Differential Geometry: Qual. Review

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1.1 Theorems and Definitions.

Definition. Suppose M is a topological space. We say that M is a **topological manifold of dimension** n or a **topological n-manifold** if it has the following properties:

- M is a Hausdorff space, i.e. for every pair of distinct points $p, q \in M$, there are disjoint open subsets $U, V \subset M$ such that $p \in U$ and $q \in V$.
- M is second-countable, i.e. there exists a countable basis for the topology of M.
- M is locally Euclidean of dimension n, i.e. each point of M has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$. ▲

Definition. Let M be a topological n-manifold. A **coordinate chart** (or just a chart) on M is a pair (U, φ) where U is an open subset of M and $\varphi : U \to \varphi(U)$ is a homeomorphism from U to an open subset $\varphi(U) \subset \mathbb{R}^n$.

Given a chart (U, φ), we call the set U a **coordinate domain**. The map φ is called a **(local) coordinate map**, and the component functions $x^1, \ldots, x^n$ of φ, defined by

$$\varphi(p) = (x^1(p), \ldots, x^n(p)),$$

are called l**ocal coordinates** on U. ▲

Definition. Let M be a topological n-manifold. If (U, φ) and (V, ψ) are two charts such that $U \cap V \neq \varnothing$, the composite map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

is called the **transition map from φ to ψ**.

Two charts (U, φ) and (V, ψ) are said to be **smoothly compatible** if either $U \cap V = \varnothing$ or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism.

An **atlas** for M is a collection of charts whose domains cover M. An atlas $\mathcal{A}$ is called a **smooth atlas** if any two charts in $\mathcal{A}$ are smoothly compatible with each other. A smooth atlas $\mathcal{A}$ on M is maximal if it is not properly contained in any larger smooth atlas.

A **smooth manifold** is a pair $(M, \mathcal{A})$ where M is a topological manifold and $\mathcal{A}$ is a maximal smooth atlas. ▲

Definition. Suppose M is a smooth n-manifold, k is a nonnegative integer, and $f : M \to \mathbb{R}^k$ is any function. We say that f is a **smooth function** if for every $p \in M$, there exists a smooth chart (U, φ) for M whose domain contains p and such that the composite function $f \circ \varphi^{-1}$ is smooth on the open subset $\varphi^{-1}(U) \subseteq \mathbb{R}^n$.

When $k = 1$, we denote by $\mathcal{C}^\infty(M)$ the space of all **real valued smooth functions on M.** ▲

Definition. Suppose M is a smooth n-manifold and $p \in M$. A **derivation at** p is a linear map $w : \mathcal{C}^\infty(M) \to \mathbb{R}$ such that

$$w(f \cdot g) = w(f)g(p) + f(p)w(g)$$

for any $f, g \in \mathcal{C}^\infty(M)$. We denote the space of all derivations at p by T_pM, this is the tangent space to M at point p. ▲

Theorem. *Suppose M is a smooth n-manifold and $p \in M$. Let (U, φ) be a chart such that $p \in U$ with local coordinates $(x^1, \ldots, x^n)$. Then, the map $\frac{\partial}{\partial x^i}\big|_p : \mathcal{C}^\infty(M) \to \mathbb{R}$ given by*

$$\frac{\partial}{\partial x^i}\Big|_p (f) := \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}(\varphi(p))$$

is a derivation at p. Furthermore, $T_pM \cong \mathbb{R}^n$ and any derivation at p has the form

$$\sum_{i=1}^n a^i \frac{\partial}{\partial x^i}\Big|_p$$

for some $a^1, \ldots, a^n \in \mathbb{R}$.

Definition. Let M, N be smooth manifolds, and let $F : M \to N$ be any map. We say that F is a **smooth map** if for every $p \in M$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and the composite map $\psi \circ F \circ \varphi^{-1}$is smooth from $\varphi(U)$ to $\psi(V)$. ▲

Definition. Let M, N be smooth manifolds. If $F : M \to N$ is smooth and $p \in M$, we define the differential $dF(p) : T_pM \to T_{F(p)}N$ as

$$[dF(p)(w)](g) = w(g \circ F)$$

for any $w \in T_pM$ and $g \in \mathcal{C}^\infty(N)$. ▲

Lemma. *Let M, N and P be smooth manifolds and $F : M \to N$ smooth. If $p \in M$, then*

- *$dF(p)(w)$ is indeed a derivation at $F(p)$.*
- *$dF(p) : T_pM \to T_{F(p)}N$ is a linear map.*
- *If $G : N \to P$ is smooth, then*

$$d(G \circ F)(p) = dG(F(p)) \circ dF(p)$$

- $d\mathrm{Id}_M(p) = \mathrm{Id}_{T_pM}$
- *If F is a diffeomorphism, then $dF(p)$ is an isomorphism and*

$$[dF(p)]^{-1} = dF^{-1}(p)$$

Definition. Let M, N be smooth manifolds and $F : M \to N$ a smooth map. Given $q \in N$, we say that q is a **regular value** of F if for every point $p \in F^{-1}(\{q\})$ we have $dF(p) : T_pM \to T_{F(p)}N$ is a surjective linear map ▲

Theorem. *(Regular Value Theorem) Suppose M is a smooth m-manifold and N a smooth n-manifold. Let $F : M \to N$ be smooth and let $q \in N$ be a regular value of F. Then, $F^{-1}(\{q\})$ is a smooth submanifold of M with dimension $m - n$.*

Definition. Let M be a topological manifold. A **real vector bundle of rank k over M** is a topological space E together with a surjective continuous map $\pi : E \to M$ such that

(i) For each $p \in M$, the fiber $E_p = \pi^{-1}(\{p\})$ is a k-dimensional real vector space.

(ii) For each $p \in M$, there is $U \subset M$ with $p \in U$ and a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$, called a local trivialization of E over U, such that

  . if $\pi_U : U \times \mathbb{R}^k \to U$ is the canonical projection, then $\pi_U \circ \Phi = \pi$.
  . For each $q \in U$, $\Phi\big|_{E_q} : E_q \xrightarrow{\sim} \{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$ is a vector space isomorphism

If M, E are smooth manifolds, π is smooth and the maps Φ can be chosen to be diffeomorphisms, then E is called a **smooth vector bundle.**

A **line bundle over** M is a vector bundle of rank 1. ▲

Definition. A real vector bundle (E, π) of rank k over M is said to be **trivial** if there is a global trivialization, that is if there is a homeomorphism $\Phi : E \overset{\sim}{\to} M \times \mathbb{R}^k$. If (E, π) is a smooth vector bundle and the global trivialization is a diffeomorphism, we say that (E, π) is smoothly trivial. ▲

Definition. Consider a real vector bundle (E, π) of rank k over M. A **section** of E is a continuous map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{id}_M$. A **local section** of E is a continuous map $\sigma : U \to E$ such that $\pi \circ \sigma = \mathrm{id}_U$ for some open set $U \subset M$. If E is a smooth vector bundle and the map σ is smooth, then we get a **smooth (local) section**. We usually denote the space of smooth sections of E as $\Gamma(E)$. ▲

Definition. Consider a real vector bundle (E, π) of rank k over M and $U \subset M$ an open subset. A r-tuple of local sections over U, say $(\sigma_1, \ldots, \sigma_r)$, is said to be **linearly independent** if for each $p \in U$, the r-tuple $(\sigma_1(p), \ldots, \sigma_r(p))$ is linearly independent in E_p. Similarly, $(\sigma_1, \ldots, \sigma_r)$ is said to **span** E if for each $p \in U$, the r-tuple $(\sigma_1(p), \ldots, \sigma_r(p))$ span E_p.

A **local frame for** E **over** U is an ordered k-tuple $(\sigma_1, \ldots, \sigma_k)$ of linearly independent local sections over U that span E; thus $(\sigma_1(p), \ldots, \sigma_k(p))$ is a basis for the fiber E_p for each $p \in U$. It is called a **global frame** if $U = M$. If E is a smooth vector bundle, a local or global frame is a **smooth frame** if each σ_i is a smooth section. ▲

Theorem. *Every smooth (local) frame for a smooth vector bundle is associated with a smooth (local) trivialization in the following way:*

- *If $s_1, \ldots, s_k : U \to E$ is a frame over $U \subseteq M$, we get a trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ by*

$$\Phi(x) := (\pi(x), \lambda^1(x), \ldots, \lambda^k(x))$$

  where $x = \lambda^i(x) s_i(\pi(x)) \in E_{\pi(x)}$.

- *Conversely, if $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ is a trivialization and $e_1, \ldots, e_k$ is the standard basis for $\mathbb{R}^k$, we define sections $\sigma_i : U \to E$ by*

$$\sigma_i(p) := \Phi^{-1}(p, e_i)$$

  Then, $s_1, \ldots, s_k$ is a frame over U.

Corollary. *A smooth vector bundle is smoothly trivial if and only if it admits a smooth global frame.*

Definition. Let M be a smooth n-manifold. We define the tangent bundle of M, denoted by TM, as follows

$$TM = \coprod_{p \in M} T_pM = \{(p, w) : w \in T_pM\}$$

▲

Lemma. *Let M be a smooth n-manifold. The tangent bundle TM together with the natural projection map $\pi : TM \to M$, is a real smooth vector bundle of rank n over M.*

Definition. Let M be a smooth n-manifold. A vector field is a section of the tangent bundle TM. That is, a **vector field** is a continuous map $X : M \to TM$ so that $\pi \circ X = \mathrm{id}_M$. ▲

Definition. Let M be a smooth n-manifold and (U, x^i) be any smooth coordinate chart. We define a map $\frac{\partial}{\partial x^i} : M \to TM$ by

$$\frac{\partial}{\partial x^i}(p) := \left(p, \frac{\partial}{\partial x^i}\Big|_p \right)$$

▲

Theorem. *Let M be a smooth n-manifold and (U, x^i) be any smooth coordinate chart. Then, $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^k})$ is a local frame for TM over U.*

Definition. Let M be a smooth n-manifold and (U, x^i) be any smooth coordinate chart. If $X : M \to TM$ is a vector field and $p \in U$, we have

$$X(p) = \left(p, X^i(p) \frac{\partial}{\partial x^i}\Big|_p \right)$$

The n smotth maps $X^i : U \to \mathbb{R}$ are called the **component functions** of X in the given chart. In this case, we abuse notation and say that X can be written locally as

$$X = X^i \frac{\partial}{\partial x^i},$$

so that $X(p)$ is representing the element $X^i(p) \frac{\partial}{\partial x^i}\Big|_p \in T_pM$. ▲

Definition. Let M be a smooth n-manifold. A **smooth vector field** X on M can be alternatively defined as a linear map $X : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ such that $X(fg) = X(f)g + fX(g)$ for any $f, g \in \mathcal{C}^\infty(M)$. Thus, if for a smooth coordinate chart (U, x^i) we have $X = X^i \frac{\partial}{\partial x^i}$, then for any $f \in \mathcal{C}^\infty(M)$, the function $X(f) : M \to \mathbb{R}$ is given by

$$X(f)(p) = X^i(p) \frac{\partial}{\partial x^i}\Big|_p (f)$$

▲

Lemma. *Let M be a smooth n-manifold. The space of smooth vector fields, denoted by $\Gamma(TM)$ is a Lie algebra. That is, for any $X, Y \in \Gamma(TM)$ the Lie bracket $[X, Y] := XY - YX$ is again a smooth vector field.*

Lemma. *Let M be a smooth n-manifold and $X, Y \in \Gamma(TM)$ with coordinate expressions $X = X^i \frac{\partial}{\partial x^i}, Y = Y^i \frac{\partial}{\partial x^i}$ in terms of some smooth local coordinates (x^i) for M. Then $[X, Y] \in \Gamma(TM)$ has the following coordinate expression:*

$$[X, Y] = \big(X(Y^i) - Y(X^i)\big) \frac{\partial}{\partial x^i}$$

Theorem. *(Properties of the Lie Bracket). The Lie bracket satisfies the following identities for all $X, Y, Z \in \Gamma(TM)$*

- *BILINEARITY: For $a, b \in \mathbb{R}$,*

  $$[aX + bY, Z] = a[X, Z] + b[Y, Z] \quad \textit{and} \quad [X, aY + bZ] = a[X, Y] + b[X, Z]$$

- *ANTISYMMETRY:*

  $$[X, Y] = -[X, Y]$$

- *JACOBI IDENTITY:*

  $$\Big[X, [Y, Z]\Big] + \Big[Y, [Z, X]\Big] + \Big[Z, [X, Y]\Big] = 0$$

- *for $f, g \in \mathcal{C}^\infty(M)$,*

  $$[fX, gY] = fg[X, Y] + (fX(g))Y - (gY(f))X$$

Definition. Let M be a smooth n-manifold. A **Riemannian metric** on M is a family of (positive definite) inner products

$$g = \{g_p \colon T_pM \times T_pM \longrightarrow \mathbb{R}\}_{p \in M},$$

such that, for all smooth vector fields $X, Y : M \to TM$, the map

$$p \mapsto g_p(X(p), Y(p))$$

defines a smooth function $M \to \mathbb{R}$.

Given local coordinates (x^i) we have

$$g_{ij}(p) := g_p\left(\frac{\partial}{\partial x^i}(p), \frac{\partial}{\partial x^j}(p)\right)$$

Since $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n})$ is a local frame for TM, we also say that

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

so we may think of g as a smooth positive definite inner product on $\Gamma(TM)$.

A smooth manifold M together with a given Riemannian metric g is called a **Riemannian manifold** and denoted by (M, g). ▲

Theorem. *(Existence of Riemannian Metrics). Every smooth manifold with or without boundary admits a Riemannian metric.*

Remark. Let $\sigma : [a, b] \to M$ be a smooth parametrized curve on a smooth n-manifold M . At any time $t \in [a, b]$, the velocity $\dot{\sigma}(t)$of σ acts on functions by

$$\dot{\sigma}(t)f = \frac{d}{dt}(f \circ \sigma)(t)$$

If we write the local coordinate representation of σ as $\sigma(t) = (\sigma^1(t), \dots, \sigma^n(t))$, then $\dot{\sigma}(t) = \dot{\sigma}^i(t)\frac{\partial}{\partial x^i}$ (A dot always denotes the ordinary derivative with respect to t.) Thus, we think of $\dot{\sigma}(t)$ as an element of $T_{\sigma(t)}M$. ▼

Definition. Let (M, g) be a connected Riemannian manifold and let $\sigma : [a, b] \to M$ be a smooth parametrized curve in M. The length of σ is defined as

$$L(\sigma) := \int_a^b \sqrt{g_{\sigma(t)}\big(\dot{\sigma}(t), \dot{\sigma}(t)\big)}dt$$

The **distance function** $d_g : M \times M \to \mathbb{R}$ is defined by

$$d_g(p, q) := \inf\{L(\sigma) : \sigma \in \mathcal{C}^\infty([0, 1], M), \sigma(0) = p, \sigma(1) = q\}$$

▲

Theorem. *Let (M, g) be a connected Riemannian manifold. Then,*

- *(M, d_g) is a metric space.*

- *The topology induced by d_g is the same as the original topology on M.*
- *If d_g is complete, then between any two points in M, there exists a curve that minimizes the distance.*

Definition. Let M be a smooth manifold and $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ an open cover of M indexed by a set A. A **smooth partition of unity** subordinated to $\mathcal{U}$ is a collection $\{\phi_\alpha\}_{\alpha \in A}$ in $\mathcal{C}^\infty(M)$ such that

- $0 \leq \phi_\alpha(p) \leq 1$ for all $p \in M$ and all $\alpha \in A$.
- $\operatorname{supp}(\phi_\alpha) := \overline{\{p \in M : \phi_\alpha(p) \neq 0\}} \subset U_\alpha$
- The family of supports $\{\operatorname{supp}(\phi_\alpha)\}_{\alpha \in A}$ is locally finite, meaning that every point has a neighborhood that intersects $\operatorname{supp}(\phi_\alpha)$ for only finitely many values of α
- $\sum_n \phi_{\alpha \in A}(p) = 1$ for all $p \in M$.

