Oral Exam
Miscellaneous results on L^p-spaces

Alonso Delfín
Department of Mathematics
University of Oregon

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Abstract

The main goal of this document is for me to have some kind of guide for my oral exam. This document contains some useful results on L^p spaces. The principal reference are the chapters on L^p spaces of N.L. Carothers' book on Banach Space theory [2]. Another good reference is [1]. This is a work in progress, little proofreading has been done and it's possible it contains some typos/mistakes.

Contents

1 Preliminaries

1.1 L^p spaces

For a measure space $(X, \mathcal{M}, \mu)$ and $p \in [1, \infty)$ we denote by $L^p(X, \mathcal{M}, \mu)$ to the set of equivalence classes of measurable functions $f : X \to \mathbb{C}$ (equal a.e. $[\mu]$) such that

$$\|f\|_p := \left(\int_X |f|^p d\mu \right)^{1/p} < \infty$$

If $\mathcal{M} = 2^X$ and ν is counting measure, we simply write $\ell^p(X)$ for $L^p(X, 2^X, \nu)$. These are Banach spaces under the norm $f \mapsto \|f\|_p$. For $p = \infty$ we set $L^\infty(X, \mathcal{M}, \mu)$ as the set of equivalence classes of measurable functions $f : X \to \mathbb{C}$ (equal a.e. $[\mu]$) such that

$$\|f\|_\infty := \inf\{\alpha > 0 : |f| \leq \alpha \text{ a.e. } [\mu]\} < \infty$$

This is again a Banach space with norm $f \mapsto \|f\|_\infty$. When $\mathcal{M} = 2^X$ and ν is counting measure, $\ell^\infty(X) := L^\infty(X, 2^X, \nu)$ is the space of bounded sequence equipped with the usual sup-norm.

For a general measure space $(X, \mathcal{M}, \mu)$ we sometimes simply write $L^p(X, \mu)$ avoiding the reference to the σ-algebra $\mathcal{M}$.

Three very important examples are

1. $L^p([0, 1])$ which is short for $L^p([0, 1], \mathcal{B}, m)$ where $\mathcal{B}$ is the Borel σ-algebra and m is Lebesgue measure.
2. ℓ^p which is short for $\ell^p(\mathbb{Z}_{>0})$.
3. ℓ^p_n which is short for $\ell^p(\{1, \dots, n\})$

A measure space $(X, \mathcal{M}, \mu)$ is said to be **complete** whenever any subset of a measure zero set is measurable. There is a deep result in abstract measure theory (Maharam's theorem [4]) that says that every complete measure space can be decomposed into copies of $[0, 1]$ (its nonatomic parts) and copies of discrete spaces with counting measures on them (its purely atomic parts). From a Banach space point of view, this means that if $(X, \mathcal{M}, \mu)$ is complete, then $L^p(X, \mathcal{M}, \mu)$ can be written as a direct sum of copies of $L^p([0, 1])$ and ℓ^p. For this reason, the results presented in this document will focus principally on $L^p([0, 1])$ and ℓ^p.

1.2 Bases in Banach Spaces

If E is a normed vector space; we say that $(\xi_n)_{n=1}^\infty$, a nonzero sequence in E, is a **Schauder basis** for E if for each $\xi \in E$, there is a unique sequence of complex numbers $(a_n)_{n=1}^\infty$ such that

$$\xi = \sum_{n=1}^{\infty} a_n \xi_n$$

(that is the series converges in norm to ξ). If $(\xi_n)_{n=1}^\infty$ is a basis for E, then

$$\operatorname{span}(\xi_n)_{n \in \mathbb{Z}_{>0}} := \left\{ \sum_{n=1}^{k} a_n x_n : a_1, \dots, a_k \in \mathbb{C}; n \in \mathbb{Z}_{>0} \right\}$$

is a dense subspace of E. It follows that any normed space with a Schauder basis is separable. However, Per Enflo constructed in 1973 a separable Banach space that does not have a (Schauder) basis.

Fix $p \in [1, \infty)$. For each $n \in \mathbb{Z}_{>0}$, let $\delta^n := (\delta^n_k)_{k \in \mathbb{Z}_{>0}} \in \ell^p$ where $\delta^n_k = \delta_{k,n}$. Then, $(\delta^n)_{n=1}^\infty$ is a Schauder basis for ℓ^p.

If E is a normed vector space; we say that $(\xi_n)_{n=1}^\infty$ is a **basic sequence** if $(\xi_n)_{n=1}^\infty$ is a basis for its closed linear span, a space we denote by $\overline{\operatorname{span}(\xi_n)}$. Turns out that every infinite dimensional Banach space contains a basic sequence. This was shown back in 1933 by Mazur.

Given a basis $(\xi_n)_{n=1}^\infty$ of a normed space E we get coordinate functionals $\omega_n : E \to \mathbb{C}$ by letting

$$\omega_n \left(\sum_{k=1}^{\infty} a_k \xi_k \right) := a_n$$

Each $\omega_n \in E^*$ and $\omega_n(\xi_m) = \delta_{n,m}$. We also get idempotents $s_n : E \to E$ given by

$$s_n \left(\sum_{k=1}^{\infty} a_k \xi_k \right) := \sum_{k=1}^{n} a_k \xi_k$$

Each $s_n \in \mathcal{L}(E)$, $s_n\xi \to \xi$ as $n \to \infty$ for any $\xi \in E$ and $K_{(\xi_n)} := \sup_{n\in\mathbb{Z}_{>0}} \|s_n\| < \infty$. The number $K_{(\xi_n)}$ is called the **basis constant** of the basis $(\xi_n)_{n=1}^{\infty}$. An important remark is that the range of each s_n is $\mathrm{span}(\xi_k)_{k=1}^{n}$ and that s_n acts as the identity on this space. This gives that $\|s_n\| \geq 1$ for all n and therefore $K_{(\xi_n)} \geq 1$.

Notice that any $\xi \in E$ is uniquely written as

$$\xi = \sum_{n=1}^{\infty} \omega_n(\xi)\xi_n = \sum_{n=1}^{\infty} \langle \xi, \omega_n \rangle \xi_n,$$

where $\langle \cdot, \cdot \rangle : E \times E^* \to \mathbb{C}$ is the dual pairing.

To recognize a sequence of elements in a Banach space as a basic sequence we use the following test, also known as *Grunblum's criterion*:

Proposition 1.1. *A sequence $(\xi_n)_{n=1}^{\infty}$ of nonzero elements of a Banach space E is basic if and only if there is a positive constant K such that*

$$\Big\| \sum_{k=1}^{m} a_k\xi_k \Big\| \leq K \Big\| \sum_{k=1}^{n} a_k\xi_k \Big\|$$

for every sequence of scalars (a_k) and all integers m, n such that $m \leq n$

Proof. Suppose first that $(\xi_n)_{n=1}^{\infty}$ is a basic sequence. Then, $(\xi_n)_{n=1}^{\infty}$ is a basis for $\overline{\mathrm{span}(\xi_n)}$. Define $s_m : \overline{\mathrm{span}(\xi_n)} \to \overline{\mathrm{span}(\xi_n)}$ as above. Then, for any $m \leq n$ we have

$$\Big\| \sum_{k=1}^{m} a_k\xi_k \Big\| = \Big\| s_m\Big(\sum_{k=1}^{n} a_k\xi_k \Big) \Big\| \leq K_{(\xi_n)} \Big\| \sum_{k=1}^{n} a_k\xi_k \Big\|$$

Conversely, assume that $\Big\| \sum_{k=1}^{m} a_k\xi_k \Big\| \leq K \Big\| \sum_{k=1}^{n} a_k\xi_k \Big\|$ for some $K \in (0, \infty)$. Notice first that the elements in $(\xi_n)_{n=1}^{\infty}$ are linearly independent. Indeed, if $\sum_{k=1}^{n} a_k\xi_k = 0$ then for any $m \leq n$

$$|a_m|\|\xi_m\| = \Big\| \sum_{k=1}^{m} a_k\xi_k - \sum_{k=1}^{m-1} a_k\xi_k \Big\| \leq 2K \Big\| \sum_{k=1}^{n} a_k\xi_k \Big\| = 0,$$

whence $a_m = 0$. Then, for each $m \in \mathbb{Z}_{>0}$ the map $t_m : \mathrm{span}(\xi_n) \to \mathrm{span}(\xi_n)_{n=1}^{m}$ given by

$$t_m\left(\sum_{k=1}^{n} a_k\xi_k \right) := \sum_{k=1}^{\min(m,n)} a_k\xi_k$$

is a well defined linear map with $\|t_m(\xi)\| \leq K\|\xi\|$ for any $\xi \in \mathrm{span}(\xi_n)$. Therefore, for each $m \in \mathbb{Z}_{>0}$ there is a unique extension $t_m : \overline{\mathrm{span}(\xi_n)} \to \mathrm{span}(\xi_n)_{n=1}^{m}$ with $\|t_m\| \leq K$. We claim that $t_m\xi \to \xi$ as $m \to \infty$ for any $\xi \in \overline{\mathrm{span}(\xi_n)}$. Let $\varepsilon > 0$ and choose $\eta := \sum_{k=1}^{n} a_k\xi_k \in \mathrm{span}(\xi_n)$ such that $\|\eta - \xi\| < \frac{\varepsilon}{K+1}$. Then, for $m \geq n$ we have $t_m(\eta) = \eta$ and therefore

$$\|t_m\xi - \xi\| \leq \|t_m\xi - \eta\| + \|\eta - \xi\| < K\frac{\varepsilon}{K+1} + \frac{\varepsilon}{K+1} = \varepsilon$$

