \\ 0 & \text{if } k \in X_n \setminus F \end{cases}.$$

A direct computation shows that $s(E, F, S, \mathbf{c}, g) \in \mathrm{RI}(\ell_d^p, \ell_n^p)$.

MAIN THEOREM

Our main result is that, when $p \in [1, \infty) \setminus \{2\}$, then $s \in \mathrm{RI}(\ell_d^p, \ell_n^d)$ if and only if there is a set transformation (E, F, S), a vector $\mathbf{c} \colon F \to \mathbb{R}_{>0}$, and a function $g \colon F \to \mathbb{T}$ such that

$$s = s(E, F, S, \mathbf{c}, g)$$

APPLICATION: COUNTING SET TRANSFORMATIONS

A key known result (Clarkson's) gives that two vectors $\xi, \eta \in \ell_d^p$, $p \neq 2$, have disjoint support if and only if $\|\xi + \eta\|_p^p + \|\xi - \eta\|_p^p = 2\|\xi\|_p^p + 2\|\eta\|_p^p$. Therefore, by applying our theorem we get

$$\mathrm{card}(\mathrm{ST}(d, n)) = (d + 1)^n - 1$$