▲

Theorem. *(Existence of Partitions of Unity). Suppose M is a smooth manifold, and $\mathcal{U}$ is any indexed open cover of M. Then there exists a smooth partition of unity subordinate to $\mathcal{U}$.*

Theorem. *Suppose M is a smooth manifold, $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ is any indexed open cover of M, and let g_α be a Riemannian metric on U_α. If $\{\phi_\alpha\}_{\alpha \in A}$ is a smooth partition of unity subordinated to $\mathcal{U}$, then*

$$\sum_{\alpha \in A} \phi_\alpha g_\alpha$$

is a Riemannian metric on M.

Definition. Let M be a smooth manifold. A **connection** on M is a map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ (we put $\nabla_X Y := \nabla(X, Y)$) such that

- ∇ is bilinear.
- $\nabla_{fX} Y = f \nabla_X Y$ for any $X, Y \in \Gamma(TM)$ and $f \in \mathcal{C}^\infty(M)$.
- $\nabla_X fY = X(f)Y + f \nabla_X Y$ for any $X, Y \in \Gamma(TM)$ and $f \in \mathcal{C}^\infty(M)$.

▲

Theorem. *Let ∇ be a connection on a smooth manifold M. Then ∇ is local, that is for $X, Y \in \Gamma(TM)$ we have that*

- *If $U \subset M$ is such that $X\big|_U = 0$, then $(\nabla_X Y)\big|_U = 0$*
- *If $V \subset M$ is such that $Y\big|_V = 0$, then $(\nabla_X Y)\big|_V = 0$*

Remark. From now on, the following the convenient notation will be used

$$\partial_{x^i} := \frac{\partial}{\partial x^i}$$

▼

Definition. Since any connection ∇ is local, in local coordinates $(U, (x^i))$ we have

$$\nabla_{\partial_{x^i}} \partial_{x^j} = {\Gamma_{ij}}^k \partial_{x^k}$$

The smooth functions ${\Gamma_{ij}}^k : U \to \mathbb{R}$ are known as **Christoffel symbols** of ∇ with respect to the frame $(\partial_{x^1}, \ldots, \partial_{x^n})$. ▲

Theorem. *Suppose M is a smooth manifold, $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ is any indexed open cover of M, and let ${}^\alpha\nabla$ be a connection on U_α. If $\{\phi_\alpha\}_{\alpha \in A}$ is a smooth partition of unity subordinated to $\mathcal{U}$, then*

$$\sum_{\alpha \in A} \phi_\alpha {}^\alpha\nabla$$

is a connection on M.

Definition. Let M be a smooth manifold, g a Riemannian metric on M and ∇ a connection on M. We define $r_g : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \mathcal{C}^\infty(M)$ by

$$r_g(X, Y, Z) := X\big(g(Y, Z)\big) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$$

If $r_g \equiv 0$, we say that ∇ is **Riemannian**. ▲

Definition. Let M be a smooth manifold and ∇ a connection on M. We define the **torsion tensor** $T : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ by

$$T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$$

If $T \equiv 0$, we say that ∇ is **torsion free**. ▲

Lemma. *Both r_g and T are tensorial, that is for any $f \in \mathcal{C}^\infty(M)$*

$$r_g(fX, Y, Z) = r_g(X, fY, Z) = r_g(X, Y, fZ) = fr_g(X, Y, Z)$$

$$T(fX, Y) = T(X, fY) = fT(X, Y)$$

Corollary. *Both r_g and T are local.*

Lemma. *On local coordinates (x^i), torsion free is equivalent to ${\Gamma_{ij}}^k = {\Gamma_{ji}}^k$ for all i, j, k.*

Theorem. *Let M be a smooth manifold and ∇ a connection on M. The following are equivalent*

(i) *∇ is torsion free.*

(ii) *Given any point $p \in M$, there exist local coordinates centered at p so that ${\Gamma_{ij}}^k(p) = 0$.*

Theorem. *Let (M, g) be a Riemannian manifold. Then, there exists a unique connection called* ***the Levi-Civita connection*** *so ∇ is torsion free and Riemannian.*

Definition. Let (M, g) be a Riemannian manifold and ∇ a connection on M. We define the **Christoffel symbols** of the second kind as

$$\Gamma_{ijk} := g(\nabla_{\partial_{x^i}} \partial_{x^j}, \partial_{x^k})$$

for local coordinates (x^i). ▲

Lemma. *Let (M, g) be a Riemannian manifold and ∇ a connection on M. Then,*

$${\Gamma_{ij}}^k = g^{kl}\Gamma_{ijl}$$

where (g^{kl}) is the inverse matrix of (g_{ij})

Remark. From now on, the following the convenient notation will be used

$$g_{ij/k} := \partial_{x^k} g_{ij}$$

▼

Lemma. *Torsion free is equivalent to $\Gamma_{ijk} = \Gamma_{jik}$ for all i, j, k. Riemannian connection is equivalent to $\Gamma_{ijk} + \Gamma_{ikj} = g_{jk/i}$*

Theorem. *Let (M, g) be a Riemannian manifold and ∇ the Levi-Civita connection. Then, the* ***Kozul formula*** *holds, that is*

$$\Gamma_{ijk} = \frac{1}{2}\left(g_{jk/i} + g_{ik/j} - g_{ij/k}\right)$$

Example. Take $M = \mathbb{R}^n$ and let $X = X^i\partial_{x^i}, Y = Y^i\partial_{x^i}$ be vector fields on $\mathbb{R}^n$. Then, the **euclidian connection** given by

$$\nabla^{\mathrm{e}}_X Y := X(Y^i)\partial_{x^i} = X^j\partial_{x^j}(Y^i)\partial_{x^i}$$

is the Levi-Civita connection for $\mathbb{R}^n$. Furthermore, in this case ${\Gamma_{ij}}^k = 0$ for all i, j, k. ▼

Example. Let $S \subset \mathbb{R}^3$ be a surface (i.e. a smooth 2-manifold). If X, Y are vector fields in $\mathbb{R}^3$ that are tangent to S, then $\nabla^{\mathrm{e}}_X Y$ is not necessarily tangent to S. However, if for any point $p \in S$, $\pi_S : T_p\mathbb{R}^3 \to T_pS$ is the orthogonal projection back to the surface, then $\pi_S(\nabla^{\mathrm{e}}_X Y)$ is tangent to S.

In fact, we have that $\pi_S \circ \nabla^{\mathrm{e}}$ is the Levi-Civita connection of S. ▼

Definition. Let (M, g) be a Riemannian manifold and ∇ the Levi-Civita connection. A smooth curve $\sigma : [a, b] \to M$ is a geodesic if

$$\nabla_{\dot\sigma}\dot\sigma = 0$$

▲

Theorem. *Let (M, g) be a Riemannian n-manifold and ∇ the Levi-Civita connection. Consider a smooth curve $\sigma : [a, b] \to M$ with local coordinate representation given by $\sigma(t) = (x^1(t), \ldots, x^n(t))$ (thus $\dot\sigma = \dot x^i\partial_{x^i}$). Then the geodesic equation above is*

$$\ddot x^k + \sum_{ij} \dot x^i \dot x^j {\Gamma_{ij}}^k = 0$$

for all $k = 1, \ldots, n$.

Theorem. *Let (M, g) be a Riemannian n-manifold, ∇ the Levi-Civita connection, and $\sigma : [a, b] \to M$ a smooth curve. Suppose that $M \subset \mathbb{R}^m$ for some m and let ∇^e be the euclidean connection in $\mathbb{R}^m$. Then, if for any point $p \in M$, $\pi_M : T_p\mathbb{R}^m \to T_pM$ is the ortogonal projection back to M, we have*

$$\nabla_{\dot\sigma}\dot\sigma = \pi_M(\nabla^e_{\dot\sigma}\dot\sigma) = \pi_M\ddot\sigma$$

and therefore σ is a geodesic if and only if $\ddot\sigma \perp M$.

Theorem. *(Geodesics travel at constant velocity) Let* (M, g) *be a Riemannian n-manifold,* ∇ *the Levi-Civita connection, and* $\sigma : [a, b] \to M$ *a geodesic. Then,* $\frac{d}{dt} g(\dot{\sigma}(t), \dot{\sigma}(t)) \overset{R}{=} 2g(\nabla_{\dot{\sigma}}\dot{\sigma}, \dot{\sigma}) = 0$.

Lemma. *Let* (M, g) *be a Riemannian n-manifold,* ∇ *the Levi-Civita connection, and* $\gamma : [a, b] \to M$ *a smooth curve. Then,* γ *is an umparametrized geodesic iff*

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \alpha\dot{\gamma}$$

for some smooth function $\alpha : [a, b] \to \mathbb{R}$.

Lemma. *Let* (M, g) *be a Riemannian manifold and* $\sigma : I \to M$ *a geodesic. Then if* $\lambda > 0$ *is such taht* $\tilde{\sigma}(t) := \sigma(\lambda t)$ *is defined,* $\tilde{\sigma}$ *is also a geodesic.*

Definition. Let (M, g) be a Riemannian manifold. M is said to be geodesically complete if all geodesics extend for infinite time. ▲

Theorem. *Let* (M, g) *be a Riemannian manifold,* ∇ *the Levi-Civita connection, and suppose that* M *is compact. Then,* M *is geodesically complete.*

Lemma. *Let* (M, g) *be a Riemannian n-manifold. If* $p \in M$ *and* $w \in T_pM$. *There exists a unique geodesic* $\sigma_w : I \to M$ *such that*

- $[0, 1] \subset I$
- $\sigma_w(0) = p$
- $\dot{\sigma}_w(0) = w$

Furthermore, if $\varepsilon > 0$ *and the geodesic* σ_v *is defined for* $|t| < \varepsilon$*, then for* $\lambda > 0$ *the curve* $\gamma = \sigma_{\lambda v}$ *is a geodesic defined for* $|t| < \varepsilon/\lambda$ *and* $\gamma(t) = \sigma_v(\lambda t)$ *for* $|t| < \varepsilon/\lambda$.

Remark. Intuitively, the second part of the previous lemma means that since the speed of a geodesic is constants we an go over its trace within a prescribed time by adjusting our speed appropriately. ▼

Definition. Let (M, g) be a Riemannian manifold. If $p \in M$ and $w \in T_pM$ is non zero, we set

$$\exp_p(w) := \sigma_w(1) \quad \text{and} \quad \exp_p(0) := p$$

Geometrically, the construction corresponds to laying off (if possible) a length equal to $\|w\|_{g_p} := g_p(w,w)$ along the geodesic that passes trough p in the direction of v; the point of M thus obtained is denoted by $\exp_p(w)$. ▲

Theorem. *Let (M,g) be a Riemannian manifold. Given $p \in M$ there is $\varepsilon > 0$ such that the map $\exp_p : B_\varepsilon(0) \to M$ is a diffeomorphism. Here $B_\varepsilon(0) = \{w \in T_pM : \|w\|_{g_p} < \varepsilon\}$. Furthermore, $\exp_p(tw) = \sigma_w(t)$ and $d\exp_p(0) : T_pM \to T_pM$ is the identity map.*

Corollary. *Let (M,g) be a connected Riemannian manifold. For $p \in M$ and $w \in T_pM$, we let $q := \exp_p(w)$. Then, the curve $\gamma : [0,1] \to M$ given by $\gamma(t) := \exp_p(tw)$ is the geodesic from p to q of length $\|w\|_{g_p}$. Furthermore, if σ is any curve from p to q, then $L(\sigma) \geq L(\gamma)$, and equality holds iff σ is a reparametrization of γ. Also,*

$$d_g(p,q) = \|w\|_{g_p}$$

Theorem. ***(Hopf Rinow)*** *Let (M,g) be a connected Riemannian manifold. Then the following statements are equivalent:*

(1) (M,d_g) is a complete metric space.

(2) M is geodesically complete.

(3) All geodesics through given base point extend for infinite time

Furthermore, any of these equivalent conditions imply that for any two points $p,q \in M$ there exists a geodesic σ from p to q such that $L(\sigma) = d_g(p,q)$.

Definition. Let M,N be smooth manifolds and $F : M \to N$ a smooth map. The push forward $dF : TM \to TN$ is defined by

$$dF(p,w) := (F(p),[dF(p)](w))$$

The pull back $F^* : \mathcal{C}^\infty(N) \to \mathcal{C}^\infty(M)$ is given by

$$F^*(h) := h \circ F$$

for any $h \in \mathcal{C}^\infty(N)$ ▲

Definition. Let (M,g^M) and (N,g^N) be two Riemannian manifolds, and $F : M \to N$ be a diffeomorphism. Then, F is called a **Riemannian isometry,** if

$$g^M = F^* g^N$$

or pointwise

$$g_p^M(u,v) = g_{F(p)}^N(dF(p)(u), dF(p)(v)) \qquad \forall p \in M, \forall u,v \in T_pM.$$

Moreover, a smooth map $F : M \to N$ is called a local isometry at $p \in M$ if there is a neighbourhood $U \subset M$, $p \in U$, such that $F : U \to F(U)$ is a diffeomorphism satisfying the previous relation. ▲

Lemma. *Let (M, g^M) and (N, g^N) be two Riemannian manifolds. If $F : M \to N$ is a Riemannian isometry, then $F : (M, d_{g^M}) \to (N, d_{g^N})$ is an isometry of the underlying metric space. That is, for any $p, q \in M$*

$$d_{g^M}(p,q) = d_{g^N}(F(p), F(q))$$

Conversely, if F is an isometry of the underlying metric space and bijective, then F is a Riemannian isometry.

Definition. A Riemannian manifold (M, g) is said to be **homogeneous**, if given any $p, q \in M$ there is a Riemannian isometry $\Phi : M \to M$ such that $\Phi(p) = q$. ▲

Theorem. *Let (M, g) be homogeneous. Then M is geodesically complete.*

Theorem. *A Riemannian manifold (M, g) and $\Phi : M \to M$ a Riemannian isometry. Let $\mathcal{F} = \{p \in M : \Phi(p) = p\}$ the fixed point set of Φ. Then $\mathcal{F}$ is a smooth submanifold of M that is totally geodesic. That is, for all geodesics σ such that $\sigma(0) \in \mathcal{F}$, we have $\sigma(t) \in \mathcal{F}$ for all t.*

2 Winter 2018: Math 638

2.1 Theorems and Definitions.

Definition. Let (M, g) be a Riemannian manifold and ∇ the Levi-Civita connection. For $X, Y \in \Gamma(TM)$, we set

$$R(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

Thus, $R(X,Y) : \Gamma(TM) \to \Gamma(TM)$. For $X, Y, Z, W \in \Gamma(TM)$, we set

$$R(X,Y,Z,W) := g\big(R(X,Y)Z, W\big)$$

▲

Lemma. *(Curvature Properties)*

(i) *The map* $(X,Y,Z) \mapsto R(X,Y)Z$ *is multilinear and tensorial.*

(ii) $R(X,Y) = -R(Y,X)$.

(iii) *The Bianchi identity holds, that is*

$$R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0$$

(iv) $R(X,Y)$ *is skew symmetric, that is*

$$g\big(R(X,Y)Z, W\big) = -g\big(Z, R(X,Y)W\big).$$

Equivalently $R(X,Y,Z,W) = -R(X,Y,W,Z)$.

(v) $R(X,Y,Z,W) = R(Z,W,X,Y)$.

(vi) *The identities above are the only universal curvature identities for the Levi-Civita connection, all others are consequences of these ones.*

Definition. Let (M, g) be a Riemannian n-manifold and ∇ the Levi-Civita connection. If (U, x^i) are local coordinates we set

$$R_{ijkl} := R(\partial_{x^i}, \partial_{x^j}, \partial_{x^k}, \partial_{x^l}) = g\big(R(\partial_{x^i}, \partial_{x^j})\partial_{x^k}, \partial_{x^l}\big)$$

for all $i, j, k, l \in \{1, \ldots, n\}$. ▲

Theorem. *Let (M,g) be a Riemannian n-manifold*

and ∇ the Levi-Civita connection. If (U,x^i) are local coordinates around $p \in U$ such that

$$g_{ij/k}(p) = 0$$

for all $i,j,k \in \{1,\dots,n\}$, then

$$R_{ijkl}(p) = \frac{1}{2}\Big(g_{jl/ik} + g_{ik/jl} - g_{il/jk} - g_{jk/il}\Big)(p)$$

for all $i,j,k,l \in \{1,\dots,n\}$.