This proves our claim. Since each t_m has range $\mathrm{span}(\xi_n)_{n=1}^{m}$, it follows that any element in $\overline{\mathrm{span}(\xi_n)}$ can be written uniquely in the form $\sum_{n=1}^{\infty} a_n\xi_n$. Hence, $(\xi_n)_{n=1}^{\infty}$ is a basic sequence as we needed to prove. ∎

Let E and F be Banach spaces. Two basic sequences $(\xi_n)_{n=1}^{\infty}$ in E and $(\eta_n)_{n=1}^{\infty}$ in F are said to be **isomorphically equivalent** if for any sequence of scalars $(a_n)_{n=1}^{\infty}$

$$\sum_{n=1}^{\infty} a_n\xi_n \quad \text{converges if and only if} \quad \sum_{n=1}^{\infty} a_n\eta_n \quad \text{converges}$$

It is an easy consequence of the closed graph theorem that if $(\xi_n)_{n=1}^{\infty}$ and $(\eta_n)_{n=1}^{\infty}$ are isomorphically equivalent, then the spaces $\overline{\mathrm{span}(\xi_n)}$ and $\overline{\mathrm{span}(\eta_n)}$ must be isomorphic. Equivalently, $(\xi_n)_{n=1}^{\infty}$ and $(\eta_n)_{n=1}^{\infty}$ are equivalent if there are constants $C_1, C_2 \in (0, \infty)$ such that

$$C_1 \Big\| \sum_{n=1}^{\infty} a_n\eta_n \Big\| \leq \Big\| \sum_{n=1}^{\infty} a_n\xi_n \Big\| \leq C_2 \Big\| \sum_{n=1}^{\infty} a_n\eta_n \Big\|$$

for all scalars $(a_n)_{n=1}^{\infty}$. When $C_1 = C_2 = 1$, we say that the basic sequences are **isometrically isomorphic**.

The following theorem gives a test to check whether a sequence is a basic sequence isomorphically equivalent to a given basic sequence.

Theorem 1.2. *(Principle of small perturbations) Let $(\xi_n)_{n=1}^\infty$ be a basic sequence in a Banach space E. If $(\eta_n)_{n=1}^\infty$ is a sequence in E such that*

$$2K_{(\xi_n)} \sum_{n=1}^{\infty} \frac{\|\xi_n - \eta_n\|}{\|\xi_n\|} = \delta < 1$$

Then $(\eta_n)_{n=1}^\infty$ is a basic sequence equivalent to $(\xi_n)_{n=1}^\infty$.

Proof. Let $\omega_n : \overline{\operatorname{span}(\xi_n)} \to \mathbb{C}$ the coordinate functionals. By Hahn Banach these maps extend to linear functionals $\omega_n : E \to \mathbb{C}$. The map $t : E \to E$ given by

$$t(\xi) = \xi + \sum_{n=1}^{\infty} \omega_n(\xi)(\xi_n - \eta_n)$$

is linear and bounded by $1+\delta$. It's also easy to check that $\|t-1\| < \delta < 1$, whence t is invertible. Since t restricts to an isomorphism $\overline{\operatorname{span}(\xi_n)} \to \overline{\operatorname{span}(\eta_n)}$, the result follows. ■

Let $(\xi_n)_{n=1}^\infty$ be a basic sequence in a Banach space E and $\lambda_1 < \gamma_1 < \lambda_2 < \gamma_2 < \cdots$ an increasing sequence of integers. For each $k \in \mathbb{Z}_{>0}$ let

$$\eta_k := \sum_{j=\lambda_k}^{\gamma_k} b_j \xi_j$$

be any non-zero vector in $\operatorname{span}(\xi_{\lambda_k}, \ldots, \xi_{\gamma_k})$. Then $(\eta_k)_{k=1}^\infty$ is said to be a **block basic sequence** with respect to $(\xi_n)_{n=1}^\infty$. The next lemma gives that any block basic sequence of $(\xi_n)_{n=1}^\infty$ is also a basic sequence:

Lemma 1.3. *Let $(\eta_k)_{k=1}^\infty$ be a block basic sequence with respect to the basic sequence $(\xi_n)_{n=1}^\infty$. Then, $(\eta_k)_{k=1}^\infty$ is a basic sequence with basic constant at most $K_{(\xi_n)}$.*

Proof. We prove this using Grunblum's criterion. Let $m \le n$,

$$\Big\|\sum_{k=1}^{m} a_k\eta_k\Big\| = \Big\|\sum_{k=1}^{m} a_k \sum_{j=\lambda_k}^{\gamma_k} b_j\xi_j\Big\| = \Big\|\sum_{j=1}^{\gamma_m} c_j\xi_j\Big\| \le K_{(\xi_n)}\Big\|\sum_{j=1}^{\gamma_n} c_j\xi_j\Big\| = K_{(\xi_n)}\Big\|\sum_{k=1}^{n} a_k \sum_{j=\lambda_k}^{\gamma_k} b_j\xi_j\Big\| = K_{(\xi_n)}\Big\|\sum_{k=1}^{n} a_k\eta_k\Big\|$$

where each c_j should be carefully chosen to equal either $a_k b_j$ or 0. ■

A really useful way to produce basic block sequences comes from The Bessaga–Pełczyńki Selection Principle, which is in turn an application of the Principle of small perturbations stated above.

Proposition 1.4. *Let $(\xi_n)_{n=1}^\infty$ be a basis in a Banach space E. Suppose $(\upsilon_n)_{n=1}^\infty$ is a sequence in E such that*

- $\inf_{n\in\mathbb{Z}_{>0}} \|\upsilon_n\| > 0$
- $\lim_{n\to\infty} \omega_k(\upsilon_n) = 0$ *for all* $k \in \mathbb{Z}_{>0}$

Then, $(\upsilon_n)_{n=1}^\infty$ contains a subsequence that is isomorphically equivalent to some block basic sequence $(\eta_k)_{k=1}^\infty$ of $(\xi_n)_{n=1}^\infty$.

Proof. We use a "gliding hump" argument. Let $\alpha := \inf_{n\in\mathbb{Z}_{>0}} \|\upsilon_n\|$ and $0 < \varepsilon < 1$. For each $n \in \mathbb{Z}_{>0}$ we have idempotents $s_n : E \to E$ with respect to the basis $(\xi_n)_{n=1}^\infty$ and for this proof set $s_0 := 0$. Put $K := K_{(\xi)_n} = \sup_n \|s_n\|$ and recall that $K \ge 1$. Let $n_1 = 1$ and $\lambda_0 = 0$ and choose $\lambda_1 > 0$ such that

$$\|s_{\lambda_1}\upsilon_{n_1} - \upsilon_{n_1}\| < \frac{\alpha\varepsilon}{2K}$$

Since $s_{\lambda_1}\upsilon_n = \sum_{k=1}^{\lambda_1} \omega_k(\upsilon_n)\xi_k$ we have by hypothesis that $\|s_{\lambda_1}\upsilon_n\| \to 0$ as $n \to \infty$, so we can choose $n_2 > n_1$ such that $\|s_{\lambda_1}\upsilon_{n_2}\| < \frac{\alpha\varepsilon^2}{2K}$. Now choose $\lambda_2 > \lambda_1$ such that

$$\|s_{\lambda_2}\upsilon_{n_2} - \upsilon_{n_2}\| < \frac{\alpha\varepsilon^2}{2K}$$

As before, we can choose $n_3 > n_2$ such that $\|s_{\lambda_2}\upsilon_{n_3}\| < \frac{\alpha\varepsilon^3}{2K}$. We proceed inductively and get a subsequence $(\upsilon_{n_k})_{k=1}^\infty$ and a strictly increasing sequence $(\lambda_k)_{k=0}^\infty$ such that

$$\|s_{\lambda_{k-1}}\upsilon_{n_k}\| < \frac{\alpha\varepsilon^k}{2K} \quad \text{and} \quad \|s_{\lambda_k}\upsilon_{n_k} - \upsilon_{n_k}\| < \frac{\alpha\varepsilon^k}{2K} \ \forall\ k \ge 1$$

Then, for each $k \ge 1$ define $\eta_k = s_{\lambda_k}\upsilon_{n_k} - s_{\lambda_{k-1}}\upsilon_{n_k}$. It's clear that $(\eta_k)_{k=1}^\infty$ is a block sequence of $(\xi_n)_{n=1}^\infty$ and therefore $(\eta_k)_{k=1}^\infty$ has basis constant at most K. Moreover,

$$\|\eta_k - \upsilon_{n_k}\| \le \|s_{\lambda_k}\upsilon_{n_k} - \upsilon_{n_k}\| + \|s_{\lambda_{k-1}}\upsilon_{n_k}\| < \frac{\alpha\varepsilon^k}{K},$$

and since $\varepsilon \in (0,1)$ and $K \geq 1$ we also have

$$\|\eta_k\| > \|v_{n_k}\| - \frac{\alpha\varepsilon^k}{K} \geq \alpha - \frac{\alpha\varepsilon^k}{K} \geq \alpha(1-\varepsilon)$$

Then,

$$2K\sum_{k=1}^{\infty}\frac{\|\eta_k - v_{n_k}\|}{\|\eta_k\|} < \frac{2}{1-\varepsilon}\sum_{k=1}^{\infty}\varepsilon^k = \frac{2\varepsilon}{(1-\varepsilon)^2}$$

In particular, if we choose any $\varepsilon \in (0, \frac{1}{4}]$ we get $\frac{2\varepsilon}{(1-\varepsilon)^2} \leq \frac{8}{9} < 1$ so the Principle of small perturbations shows that $(v_{n_k})_{k=1}^{\infty}$ is a basic sequence that is isomorphically equivalent to $(\eta_k)_{k=1}^{\infty}$. ■

Corollary 1.5. *Let $(\xi_n)_{n=1}^{\infty}$ be a basis in a Banach space E and F an infinite dimensional subspace of E. Then, F contains a basic sequence that's isomorphically equivalent to a block basic sequence of $(\xi_n)_{n=1}^{\infty}$.*

Proof. By the Bessaga–Pełczyński Selection Principle, it suffices to find a sequence $(v_n)_{n=1}^{\infty}$ in F for which $\inf_{n\in\mathbb{Z}_{>0}}\|v_n\| > 0$ and $\lim_{n\to\infty}\omega_k(v_n) = 0$ for all $k \in \mathbb{Z}_{>0}$. Well, for each $n \in \mathbb{Z}_{>0}$ consider the map $\psi_n : F \to \mathbb{C}^n$ given by

$$\psi_n(v) = (\omega_1(v), \ldots, \omega_n(v))$$

Since F is infinite dimensional but $\mathbb{C}^n$ isn't, the map ψ_n has a non-trivial kernel and therefore we can choose $v_n \in F$ such that $\|v_n\| = 1$ and $\omega_j(v_n) = 0$ for $1 \leq j \leq n$. This gives the desired sequence and we are done. ■

1.3 Complements and Idempotents

If E is a normed vector space; we say that a closed subspace $M \subset E$ is **complemented** if there is a closed subspace $N \subset E$ such that $E = M + N$ and $M \cap N = \varnothing$ (this will be written as the inner direct sum $E = M \oplus N$). Equivalently, a closed subspace $M \subset E$ is complemented if there is a continuous linear idempotent $s : E \to E$ with range M.

2 Basic Inequalities

2.0.1 Hölder's Inequality for $1 < p \leq q < \infty$.

Let $f \in L^p(X,\mu)$ and $g \in L^q(X,\mu)$ where $1 < p \leq q < \infty$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, $fg \in L^1(X,\mu)$ and $\|fg\|_1 \leq \|f\|_p\|g\|_q$. Equality occurs if and only if there are constants a, b (not both 0) such that $a|f|^p = b|g|^q$.

2.0.2 Hölder's Inequality for $p = 1$, $q = \infty$.

Let $f \in L^1(X,\mu)$ and $g \in L^\infty(X,\mu)$. Then, $fg \in L^1(X,\mu)$ and $\|fg\|_1 \leq \|f\|_1\|g\|_\infty$. Equality occurs if and only if $|g| = \|g\|_\infty$ a.e. on $\mathrm{supp}(f)$.

If $\mu(X) < \infty$ a direct consequence of Hölder's inequality gives that if $1 \leq p \leq q \leq \infty$ then $L^q(X,\mu) \subseteq L^p(X,\mu)$. Furthermore, $\|f\|_p \leq \|f\|_q \mu(X)^{\frac{1}{p}-\frac{1}{q}}$ for any $f \in L^q(X,\mu)$.

If $\mu(X) = 1$, we get $\|f\|_p \leq \|f\|_q$ and therefore the inclusion map $L^q(X,\mu) \hookrightarrow L^p(X\ \mu)$ has norm 1.

2.0.3 Generalized Hölder's inequality.

Let $1 \leq p, q, r < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $f \in L^p(X,\mu)$ and $g \in L^q(X,\mu)$. Then $fg \in L^r(X,\mu)$ and $\|fg\|_r \leq \|f\|_p\|g\|_q$.

2.0.4 Liapounov's inequality.

Let $1 \leq p < q < \infty$ and $f \in L^p(X,\mu) \cap L^q(X,\mu)$. Then $f \in L^r(X,\mu)$ (where $r := (1-\lambda)p + \lambda q$, with $\lambda \in (0,1)$) and $\|f\|_r^r \leq \|f\|_p^{(1-\lambda)p} \cdot \|f\|_q^{\lambda q}$.

2.0.5 Minkowski's Inequality for $1 < p < \infty$.

If $f, g \in L^p(X,\mu)$ for $1 < p < \infty$, then $\|f+g\|_p \leq \|f\|_p + \|g\|_p$. Equality occurs if and only if there are non negative constants a, b (not both 0) such that $af = bg$.

2.0.6 Minkowski's Inequality for $p \in \{1, \infty\}$.

If $f, g \in L^p(X, \mu)$ for $p \in \{1, \infty\}$, then $\|f+g\|_p \leq \|f\|_p + \|g\|_p$. For $p = 1$, equality occurs if and only if $|f+g| = |f|+|g|$. For $p = \infty$, equality occurs if and only if f and g have the same sign on some set of positive measure where both functions "attain" their norms.

Let $I \subset \mathbb{R}$ be an interval (possibly infinite or half infinite, and any of open, closed, or half open). A function $\varphi : I \to \mathbb{R}$ is **convex on** I if

$$(1-\lambda)\varphi(x) + \lambda\varphi(y) \leq \varphi((1-\lambda)x + \lambda y)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

2.0.7 Jensen's Inequality

Let $(X, \mathcal{M}, \mu)$ be such that $\mu(X) = 1$. If $f \in L^1(X, \mu)$ is real valued and $\varphi : \mathbb{R} \to \mathbb{R}$ is convex, then

$$\varphi\left(\int_X f d\mu\right) \leq \int \varphi \circ f d\mu$$

2.0.8 ℓ^p Inequalities

Let $1 \leq p \leq q \leq \infty$. Then, $\ell^1 \subseteq \ell^p \subseteq \ell^q \subseteq \ell^\infty$ and $\|x\|_1 \geq \|x\|_p \geq \|x\|_q \geq \|x\|_\infty$, so all the inclusions are norm 1.

2.0.9 ℓ^p_n Inequalities

For $1 \leq p \leq q < \infty$, combining previous results we get $\|x\|_q \leq \|x\|_p \leq n^{\frac{1}{p}-\frac{1}{q}}\|x\|_q$. This gives that, even though ℓ^p_n and ℓ^q_n are equal as sets (in fact both are identified with $\mathbb{C}^n$), they are not isometrically isomorphic.

3 Support, Disjointness and Isometries

Recall that for a function $f : X \to \mathbb{C}$, its **support** is $\text{supp}(f) := \{x \in X : f(x) \neq 0\}$. If f is measurable, then $\text{supp}(f)$ is a measurable set. Two functions $f, g : X \to \mathbb{C}$ are **disjointly supported** if $\text{supp}(f) \cap \text{supp} g = \varnothing$; that is, if $fg = 0$.

3.0.1 L^p functions have σ-finite support.

An important fact is that if $f \in L^p(X, \mu)$ for $1 \leq p < \infty$, then $\text{supp}(f)$ is σ-finite. To see this, for $f \in L^p(X, \mu)$ and $n \in \mathbb{Z}_{>0}$ define

$$E_n := \left\{x \in X : |f(x)|^p \geq \frac{1}{n}\right\}$$

Clearly $\text{supp}(f) = \bigcup_{n=1}^{\infty} E_n$. Furthermore,

$$\mu(E_n) = \int_{E_n} \frac{n}{n} d\mu = n \int_{E_n} \frac{1}{n} d\mu \leq n \int_{E_n} |f|^p d\mu \leq n\|f\|_p^p < \infty$$

Therefore, $\text{supp}(f)$ is σ-finite.