Lemma. *(Curvature Symmetries)*

(i) $R_{ijkl} = -R_{jikl}$

(ii) $R_{ijkl} + R_{kijl} + R_{jkil} = 0$

(iii) $R_{ijkl} = -R_{ijlk}$

(iv) $R_{ijkl} = R_{klij}$

(v) These are the only universal curvature symmetries. The rest are algebraic consequences of these, that is, given a collection A_{ijkl} satisfying (i)-(iv) above, there exists a Riemannian metric g such that $R_{ijkl}(p) = A_{ijkl}$.

Definition. Let (M,g) be a Riemannian n-manifold. The **scalar curvature** is given by

$$\tau := g^{il}g^{jk}R_{ijkl}$$

where (g^{kl}) is the inverse matrix of (g_{ij}). The Gaussian curvature is $\frac{\tau}{2}$. ▲

Definition. Let (M,g) be a Riemannian manifold. We define

$$\rho(X,Y) := \mathrm{tr}(Z \mapsto R(Z,X)Y)$$

The **Ricci tensor** is then given on local coordinates (x^i) by

$$\rho_{ij} := \rho(\partial_{x^i}, \partial_{x^j}) = g^{kl}R_{iklj}$$

▲

Remark. Notice that

$$\mathrm{tr}_g(\rho) = g^{ij}\rho_{ij} = g^{ij}g^{kl}R_{iklj} = \tau$$

▼

Theorem. *(Myers) Let (M, g) be a geodesically complete and connected Riemannian manifold. Suppose there exist $\varepsilon > 0$ such that $\rho(X, X) \geq \varepsilon g(X, X)$ for all vector fields X (i.e. the Ricci tensor is uniformly positive). Then M is compact and the fundamental group $\pi_1 M$ is finite.*

Definition. Suppose M is a hypersurface, that is an n-manifold embedded in $\mathbb{R}^{n+1}$ with Riemannian metric given by the induced metric. Take local coordinates $(u^1, \ldots, u^n)$ and local parametrization $F : U \to \mathbb{R}^m$

$$F(u^1, \ldots, u^n) = (x^1, \ldots, x^{n+1})$$

such that $\operatorname{rank}(\frac{\partial x^i}{\partial u^j}) = n$. The **first fundamental form** is the matrix given by

$$g_{ij} = \partial_{u^i} F \cdot \partial_{u^j} F \cdot$$

The **second fundamental form** is defined as

$$L(X, Y) = (\nabla^e_X Y) \cdot N$$

where $X, Y \in \Gamma(TM)$, ∇^e is the euclidean connection in $\mathbb{R}^{n+1}$ and N is the unit normal to M (two choices of orientation $\pm N$) ▲

Lemma. *The second fundamental form is symmetric and tensorial. That is, for any $X, Y \in \Gamma(TM)$ and $f \in \mathcal{C}^\infty(M)$ we have*

- $L(X, Y) = L(Y, X)$
- $L(fX, Y) = L(X, fY) = fL(X, Y)$

Furthermore, if $Z, W \in \Gamma(TM)$, then

$$R(X, Y, Z, W) = L(X, W)L(Y, Z) - L(X, Z)L(Y, W)$$

Example. Suppose $S \subset \mathbb{R}^3$ is a surface with local parametrization given by $T(u_1, u_2)$. Then, the first fundamental form is given by

$$g_{ij} := \partial_{u_i} T \cdot \partial_{u_j} T$$

The second fundamental form is given by

$$L_{ij} := \partial^2_{u_i u_j} T \cdot N,$$

where N is the unit vector normal to $\partial_{u_1}T$ and $\partial_{u_2}T$. Further, thanks to Gauss's Teorema Egregium, it follows that the Gaussian curvature is

$$K = \frac{\det(L_{ij})}{\det(g_{ij})} = \frac{L_{11}L_{22} - (L_{12})^2}{g_{11}g_{22} - (g_{12})^2}$$

▼

Theorem. *Let (M, g) be a Riemannian n-manifold. Then,*

$$\frac{\mathrm{vol}(B_r^M(p))}{\mathrm{vol}(B_r^{\mathbb{R}^n}(0))} = 1 - \frac{\tau(p)}{6(n+2)}r^2 + \mathcal{O}(r^4)$$

Definition. Let (M, g) be a Riemannian manifold. If $\{X, Y\}$ is the basis of a 2-plane Π in $\Gamma(TM)$, we define the **sectional curvature** by

$$R(\Pi) := \frac{R(X, Y, Y, X)}{g(X, X)g(Y, Y) - g(X, Y)^2} = \frac{g\big(R(X, Y)Y, X\big)}{g(X, X)g(Y, Y) - g(X, Y)^2}$$

This definition is independent of the chosen basis, it only depends on $\Pi \subset \Gamma(TM)$. ▲

Theorem. *Let (M, g) be a (pseudo)Riemannian n-manifold with constant sectional curvature, that is $R(\Pi) = c$ for any 2-plane $\Pi \subset \Gamma(TM)$. Then,*

- *if $c = 0$, (M, g) is locally isometric to $\mathbb{R}^n$ with flat metric.*
- *if $c > 0$, (M, g) is locally isometric to the sphere of radius $\frac{1}{\sqrt{c}}$*

$$S^n\left(\frac{1}{\sqrt{c}}\right) := \left\{x \in \mathbb{R}^{n+1} : \|x\|^2 = \frac{1}{c}\right\}$$

- *if $c < 0$, (M, g) is locally isometric to the pseudo-sphere of radius $\frac{1}{\sqrt{-c}}$*

$$\mathcal{S}^n\left(\frac{1}{\sqrt{-c}}\right) := \left\{x \in \mathbb{R}^{n+1} : x_1^2 + \ldots + x_n^2 - x_{n+1}^2 = \frac{1}{c}\right\}$$

Lemma. *Let (M, g) be a (pseudo)Riemannian n-manifold with constant sectional curvature c. If $\{e_1, \ldots, e_n\}$ is an orthonormal basis for T_pM, then*

$$R(e_i, e_j, e_k, e_l) = c\,(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})$$

That is, the only non zero curvatures are $R(e_i, e_j, e_j, e_i) = c$ and $R(e_i, e_j, e_i, e_j) = -c$ and the induced symmetries.

Definition. Let (M, g) be a Riemannian manifold and ∇ the Levi-Civita connection. For $X \in \Gamma(TM)$, we define the **Jacobi operator** $J(X) : \Gamma(TM) \to \Gamma(TM)$ by

$$J(X)Y := R(Y, X)X$$

▲

Lemma.

- *$J(X)X = 0$ for any $X \in \Gamma(TM)$.*
- *$J(X)$ is self adjoint, that is for any $Y, Z \in \Gamma(TM)$*

$$g(J(X)Y, Z) = g(Y, J(X)Z)$$

Definition. Let $\sigma : I \to M$ a geodesic. If Y is a vector field we set

$$\ddot{Y} := \nabla_{\dot{\sigma}} \nabla_{\dot{\sigma}} Y$$

Then, we say that Y is a **Jacobi vector field along** σ if

$$\ddot{Y} + J(\dot{\sigma})Y = 0$$

▲

Example. Suppose $T : [a, b] \times [0, \varepsilon] \to M$ is so that the curves $s \mapsto T(s, t)$ are geodesics for all t. Then $Y := \frac{\partial}{\partial t}$ are Jacobi vecotor fields, that is

$$\ddot{Y} + J\left(\frac{\partial}{\partial s}\right) Y = 0$$

▼

Lemma. *Let $\sigma : I \to M$ a geodesic and $p := \sigma(0) \in M$.*

- *Consider Y a Jacobi vector fields along σ. If $Y(0) \perp \dot{\sigma}(0)$ and $\dot{Y}(0) \perp \dot{\sigma}(0)$. Then $Y(t) \perp \dot{\sigma}(t)$ for all t.*
- *If Y, Z are Jacobi vector fields along σ, so is $aY + bZ$ for $a.b \in \mathbb{R}$.*
- *$\dot{\sigma}(t)$ and $t\dot{\sigma}(t)$ are Jacobi vector fields along σ.*
- *Given $w_0, w_1 \in T_pM$, there exist a unique Jacobi vector field Y along γ such that $Y(0) = w_0$ and $\dot{Y}(0) = w_1$.*

Theorem. *Let (M_1, g_1) and (M_2, g_2) be Riemannian n-manifolds both with constant sectional curvature c. If $\Phi : T_{p_1}M_1 \to T_{p_2}M_2$ is an isometry for some $p_1 \in M_1$ and $p_2 \in M_2$, then there exist an isometry from a neighborhood of p_1 to a neighborhood of p_2.*

Corollary. *Let (M, g) be a Riemannian manifold with constant sectional curvature, that is $R(\Pi) = c$ for any 2-plane $\Pi \subset \Gamma(TM)$. Then,*

- *if $R(\Pi) = 0$, (M, g) is locally isometric to $\mathbb{R}^n$ with flat metric.*
- *if $R(\Pi) = c$, (M, g) is locally homogeneous.*

Definition. Let (M, g) be a Riemannian manifold. We define ∇R as follows

$$(\nabla_X R)(Y, Z)W = \nabla_X\big(R(Y, Z)W\big) - R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W - R(Y, Z)\nabla_X W$$

▲

Lemma. *Let (M, g) be a Riemannian manifold. Then ∇R is a tensor, that is, for any $f \in \mathcal{C}^\infty(M)$*

$$(\nabla_{fX} R)(Y, Z)W = (\nabla_X R)(fY, Z)W = (\nabla_X R)(Y, fZ)W = (\nabla_X R)(Y, Z)fW = f(\nabla_X R)(Y, Z)W$$

Thus, $(\nabla_{fX} R)(Y, Z)W)(p)$ only depends on $X(p), Y(p), Z(p), W(p)$.

Theorem. *Let (M_1, g_1) and (M_2, g_2) be Riemannian n-manifolds. Suppose that $\nabla R_1 \equiv 0$, $\nabla R_2 \equiv 0$ and that there is an isometry $\Phi : T_{p_1}M_1 \to T_{p_2}M_2$ is an isometry for some $p_1 \in M_1$ and $p_2 \in M_2$, such that*

$$R_2(\Phi X, \Phi Y, \Phi Z, \Phi W) = R_1(X, Y, Z, W),$$

that is $\Phi^* R_2 = R_1$. Then, there is a neighborhood U of p and a local isometry $\phi : U \to U$

Theorem. *Let (M, g) be a Riemannian manifold. The following are equivalent*

(i) $\nabla R \equiv= 0$

(ii) For any $p \in M$, there is a neighborhood U of p and a local isometry $\phi : U \to U$ such that $\phi(p) = p$ and $\phi_* = -\mathrm{id}_{T_pM}$

Definition. Let (M, g) be a Riemannian manifold. If any of the equivalent conditions in the theorem above is satisfied, (M, g) is said to be a **local symmetric space** ▲

Corollary. *If M has constant sectional curvature, then M is a local symmetric space.*

Theorem. *Let M be a geodesically complete and connected Riemannian manifold. If $R(\Pi) \leq 0$, then, $\exp_p : T_pM \to M$ is a covering projection (i.e. local diffeomorphism). Furthermore, if M is simply connected, $\exp_p : T_pM \to M$ is a diffeomorphism.*

Theorem. *Let M be a simply connected Riemannian compact n-manifold such that $R(\Pi) > 0$. Define*

$$P(M) := \frac{\sup_\Pi R(\Pi)}{\inf_\Pi R(\Pi)}$$

Then,

- *If $P(M) = 1$, then M has constant sectional curvature and is isometric to $S^n(r)$ for some r.*
- *If $P(M) < 4$, then M is homeomorphic to S^n.*
- *If there is $\delta(n) > 0$ such that $P(M) < 1 + \delta(n)$, then M is diffeomorphic to S^n*
- *If $P(M) = 4$, then M is either homeomorphic to S^n, or isometric to $\mathbb{C}P^{n/2}$, $\mathbb{H}P^{n/4}$ or the Cayley plane.*

Theorem. *Let (M, g) be a Riemannian n-manifold, $p \in M$ and $w \in T_pM$ such that $\|w\|_{g_p} = 1$. Then,*

$$S(w^\perp) := \{v \in T_pM : \|w\|_{g_p} = 1, v \perp w\} \cong S^{n-2}$$

Further, there is $c(n)$ such that

$$\rho(w, w) = c(m) \int_{S(w^\perp)} R(\text{span}\{w, v\}) dv$$

That is $\rho(w, w)$ is the normalized average of the sectional curvatures of the 2-panes containing w. Here dv is the usual measure on S^{n-2}.

Proof. Choose an orthonormal basis $\{e_1, \ldots, e_n\}$ for T_pM so that $e_1 = w$ and

$$v = x^2 e_2 + \cdots + x^n e_n$$

such that $(x^2)^2 + \cdots + (x^n)^2 = 1$. Then.

$$R(\text{span}\{w, v\}) = R(v, w, w, v) = x^i x^j R(e_i, w, w, e_j)$$

Hence,

$$\begin{aligned}
\int_{S(w^\perp)} R(\text{span}\{w, v\}) dv &= \int_{\|x\|=1} x^i x^j R(e_i, w, w, e_j) dx \\
&= \sum_{ij} R(e_i, w, w, e_j) \int_{S^{n-2}} \frac{\|x\|^2}{n-1} dx \\
&= \rho(w, w) \int_{S^{n-2}} \frac{\|x\|^2}{n-1} dx
\end{aligned}$$

Thus, we set $c(n) = \frac{1}{\int_{S^{n-2}} \frac{\|x\|^2}{n-1} dx}$ and we are done. ■

Theorem. *Let (M, g) be a Riemannian n-manifold, $p \in M$ and $w \in T_pM$. Then, there is $\widetilde{c}(n)$ such that*

$$\tau = \widetilde{c}(n) \int_{S^{n-1}} \rho(w, w) dw$$

That is, the scalar curvature is a normalized average of the Ricci tensor.