3.0.2 An isometric copy of ℓ^p inside L^p

Let $p \in [1, \infty)$. Suppose that $(f_n)_{n=1}^\infty$ is a sequence of disjointly supported non-zero functions in $L^p(X, \mu)$. For each $n \in \mathbb{Z}_{>0}$ define

$$g_n := \frac{f_n}{\|f_n\|}.$$

Then, $(g_n)_{n=1}^\infty$ is a sequence of disjointly supported norm one functions in $L^p(X, \mu)$. As usual, we denote by $(\delta^n)_{n=1}^\infty$ the canonical basis for ℓ^p. Now define $\Phi : \text{span}(\delta^n) \to \text{span}(g_n)$ by letting $\Phi(\delta^n) := g_n$ and extending by linearity. Notice that, since the functions g_n are disjointly supported, then

$$\left\|\Phi\left(\sum_{n=1}^k a_n\delta^n\right)\right\|_p^p = \left\|\sum_{n=1}^k a_n g_n\right\|_p^p = \int_X \left|\sum_{n=1}^k a_n g_n\right|^p d\mu = \sum_{n=1}^k |a_n|^p \|g_n\|_p^p = \sum_{n=1}^k |a_n|^p = \left\|\sum_{n=1}^k a_n\delta^n\right\|_p^p$$

Thus, by density Φ extends to a linear map $\Phi : \ell^p \to \overline{\text{span}(g_n)}$. Clearly Φ is an isometric surjection and therefore $\overline{\text{span}(g_n)}$ is an isometric copy of ℓ^p inside $L^p(X, \mu)$.

3.0.3 ℓ^p is complemented in L^p

We can say more about the above scenario: the isometric copy of ℓ^p in $L^p(X,\mu)$ is a complemented subspace. To prove this, let (g_n) be as above. It suffices to find a continuous linear idempotent in $\mathcal{L}(L^p(X,\mu))$ with range $\overline{\operatorname{span}(g_n)}$. Well, for each $n \in \mathbb{Z}_{>0}$ we define $E_n := \operatorname{supp}(g_n)$ and

$$h_n := \begin{cases} \frac{|g_n|}{g_n} \cdot |g_n|^{p-1} & \text{on } E_n \\ 0 & \text{on } X \setminus E_n \end{cases}$$

If q is such that $\frac{1}{p} + \frac{1}{q} = 1$, then $h_n \in L^q(X,\mu)$ and $\|h_n\|_q^q = \|g_n\|_p^p = 1$. Furthermore, notice that $\operatorname{supp}(h_n) = E_n$ and that $g_n h_n = |g_n|^p$. For any $f \in L^p(X,\mu)$ put

$$s(f) := \sum_{n=1}^{\infty} \left(\int_{E_n} f h_n d\mu \right) g_n$$

We note that $s(f) \in L^p(X,\mu)$; indeed disjointness of the norm one functions g_n, Hölder inequality and the fact that $\|h_n\|_q = 1$ all together give

$$\|s(f)\|_p^p = \sum_{n=1}^{\infty} \left| \int_{E_n} f h_n d\mu \right|^p \leq \sum_{n=1}^{\infty} \left(\int_{E_n} |f|^p d\mu \right) \left(\int_{E_n} |h_n|^q \right)^{p/q} \leq \sum_{n=1}^{\infty} \left(\int_{E_n} |f|^p d\mu \right) \leq \|f\|_p^p$$

Thus, $s \in \mathcal{L}(L^p(X,\mu))$. It's clear that s has range $\overline{\operatorname{span}(g_n)}$ so we only need to show that s is an idempotent. To do so it's enough to check that s is the identity on $\operatorname{span}(g_n)$. Well, for any $k \in \mathbb{Z}_{>0}$ we have

$$s(g_k) = \sum_{n=1}^{\infty} \left(\int_{E_n} g_k h_n d\mu \right) g_n = \left(\int_{E_k} g_k h_k d\mu \right) g_k = \left(\int_{E_k} |g_k|^p d\mu \right) g_k = \|g_k\|_p^p g_k = g_k.$$

It now follows that s is the identity on $\overline{\operatorname{span}(g_n)}$, as desired.

3.0.4 Disjointly suported sequences in $L^p([0,1])$

The following lemma will be useful to check whether a sequence in $L^p([0,1])$ is isomorphically equivalent to a disjointly supported sequence.

Lemma 3.1. *Let $p \in [1,\infty)$ and $(f_n)_{n=1}^{\infty}$ be a sequence of norm one functions in $L^p([0,1])$. If $m(\operatorname{supp}(f_n)) \to 0$, then there is a subsequence of (f_n) that's isomorphically equivalent to a disjointly supported sequence in $L^p([0,1])$.*

Proof. For each $n \in \mathbb{Z}_{>0}$, we have a measure μ_n (which is absolutely continuous with respect to m) given by

$$\mu_n(A) := \int_A |f_n|^p dm$$

Hence, for each $\varepsilon > 0$ there is $\delta_n(\varepsilon) > 0$ such that $\mu_n(A) < \varepsilon$ whenever $m(A) < \delta_n(\varepsilon)$. Let $A_n = \operatorname{supp}(f_n)$, by hypothesis $m(A_n) \to 0$ and therefore there is n_1 such that $m(A_n) < \delta_{n_1}(4^{-2p})$ for all $n \geq n_1$, whence $\mu_{n_1}(A_n) < 4^{-2p}$ for any $n \geq n_1$. Similarly, choose $n_2 > n_1$ so that $m(A_n) < \min\{\delta_{n_1}(4^{-3p}), \delta_{n_2}(4^{-3p})\}$ for any $n \geq n_2$. Therefore $\mu_{n_1}(A_n) < 4^{-3p}$ and $\mu_{n_2}(A_n) < 4^{-3p}$ for $n \geq n_2$. We proceed inductively and find a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ such that for each k

$$\mu_{n_k}(A_{n_j}) < 4^{-jp} \quad \forall\ j > k$$

For each $k \in \mathbb{Z}_{>0}$ we define $B_k := A_{n_k} \setminus \bigcup_{j=k+1}^{\infty} A_{n_j}$. Notice that the sets B_k are mutually disjoint. Define $g_k = f_{n_k} \chi_{B_k}$ so that $(g_k)_{k=1}^{\infty}$ is a disjointly supported sequence in $L^p([0,1])$. Then, for any $k \in \mathbb{Z}_{>0}$

$$\|f_{n_k} - g_k\|_p^p = \int_{[0,1]} |f_{n_k}|^p (1 - \chi_{B_k}) dm = \int_{A_{n_k} \setminus B_k} |f_{n_k}|^p dm \leq \sum_{j=k+1}^{\infty} \mu_{n_k}(A_{n_j}) < \sum_{j=k+1}^{\infty} 4^{-jp} < 4^{-kp}$$

and so $\|g_k\|_p > 1 - 4^{-k} \geq \frac{3}{4}$. We already know that $(g_k)_{k=1}^{\infty}$ is a basic sequence equivalent to the usual basis of ℓ^p and therefore has basis constant equal to 1. Thus, since

$$2\sum_{k=1}^{\infty} \frac{\|g_k - f_{n_k}\|_p}{\|g_k\|_p} < 2\sum_{k=1}^{\infty} \frac{4^{-k}}{\frac{3}{4}} = \frac{8}{3} \cdot \frac{1}{3} < 1$$

the principle of small perturbations proves that $(f_{n_k})_{k=1}^{\infty}$ is isomorphically equivalent to $(g_k)_{k=1}^{\infty}$ ■

3.0.5 A Test for Disjointness on L^p for $p \in [1,\infty) \setminus \{2\}$

Let $p \in [1,\infty)$. If f and g are disjointly supported functions in $L^p(X,\mu)$ then it's clear that $\|f+g\|_p^p = \|f\|_p^p + \|g\|_p^p = \|f-g\|_p^p$. Turns out that we can rephrase this fact as a "test" for disjointness of two functions in $L^p(X.\mu)$. We need to prove a useful inequality first:

Lemma 3.2. *Let $a, b \in \mathbb{C}$. Then*

$$|a+b|^p + |a-b|^p \begin{cases} \leq 2(|a|^p + |b|^p) & \text{if } p \in [1,2) \\ = 2(|a|^p + |b|^p) & \text{if } p = 2 \\ \geq 2(|a|^p + |b|^p) & \text{if } p \in (2,\infty) \end{cases}$$

For $p \neq 2$, equality can only occur if $ab = 0$.

Proof. The equality for $p = 2$ is obvious. The inequality for $p = 1$ follows from the triangle inequality. For $p \in (1,2)$ we apply Hölder inequality for $(|a+b|^p, |a-b|^p) \in \ell_2^1$ for the conjugate exponents $\frac{2}{p}$ and $\frac{2}{2-p}$

$$\begin{aligned} |a+b|^p + |a-b|^p = \|(|a+b|^p, |a-b|^p)\|_1 &\leq \|(|a+b|^p, |a-b|^p)\|_{\frac{2}{p}} \|(1,1)\|_{\frac{2}{2-p}} \\ &= (|a+b|^2 + |a-b|^2)^{p/2} (1+1)^{\frac{2-p}{2}} \\ &= (2|a|^2 + 2|b|^2)^{p/2} (2)^{1-\frac{p}{2}} \\ &= 2(|a|^2 + |b|^2)^{p/2} \\ &\leq 2(|a|^2 + |b|^2), \end{aligned}$$

where the last inequality follows because $\frac{p}{2} < 1$. Similarly, for $p \in (2,\infty)$ we now apply Hölder inequality for $(|a+b|^2, |a-b|^2) \in \ell_2^1$ for the conjugate exponents $\frac{p}{2}$ and $\frac{p}{p-2}$

$$\begin{aligned} 2(|a|^2 + |b|^2) = \|(|a+b|^2, |a-b|^2)\|_1 &\leq \|(|a+b|^2, |a-b|^2)\|_{\frac{p}{2}} \|(1,1)\|_{\frac{p}{p-2}} \\ &= (|a+b|^p + |a-b|^p)^{2/p} (1+1)^{\frac{p-2}{p}} \\ &= (|a+b|^p + |a-b|^p)^{2/p} (2)^{1-\frac{2}{p}} \\ &\leq |a+b|^p + |a-b|^p \end{aligned}$$

where the last inequality follows because $\frac{2}{p} < 1$. Finally, if $p \neq 2$ notice that the the only way that all the inequalities used above become equality is when either a or b are 0; that is when $ab = 0$, as wanted. ∎

We can now state and prove a test for disjointness:

Theorem 3.3. *Let $p \in [1,\infty) \setminus \{2\}$ and $f, g \in L^p(X,\mu)$. Then f and g are disjointly supported if and only if*

$$\|f+g\|_p^p + \|f-g\|_p^p = 2(\|f\|_p^p + \|g\|_p^p)$$

Proof. The *only if* part was already discussed. Assume that $\|f+g\|_p^p + \|f-g\|_p^p = 2(\|f\|_p^p + \|g\|_p^p)$. We wish to show that $fg = 0$ a.e. $[\mu]$. Define $h := |f+g|^p + |f-g|_p^p - 2(|f|^p + |g|^p)$. By the previous lemma we have that either $h \geq 0$ or $h \leq 0$. By hypothesis

$$\int_X h\,d\mu = \|f+g\|_p^p + \|f-g\|_p^p - 2(\|f\|_p^p + \|g\|_p^p) = 0$$

Therefore $h = 0$ a.e $[\mu]$, but by the previous lemma this means $fg = 0$ a.e. $[\mu]$ as wanted. ∎

Example 3.4. If f and g have disjoint support then $\|f+g\|_p = \|f\|_p + \|g\|_p$. However, the converse of this is false. For example, for $p = 1$, any non-negative functions satisfy $\|f+g\|_1 = \|f\|_1 + \|g\|_1$. For any $p \in [1,\infty]$ if f is a non-negative multiple of g, it's also clear that $\|f+g\|_p = \|f\|_p + \|g\|_p$.

Corollary 3.5. *Let $p \in [1,\infty) \setminus \{2\}$ and $t \in \mathcal{L}(L^p(X,\mu), L^p(Y,\nu))$ an isometry. Then t maps disjointly supported functions to disjointly supported functions.*

Proof. Let f, g be disjointly supported. Then, since t is an isometry

$$\|t(f)+t(g)\|_p^p + \|t(f)-t(g)\|_p^p = \|t(f+g)\|_p^p + \|t(f-g)\|_p^p = \|f+g\|_p^p + \|f-g\|_p^p = 2(\|f\|_p^p + \|g\|_p^p) = 2(\|t(f)\|_p^p + \|t(g)\|_p^p).$$

The previous theorem gives that $t(f)$ and $t(g)$ have disjoint support, as wanted. ∎

3.0.6 Lamperti's Theorem

Lamperti's Theorem characterizes isometries between L^p spaces for $p \neq 2$. Turns out that any such isometry actually comes from a map between the measurable sets. We need a precise definition of this. The main references here are [3] and [5].

Definition 3.6. Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ be measure spaces. A map $S : \mathcal{B} \to \mathcal{C}$ defines modulo null sets, is a**measurable set transformation** (also called regular set isomorphism in [3]) if

(i) $S(X \setminus E) = S(X) \setminus S(E)$ for all $E \in \mathcal{B}$.

(ii) $S(\bigcup_{n=1}^{\infty} E_n) = \bigcup_{n=1}^{\infty} S(E_n)$ for disjoint $E_n \in \mathcal{B}$.

(iii) $\nu(S(E)) = 0$ if and only if $\mu(E) = 0$.

We are using the usual abuse of notation: When we say $S(E) = F$ we mean that the class $[E] = \{E' \in \mathcal{B} : \mu(E' \triangle E) = 0\}$ gets mapped by S to the class $[F] = \{F' \in \mathcal{C} : \nu(C' \triangle C) = 0\}$.

We put $L^0(X, \mathcal{B}, \mu)$ for the set of complex valued functions modulo functions that vanish a.e. $[\mu]$. A measurable set transformation $S : \mathcal{B} \to \mathcal{C}$ induces a "pushforward" of functions $S_* : L^0(X, \mu) \to L^0(Y, \nu)$, which is the unique linear map with the following properties

1. $S_*(\chi_E) = \chi_{S(E)}$ for all $E \in \mathcal{B}$
2. If $(\xi_n)_{n=1}^{\infty}$ is a sequence in $L^0(X, \mu)$ such that $\xi_n \to \xi$ a.e $[\mu]$, then $S_*(\xi_n) \to S(\xi)$ a.e. $[\nu]$.
3. $S_*(\xi \cdot \eta) = S_*(\xi) \cdot S_*(\eta)$ and $S_*(\bar{\xi}) = \overline{S_*(\xi)}$.
4. $S_*(L^0(X, \mu)) = L^0(Y, \nu|_{\mathrm{ran}(S)})$
5. S_* is injective if and only if S_* is injective.
6. $S(\xi^{-1}(B)) = S_*(\xi)^{-1}(B)$ for any Borel set B in $\mathbb{C}$.
7. If $T : \mathcal{C} \to \mathcal{D}$ is another measurable set transformation, then $(T \circ S)_* = T_* \circ S_*$.

We put $\mathrm{ACM}(\mathcal{B}, \mu)$ for the set of measures in $\mathcal{B}$ that are absolutely continuous with respect to μ. If $S : \mathcal{B} \to \mathcal{C}$ is an injective measurable set transformation, it induces a "pushforward" of measures $S_* : \mathrm{ACM}(\mathcal{B}, \mu) \to \mathrm{ACM}(\mathcal{C}, \nu|_{\mathrm{ran}(S)})$ that has the following main properties:

1. For $\lambda \in \mathrm{ACM}(\mathcal{B}, \mu)$ and any non negative $\xi \in L^0(X, \mu)$ or $\xi \in L^1(X, \mu)$ we have
$$\int_X \xi d\lambda = \int_Y S_*(\xi) dS_*(\lambda)$$
2. If $\lambda \in \mathrm{ACM}(\mathcal{B}, \mu)$ is σ-finite, then so is $S_*(\lambda)$.
3. If $\lambda, \rho \in \mathrm{ACM}(\mathcal{B}, \mu)$ are σ-finite measures that are mutually absolutely continuous, then $S_*(\lambda)$ and $S_*(\rho)$ are mutually absolutely continuous and
$$\frac{dS_*(\rho)}{dS_*(\lambda)} = S_* \left(\frac{d\rho}{d\lambda} \right)$$

Lemma 3.7. *Let $p \in [1, \infty)$, let μ be σ finite and let $S : \mathcal{B} \to \mathcal{C}$ an injective measurable set transformation such that $\nu|_{\mathrm{ran}(S)}$ is σ-finite. Suppose that there is $g \in L^0(Y, \nu)$ such that $|g| = 1$ a.e. $[\nu]$ and put $s : L^p(X, \mu) \to L^p(Y, \nu)$ by*
$$s(\xi) := \left[\frac{dS_*(\mu)}{d\nu|_{\mathrm{ran}(S)}} \right]^{1/p} S_*(\xi) g$$
Then $s \in \mathcal{L}(L^p(X, \mu), L^p(Y, \nu))$ is an isometry.

Proof. That $s \in \mathcal{L}(L^p(X, \mu), L^p(Y, \nu))$ follows because S_* is a linear map. The isometry part is a computation
$$\|s(\xi)\|_p^p = \int_Y |S_*(\xi)|^p dS_*(\mu) = \int_Y S_*(|\xi|^p) dS_*(\mu) = \int_X |\xi|^p d\mu = \|\xi\|_p^p$$
■

Lamperti's theorem gives a converse of the previous lemma for $p \neq 2$. The proof relies heavily on Corollary 3.5.