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for T_pM. Let $w = x^i e_i$. Then,

$$\rho(w, w) = x^i x^j \rho(e_i, e_j)$$

Thus,

$$\begin{aligned}
\int_{S^{n-1}} \rho(w, w) dw &= \sum_{i,j} \rho(e_i, e_j) \int_{S^{n-1}} x^i x^j dx \\
&= \left(\int_{S^{n-1}} \|x\|^2 dx \right) \sum_{i,j} \rho(e_i, e_j) \\
&= \left(\int_{S^{n-1}} \|x\|^2 dx \right) \tau
\end{aligned}$$

Thus, we set $\widetilde{c}(n) = \frac{1}{\int_{S^{n-1}} \|x\|^2 dx}$ and we are done. ■

Vector Calculus:

Consider $M = \mathbb{R}^3$ as a smooth manifold with usual coordinates (x, y, z). Set

$$\nabla := (\partial_x, \partial_y, \partial_z)$$

If $f : \mathbb{R}^3 \to \mathbb{R}$ is in $\mathcal{C}^\infty(\mathbb{R}^3)$

$$\operatorname{grad}(f) := \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

If $\vec{F} \in \Gamma(\mathbb{R}^3)$ is a smooth vector field, $\vec{F} = (F_1, F_2, F_3)$, $F_i \in \mathcal{C}^\infty(\mathbb{R}^3)$,

$$\operatorname{curl}(\vec{F}) := \nabla \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

If $\vec{G} \in \Gamma(\mathbb{R}^3)$ is a smooth vector field, $\vec{G} = (G_1, G_2, G_3)$, $G_i \in \mathcal{C}^\infty(\mathbb{R}^3)$,

$$\operatorname{div}(\vec{G}) := \nabla \cdot \vec{G} = \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z}$$

Clearly $\ker(\text{grad}) = \{\text{constant functions}\} \cong \mathbb{R}$. It's also well known that $\operatorname{im}(\text{grad}) = \ker(\text{curl})$, $\operatorname{im}(\text{curl}) = \ker(\text{div})$, and $\operatorname{im}(\text{div}) = \mathcal{C}^\infty(\mathbb{R}^3)$. Hence, the following sequence is exact

$$0 \to \mathbb{R} \hookrightarrow \mathcal{C}^\infty(\mathbb{R}^3) \xrightarrow{\text{grad}} \Gamma(\mathbb{R}^3) \xrightarrow{\text{curl}} \Gamma(\mathbb{R}^3) \xrightarrow{\text{div}} \mathcal{C}^\infty(\mathbb{R}^3) \to 0$$

Theorem. *1) The Fundamental Theorem of Calculus: If f' is a real-valued continuous function on $[a, b]$, then*

$$f(b) - f(a) = \int_a^b f'(x)\ dx$$

2) Green's Theorem: Let R be a bounded region in $\mathbb{R}^2$ with piecewise smooth boundary ∂R. Orient the boundary to keep the region on the left. Let $F = (P, Q)$ be a smooth vector field defined on all of R (i.e. $P, Q : R \to \mathbb{R}$ are smooth). Then

$$\oint_{\partial R} P\ dx + Q\ dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dxdy$$

3) Stokes' Theorem (Curl). Let S be a smooth bounded oriented surface in $\mathbb{R}^3$. Orient the boundary ∂S to keep S on the left. Let $\vec{F} = (F_1, F_2, F_3)$, be a smooth vector field on S (i.e. $F_i : S \to \mathbb{R}$ are smooth) and $\vec{N}$ the unit normal giving the orientation of S,

$$\oint_{\partial S} F_1\ dx + F_2\ dy + F_3\ dz = \iint_S \operatorname{curl}(\vec{F}) \cdot \vec{N}\ dA$$

4) Gauss's Theorem (Diveregence). Let R be a bounded region in $\mathbb{R}^3$ with smooth boundary and $\vec{F} = (F_1, F_2, F_3)$, be a smooth vector field on R (i.e. $F_i : R \to \mathbb{R}$ are smooth). Orient the boundary using the unit outward normal $\vec{N}$. Then

$$\iint_{\partial R} \vec{F} \cdot \vec{N} \; dA = \iiint_R \operatorname{div}(\vec{F}) \; dV$$

Differential Forms:

Let M be a smooth n-manifold. Recall that the **tangent bundle** TM is a smooth vector bundle over M whose fibers over p are T_pM. For local coordinates (U, x^i) a local frame for TM over U is given by $(\partial_{x^1}, \ldots, \partial_{x^n})$.

The **cotangent bundle** T^*M is the dual bundle of TM. That is, T^*M is a is a smooth vector bundle over M whose fibers over p are the dual spaces

$$T_p^*M := (T_pM)^* = \operatorname{Hom}(T_pM, \mathbb{R})$$

Thus, for local coordinates (U, x^i), a local frame for T^*M over U is given by $(dx^1, \ldots, dx^n)$ where $\{dx^1(p), \ldots, dx^n(p)\} \subset T_p^*M$ is the dual basis to $\{\partial_{x^1}(p), \ldots, \partial_{x^n}(p)\} \subset T_pM$. The smooth sections of T^*M are called **1-forms** on M.

Definition. Let M be a smooth n-manifold. If $f \in \mathcal{C}^\infty(M)$, we locally define $df \in \Gamma(T^*M)$ as

$$df := \partial_{x^i} f \; dx^i$$

▲

Remark. If $X \in \Gamma(TM)$ is locally written as $X = X^i \partial_{x^i}$, we have

$$[df(p)](X(p)) = [\partial_{x^i} f(p) \; dx^i(p)](X^j(p)\partial_{x^j}(p)) = X^k(p)\partial_{x^k} f(p) = X(f)(p)$$

▼

Definition. Let M, N be smooth manifolds and $F : M \to N$ a smooth map. Recall that $F^* : \mathcal{C}^\infty(N) \to \mathcal{C}^\infty(M)$ is given by $F^*h = h \circ F$ and that $F_* : T_pM \to T_{F(p)}N$ (note that F_* was also called $dF(p)$ in the Fall) is given by $F_*(X_p)(h) = X_p(F^*h)$ for any $X_p \in T_pM$ and any $h \in \mathcal{C}^\infty(N)$. We define the **pullback of 1-forms** $F^* : T_{F(p)}^*N \to T_p^*M$ by

$$F^*(\omega_{F(p)})(X_p) = \omega_{F(p)}(F_*(X_p))$$

for any $\omega_{F(p)} \in T_{F(p)}^*N$ and $X_p \in T_pM$ ▲

Lemma. *Let M, N be smooth manifolds and $F : M \to N$ a smooth map. If $h \in \mathcal{C}^\infty(N)$, then*

$$d(F^*h) = F^*(dh)$$

Proof. Let $(U, x^1, \dots, x^n)$ be local coordinates in M, $p \in U$ and let $(V, y^1, \dots, y^m)$ local coordinates in N such that $F(U) \subset V$. We abuse notation by setting $dx^i := dx^i(p)$ and $dy^j := dy^j(F(p))$. Notice that each $dy^j \in T^*_{F(p)}N$ and therefore

$$F^*(dy^j) = \sum_{i=1}^{n} a^j_i dx^i \in T^*_p M$$

Thus,

$$[F^*(dy^j)](\partial_{x^k}) = \left[\sum_{i=1}^{n} a^j_i dx^i\right](\partial_{x^k}) = \sum_{i=1}^{n} a^j_i \delta_{i,k} = a^j_k$$

On the other side, by definition of pullback $[F^*(dy^j)](\partial_{x^k}) = dy^j(F_*\partial_{x^k})$. But, for any $h \in \mathcal{C}^\infty(N)$ we have

$$[F_*\partial_{x^k}](h) = \partial_{x^k}(h \circ F) = \sum_{j=1}^{m} \frac{\partial h}{\partial y^j}\frac{\partial y^j}{\partial x^k} = \sum_{j=1}^{m} \frac{\partial y^j}{\partial x^k}\partial_{y^j}(h),$$

that is

$$F_*\partial_{x^k} = \sum_{i=1}^{m} \frac{\partial y^i}{\partial x^k}\partial_{y^i}$$

Hence,

$$[F^*(dy^j)](\partial_{x^k}) = dy^j(F_*\partial_{x^k}) = dy^j\left(\sum_{i=1}^{m} \frac{\partial y^i}{\partial_{x^k}}\partial_{y^i}\right) = \sum_{i=1}^{m} \frac{\partial y^i}{\partial x^k}\delta_{j,i} = \frac{\partial y^j}{\partial x^k}$$

Thus, we've shown that $a^j_k = \frac{\partial y^j}{\partial x^k}$. Finally, if $h \in \mathcal{C}^\infty(N)$ we know that $dh = \partial_{y^j} h dy^j$ and therefore

$$\begin{aligned} F^*(dh) = \sum_{j=1}^{m} \partial_{y^j} h F^*(dy^j) &= \sum_{j=1}^{m} \partial_{y^j} h \left(\sum_{i=1}^{n} \frac{\partial y^j}{\partial x^k} dx^i\right) \\ &= \sum_{i,j} \frac{\partial h}{\partial y^j}\frac{\partial y^j}{\partial_{x^k}} dx^i = \sum_{i=1}^{n} \partial_{x^i}(F^*h) dx^i = d(F^*h) \end{aligned}$$

■

Brief review of Exterior algebras:

Recall that if V is a real vector space, we set $T^p(V) := V \otimes \cdots \otimes V$ (p times) for any $p \geq 0$. In particular, $T^0(V) = \mathbb{R}$, $T^1(V) = V$,$T^2(V) = V \otimes V$, and so on. Set

$$T(V) := \bigoplus_{p=0}^{\infty} T^p(V)$$

We call $T(V)$ the tensor algebra on V, and it is the universal unital algebra generated by V. If $J(V)$ is the two-sided ideal of the tensor algebra generated by the elements $\{v \otimes w + w \otimes v : v, w \in V\}$, we define the **exterior algebra** $\Lambda(V)$ to be the quotient

$$\Lambda(V) := T(V)/J(V)$$

We write $v_1 \wedge \cdots \wedge v_n$ for the image in $\Lambda(V)$ of the pure tensor $v_1 \otimes \cdots \otimes v_n$ in $T(V)$. In fact, $\Lambda(V)$ is the universal unital algebra generated by V subject to the relations $v \wedge w + w \wedge v = 0$. Hence, $\Lambda(V)$ has the following anti-symmetric property

$$v_1 \wedge \cdots \wedge v_i \wedge v_{i+1} \wedge \cdots \wedge v_n = -v_1 \wedge \cdots \wedge v_{i+1} \wedge v_i \wedge \cdots \wedge v_n$$

for any $1 \leq i \leq n-1$ and $v_1, \ldots, v_n \in V$. We let $\Lambda^p(V)$ be the image of $T^n(V)$ under the quotient map $T(V) \twoheadrightarrow \Lambda(V)$.

Lemma. *Let V be a real vector space with basis $\{e_1, \ldots, e_n\}$. Then,*

- $\Lambda^p(V) = 0$ *for any $p > n$.*
- $\Lambda^n(V) = \mathrm{span}\{e_1 \wedge \cdots \wedge e_n\}$ *and therefore* $\dim(\Lambda^n(V)) = 1$
- *In general, for $p \leq n$*

  $$\Lambda^p(V) = \mathrm{span}\{e_{i_1} \wedge \cdots \wedge e_{i_p} : 1 \leq i_1 < i_2 < \cdots < i_p \leq n\}$$

  and therefore $\dim(\Lambda^p(V)) = \binom{n}{p}$.

Back to differential forms:

Let M be a smooth n-manifold. Recall that we defined 1-forms to be smooth sections to the bundle T^*M. Therefore, any 1-form ω can written as

$$\omega = f_i dx^i,$$

where each $f_i \in \mathcal{C}^\infty(M)$.

Definition. In general, If M is a smooth n-manifold, we define a p**-form** on M as a smooth section of the vector bundle

$$\Lambda^p(T^*M) \xrightarrow{\pi} M$$

That is, a p-form is a smooth map $\omega : M \to \Lambda^p(T^*M)$, such that $\pi \circ \omega = \mathrm{id}_M$. So, for any $x \in M$ we, have a map

$$\omega(x) : \underbrace{T_xM \times \cdots \times T_xM}_{p \text{ times}} \to \mathbb{R}.$$

Sometimes $\omega(x)$ is denoted by ω_x or simply by ω when x is clear from context. We denote by $\Omega^p(M) := \Gamma(\Lambda^p(T^*M))$ the space of p-forms in M. Notice that $\Omega^0(M) = \mathcal{C}^\infty(M)$ and that $\Omega^q(M) = 0$ for any $q > n$. ▲

Remark. Consider sets $I = \{i_1, \ldots, i_p : 1 \leq i_1 < \cdots < i_p \leq n\}$ and let $dx^I := dx^{i_1} \wedge \cdots \wedge dx^{i_p}$. Then, any p form ω can written as

$$\omega = \sum_{|I|=p} f_I dx^I,$$

where each $f_I \in \mathcal{C}^\infty(M)$. ▼

Remark. If $\omega \in \Omega^p(M)$ and $\theta \in \Omega^q(M)$, then we get $\omega \wedge \theta \in \Omega^{p+q}(M)$ so that $(\omega \wedge \theta)(x) = \omega(x) \wedge \theta(x)$ for any $x \in M$. Thus,

$$\omega \wedge \theta = (-1)^{pq} \theta \wedge \omega$$

▼

Definition. Let M, N be smooth manifolds. If $F : M \to N$ a smooth map. it induces a **pullback of** p**-forms** $F^* : \Omega^p(N) \to \Omega^p(M)$, so that if $\omega \in \Omega^p(N)$, then $F^*\omega \in \Omega^p(M)$ is given by

$$(F^*\omega)_x(X_1, \ldots, X_p) := \omega_{F(x)}(F_*X_1, \ldots, F_*X_p)$$

for any $x \in M$ and $X_1, \ldots, X_p \in T_xM$ ▲

Definition. Let M be a smooth n-manifold. For $p \geq 0$ we define a map $d : \Omega^p(M) \to \Omega^{p+1}(M)$ by

$$d\left(\sum_{|I|=p} f_I dx^I\right) := \sum_{|I|=p} df_I \wedge dx^I = \sum_{|I|=p} \left(\sum_{i=1}^{n} \partial_{x^i} f_I \; dx^i\right) \wedge dx^I$$

▲

Lemma. *Let M_1, M_2 be smooth manifolds and $d_{M_i} : \Omega^p(M_i) \to \Omega^{p+1}(M_i)$ the map defined above. (when M_i is clear we denote $d := d_{M_i}$.)*

- *For any $\omega \in \Omega^p(M_i)$, $d(d\omega) = 0 \in \Omega^{p+2}(M_i)$, i.e. $d^2 = 0$.*
- *If $F : M_1 \to M_2$ is a smooth map, then $F^*(d_{M_2}\omega) = d_{M_1}(F^*\omega)$ for any $\omega \in \Omega^p(M_2)$.*
- *If $U \subset M_i$, $d(\omega|_U) = (d\omega)|_U$.*
- *For any $\omega \in \Omega^p(M_i)$ and $\theta \in \Omega^q(M_i)$ we have*

$$d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^p \omega \wedge d\theta$$

Remark. Let M be a smooth n-manifold. Notice that since $d^2 = 0$, we get a complex, known as the **de Rham complex**, given by

$$\mathcal{C}^\infty(M) = \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M)$$

▼

Orientable Manifolds:

Lemma. *Let M be a smooth n-manifold. The following are equivalent and if either are satisfied M is said to be **orientable***

(1) There is an atlas $\{(U_\alpha, \varphi_\alpha)\}$ so that the transition functions satisfy

$$\det \varphi'_{\alpha,\beta} > 0$$

*(2) There exist a smooth n-form which never vanishes, such form is known as **the oriented volume form**.*

Example. $M = S^n \subset \mathbb{R}^{n+1}$ is orientable and the oriented volume form is given by

$$\omega_n := \sum_{i=1}^{n+1} (-1)^{i+1} x^i \; dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge x^{n+1}$$

where the hat means that dx^i is omitted.

If $M = S^1$, we have $\omega_1 = x \; dy - y \; dx$. Thus, if $\theta \mapsto (\cos(\theta), \sin(\theta))$ parametrizes S^1 we get

$$\omega_1 = \cos(\theta) d(\sin(\theta)) - \sin(\theta) d(\cos(\theta)) = (\cos^2\theta + \sin^2\theta) d\theta = d\theta$$

If $M = S^2$, we have $\omega_2 = x\ dy \wedge dz - y\ dx \wedge dz + z\ dx \wedge dy$. Thus, if $(\phi, \theta) \mapsto (\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi))$ parametrizes S^2 we get

$$\omega_2 = \sin\phi\ d\phi \wedge d\theta$$

▼

Example.

- $\mathbb{R}P^n := S^n / x \sim -x$ is orientable if and only if n is odd.
- $(S^n \times S^n)/(x, y) \sim (-y, x)$ is not orientable.
- $(S^n \times S^n)/(x, y) \sim (-x, -y)$ is orientable.