Theorem 3.8. *(Lamperti) Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ be σ-finite measure spaces and let $s \in \mathcal{L}(L^p(X, \mu), L^p(Y, \nu))$ be an isometry. Then, there is a measurable set $F \in \mathcal{C}$, a function $g \in L^0(F, \mathcal{C}|_F, \nu|_F)$ with $|g| = 1$ a.e. $[\nu|_F]$ and a bijective measurable set transformation $S : \mathcal{B} \to \mathcal{C}|_F$ such that*
$$s(\xi) = \begin{cases} \left[\frac{dS_*(\mu)}{d\nu|_F} \right]^{1/p} S_*(\xi) g & \text{on } F \\ 0 & \text{on } Y \setminus F \end{cases}$$

Sketch of Proof. $S(E) := \mathrm{supp}(s(\chi_E))$ works. To get g, for $\mu(X) < \infty$, check $h := s(\mathbf{1})$ is so that $|h|^{1/p}$ is $\frac{dS_*(\mu)}{d\nu|_F}$. "□"

4 Isomorphic Types of L^p spaces

4.0.1 First we look at ℓ^p.

We start by showing that the spaces ℓ^p and ℓ^q are not isomorphic for $p \neq q$ in $[1, \infty)$. This will be Corollary 4.3. First we need a lemma.

Lemma 4.1. *Let $p \in [1, \infty)$ and as usual we denote $(\delta^n)_{n=1}^{\infty}$ the standard basis in ℓ^p. Suppose $(\eta_k)_{k=1}^{\infty}$ is a **seminormalized** block basic sequence in ℓ^p with respect to $(\delta^n)_{n=1}^{\infty}$ (seminormalized means that $\inf_k \|\eta_k\| > 0$ and $\sup_k \|\eta_k\| < \infty$). Then $(\eta_k)_{k=1}^{\infty}$ is isomorphically equivalent to $(\delta^n)_{n=1}^{\infty}$.*

Proof. There is an increasing sequence $\lambda_1 < \gamma_1 < \lambda_2 < \gamma_2 < \cdots$ an increasing sequence of positive integers such that for each $k \in \mathbb{Z}_{>0}$, we have $\eta_k = \sum_{j=\lambda_k}^{\gamma_k} b_j \delta^j$. Then, for any $m \in \mathbb{Z}_{>0}$

$$\Big\| \sum_{k=1}^{m} a_k \eta_k \Big\|_p^p = \Big\| \sum_{k=1}^{m} \sum_{j=\lambda_k}^{\gamma_k} a_k b_j \delta^j \Big\|_p^p = \sum_{j=1}^{\gamma_m} |c_j|^p = \sum_{k=1}^{m} |a_k|^p \left(\sum_{j=\lambda_k}^{\gamma_k} |b_j|^p \right) = \sum_{k=1}^{m} |a_k|^p \|\eta_k\|_p^p$$

where each c_j should be carefully chosen to equal either $a_k b_j$ or 0. Let $C_1 = \inf_k \|\eta_k\|$ and $C_2 = \sup_k \|\eta_k\|$. Then, for any $m \in \mathbb{Z}_{>0}$

$$C_1^p \Big\| \sum_{k=1}^{m} a_k \delta^k \Big\|_p^p = \sum_{k=1}^{m} |a_k|^p C_1^p \leq \sum_{k=1}^{m} |a_k|^p \|\eta_k\|_p^p \leq \sum_{k=1}^{m} |a_k|^p C_2^p = C_2^p \Big\| \sum_{k=1}^{m} a_k \delta^k \Big\|_p^p$$

Since we already saw above that the middle term is $\| \sum_{k=1}^{m} a_k \eta_k \|_p^p$, the desired result follows. ■

Theorem 4.2. *Let $1 \leq p < q < \infty$ and $t \in \mathcal{L}(\ell^q, \ell^p)$. Then $\|t(\delta^n)\|_p \to 0$ as $n \to \infty$.*

Proof. We claim that $t(\delta^n) \to 0$ weakly in ℓ^p. Indeed, take any $\varphi \in (\ell^p)^*$. We have to show that $\varphi(t(\delta^n)) \to 0$ as $n \to \infty$. Well, since t is bounded we clearly have that $\varphi \circ t \in (\ell^q)^*$ and therefore we might see $\varphi \circ t$ as an element of $\ell^{q'}$ where q' is the Hölder conjugate for q. Then,

$$\varphi(t(\delta^n)) = \langle \delta^n, \varphi \circ t \rangle = \sum_{k=1}^{\infty} \delta_k^n (\varphi \circ t)_k = (\varphi \circ t)_n$$

Since $\varphi \circ t \in \ell^{q'}$ and $1 < q' < \infty$ it follows that $(\varphi \circ t)_n \to 0$ as $n \to \infty$, so the claim is proved. Now suppose for a contradiction that $\|t(\delta^n)\|_p \not\to 0$. Then, both hypotheses of Proposition 1.4 (the Bessaga–Pełczyńki Selection Principle) are met and therefore there is a subsequence $(t(\delta^{n_k}))_{k=1}$ that is isomorphically equivalent to some block basic sequence $(\eta_k)_{k=1}^{\infty}$ of $(\delta^k)_{k=1}^{\infty}$ regarded a the basis of ℓ^p. Even better, since $(t(\delta^{n_k}))_{k=1}^{\infty}$ is seminormalized, then so is $(\eta_k)_{k=1}^{\infty}$. Thus, by the previous lemma, we actually have that $(\eta_k)_{k=1}^{\infty}$ is isomorphically equivalent to $(\delta^k)_{k=1}^{\infty}$. To sum up, we have that $(t(\delta^{n_k}))_{k=1}$ is isomorphically equivalent to the basis $(\delta^k)_{k=1}^{\infty}$ in ℓ^p. Then, there is a constant C such that

$$\left(\sum_{k=1}^{\infty} |a_k|^p \right)^{1/p} = \Big\| \sum_{k=1}^{\infty} a_k \delta^k \Big\|_p \leq C \Big\| \sum_{n=1}^{\infty} a_k t(\delta^{n_k}) \Big\|_p \leq C\|t\| \Big\| \sum_{n=1}^{\infty} a_k \delta^{n_k} \Big\|_q = C\|t\| \left(\sum_{k=1}^{\infty} |a_k|^q \right)^{1/q}$$

for all $(a_k)_{k=1}^{n} \in \ell^p \subset \ell^q$. In particular, for all $n \in \mathbb{Z}_{>0}$ we get $n^{1/p} \leq C\|t\| n^{1/q}$, which implies that $n^{\frac{1}{p} - \frac{1}{q}} \leq C\|t\|$ for all $n \in \mathbb{Z}_{>0}$, which is impossible because $p < q$. So we must have $\|t(\delta^n)\|_p \to 0$, as wanted. ■

Corollary 4.3. *Let $1 \leq p < q < \infty$. Then ℓ^p is not isomorphic to ℓ^q.*

Proof. Suppose that there is an isomorphism $t : \ell^q \to \ell^p$. Then, by the previous theorem

$$1 = \|\delta^n\|_q = \|t^{-1}(t(\delta^n))\|_q \leq \|t^{-1}\| \|t(\delta^n)\|_p \to 0 \text{ as } n \to \infty,$$

which is absurd. ■

We can say even more:

Corollary 4.4. *Let $p, q \in [1, \infty)$ with $p \neq q$. Then there's no subspace of ℓ^q isomorphic to ℓ^p.*

Proof. Suppose $F \subset \ell^q$ is isomorphic to ℓ^p via $t : F \to \ell^p$. Then, as in the proof of Corollary 1.5, we find $(\upsilon_n)_{n=1}^{\infty}$ a weakly null normalized sequence in F. As in the proof of Theorem 4.2 we get $\|t(\upsilon_n)\| \to 0$ as $n \to \infty$. But again

$$1 = \|\upsilon_n\|_q = \|t^{-1}(t(\upsilon_n))\|_q \leq \|t^{-1}\| \|t(\upsilon_n)\|_p \to 0 \text{ as } n \to \infty,$$

which is absurd. ■

We actually get a stronger result

Theorem 4.5. *Let $1 \leq p < q < \infty$ and $t \in \mathcal{L}(\ell^q, \ell^p)$. Then t is compact.*

Proof. Since ℓ^q is reflexive, its closed unit ball is weak-compact. By Theorem 4.2 t is weak to weak-to-norm continuous. Then, the the image of the unit ball under t is a compact subset of ℓ^p. ■

4.0.2 An isomorphic copy of ℓ^2 in L^p.

On the previous section we saw that $L^p(X, \mu)$ contains an isometric copy of ℓ^p. In this section we will answer whether $L^p[0,1]$ contains a copy of ℓ^q or $L^q[0,1]$ when $p \neq q$. In particular, we will show that $L^p[0,1]$ is not isomorphic to $L^q[0,1]$ when $p \neq q$. First we start by showing that $L^p[0,1]$ has an isomorphic copy of ℓ^2 for any $p \in [1, \infty)$. The main ingredients are the Rademacher functions and Khinchine's inequality:

Definition 4.6. For each $n \in \mathbb{Z}_{>0}$ we define $r_n : [0,1] \to \mathbb{C}$ by

$$r_n(t) := \operatorname{sgn}(\sin(2^n \pi t))$$

We note that each r_n takes values in $\{-1, 1\}$. One checks that $(r_n)_{n=1}^{\infty}$ is an orthonormal sequence in $L^2[0,1]$.