▼

Theorem. *If M, N are orientable smooth manifolds, then $M \times N$ is orientable. If either M or N is not orientable, then $M \times N$ is not orientable.*

Definition. Let M be a smooth n-manifold and $X \in \Gamma(TM)$ a smooth vector field. We define a map $\iota_X : \Omega^p(M) \to \Omega^{p-1}(M)$ as follows

$$\iota_X(\omega)_x(X_1, \ldots, X_{p-1}) := \omega_x(X(x), X_1, \ldots, X_{p-1})$$

for $X_1, \ldots, X_{p-1} \in T_xM$. ▲

Lemma. *Let M be a smooth n-manifold, $X, Y \in \Gamma(TM)$, $f \in \mathcal{C}^\infty(M)$, $\omega \in \Omega^p(M)$ and $\eta \in \Omega^q(M)$.*

- $\iota_X(df)_x = X(f)(x)$.
- $\iota_X(f\omega) = f\iota_X(\omega)$
- $\iota_X(\omega \wedge \eta) = \iota_X(\omega) \wedge \eta + (-1)^p \omega \wedge \iota_X(\eta)$
- $\iota_{fX}(\omega) = f\iota_X(\omega)$
- $\iota_{X+Y}(\omega) = \iota_X(\omega) + \iota_Y(\omega)$

Example. $M = \mathbb{R}^2$

$$\iota_{\partial_x}(dx \wedge dy) - \iota_{\partial_y}(dy \wedge dx) = -dx + dy$$

$M = \mathbb{R}^3$

$$\iota_{x\partial_x}(dx \wedge dy \wedge dz) = x\ dy \wedge dz$$
$$\iota_{x\partial_y}(dx \wedge dy \wedge dz) = -x\ dx \wedge dz$$

▼

Theorem. *Let M be an oriented n-Riemannian manifold and $M_1 \subset M$ a closed oriented $n-1$ submanifold (for example $M_1 = \partial M$). Let ω_M be the oriented volume form of M and ω_{M_1} the one of M_1. Let $\vec{N}$ be the unit normal vector field compatible with the orientation of M and M_1. Then,*

$$\omega_{M_1} = \iota_{\vec{n}}(\omega_M)$$

Example. If $M = \mathbb{R}^n$, $M_1 = S^{n-1}$. We have $\omega_{\mathbb{R}^n} = dx^1 \wedge \cdots \wedge dx^n$ and clearly $\vec{N} = x^1\partial_{x^1} + \cdots + x^n\partial_{x_n}$. Then,

$$\begin{aligned}
\omega_{S^{n-1}} &= \iota_{\vec{N}}(\omega_{\mathbb{R}^n}) \\
&= \iota_{x^1\partial_{x^1}+\cdots+x^n\partial_{x_n}}(dx^1 \wedge \cdots \wedge dx^n) \\
&= \sum_{i=1}^{n} \iota_{x^i\partial_{x^i}}(dx^1 \wedge \cdots \wedge dx^n) \\
&= \sum_{i=1}^{n} (-1)^{i+1} x^i\ dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n
\end{aligned}$$

▼

Integration of differential forms:

Definition. Let $D, D' \subset \mathbb{R}^n$ and $\varphi : D \to D'$ a diffeomorphism. Then, φ is saif to be **orientation preserving** if

$$\det \varphi' > 0$$

▲

Remark. If $\vec{y} = (y^1, \ldots, y^n)$ are coordinates in D and

$$\varphi(y^1, \ldots, y^n) = \big(x^1(\vec{y}), \ldots, x^n(\vec{y})\big)$$

Then,

$$\det \varphi' = \det \left(\frac{\partial x^i}{\partial y^j} \right)$$

▼

Theorem. *(Change of Variables: Multivariable Calculus) Let $D, D' \subset \mathbb{R}^n$, $\varphi : D \to D'$ an orientation preserving diffeomorphism and $f : D' \to \mathbb{R}$ a smooth map. Then,*

$$\int_{D'} f\ dx^1 \cdots dx^n = \int_D \varphi^*(f) \det \varphi'\ dy^1 \cdots dy^n$$

Theorem. *Let $D, D' \subset \mathbb{R}^n$, $\varphi : D \to D'$ an orientation preserving diffeomorphism and $\omega \in \Omega^n(D')$ an n-form given by $\omega = f\ dx^1 \wedge \cdots \wedge dx^n$. Then*

$$\varphi^*\omega = \varphi^*(f) \det \varphi'\ dy^1 \wedge \cdots \wedge dy^n \in \Omega^n(D)$$

Definition. Let $D \subset \mathbb{R}^n$ and $\omega \in \Omega^n(D)$ an n-form given by $\omega = f\ dx^1 \wedge \cdots \wedge dx^n$. We define

$$\int_D \omega := \int_D f\ dx^1 \cdots dx^n$$

▲

Corollary. *(Change of Variables: Differential Forms) Let $D, D' \subset \mathbb{R}^n$, $\varphi : D \to D'$ an orientation preserving diffeomorphism and $\omega \in \Omega^n(D')$ an n-form given by $\omega = f\ dx^1 \wedge \cdots \wedge dx^n$. Then. Then,*

$$\int_{D'} \omega = \int_D \varphi^*\omega$$

Definition. Let M be an orientable smooth n-manifold and $\omega \in \Omega^n(M)$ an n-form such that

$$\text{supp } \omega := \overline{\{x \in M : \omega(x) \neq 0\}}$$

is compact. Pick an atlas $\{(U_\alpha, \varphi_\alpha)\}$ for which the transition functions $\varphi_{\alpha,\beta} : D_\beta \to D_\alpha$ are orientation preserving, where $D_\alpha := \varphi(U_\alpha) \subset \mathbb{R}^n$. Let $\psi_\alpha := \varphi_\alpha^{-1} : D_\alpha \to U_\alpha$. Then, $\psi_\alpha^* : \Omega^n(U_\alpha) \to \Omega^n(D_\alpha)$. There is $N \in \mathbb{N}$ such that supp $\omega \subset U_{\alpha_1}, \ldots, U_{\alpha_N}$. Let $\{\phi_k\}_{k=1}^N$ be a partition of unity subordinated to $\{U_{\alpha_k}\}_{k=1}^N$. We define the integral of ω by

$$\int_M \omega := \sum_{k=1}^N \int_{D_{\alpha_k}} \psi_{\alpha_k}^*(\phi_k \cdot \omega)$$

▲

Remark. The definition above is independent from the choice of atlas and from the partition of unity. ▼

Theorem. *(**Stokes' Theorem**) Let M be a smooth oriented n-manifold with boundary ∂M and $\omega \in \Omega^{n-1}(M)$ an $(n-1)$-form with compact support. Then, ∂M has a natural orientation so that*

$$\int_{\partial M} \omega = \int_M d\omega$$

Remark. The statement of Stokes' Theorem requires a bit of interpretation. On the left-hand side ω is to be interpreted as $\iota_{\partial M}^* \omega$ where $\iota_{\partial M} : \partial M \hookrightarrow M$ is the canonical inclusion. If $\partial M = \varnothing$ then the left-hand side is to be interpreted as zero. When M is 1-dimensional, the left-hand integral is really just a finite sum. ▼

Vector Calculus vs Differential forms:

Each one of the flowing squares commutes:

$$\begin{tikzcd}
f \in \mathcal{C}^\infty(\mathbb{R}^3) \arrow[r] \arrow[d, "\mathrm{grad}"'] & f \in \Omega^0(\mathbb{R}^3) \arrow[d, "\mathrm{d}"] \\
(F_1, F_2, F_3) \in \Gamma(\mathbb{R}^3) \arrow[r] \arrow[d, "\mathrm{curl}"'] & F_1 dx + F_2 dy + F_3 dz \in \Omega^1(\mathbb{R}^3) \arrow[d, "\mathrm{d}"] \\
(G_1, G_2, G_3) \in \Gamma(\mathbb{R}^3) \arrow[r] \arrow[d, "\mathrm{div}"'] & G_1 dy \wedge dz + G_2 dz \wedge dx + G_3 dx \wedge dy \in \Omega^2(\mathbb{R}^3) \arrow[d, "\mathrm{d}"] \\
g \in \mathcal{C}^\infty(\mathbb{R}^3) \arrow[r] & g dx \wedge dy \wedge dz \in \Omega^3(\mathbb{R}^3)
\end{tikzcd}$$

Indeed, recall that $\mathrm{grad}(f) = (\partial_x f, \partial_y f, \partial_x f)$, while

$$df = \partial_x f \ dx + \partial_y f \ dy + \partial_z f \ dz$$

Also $\mathrm{curl}(F_1, F_2, F_2) = (\partial_y F_3 - \partial_z F_2, \partial_z F_1 - \partial_x F_3, \partial_x F_2 - \partial_y F_1)$, whereas

$$\begin{aligned}
&d(F_1 dx + F_2 dy + F_3 dz) \\
&= (\partial_y F_3 - \partial_z F_2) dy \wedge dz + (\partial_z F_1 - \partial_x F_3) dz \wedge dx + (\partial_x F_2 - \partial_y F_1) dx \wedge dy
\end{aligned}$$

Finally, we had $\mathrm{div}(G_1, G_2, G_3) = \partial_x G_1 + \partial_y G_2 + \partial_z G_3$, while

$$d(G_1 dy \wedge dz + G_2 dz \wedge dx + G_3 dx \wedge dy) = (\partial_x G_1 + \partial_y G_2 + \partial_z G_3) dx \wedge dy \wedge dz$$

De Rham Cohomology

Definition. Let M be a smooth n-manifold. A form $\omega \in \Omega^p(M)$ is said to be **closed** if $d\omega = 0$. A form $\omega \in \Omega^p(M)$ is called **exact** if $\omega = d\eta$ for some $\eta \in \Omega^{p-1}(N)$. Notice that, since $d^2 = 0$, any exact form is closed.

One defines the The form $\omega \in \Omega^p(M)$ group $H^p_{\text{dR}}(M)$ to be the set of closed forms in $\Omega^p(M)$ modulo the exact forms, i.e.

$$H^p_{\text{dR}}(M) := \frac{\ker(d : \Omega^p(M) \to \Omega^{p+1}(M))}{\operatorname{im}(d : \Omega^{p-1}(M) \to \Omega^p(M))}$$

▲

Remark. Recall that for a smooth function $F : N \to M$, the pullback $F^* : \Omega^p(M) \to \Omega^p(N)$ is so that

$$d(F^*\omega) = F^*(d\omega),$$

Therefore, F^* descends to a linear map $[F^*] : H^p_{\text{dR}}(M) \to H^p_{\text{dR}}(N)$ given by

$$[F^*][\omega] := [F^*\omega]$$

which is well defined. ▼

Example. Notice that $H^0_{\text{dR}}(M) = \{\text{locally constant functions on } M\}$. Thus, if M has k connected components, we have $H^0(M) \cong \mathbb{R} \oplus \cdots \oplus \mathbb{R}$, k times. In fact, if $M = \bigsqcup_{i=1}^k M_k$ and

$$\mathbb{1}_{M_i}(x) := \begin{cases} 1 & \text{if } \ x \in M_i \\ 0 & \text{if } \ x \notin M_i \end{cases}$$

Then $\{[\mathbb{1}_{M_i}]\}$ is a basis for $H^0_{\text{dR}}(M)$. Moreover, for any p,

$$H^p_{\text{dR}}(M) = \bigoplus_{i=1}^k H^p_{\text{dR}}(M_i)$$

▼

Example. If M is a compact smooth oriented n-manifold without boundary, Stokes theorem applied to the volume form ω_M implies that

$$0 \neq [\omega_M] \in H^n(M)$$

▼

Example.

$$H^p_{\mathrm{dR}}(\{\mathrm{pt}\}) \cong \begin{cases} \mathbb{R} & \text{if } \ p = 0 \\ 0 & \text{if } \ p > 0 \end{cases}$$

▼

Example.

$$H^p_{\mathrm{dR}}(S^n) \cong \begin{cases} \mathbb{R} & \text{if } \ p = 0, n \\ 0 & \text{if } \ 0 < p < n \end{cases}$$

In fact, $H^0_{\mathrm{dR}}(S^n) = \mathbb{R}[\mathbb{1}_{S^n}]$ and $H^n_{\mathrm{dR}}(S^n) = \mathbb{R}[\omega_{S^n}]$. This is commonly written as $H^*_{\mathrm{dR}}(S^n) = \mathrm{span}\{[\mathbb{1}_{S^n}], [\omega_{S^n}]\} = \mathbb{R}[\mathbb{1}_{S^n}] \oplus \mathbb{R}[\omega_{S^n}]$. ▼

Lemma. *Let M, N be smooth manifolds and $F : M \times [0,1] \to N$ be smooth. Define $F_0, F_1 : M \to N$ by $F_0(x) := F(x,0)$ and $F_1(x) := F(x,1)$. Then,*

$$[F_0^*] = [F_1^*] : H^p_{\mathrm{dR}}(N) \to H^p_{\mathrm{dR}}(N)$$

That is, the pullback of two homotopic maps is equal.

Definition. Let M be a smooth manifold and $N \subset M$ a smooth submanifold. We say that N is a **deformation retract of** M, denoted by $N \searrow M$, if there is a smooth map $H : M \times [0,1] \to M$ such that

- $H(x,0) = x$ for all $x \in M$.
- $H(y,t) = y$ for all $y \in N$ and all $t \in [0,1]$.
- $H(x,1) \in N$ for all $x \in M$.

▲

Lemma. *Let M be a smooth manifold and $N \subset M$ such that $N \searrow M$. Then, for any p*

$$H^p_{\mathrm{dR}}(M) \cong H^p_{\mathrm{dR}}(N)$$

Corollary. $H^p_{\mathrm{dR}}(\mathbb{R}^n \setminus \{0\}) \cong H^p_{\mathrm{dR}}(S^{n-1})$

Theorem. *(Mayer Vietoris) Let M be a smooth manifold with or without boundary, and let U_1, U_2 be open subsets of M whose union is M.*

$$\begin{tikzcd}
& U_1 \arrow[dr, hook, "\jmath_1"] & \\
U_1 \cap U_2 \arrow[ur, hook, "\imath_1"] \arrow[dr, hook, "\imath_2"'] & & M \\
& U_2 \arrow[ur, hook, "\jmath_2"'] &
\end{tikzcd}$$

For each p, there is a linear map $\delta : H^p_{\mathrm{dR}}(U_1 \cap U_2) \to H^{p+1}_{\mathrm{dR}}(M)$ such that the following sequence, called the Mayer–Vietoris sequence for the open cover $\{U_1, U_2\}$, is exact:

$$\cdots H^p_{\mathrm{dR}}(M) \xrightarrow{\jmath_1^* \oplus \jmath_2^*} H^p_{\mathrm{dR}}(U_1) \oplus H^p_{\mathrm{dR}}(U_2) \xrightarrow{\imath_1^* - \imath_2^*} H^p_{\mathrm{dR}}(U_1 \cap U_2) \xrightarrow{\delta} H^{p+1}_{\mathrm{dR}}(M) \to \cdots$$

Example.

$$H^p_{\mathrm{dR}}(\mathbb{C}P^n) \cong \begin{cases} \mathbb{R} & \text{if } p \text{ even }, 0 \le p \le 2n \\ 0 & \text{else} \end{cases}$$

▼

Lemma. *Let G be a finite group which acts without fixed points on a compact smooth manifold $\tilde{M}$ without boundary. Let $M := \tilde{M}/G$ and let $\pi : \tilde{M} \to M$ be the associated covering projection. Then*

(1) $\pi^* : H^p_{\mathrm{dR}}(M) \to H^p_{\mathrm{dR}}(\tilde{M})$ *is injective.*

(2) $H^p_{\mathrm{dR}}(M) \cong \mathrm{im}(\pi^*) = \{[\omega] \in H^p_{\mathrm{dR}}(\tilde{M}) : [g^*][\omega] = [\omega]$ *for all* $g \in G\}$. *Where For each $g \in G$, the map $g : \tilde{M} \to \tilde{M}$ is so that $g(x) = g \cdot x$ for $x \in \tilde{M}$.*

Lemma. *Let M be a smooth manifold and let S^a (for $a \ge 1$) be the unit sphere in $\mathbb{R}^{a+1}$. Let P be any point of S^a and let $\iota_P(x) := (P, x)$ define an inclusion of M in $S^a \times M$. There is a natural short exact sequence*

$$0 \to H^{p-a}_{\mathrm{dR}}(M) \xrightarrow{\delta} H^p_{\mathrm{dR}}(M \times S^a) \xrightarrow{\iota_P^*} H^p_{\mathrm{dR}}(M) \to 0$$

Corollary. *Let ω_i be the volume form on S^{a_i}. Then,*

$$H^*_{\mathrm{dR}}(S^{a_1} \times \cdots \times S^{a_k}) = \mathrm{span}\{[\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}] : 1 \le i_1 < \cdots < i_p \le k\}$$

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3.1 Theorems and Definitions.

Let $(\cdot, \cdot)$ be a positive definite inner product on a finite dimensional real vector space. Let $\Lambda^p(V)$ be the exterior p-th algebra on V. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V. If $I = \{1 \leq i_1 < \cdots < i_p \leq n\}$, let $e_I := e_{i_1} \wedge \cdots \wedge e_{i_p}$. Recall that $\{e_I\}$ forms an orthonormal basis for $\Lambda^p(V)$. We have that $\Lambda^p(V)$ inherits a natural inner product as follows

$$(e_I, e_J) := \det\Big((e_i, e_j)\Big)_{i \in I, j \in J}$$

Definition. Let $(\cdot, \cdot)$ be a positive definite inner product on a finite dimensional real vector space. Let $\Lambda^p(V)$ be the p-th exterior algebra on V. If $\xi \in V$, let $\operatorname{ext}(\xi) : \Lambda^p(V) \to \Lambda^{p+1}(V)$ be the linear map

$$\operatorname{ext}(\xi) : \omega \mapsto \xi \wedge \omega.$$

Let $\operatorname{int}(\xi) := \operatorname{ext}(\xi)^\star : \Lambda^{p+1}(V) \to \Lambda^p(V)$, i.e.

$$(\xi \wedge \omega, \phi) = (\omega, \operatorname{int}(\xi)\phi)$$

for all $\omega \in \Lambda^p(V)$ and $\phi \in \Lambda^{p+1}(V)$. ▲

Lemma. *Let $(\cdot, \cdot)$ be a positive definite inner product on a finite dimensional real vector space. If $\xi, \eta \in V$, then*

$$\operatorname{ext}(\xi)\operatorname{int}(\eta) + \operatorname{int}(\eta)\operatorname{ext}(\xi) = (\eta, \xi)\,\mathrm{id}_{\Lambda^p(V)}$$

Definition. Let M be a Riemannian m-manifold. Let $g_{jk} := g(\partial_{x^i}, \partial_{x^j})$, and let $\mathfrak{g} = \sqrt{\det(g_{ij})}$. We have

$$\mathrm{dvol} = \mathfrak{g} dx^1 \ldots dx^m.$$

The **co-derivative** $\delta : \Omega^p(M) \to \Omega^{p-1}(M)$ is defined by

$$(d\omega, \sigma)_{L^2} := \int g(d\omega, \sigma)\mathfrak{g} dx^1 \ldots dx^m = \int g(\omega, \delta\sigma)\mathfrak{g} dx^1 \ldots dx^m \;\; =: (\omega, \delta\sigma)_{L^2}$$

The **Laplacian operator** on $p$forms is then given by $\Delta = \delta d + d\delta$. A p-form ω is said to be harmonic if $\Delta\omega = 0$. ▲

Remark. If $\omega \in \Omega^p(M)$ is harmonic, i.e. $\Delta\omega = 0$, then

$$\begin{aligned}(\Delta\omega, \omega)_{L^2} = 0 &\Leftrightarrow (\delta(d\omega), \omega)_{L^2} + (d(\delta\omega), \omega)_{L^2} = 0 \\ &\Leftrightarrow (d\omega, d\omega)_{L^2} + (\delta\omega, \delta\omega)_{L^2} = 0 \\ &\Leftrightarrow \|d\omega\|_{L^2} + \|\delta\omega\|_{L^2} = 0\end{aligned}$$

Thus, $d\omega = 0$ and $\delta\omega$. Hence $\ker(\Delta) = \ker(d) \cap \ker(\delta)$. In particular, if $\omega \in \ker(\Delta)$, then $[\omega] \in H^p_{\mathrm{dR}}(M)$. ▼

Theorem. ***(Hodge's Theorem)*** *Let M be a compact Riemannian manifold without boundary. Then the map $\omega \mapsto [\omega]$ from $\ker(\Delta)$ to $H^p_{\mathrm{dR}}(M)$ is an isomorphism of vector spaces.*

Corollary. ***(Kunneth Formula)*** *Let M_1, M_2 be compact Riemannian manifolds without boundary. Then,*

$$H^p_{\mathrm{dR}}(M_1 \times M_2) = \bigoplus_{a+b=p} H^a_{\mathrm{dR}}(M_1) \wedge H^b_{\mathrm{dR}}(M_2)$$

Definition. Let M be a smooth Riemannian m-manifold and $\omega \in \Omega^p(M)$ given by

$$\omega = \sum_{1 \le i_1 < \cdots < i_p \le m} f_{i_1,\ldots,i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

The **Levi-Civita connection on p-forms** is defined by the formula

$$\begin{aligned}\nabla_{\partial_{x^i}}\omega = &\sum_{1 \le i_1 < \cdots < i_p \le m} (\partial_{x^i} f_{i_1,\ldots,i_p}) dx^{i_1} \wedge \cdots \wedge dx^{i_p} \\ -&\sum_{1 \le i_1 < \cdots < i_p \le m} \sum_{\nu=1}^{p} \sum_{k=1}^{m} {\Gamma_{ik}}^{i_\nu} f_{i_1,\ldots,i_p} dx_{i_1} \wedge \cdots \wedge dx^{i_{\nu-1}} \wedge dx^k \wedge dx^{i_{\nu+1}} \wedge \cdots \wedge dx^{i_p}\end{aligned}$$

In particular,

$$\nabla_{\partial_{x^i}} dx^j = -{\Gamma_{ik}}^j dx^k$$

▲

Lemma. *Let M be a Riemannian m-manifold. Then*

- $d = \sum_{i=1}^{m} \mathrm{ext}(dx^i) \nabla_{\partial_{x^i}}$

- $\delta = -\sum_{i=1}^{m} \mathrm{int}(dx^i)\nabla_{\partial_{x^i}}$

Definition. Let M be a compact oriented m-manifold without boundary. Let $(\cdot,\cdot)$ be the pointwise inner product on p-forms. Let orn $\in \Omega^m(M)$ be the oriented volume form. The **Hodge $\star$ operator**, $\star_p : \Omega^p(M) \to \Omega^{m-p}(M)$, is characterized by the property:

$$(\omega_p, \tilde{\omega}_p)\,\mathrm{orn} = \omega_p \wedge \star_p \tilde{\omega}_p.$$

▲

Remark. Let M be a compact oriented m-manifold without boundary. For any $\omega, \eta \in \Omega^p(M)$ we have

$$(\omega, \eta)_{L^2} = \int_M \omega \wedge \star\eta$$

▼

Lemma. *Let M be a compact oriented m-manifold without boundary. Then $\star_{m-p}\star_p = \varepsilon_p\,\mathrm{id}_p$ and that $\star_p\Delta_p = \epsilon_p\Delta_{m-p}\star_p$ where $\varepsilon_p^2 = 1$ and $\epsilon_p^2 = 1$, i.e. $\varepsilon_p = \pm 1_p$ and $\epsilon_p = \pm 1_p$ are an appropriate choice of signs.*

Corollary. *Let M be a compact oriented m-manifold without boundary. Then, $(\star_p)^{-1} = \varepsilon_p \star_{m-p}$*

Remark. Let M be a compact oriented m-manifold without boundary. If $d : \Omega^p(M) \to \Omega^{p+1}(M)$ and $\delta : \Omega^p(M) \to \Omega^{p-1}(M)$ are the differential and codifferential maps respectively, then

$$\star_{m-p-1} d \star_p = \pm\delta$$

▼

Theorem. ***(Poincaré Duality)*** *Let M be a compact oriented m-manifold without boundary, then*

$$H^p_{\mathrm{dR}}(M) \cong H^{m-p}_{\mathrm{dR}}(M)$$

where the isomorphism from $H^p_{\mathrm{dR}}(M)$ to $H^{m-p}_{\mathrm{dR}}(M)$ is induced by $\star_p$.

Corollary. *Let M be a connected, compact and oriented m-manifold without boundary, then*

$$H^m_{\mathrm{dR}}(M) \cong H^0_{\mathrm{dR}}(M) \cong \mathbb{R}$$

Theorem. *Let M be a compact oriented m-manifold without boundary. The map $\mathcal{I} : H^p(M) \otimes H^{m-p}(M) \to \mathbb{R}$ given by*

$$\mathcal{I}(\omega_p \otimes \tilde{\omega}_{m-p}) = \int_M \omega_p \wedge \tilde{\omega}_{m-p}$$

is well defined. Furthermore, it is a perfect pairing, that is given a non zero $[\omega_p] \in H^p(M)$ there is $[\tilde{\omega}_{m-p})] \in H^{m-p}(M)$ such that $\mathcal{I}(\omega_p \otimes \tilde{\omega}_{m-p}) \neq 0$

Corollary.

$$H^*(\mathbb{C}P^n) = \operatorname{span}\{[\mathbb{1}], [x_2], [x_2^2], \dots, [x_2^{2n}]\}$$

Brief Review of Integral Curves and Flows:

Definition. Let M be a smooth manifold. A smooth curve $\sigma : I \to M$ is said to be **an integral curve of** X if

$$\dot{\sigma}(t) = X_{\sigma(t)} \quad \forall\ t \in I$$

If $0 \in I$, the point $\sigma(0)$ is called the **starting point** of σ. ▲

Lemma. *Let X be a smooth vector field on a smooth manifold M. For each point $p \in M$, there exist $\varepsilon > 0$ and a smooth curve $\sigma : (-\varepsilon, \varepsilon) \to M$ that is an integral curve of X starting at p.*

Lemma. *Let X be a smooth vector field on a smooth manifold M, let $I \subset \mathbb{R}$ be an interval, and let $\sigma : I \to M$ be an integral curve of V.*

- *For any $a \in \mathbb{R}$, the curve $\tilde{\sigma} : \tilde{I} \to M$ defined by $\tilde{\sigma}(t) := \sigma(at)$ is an integral curve of the vector field aX, where $\tilde{I} = \{t \in \mathbb{R} : at \in I\}$.*
- *For any $b \in \mathbb{R}$, the curve $\hat{\sigma} : \hat{I} \to M$ defined by $\hat{\sigma}(t) := \sigma(t + b)$ is an integral curve of the vector field X, where $\hat{I} = \{t \in \mathbb{R} : t + b \in I\}$.*

Remark. Let M be a smooth manifold and $X \in \Gamma(TM)$, and suppose that for each point $p \in M$, X has a unique integral curve starting at p and defined for all $t \in \mathbb{R}$, which we denote by $\Phi^{(p)} : \mathbb{R} \to M$. Then, for each $t \in \mathbb{R}$, we can define a map $\Phi_t : M \to M$ by sending each $p \in M$ to the point obtained by following for time t the integral curve starting at p:

$$\Phi_t(p) := \Phi^{(p)}(t)$$

The second part on the previous lemma implies that for any $s \in \mathbb{R}$, the map $t \mapsto \Phi^{(p)}(t+s)$ is an integral curve of X starting at $q := \Phi^{(p)}(s)$. Since we are assuming uniqueness of integral curves, we just proved

$$\Phi^{(q)}(t) = \Phi^{(p)}(t+s),$$

or equivalently $\Phi_t(\ \Phi_s(p)\ ) = \Phi_{t+s}(p)$, which gives

$$\Phi_t \circ \Phi_s = \Phi_{t+s}$$

▼

Definition. Motivated by the previous remark, we define a **global flow on** M to be a continuous map $\Phi : \mathbb{R} \times M \to M$ satisfying the following properties

- $\Phi(0,p) = p$ for all $p \in M$.
- $\Phi(t, \Phi(s,p)) = \Phi(t+s,p)$ for all $s,t \in \mathbb{R}$, $p \in M$.

▲

Remark. We've already seen how a vector field $X \in \Gamma(TM)$ can give rise to a (global) smooth flow by considering the integral curves of the vector field, such flow will be denoted by Φ^X. Conversely, if $\Phi : \mathbb{R} \times M \to M$ is a smooth global flow, for each $p \in M$ we define $\Phi^{(p)} : \mathbb{R} \to M$ by $\Phi^{(p)}(t) := \Phi(t,p)$ and $X_p \in T_pM$ by

$$X_p := \dot{\Phi^{(p)}}(0)$$

then get a vector field X by considering the map $p \mapsto X_p$. Furthermore, each curve $\Phi^{(p)}$ is an integral curve of X. ▼

Lemma. *Let $X, Y \in \Gamma(TM)$. For each $t \in \mathbb{R}$ we define $\Phi_t^X : M \to M$ by $\Phi_t^X(p) := \Phi^X(t,p)$. Then.*

$$(\Phi_{-t}^Y \circ \Phi_{-t}^X \circ \Phi_t^Y \circ \Phi_t^X)(p) = p + t^2[X,Y]_p + \mathcal{O}(t^3)$$

Lie Groups and Lie Algebras:

Lie Groups:

Definition. A **Lie Group** G is

- A smooth manifold with element $1 \in G$.
- A multiplication map $m : G \times G \to G$ such that m is smooth.
- An inversion map $\text{inv} : G \to G$ such that inv is smooth.
- (G, m, inv) give G the structure of a group.

▲

Example. The following are all matrix Lie groups

- $\text{GL}(n, \mathbb{R}) := \{A \in M_n(\mathbb{R}) : \det(A) \neq 0\}$
- $\text{O}(n) := \left\{A \in \text{GL}(n, \mathbb{R}) : AA^T = I_n\right\}$
- $\text{SL}(n, \mathbb{R}) := \{A \in \text{GL}(n, \mathbb{R}) : \det(A) = 1\}$
- $\text{SO}(n) := \text{SL}(n, \mathbb{R}) \cap \text{O}(n)$
- $\text{GL}(n, \mathbb{C}) := \{A \in M_n(\mathbb{C}) : \det(A) \neq 0\}$
- $\text{U}(n) := \{A \in \text{GL}(n, \mathbb{C}) : AA^* = I_n\}$
- $\text{SL}(n, \mathbb{C}) := \{A \in \text{GL}(n, \mathbb{C}) : \det(A) = 1\}$
- $\text{SU}(n) := \text{SL}(n, \mathbb{C}) \cap \text{U}(n)$

▼

Lie Algebras:

Definition. A **Lie algebra (over $\mathbb{R}$)** is a real vector space $\mathfrak{g}$ endowed with a map called the bracket from $\mathfrak{g} \times \mathfrak{g}$ to $\mathfrak{g}$, usually denoted by $(X, Y) \mapsto [X, Y]$ that satisfies the following properties for all $X, Y, Z \in \mathfrak{g}$

- BILINEARITY: For $a, b \in \mathbb{R}$,

  $$[aX + bY, Z] = a[X, Z] + b[Y, Z] \quad \text{and} \quad [X, aY + bZ] = a[X, Y] + b[X, Z]$$

- ANTISYMMETRY:

  $$[X, Y] = -[X, Y]$$

- JACOBI IDENTITY:

  $$\Big[X, [Y, Z]\Big] + \Big[Y, [Z, X]\Big] + \Big[Z, [X, Y]\Big] = 0$$

If $\mathfrak{g}$ is a Lie algebra, a linear subspace $\mathfrak{h} \subset \mathfrak{g}$ is called a **Lie subalgebra** of $\mathfrak{g}$ if it is closed under brackets. In this case $\mathfrak{h}$ is itself a Lie algebra with the restriction of the same bracket.