Furthermore, the following Proposition, whose proof we omit, tells us that for each $p \in [1, \infty)$ the subspace $\overline{\operatorname{span}(r_n)} \subset L^p([0,1])$ is isomorphic to ℓ^2

Proposition 4.7. *(Khinchine's inequality) Given $p \in (0, \infty)$ there exist constants $A_p, B_p \in (0, \infty)$ such that*

$$A_p \left(\sum_{k=1}^{n} |a_k|^2 \right)^{1/2} \leq \left\| \sum_{k=1}^{n} a_k r_k \right\|_p \leq B_p \left(\sum_{k=1}^{n} |a_k|^2 \right)^{1/2}$$

for any $n \in \mathbb{Z}_{>0}$ and any scalars $a_1, \ldots, a_n \in \mathbb{C}$.

For $p = \infty$, we use the fact

$$\max_{t \in [0,1]} \left| \sum_{k=1}^{n} a_k r_k \right| = \sum_{k=1}^{n} |a_k|$$

to deduce that $\overline{\operatorname{span}(r_n)}$ is an isometric copy of ℓ^1 in $L^\infty([0,1])$.

Corollary 4.8. *Let $p \in [1, \infty) \setminus \{2\}$. Then, the spaces $L^p[0,1]$ and ℓ^p are not isomorphic.*

Proof. We just saw that $L^p([0,1])$ contains an isomorphic copy of ℓ^2 and we already know from Corollary 4.4 that ℓ^p does not contain an isomorphic copy of ℓ^2. ■

4.0.3 Now we look at L^p.

Our next main goal is to prove that $L^p([0,1])$ is not isomorphic to $L^q([0,1])$ for $p \neq q$. First we need to answer whether $L^p[0,1]$ has an isomorphic copy of $L^p[0,1]$ or ℓ^q for $q \neq p$. The answer and techniques will be different for $p \in [1,2]$ than for $p \in (2, \infty)$ (that is depending on the type and cotype of L^p). The answer for $p \in (2, \infty)$ is negative and after proving this fact we will be able to tackle out main goal. The results in this section are due to Kadec and Pełczyńki from 1962.

The principal objects we need are subsets of $L^p[0,1]$, defined for any $p \in (0, \infty)$ and any $\varepsilon \in (0,1)$, by

$$M(p, \varepsilon) := \{f \in L^p([0,1]) : m(E(f, \varepsilon)) \geq \varepsilon\},$$

where $E(f, \varepsilon) := \{x \in [0,1] : |f(x)| \geq \varepsilon \|f\|_p\}$. Below we state the general properties of these objects, which are easy consequences of the definition

- $M(p, \varepsilon_1) \supset M(p, \varepsilon_2)$ when $\varepsilon_1 < \varepsilon_2$.
- $\bigcup_{\varepsilon \in (0,1)} M(p, \varepsilon) = L^p([0,1])$.
- Any finite subset of $L^p([0,1])$ is contained in $M(p, \varepsilon)$ for some $\varepsilon \in (0,1)$.
- $M(p, \varepsilon)$ is closed under scalar multiplication.
- $M(p, \varepsilon)$ doesn't contain any of "spikes" functions $g_\delta := \delta^{-1/p} \chi_{[0,\delta]}$; where $\delta \in (0, \varepsilon)$.

The following lemma will help us classify subspaces of $L^p([0,1])$ that are entirely contained in $M(p, \varepsilon)$ for some $\varepsilon > 0$.

Lemma 4.9. *For a subset $A \subset L^p([0,1])$, the following are equivalent*

(i) $A \subset M(p,\varepsilon)$ *for some* ε.

(ii) For $q < p$ *there is a constant* $C_q < \infty$ *such that*

$$\|f\|_q \leq \|f\|_p \leq C_q \|f\|_q$$

for all $f \in A$.

(iii) For some $q < p$ *there is a constant* $C_q < \infty$ *such that*

$$\|f\|_q \leq \|f\|_p \leq C_q \|f\|_q$$

for all $f \in A$.

Proof. $[(i) \Rightarrow (ii)]$ Take any $f \in A \subset M(p,\varepsilon)$. For any $q < p$ Hölder gives $\|f\|_q \leq \|f\|_p$. Since $f \in M(p,\varepsilon)$ we have

$$\|f\|_q^q \geq \int_{E(f,\varepsilon)} |f|^q dm \geq \varepsilon^q m(E(f,\varepsilon)) \geq \varepsilon^{q+1} \|f\|_p^q$$

Thus, $C_q := \varepsilon^{-\frac{q+1}{q}}$ works.

$[(ii) \Rightarrow (iii)]$ is obvious.

$[(iii) \Rightarrow (i)]$ We prove the contrapositive statement instead. Assume that for all $\varepsilon \in (0,1)$ there is $f_\varepsilon \in A$ such that $f_\varepsilon \notin M(p,\varepsilon)$. Then, set $E := E(f_\varepsilon,\varepsilon)$, so that $m(E) < \varepsilon$ and $|f_\varepsilon| < \varepsilon \|f_\varepsilon\|_p$ on $[0,1] \setminus E$. Take any $q < p$, using Hölder we find

$$\|f_\varepsilon\|_q^q < \int_E |f|^q dm + \varepsilon^q \|f_\varepsilon\|_p^q < \|f_\varepsilon\|_p^q (m(E)^{\frac{p-q}{p}} + \varepsilon^q) < \|f_\varepsilon\|_p^q (\varepsilon^{\frac{p-q}{p}} + \varepsilon^q)$$

This says that for any $M > 0$ we can find $f_M \in A$ such that $\|f_M\|_p > M \|f_M\|_q$, which is the nagation of statement (iii). ■

The key application of the previous lemma is that if F is a subspace of $L^p([0,1])$ that is entirely contained in $M(p,\varepsilon)$ for some ε, then the $\|\cdot\|_p$ and $\|\cdot\|_q$ norms are equivalent for any $q \in (0,p)$. In particular for $p \in (2,\infty)$ any closed infinite dimensional subspace of $L^p([0,1])$ is completely characterize by the next theorem.

Theorem 4.10. *Let* $p \in (2,\infty)$ *and let* F *be an infinite-dimensional closed subspace of* $L^p([0,1])$*. Then, either*

1. *F is isomorphic to ℓ^2 and the L^p and L^2 topologies agree on F, or*
2. *F contains a subspace that is isomorphic to ℓ^p and complemented in L^p.*

Proof. Well suppose first that F is entirely contained in $M(p,\varepsilon)$ for some ε. Then, by the previous lemma, the 2-norm is equivalent to the p-norm and therefore F is an infinite dimensional closed subspace of $L^2([0,1])$, so F is isomorphic to ℓ^2.
Suppose otherwise that F fails to be entirely contained in $M(p,\varepsilon)$ for all $\varepsilon \in (0,1)$. Take any sequence $(\varepsilon_n)_{n=1}^\infty$ in $(0,1)$ such that $\varepsilon_n \to 0$. For each $n \in \mathbb{Z}_{>0}$ there is $f_n \in F$ with $\|f_n\|_p = 1$ such that $f_n \notin M(p,\varepsilon_n)$. Let $E_n := E(f_n,\varepsilon_n)$ so that $m(E_n) < \varepsilon_n$ and put $g_n := f\chi_{E_n}$. By construction each g_n is a "spike" function with $m(\mathrm{supp}(g_n)) < \varepsilon_n$ and $\|g_n\|_p > 1 - \varepsilon_n$. Thus, $(g_n)_{n=1}^\infty$ is seminormalized and $m(\mathrm{supp}(g_n)) \to 0$. Therefore, by Lemma 3.1 a subsequence of (g_n) is isomorphically equivalent to a disjointly supported sequence in $L^p([0,1])$. Therefore, as we already saw in Section 3.0.2, the closed span of this subsequence is isomorphic to ℓ^p and complemented in $L^p([0,1])$. Finally, since $\|f_n - g_n\| < \varepsilon_n$, we might choose $(\varepsilon_n)_{n=0}^\infty$ in such way that $\varepsilon_n \to 0$ "fast enough" so that the principle of small perturbations applies to get that $(f_n)_{n=1}^\infty$ has a subsequence isomorphically equivalent to the usual basis for ℓ^p. The desired result follows. ■

Corollary 4.11. *For* $p \in (2,\infty]$ *and* $q \in [1,\infty) \setminus \{2,p\}$*, no subspace of* $L^p([0,1])$ *is isomorphic to* $L^q([0,1])$ *or* ℓ^q.