If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, a linear map $\theta : \mathfrak{g} \to \mathfrak{h}$ is called a **Lie algebra homomorphism** if it preserves brackets:

$$[\theta X, \theta Y]_{\mathfrak{h}} = \theta[X, Y]_{\mathfrak{g}}$$

An invertible Lie algebra homomorphism is called a **Lie algebra isomorphism**. If there exists a Lie algebra isomorphism from $\mathfrak{g}$ to $\mathfrak{h}$, we say that they are isomorphic as Lie algebras. ▲

Lie Algebra of a Lie Group:

Definition. Let G be a Lie group. If $g \in G$ we have **left** and **right** multiplication by g denoted by L_g and R_g respectively. That is, $L_g(h) = gh$ and $R_g(h) = hg$. ▲

Remark. Both L_g and R_g are diffeomorphisms from G to it self and clearly $L_g^{-1} = L_{g^{-1}}$, $R_g^{-1} = R_{g^{-1}}$. Furthermore, $L_g R_h = R_h L_g$ for any $g, h \in G$. ▼

Remark. Recall that for a smooth manifold, a vector field is a smooth section of the tangent bundle TM. In what follows, if $X \in \Gamma(TM)$ is a vector field, we'll use the notation $X_p := X(p) \in T_pM$. Furthermore, if $F : M \to N$ is a diffeomorphism of smooth manifolds, we have a well defined pushforward of vector fields $F_* : \Gamma(TM) \to \Gamma(TN)$ given by

$$(F_*X)_q = F_*(X_{F^{-1}(q)}) \quad \text{for} \quad X \in \Gamma(TM),\ q \in N$$

Where F_* in the RHS is the usual differential $F_* : T_{F^{-1}(q)}M \to T_qN$. Finally, a routine computation gives that for any $X, Y \in \Gamma(TM)$

$$F_*[X, Y] = [F_*X, F_*Y]$$

All the above makes sense because F is a diffeormorphism, otherwise the pushforward of vector fields may not be well defined. ▼

Definition. Let G be a Lie group. The Lie algebra of G, denoted by $\mathfrak{g}(G) := \mathfrak{g}$, is the space of left invariant vector fields. That is,

$$\mathfrak{g} := \{X \in \Gamma(TG) : (L_g)_*X_h = X_{gh} \text{ for all } g, h \in G\}$$

Since $L_g : G \to G$ is a diffeormorphism, thanks to the remark above we have in fact

$$\mathfrak{g} = \{X \in \Gamma(TG) : (L_g)_* X = X \text{ for all } g \in G\}$$

▲

Lemma. *If $X, Y \in \mathfrak{g}$, then $[X, Y] \in \mathfrak{g}$, that is $\mathfrak{g}$ is indeed a Lie algebra.*

Proof. For any $g \in G$ we have

$$(L_g)_*[X, Y] = [(L_g)_* X, (L_g)_* Y] = [X, Y]$$

■

Lemma. *The map $X \mapsto X_1$, from $\mathfrak{g}$ to $T_1 G$ is a bijection. Thus, we usually identify $\mathfrak{g}$ with $T_1 G$.*

Proof. Suppose first that $X_1 = Y_1$ for $X, Y \in \mathfrak{g}$. Then, for any $g \in G$

$$X_g = (L_g)_* X_1 = (L_g)_* Y_1 = Y_g$$

Thus $X = Y$, proving injectivity. To show surjectivity, take any $w \in T_1 G$ and let's define X^w by

$$X^w_g := (L_g)_* w.$$

Then, $X^w \in \Gamma(TG)$. Moreover, for any $h \in G$ we get

$$(L_h)_* X^w_g = (L_h)_*(L_g)_* w = (L_h L_g)_* w = (L_{hg})_* w = X^w_{hg}$$

Hence, $X^w \in \mathfrak{g}$. By construction $X^w_1 = w$. ■

Remark. When $G = \mathrm{GL}(n, \mathbb{R})$, its Lie algebra $\mathfrak{g}$ which we've identified with $T_{I_n} G$, is now in turn identified with $M_n(\mathbb{R})$. Thus, for any $w \in M_n(\mathbb{R})$ and any $g \in \mathrm{GL}(n, \mathbb{R})$ we have

$$X^w_g = (L_g)_* w = gw$$

because L_g is a linear map and therefore $(L_g)_* = L_g$. ▼

Remark. Let $G = \mathrm{GL}(n, \mathbb{R})$, whose Lie algebra is $\mathfrak{g} \cong T_{I_n} G \cong M_n(\mathbb{R})$. If $X := X^w \in \Gamma(TG)$ for any $w \in M_n(\mathbb{R})$, then its flow Φ^X is such that $\Phi^{(I_n)}(t) := \Phi^X_t(I_n)$ is the unique integral curve to X starting at I_n. Thus, since by construction $\partial_t\{\Phi^{(I_n)}\}(0) = X_{I_n} = I_n w = w$, it follows that

$$\Phi^X_t(I_n) = \exp_{I_n}(tw)$$

For $g \in G$, notice that the curve $t \mapsto g\Phi_t^X(I_n)$ is the integral curve for X starting at g. Thus, by uniqueness of such integral curves, we ought to have $\Phi_t^X(g) = g\Phi_t^X(I_n)$, that is

$$\Phi_t^X(g) = g \exp_{I_n}(tw) = g \sum_{k=0}^{\infty} \frac{t^k w^k}{k!}$$

That is, in this case the exponential map coincides with the usual exponential of matrices. ▼

Remark. Let $G = \mathrm{GL}(n, \mathbb{R})$, whose Lie algebra is $\mathfrak{g} \cong T_{I_n}G \cong M_n(\mathbb{R})$. The vector spaces $\mathfrak{g}$ and $M_n(\mathbb{R})$ have independently defined Lie algebra structures-the first coming from Lie brackets of vector fields, and the second from commutator brackets of matrices. The next theorem shows that the natural vector space isomorphism between these spaces is in fact a Lie algebra isomorphism. ▼

Theorem. *Let $G = \mathrm{GL}(n, \mathbb{R})$ and $\mathfrak{g}$ its Lie algebra. For $w \in T_{I_n}G \cong M_n(\mathbb{R})$ let $X^w \in \mathfrak{g}$ be as in the previous proof. Then,*

$$[X^v, X^w] = X^{[v,w]}$$

for any $v, w \in M_n(\mathbb{R})$.

Theorem. *Let G be a Lie group. The map $\exp_1 : T_1G \to G$ is a local diffeomorphism such that $(\exp_1)_* = \mathrm{id}$ and for any $w \in T_1G$,*

$$\exp_1(tw)\exp_1(tw) = \exp_1((t+s)w)$$

Theorem. *If H is a closed subgroup of a Lie group G, then H is a closed submanifold of H and it's a Lie group on its own right. Furthermore, the Lie algebra of H, denoted by $\mathfrak{h}$, is given by*

$$\mathfrak{h} = \{w \in \mathfrak{g} : \exp_1(tw) \in H \quad \textit{for all} \quad t \in \mathbb{R}\}$$

Example. Recall that $\mathrm{O}(n) := \left\{A \in \mathrm{GL}(n, \mathbb{R}) : AA^T = I_n\right\}$. Then, the Lie algebra of $\mathrm{O}(n)$, which we denote by $\mathfrak{o}(n)$, is given by

$$\mathfrak{o}(n) = \{w \in M_n(\mathbb{R}) : \exp(tw) \in \mathrm{O}(n) \quad \text{for all} \quad t \in \mathbb{R}\}$$

Suppose that $w \in \mathfrak{o}(n)$. Then, for any $t \in \mathbb{R}$ we have $\exp(tw) \in \mathrm{O}(n)$ and therefore

$$\begin{aligned}
\exp(tw)\exp(tw)^T = I_n &\iff \exp(tw)\exp(tw^T) = I_n \\
&\iff \left(I_n + tw + \mathcal{O}(t^2)\right)\left(I_n + tw^T + \mathcal{O}(t^2)\right) = I_n \\
&\iff I_n + tw + tw^T + \mathcal{O}(t^2) = I_n \\
&\iff tw + tw^T = \mathcal{O}(t^2) \\
&\iff w + w^T = \mathcal{O}(t)
\end{aligned}$$

Thus, if $t = 0$ we get $w + w^T = 0$. Now suppose that $w \in M_n(\mathbb{R})$ is such that $w + w^T = 0$. Then, for any $t \in \mathbb{R}$

$$\exp(tw)\exp(tw)^T = \exp(tw)\exp(tw^T) = \exp(tw)\exp(-tw) = \exp((t-t)w) = I_n$$

Thus, we've shown

$$\mathfrak{o}(n) = \left\{w \in M_n(\mathbb{R}) : w + w^T = 0\right\}$$

▼

Example. Let $\langle \cdot, \cdot \rangle$ be an inner product in $\mathbb{R}^n$ with signature (P, N), for $P + N = n$. Let

$$O(P, N) := \{A \in \mathrm{GL}(n, \mathbb{R}) : \langle Ax, Ay \rangle = \langle x, y \rangle \text{ for any } x, y \in \mathbb{R}^n\}$$

The adjoint of a matrix $w \in M_n(\mathbb{R})$ for this inner product is defined as usual by $\langle wx, y \rangle = \langle x, w^* y \rangle$. Then, we find that the Lie algebra of $O(P, N)$ is

$$\mathfrak{o}(P, N) = \{w \in M_n(\mathbb{R}) : w + w^* = 0\}$$

▼

Example. Recall that $\mathrm{U}(n) := \{A \in \mathrm{GL}(n, \mathbb{C}) : AA^* = I_n\}$, where A^* is the conjugate transpose of A. Then,

$$\mathfrak{u}(n) = \{w \in M_n(\mathbb{C}) : w + w^* = 0\}$$

▼

Lemma. *Let $w \in M_n(\mathbb{F})$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. Then,*

$$\det(\exp(w)) = e^{\mathrm{tr}(w)}$$

Corollary. *Recall that* $\mathrm{SL}(n,\mathbb{F}) := \{A \in \mathrm{GL}(n,\mathbb{F}) : \det(A) = 1\}$, *where* $\mathbb{F}$ *is either* $\mathbb{R}$ *or* $\mathbb{C}$. *Then*

$$\mathfrak{sl}(n,\mathbb{F}) = \{w \in M_n(\mathbb{F}) : \mathrm{tr}(w) = 0\}$$

Example. Recall that $\mathrm{SO}(n) := \mathrm{SL}(n,\mathbb{R}) \cap \mathrm{O}(n)$ and $\mathrm{SU}(n) := \mathrm{SL}(n,\mathbb{C}) \cap \mathrm{U}(n)$. Then

$$\mathfrak{so}(n) = \mathfrak{sl}(n,\mathbb{R}) \cap \mathfrak{o}(n) = \{w \in M_n(\mathbb{R}) : w + w^T = 0,\ \mathrm{tr}(w) = 0\}$$

and

$$\mathfrak{su}(n) = \mathfrak{sl}(n,\mathbb{C}) \cap \mathfrak{u}(n) = \{w \in M_n(\mathbb{C}) : w + w^* = 0,\ \mathrm{tr}(w) = 0\}$$

▼

Remark. How to compute the Lie algebra of $H \times K$ in terms of their Lie algebras $\mathfrak{h}$, $\mathfrak{k}$? Suppose that H and K are both Lie subgroups of $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_m(\mathbb{R})$ respectively, so we may think of $H \times K$ as a Lie subgroup of $\mathrm{GL}_{n+m}(\mathbb{R})$. Notice that we are embedding $M_n(\mathbb{R}) \times M_m(\mathbb{R})$ into $M_{m+n}(\mathbb{R})$ via the map

$$\iota(A,B) := \begin{pmatrix} A & 0_{n\times m} \\ 0_{m\times n} & B \end{pmatrix}$$

Now, notice that the underling space for the Lie algebra of $H \times K$ is

$$T_{I_{n+m}}(H \times K) \cong T_{I_n}(H) \times T_{I_m}(K) = \mathfrak{h} \times \mathfrak{k} \subset M_n(\mathbb{R}) \times M_n(\mathbb{R}) \hookrightarrow M_{m+n}(\mathbb{R})$$

Furthermore, the Lie bracket is

$$[\iota(A_1,B_1), \iota(A_2,B_2)] = \iota([A_1,A_2],[B_1,B_2])$$

for $A_1, A_2 \in \mathfrak{h}$ and $B_1, B_2 \in \mathfrak{k}$. That is, the Lie bracket for $T_{I_{n+m}}(H \times K)$ is completely determined by the one for $\mathfrak{h}$ and the one for $\mathfrak{k}$. Finally, suppose that

$$\mathfrak{h} := \mathrm{span}_{\mathbb{R}}\{e_1, \ldots, e_h\}, \quad \mathfrak{k} := \mathrm{span}_{\mathbb{R}}\{f_1, \ldots, f_k\}.$$

Then,

$$T_{I_{n+m}}(H \times K) = \iota(\mathfrak{h} \times \mathfrak{k}) = \mathrm{span}_R\{\iota(e_1,0), \ldots, \iota(e_h,0), \iota(0,f_1), \ldots, \iota(0,f_k)\}$$

▼

Remark. Let G, H be Lie groups whose Lie algebras are $\mathfrak{g}$ and $\mathfrak{h}$. If $F : G \to H$ is a smooth map that is also a group homomorphism, for any $X \in \mathfrak{g}$ we get an element F^*X in $\mathfrak{h}$ by considering the vector field obtained by the derivation $F_*X_1 \in T_1H$. Even though F may not be a diffeomorphism, the map $F_* : \mathfrak{g} \to \mathfrak{h}$ is a well defined Lie algebra homomorphism.

We have then a functor from the category of Lie groups to the category of Lie algebras sending a group G to its Lie algebra $\mathfrak{g}(G)$ and a morphism $F : G \to H$ to $F_* : \mathfrak{g}(G) \to \mathfrak{g}(H)$. This is not a 1 to 1 functor, since non isomorphic groups can have same Lie algebras. However, isomorphic groups must have isomorphic Lie algebras. ▼

Theorem. *If $\mathfrak{g}$ is a Lie algebra, then there is a simply connected group G such that $\mathfrak{g}(G) = \mathfrak{g}$*

Theorem. *If $\theta : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, and H, G are simply connected groups such that $\mathfrak{g}(G) = \mathfrak{g}$ and $\mathfrak{g}(H) = \mathfrak{h}$, then there is a Lie group homomorphism $\Theta : G \to H$ such that $\Theta_* = \theta$.*

Lie Algebra Cohomology:

Definition. Let $\mathfrak{g}$ be a finite dimensional Lie algebra with basis given by $\{e_1, \dots, e_n\}$. The **Lie algebra structure constants** ${C_{ij}}^k \in \mathbb{R}$ are given by

$$[e_i, e_j] = {C_{ij}}^k e_k$$

Antisymmetry of the bracket implies that ${C_{ij}}^k = -{C_{ji}}^k$. ▲

Definition. Let $\mathfrak{g}$ be a finite dimensional Lie algebra with basis given by $\{e_1, \dots, e_n\}$. Let $\{e_1, \dots, e_n\} \subset \mathfrak{g}^*$ the dual basis, that is each $e^i : \mathfrak{g} \to \mathbb{R}$ is such that

$$e^i(e_j) = \delta_{ij}$$

We define a map $d : \mathfrak{g}^* \to \Lambda^2\mathfrak{g}^*$ by

$$de^k := -\sum_{i<j} {C_{ij}}^k e^i \wedge e^j$$

Extending this with the Leibnitz rule gives a map $d : \Lambda^p\mathfrak{g}^* \to \Lambda^{p+1}\mathfrak{g}^*$. Jacobi identity for the bracket implies that $d^2 = 0$ and therefore it makes sense to define

$$H^p(\mathfrak{g}) := \frac{\ker(d : \Lambda^p\mathfrak{g}^* \to \Lambda^{p+1}\mathfrak{g}^*)}{\operatorname{im}(d : \Lambda^{p-1}\mathfrak{g}^* \to \Lambda^p\mathfrak{g}^*)}$$

▲

Theorem. ***(Hodge)*** *Let G be a compact connected Lie group and $\mathfrak{g}$ its Lie algebra. Then $H^p(\mathfrak{g}) \cong H_{\mathrm{dR}}(G)$.*

Example. One can use Hodge Theorem to determine when a Lie algebra comes from a compact connected Lie group. For example, if the Lie algebra cohomology doesn't satisfy Poincaré duality, then it can't be the cohomology of a compact manifold. ▼

Remark. Notice that any Lie group G is orientable. Indeed, suppose G has dimension n as smooth manifold. Let $\{w_1, \dots, w_n\}$ be a basis for T_1G. As usual we define vector fields $X^{w_i} \in \Gamma(TG)$ by letting $X_g^{w_i} := (L_g)_* w_i$ for each $g \in G$. For each $g \in G$, $\{X_g^{w_1}, \dots, X_g^{w_n}\}$ is then a basis for T_gG. This gives a map $TG \to G \times \mathbb{R}^n$ sending $(g, a_i X_g^{w_i})$ to $(g, (a_1, \dots, a_n))$ which is clearly smooth and has a smooth inverse. Thus TG is trivial and hence G ought to be orientable (product of orientable is orientable and the tangent bundle is always orientable). ▼

Theorem. *Let G be a compact connected Lie group.*

- *G is **unimodular:** There is a bi-invariant volume form on G, that is there is $\omega \in \Omega^n(G)$ such that $(L_g)_*\omega = (R_g)_*\omega = \omega$ for all $g \in G$.*
- *There is a bi-invariant Riemannian metric on G, that is there is $(\cdot, \cdot)$, a smooth positive definite inner product on $\Gamma(TG)$, such that $((L_g)_*X, (L_g)_*Y) = ((R_g)_*X, (R_g)_*Y) = (X, Y)$ for all $X, Y \in \Gamma(TG)$, and all $g \in G$,*
- *Any harmonic p-form in bi-invariant.*

Theorem.

- *If $\mathfrak{g}$ is a finite dimensional Lie algebra, then $\mathfrak{g}$ is isomorphic to a matrix algebra.*
- *If $\mathfrak{g}$ is a finite dimensional Lie algebra, then $\mathfrak{g}$ is the Lie algebra of a matrix group.*
- *Let G and H be compact connected Lie groups, whose Lie algebras are given by $\mathfrak{g}$ and $\mathfrak{h}$. Then $G \cong H$ if and only of $\mathfrak{g} \cong \mathfrak{h}$.*

Definition. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. For each $X \in \mathfrak{g}$, we define $\mathrm{ad}(X) \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$ by

$$\mathrm{ad}(X)Y = [X, Y]$$

Jacobi identity implies that

$$\mathrm{ad}[X, Y] = [\mathrm{ad}(X), \mathrm{ad}(Y)].$$

Thus, $\mathrm{ad} : \mathfrak{g} \to \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$ is a Lie algebra homomorphism ▲

Definition. Let $\mathfrak{g}$ be a Lie algebra. The **Killing form** $\mathcal{K} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, is given by

$$\mathcal{K}(X, Y) := \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y))$$

Notice that $\mathcal{K}(X, Y) = \mathcal{K}(Y, X)$. If $\{e_1, \dots, e_n\}$ is a basis for $\mathfrak{g}$, we put $\mathcal{K}_{ij} := \mathcal{K}(e_i, e_j)$. ▲

Example. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. If $\mathcal{K}$ is non-degenerated, then G is unimodular. ▼

Remark. Let $G \leq \mathrm{GL}(n, \mathbb{R})$ be Lie group and $\mathfrak{g}$ its Lie algebra. For any $X \in \mathfrak{g}$ and any $g \in G$ we know that $(L_g)_* X = X$, furthermore we know that $(L_g)_* w = gw$ for $w \in M_n(\mathbb{R})$. So $(L_g)_*$ is actually left matrix multiplication when we regard $\mathfrak{g} \subset M_n(\mathbb{R})$. A natural question is to find how $(R_g)_*$ acts on $\mathfrak{g}$ as a subset of $M_n(\mathbb{R})$. Well, take any $X \in \mathfrak{g}$, $g \in G$, since R_g is a diiffeomorphism

$$\big((R_g)_* X\big)_{I_n} = (R_g)_* X_{R_{g^{-1}}(I_n)} = (R_g)_* X_{g^{-1}} = X_{g^{-1}} g = L_{g^{-1}} X_{I_n} g = g^{-1} X_{I_n} g$$

Thus if w corresponds to X under the identification $\mathfrak{g} \leftrightarrow T_{I_n} G$, it follows that $g^{-1} w g$ corresponds to $(R_g)_* X$ under the same identification. Under out previous notation we've shown

$$X^{g^{-1} w g} = (R_g)_* X^w$$

Therefore we say that $(R_g)_* w = g^{-1} w g$ for any $w \in T_{I_n} G$. This gives us a map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ given by $\mathrm{Ad}(g) = (R_g)_*$ for any $g \in G$. It turns out that $\mathrm{Ad}_* : \mathfrak{g} \to \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$ is actually given by ad. ▼

Theorem. *Let G be a closed connected subgroup of $\mathrm{GL}(m, \mathbb{R})$ for some m. Let $(\cdot, \cdot)$ be a left-invariant symmetric bilinear form on TG. This is defined on matrices $A, B \in T_{I_n} G \cong \mathfrak{g}$ by letting $(A, B) := (X^A, X^B)$, where as usual the vector field X^A is defined by $X^A_g := (L_g) * A$ for any $g \in G$. The following conditions are equivalent*

- $(\cdot,\cdot)$ *is bi-invariant*
- $(A,B) = (g^{-1}Ag, g^{-1}Bg)$ *for any* $A, B \in \mathfrak{g}$ *and* $g \in G$.
- $\mathrm{ad}(A)$ *is skew-adjoint, that is for any* $B, C \in \mathfrak{g}$
$$(\mathrm{ad}(A)B, C) = -(B, \mathrm{ad}(A)C)$$
- *For the following condition to be equivalent we need to add that* $(\cdot,\cdot)$ *gives Riemannian metric and* ∇ *is the Levi-Civita connection.*
- *then for any* $A, B \in \mathfrak{g}$
$$\nabla_{X^A} X^B = \frac{1}{2} X^{[A,B]}$$

Theorem. *Let* G *be a closed connected subgroup of* $\mathrm{GL}(m,\mathbb{R})$ *for some* m. *Let* $(\cdot,\cdot)$ *be a bi-invariant Riemannian metric and* ∇ *the Levi-Civita connection. Then*

- $R(X^A, X^B)X^C = -\dfrac{1}{4}\big[[A,B],C\big]$ *for any* $A, B, C \in \mathfrak{g}$
- *The curves* $g \exp^{\mathfrak{g}}(At)h$ *are geodesics in* G *for any* $g, h \in G$ *and* $A \in \mathfrak{g}$. *That is, the* $\exp^{\mathfrak{g}}$ *in Lie group sense is the exponential map in the Levi-Civita sense.*

Lemma. *Let* G *be a closed connected subgroup of* $\mathrm{GL}(m,\mathbb{R})$ *for some* m. *Let* $(\cdot,\cdot)$ *be a bi-invariant Riemannian metric and* ∇ *the Levi-Civita connection.. Then*

- G *is geodesically complete*
- $\rho = -\frac{1}{4}\mathcal{K}$, *where* ρ *is the Ricci tensor and* $\mathcal{K}$ *the Killing form.*
- $\mathcal{K}$ *is negative semi-definite.*
- *If* $\mathcal{K}$ *is negative definite and* G *is connected, then* G *is compact.*
- *If* G *is compact, then* G *is isomorphic to* $\mathrm{SO}(n)$ *for some* n.

Proof. We only prove the assertion If $\mathcal{K}$ is negative definite and G is connected, then G is compact, since it could be a **potential qual problem.** Since $\mathcal{K}$ is negative definite, we use it to get a negative definite metric on G. Thus, $\rho = -\frac{1}{4}\mathcal{K}$ is positive definite (therefore $\rho \geq \varepsilon(\cdot,\cdot)$) and therefore we can appeal to Myers's theorem from Spring, which assures us that G is compact. ■

Theorem. *If $G = \mathrm{SO}(n)$, the Killing form is negative definite.*

Example. As a corollary of the previous Lemma and Theorem, it follows that if G is compact, then its Killing form is negative semi-definite. Thus, any Lie algebra whose Killing form is not negative semi-definite can't come from a compact group. ▼

Theorem. ***(Hodge)*** *Let G be a connected compact Lie group. Then $H^*_{\mathrm{dR}}(G)$ (which is isomorphic to $H^*(\mathfrak{g})$ by a previous thm) is an exterior algebra on odd generators. That is*

$$H^*_{\mathrm{dR}}(G) = \Lambda[x_1, \ldots, x_l]$$

where each x_i has odd degree.

Example. The previous theorem implies that any Lie algebra whose cohomology is not an exterior algebra on odd generators can't come from a compact group. For example, if one gets

$$H^*(\mathfrak{g}) = \mathbb{R}[\mathbb{1}] \oplus \mathbb{R}[e^1] \oplus \mathbb{R}[e^1 \wedge e^3]$$

We have $H^*(\mathfrak{g}) = \Lambda[e^1, e^1 \wedge e^3]$ and therefore $\mathfrak{g}$ can't be the Lie algebra of a compact group. ▼

Example.

- $H^*_{\mathrm{dR}}(\mathbb{T}^n) = \Lambda[d\theta_1, \ldots, d\theta_n]$ where each $d\theta_i \in \Omega^1(S^1)$.
- $H^*_{\mathrm{dR}}(S^3) = \Lambda[\omega_{S^3}]$
- $H^*_{\mathrm{dR}}(\mathrm{U}(2)) = H^*_{\mathrm{dR}}(S^1 \times S^3) = \Lambda[\omega_{S^1}, \omega_{S^3}]$

▼

Example. Why is $S^1 \times S^2$ not a Lie group. Well it is a compact manifold, however we know that

$$H^*_{\mathrm{dR}}(S^1 \times S^2) = \operatorname{span}\{[\mathbb{1}], [\omega_{S^1}], [\omega_{S^2}], [\omega_{S^1} \wedge \omega_{S^2}]\}$$

and ω_{S^2} has degree 2. ▼

Example. In general,

$$H^*(\mathrm{U}(n)) = \Lambda[x_1, x_3, \dots, x_{2n-1}]$$

where $x_{2k-1} := [\Theta_{2k-1}]$ where Θ_{2k-1} are the Maurer-Cartan forms, which we defined as follows: Let $g : \mathrm{U}(n) \hookrightarrow M_n(\mathbb{C})$ be the natural inclusion. Then, dg and $g^{-1}dg$ are both matrices of 1-forms. We define

$$\Theta_{2k-1} := \operatorname{tr}\big((g^{-1}dg)^{2k-1}\big)$$

▼

Holomorphic Manifolds:

Definition. A **holomorphic manifold** is a manifold with an atlas of charts to $\mathbb{C}^n$, such that the transition maps are holomorphic. That is, M together with an atlas $(U_\alpha, \varphi_\alpha)$ where each $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{C}^n$ is a homeomorphism and the transition maps

$$\varphi_{\alpha,\beta} := \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

are holomorphic. Notice that holomorphic manifolds are orientable:

$$\det{}_{\mathbb{R}}\varphi'_{\alpha,\beta} = |\det{}_{\mathbb{C}}\varphi'_{\alpha,\beta}|^2 > 0$$

▲

Set $\mathrm{i} := \sqrt{-1}$. Coordinates is $\mathbb{C}^n$ are

$$z^1 = x^1 + \mathrm{i}y^1, \dots, z^n = x^n + \mathrm{i}y^n.$$

Define 1-forms

$$dz^1 = dx^1 + \mathrm{i}dy^1, \dots, dz^n = dx^n + \mathrm{i}dy^n.$$

together with their dual elements

$$\partial_{z^1} = \frac{1}{2}(\partial_{x^1} - \mathrm{i}\partial_{y^1}), \dots, \partial_{z^n} = \frac{1}{2}(\partial_{x^n} - \mathrm{i}\partial_{y^n})$$

Similarly, if

$$d\overline{z^1} = dx^1 - \mathrm{i}dy^1, \dots, d\overline{z^n} = dx^n - \mathrm{i}dy^n.$$

their dual elements are

$$\partial_{\overline{z^1}} = \frac{1}{2}(\partial_{x^1} + \mathrm{i}\partial_{y^1}), \ldots, \partial_{\overline{z^n}} = \frac{1}{2}(\partial_{x^n} + \mathrm{i}\partial_{y^n})$$

Definition. A function $f : \mathbb{C}^n \to \mathbb{C}^k$ is holomorphic iff $\partial_{\overline{z^j}} f = 0$ for all $1 \le j \le n$. ▲

Definition. For $f : \mathbb{C}^n \to \mathbb{C}^k$, define operators $\partial, \overline{\partial}$ and d by

$$\partial f = \sum_{j=1}^{n} \partial_{z^j} f \; dz^j, \quad \overline{\partial} f = \sum_{j=1}^{n} \partial_{\overline{z^j}} f \; d\overline{z^j}, \quad df = \partial f + \overline{\partial} f$$

Notice that f is holomorphic iff $\overline{\partial} f = 0$. ▲

If $I = \{1 \le i_1 < \ldots < i_p \le n\}$ and $K = \{1 \le k_1 < \ldots < k_q \le n\}$, we put

$$dz^I = dz^{i_i} \wedge \cdots \wedge dz^{i_p}, \qquad d\overline{z^K} = d\overline{z^{k_i}} \wedge \cdots \wedge d\overline{z^{k_q}}$$

Let $\Lambda^{p,q} := \mathrm{span}\{dz^I \wedge d\overline{z^K} : |I| = p, |K| = q\}$. Then, we extend the opereators define above as

$$\partial \left(\sum f_{I,K} dz^I \wedge d\overline{z^K} \right) = \sum \partial f_{I,K} dz^I \wedge d\overline{z^K}$$
$$\overline{\partial} \left(\sum f_{I,K} dz^I \wedge d\overline{z^K} \right) = \sum \overline{\partial} f_{I,K} dz^I \wedge d\overline{z^K}$$

and we get $\partial : \Lambda^{p,q} \to: \Lambda^{p+1,q}$ while $\overline{\partial} : \Lambda^{p,q} \to: \Lambda^{p,q+1}$. Further, notice that $\partial^2 = 0$, $\overline{\partial}^2 = 0$ and $\partial\overline{\partial} + \overline{\partial}\partial = 0$.

Definition. Define J to be an endomorphism of the tangent bundle so that for any $1 \le j \le n$.

$$J(\partial_{x^j}) = \partial_{y^j} \quad \text{and} \quad J(\partial_{y^j}) = -\partial_{x^j}$$

Notice that $J^2 = -\mathrm{id}$ and that $J(\partial_{z^j}) = \mathrm{i}\partial_{z^j}$ while $J(\partial_{\overline{z^j}}) = -\mathrm{i}\partial_{\overline{z^j}}$. The map J gives the **complex structure.** ▲

Definition. Let g be a Riemannian metric on M and J a complex structure on M. The metric g is said to be Hermitian if for any $X, Y \in \Gamma(TM)$,

$$g(JX, JY) = g(X.Y)$$

That is, if $J^* g = g$. ▲

Lemma. *If g_0 is an arbitrary metric, then $g := g_0 + J^* g_0$ is Hermitian. Thus, given a complex structure, Hermitian metrics always exist.*

Remark. Suppose g is Hermitian. Extending g to be complex bilinear we easily find that

$$g_{\alpha,\beta} := g(\partial_{z^\alpha}, \partial_{z^\beta}) = 0$$
$$g_{\overline{\alpha},\overline{\beta}} := g(\partial_{\overline{z^\alpha}}, \partial_{\overline{z^\beta}}) = 0$$

Moreover,

$$g_{\alpha,\overline{\beta}} := g(\partial_{z^\alpha}, \partial_{\overline{z^\beta}}) = \overline{g(\partial_{z^\alpha}, \partial_{\overline{z^\beta}})} = \overline{g_{\overline{\alpha},\beta}}$$

▼

Definition. Suppose g is Hermitian. We define Ω by $\Omega(X,Y) := g(JX,Y)$. It's easily checked that $\Omega(X,Y) = -\Omega(Y,X)$. So Ω is a 2-form. In fact, we have

$$\Omega = \mathrm{i}\left(g_{\alpha,\overline{\beta}} dz^\alpha d\overline{z^\beta}\right)$$

▲

Definition. We say that M is **Käler** if $d\Omega = 0$. In this case, in local coordinates we have

$$\Omega = \sum_{k=1}^{n} dx^k \wedge dy^k$$

▲

Lemma. *If N is a holomorphic submanifold of a Käler manifold, then N is Käler.*

Theorem. *Let M be a compact Käler manifold and $x := [\omega] \in H^2_{\mathrm{dR}}(M)$. Then, $x^n \neq 0$ and the groups $H^2_{\mathrm{dR}}(M), H^4_{\mathrm{dR}}(M), \ldots, H^{2n}_{\mathrm{dR}}(M)$ are all non zero.*

Example. For $n \geq 2$, $S^1 \times S^{2n-1}$ is a complex compact manifold. However, we know that

$$H^2_{\mathrm{dR}}(S^1 \times S^{2n-1}) = 0,$$

and therefore $S^1 \times S^{2n-1}$ is not Käler. ▼

Theorem. *If M is a Käler manifold, then Ω is harmonic.*

Example. $\mathbb{C}P^n$ is Käler. ▼

4 Quals.

4.1 Potential Qual Problems