Proof. Since we know that $L^q([0,1])$ has an isometric copy of ℓ^q, it suffices to prove that no subspace of $L^p([0,1])$ is isomorphic to ℓ^q. Suppose that $F \subset L^p([0,1])$ is isomorphic to ℓ^q. Then, F is infinite dimensional but is not isomorphic to ℓ^2 because $q \neq 2$ and by Corollary 4.4 F doesn't contain an isomorphic copy of ℓ^p because $q \neq p$. This contradicts the previous theorem. ■

For $p \in (2,\infty)$, if F is an infinite dimensional subspace of $L^p([0,1])$ that's isomorphic to ℓ^2, we also have that F is complemented in $L^p([0,1])$. Indeed, if $(g_n)_{n=1}^\infty$ is an orthonormal basis for F, the map $s : L^p([0,1]) \to L^p([0,1])$ given by

$$s(f) = \sum_{n=1}^\infty \langle f, g_n \rangle g_n$$

is an idempotent with range given by F.

The next lemma, will yield a version of Theorem 4.10 that works for any $p \in (1, \infty)$ provided that we are working with complemented subspaces.

Lemma 4.12. *Let F be a closed subspace of a Banach space E.*

- *If F is complemented in E, then F^* is isomorphic to a complemented subspace of E^*.*
- *If E is reflexive, then so is F.*

Theorem 4.13. *Let $p \in (1, \infty)$ and let F be an infinite-dimensional complemented closed subspace of $L^p([0,1])$. Then, either F is isomorphic to ℓ^2 or contains a subspace that is isomorphic to ℓ^p and complemented in $L^p([0,1])$.*

Proof. If $p \in (2, \infty)$ this is Theorem 4.10. If $p = 2$ the claim is obvious. For $p \in (1, 2)$, since $L^p([0,1])$ is reflexive, the previous lemma guarantees that F is reflexive and that F^* is isomorphic to a complemented subspace of $L^{p'}([0,1])$ where p' is the Hölder conjugate for p. Since $p \in (1, 2)$, we must have $p' \in (2, \infty)$. Therefore, by Theorem 4.10 F^* is either isomorphic to ℓ^2 or has a subspace that's isomorphic to $\ell^{p'}$ and complemented in $L^{p'}([0,1])$. If the former happens, then F has to be isomorphic to ℓ^2. If the latter happens, then $F \cong F^{**}$ has a subspace that is isomorphic to $(\ell^{p'})^* \cong \ell^p$ and complemented in $L^p([0,1])$. ■

We have finally arrived to one of our main goals:

Corollary 4.14. *For $p, q \in [1, \infty]$ with $p \neq q$, we have that $L^p([0,1])$ is not isomorphic to $L^q([0,1])$.*

Proof. Since $L^\infty([0,1])$ is not separable, it can't be isomorphic to $L^p([0,1])$ for $p \neq \infty$. Since $L^1([0,1])$ is not reflexive, it can't be isomorphic to $L^p([0,\infty])$ for $p \neq 1$. If $\ell^2 \cong L^2([0,1])$ is isomorphic to $L^p([0,1])$ for $p \neq 2$ then ℓ^2 would have an isomorphic copy of ℓ^p, which is impossible by Corollary 4.4. Now suppose that $p, q \in (1, \infty) \setminus \{2\}$ where $p \neq q$ are such that $L^p([0,1])$ is isomorphic to $L^q([0,1])$. In such case, $L^p([0,1])$ will contain a complemented isomorphic copy of ℓ^q, call it F. By the previous theorem F is either isomorphic to ℓ^2 or contains an isomorphic copy of ℓ^p. The former is impossible because $q \neq 2$ and the latter is impossible because $q \neq p$. ■

The previous Corollary settles the gives that the space $L^p([0,1])$ are are mutually nonisomorphic. We also know from Corollary 4.11 that whenever $p \in (2, \infty)$, then the only way to get an L^q space inside $L^p([0,1])$ is whenever $q = p$ or $p = 2$. We will now show that if p and q live on opposite sides of 2, then $L^p([0,1])$ can't have isomorphic copies of $L^q([0,1])$ or ℓ^q. This will leave open only the cases $1 \leq q < p < 2$ and $1 \leq p < q < 2$. We will see that the answer for the first case is again negative, but not for the second. We will need to use *unconditional convergence* of series

Definition 4.15. Let E be a Banach space. We say that a series $\sum_{n=1}^\infty \xi_n$ is unconditionally convergent if $\sum_{n=1}^\infty \xi_{\sigma(n)}$ converges for every bijection $\sigma : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$.

A clever use of the Rademacher functions gives an important result due to Orlicz back in 1930:

Theorem 4.16. *Let $p \in [1, 2)$. If $\sum_{n=1}^\infty f_n$ is unconditionally convergent in $L^p([0,1])$, then $\sum_{n=1}^\infty \|f_n\|_p^2$ is convergent.*

Corollary 4.17. *If $1 \leq p < 2 < q < \infty$ or if $1 \leq q < 2 < p < \infty$, then $L^p([0,1])$ can't have isomorphic copies of $L^q([0,1])$ or ℓ^q.*

Proof. If $1 \leq q < 2 < p < \infty$, this is simply a special case of 4.11. For $1 \leq p < 2 < q < \infty$, suffices to show that $L^p([0,1])$ doesn't contain an isomorphic copy of ℓ^q. Assume on the contrary that there is $F \subset L^p([0,1])$ and an isomorphism $t : \ell^q \to F$. Take any $(a_n)_{n=1}^\infty \in \ell^q$. We must have that $\sum_{n=1}^\infty a_n t(\delta^n)$ is unconditionally convergent in $L^p([0,1])$ and by Orlicz theroem above we get

$$\infty > \sum_{n=1}^\infty |a_n|^2 \|t(\delta^n)\|_p^2 \geq \|t^{-1}\|^{-2} \sum_{n=1}^\infty |a_n|^2$$

This proves that $\ell^q \subset \ell^2$, but this is imposible for $q > 2$, for example $(n^{-1/2})_{n=1}^\infty$ is in ℓ^q but not in ℓ^2. ■

An application of Khinchine's inequality gives the following estimate

Lemma 4.18. *If $p \in [1, \infty)$ and $f_1, \ldots, f_n \in L^p([0,1])$*

$$\int_0^1 \Big\| \sum_{k=1}^n r_k(u) f_k \Big\|_p du \leq B_p \left(\sum_{k=1}^n \|f_k\|_p^p \right)^{1/p}$$

where B_p is the constant from Khinchine's inequality 4.7.

This lemma settles the case $1 \leq p < q < 2$:

Corollary 4.19. *If $1 \leq q < p < 2$, then $L^p([0,1])$ can't have isomorphic copies of $L^q([0,1])$ or ℓ^q.*

Proof. As before, it suffices to show that $L^p([0,1])$ doesn't contain an isomorphic copy of ℓ^q. Assume on the contrary that there is $F \subset L^p([0,1])$ and an isomorphism $t : \ell^q \to F$. Using the previous lemma and that $B_p = 1$ for $p \leq 2$:

$$n^{1/q} = \int_0^1 \Big\| \sum_{k=1}^n r_k(u)\delta^k \Big\|_q du \leq \|t^{-1}\| \int_0^1 \Big\| \sum_{k=1}^n r_k(u)t(\delta^k) \Big\|_p du \leq \|t^{-1}\| \left(\sum_{k=1}^n \|t(\delta^k)\|_p^p \right) \leq \|t^{-1}\|\|t\| n^{1/p}$$

That is, $n^{\frac{1}{q}-\frac{1}{p}} \leq \|t^{-1}\|\|t\|$ for all $n \in \mathbb{Z}_{>0}$, which is impossible because $q < p$. ■

The remaining case, $1 \leq p < q < 2$, is substantially harder than the rest, but there's a payoff: For p and q in this range, $L^p([0,1])$ actually contains a closed subspace isometric to all of L^q. The proof requires several tools from probability theory that we won't present on this document.

References

[1] Fenando Albiac and Nigel J. Kalton. *Topics in Banach space theory.* Springer, 2016.

[2] Neal L. Carothers. *A short course on Banach space theory.* Cambridge University Press, 2004.

[3] Richard J. Fleming and James E. Jamison. *Isometries on Banach Spaces: Function spaces.* Chapman and Hall/CRC, 2003.

[4] Dorothy Maharam. *On homogeneous measure algebras. Proceedings of the National Academy of Sciences*, 28: pp. 108–111, 1942.

[5] N. Christopher Phillips. *Analogs of Cuntz algebras on L^p spaces. preprint (arXiv: 1201.4196 [math.FA]).*, 2012.

Department of Mathematics, University of Oregon, Eugene OR 97403-1222, USA.
E-mail address: alonsod@uoregon.edu