Language Exam
Produits Croisés d'une C^*-Algèbre par un Groupe d'Automorphismes
(G. Zeller-Meier)

Alonso Delfín
Department of Mathematics
University of Oregon

Abstract

The main goal of this document is for me to have some kind of guide for my language exam. The whole document is an overview of section 2 of [1], which is a paper written in French and is one of the earliest works introducing crossed products of C^*-algebras by discrete groups. Some notation here is slightly different from the one used in [1]. In particular, in this document all the group actions have a name. In the current literature, most notations for crossed products of a C^*-algebra A include the group G and a given name for the action of G on A. Incidentally, such action is usually denoted by α. Here, however, our common notation is $C^*(G, A, \alpha, \gamma)$ where α is instead a 2-cocycle and γ is the action of G on A. If the cocycle is trivial, we get the usual crossed product $C^*(G, A, \gamma)$.
Warning: Little proofreading has been done.

Contents

1 Notation and Preliminaries

Throughout the document, A is a C^*-algebra and G is a discrete group, that is a group equipped with the discrete topology, where $e \in G$ denotes the identity element for G. We also suppose that we are given an action of G on A; that is a homomorphism $\gamma : G \to \mathrm{Aut}(A)$. The action of $s \in G$ on an element $s \in A$ will be denoted by $\gamma_s(a)$.

For a Banach space B, we put

- $\ell^1(G, B)$ to the space of functions $f : G \to B$ such that $\|f\|_1 := \sum_{s \in G} \|f_s\| < \infty$, where $f_s := f(s)$.
- $\ell^\infty(G, B)$ to the space bounded functions $f : G \to B$, with norm $\|f\|_\infty = \sup_{s \in G} \|f_s\|$.
- $k(G, B) \subset \ell^\infty(G, B)$ to the space of functions $G \to B$ with finite support.

If $\mathcal{H}$ is a Hilbert space, $\ell^2(G, \mathcal{H})$ is the space bounded functions $f : G \to \mathcal{H}$, so that $\sum_{s \in G} \|f_s\|^2 < \infty$. Further, we write $\ell^1(G)$, $\ell^\infty(G)$, $k(G)$ and $\ell^2(G)$ instead of $\ell^1(G, \mathbb{C})$, $\ell^\infty(G, \mathbb{C})$, $k(G, \mathbb{C})$ and $\ell^2(G, \mathbb{C})$ respectively.

For $s \in G$, we denote by $\chi_s : G \to \{0, 1\}$, to the characteristic function of $\{s\}$. When B is a complex vector space, we identify $k(G, B)$ with the algebraic tensor product $k(G) \otimes B$, so that for $s \in G$ and $b \in B$, the elementary tensor $\chi_s \otimes b$ is the function $G \to B$ that vanishes everywhere, except at s, whose value is b. In other words,

$$(\chi_s \otimes b)(t) := \chi_s(t)b$$

for any $t \in G$. Similarly, if $\mathcal{H}$ is a Hilbert space, we identify $\ell^2(G, \mathcal{H})$ with the Hilbert space tensor product $\ell^2(G) \otimes \mathcal{H}$.

A G-module is an abelian group M, together with a group action of G on M, with every element of G acting as an automorphism of M. The action of s on m will be written as $\beta_s(m)$. We will write both G and M multiplicatively (the usual convention is to write M additively, but at some point we will need M to be the center of a multiplicative group). Since β_s is an automorphism of M, the action of G is compatible with the group structure on M, that is

$$\beta_s(m_1 m_2) = \beta_s(m_1)\beta_s(m_2)$$

for any $s \in G$, $m_1, m_2 \in M$. For $n \geq 0$, the set $C^n(G, M)$ of functions from $G^n \to M$ (here G^0 is $\{e\}$) is an abelian group when equipped with pointwise multiplication: $(f_1 f_2)(s_1, \ldots, s_n) := f_1(s_1, \ldots, s_n) f_2(s_1, \ldots, s_n)$. The elements of this group are called the (inhomogeneous) n-cochains. We get coboundary homomorphism $d^{n+1} : C^n(G, M) \to C^{n+1}(G, M)$ defined by

$$(d^{n+1} f)(s_1, \ldots, s_{n+1}) = \beta_{s_1}\big(f(s_2, \ldots, s_{n+1})\big) \left(\prod_{i=1}^{n} \left[f(s_1, \ldots, s_{i-1}, s_i s_{i+1}, \ldots, s_{n+1})^{(-1)^i} \right] \right) f(s_1, \ldots, s_n)^{(-1)^{n+1}}$$

One may check that $d^{n+1} d^n = 0$, so this defines the following cochain complex

$$C^0(G, M) \xrightarrow{d^1} C^1(G, M) \xrightarrow{d^2} C^2(G, M) \xrightarrow{d^3} \cdots$$

whose cohomology can be computed. Indeed, for each $n \geq 1$ we define the group of n-cocycles by $Z^n(G, M) = ker(d^{n+1})$ and the group of n-coboundaries by $B^n(G, M) = \mathrm{im}(d^n)$, so that $B^n(G, M)$ is in fact a subgroup of $Z^n(G, M)$. The n-th cohomology group of the G-module M is then defined by

$$H^n(G, M) := \frac{Z^n(G, M)}{B^n(G, M)} \quad n \geq 1$$

and $H^0(G, M) = \ker(d^1)$.

2 Crossed Products of a C^*-algebra by a discrete group of automorphisms.

L^1 algebras of some group extensions of a discrete group by a locally compact group

2.1

Let K be a locally compact group with identity denoted by 1_K and center by Z. An **extension** of a discrete group G by K is a triple (E, ι, p) where E is a locally compact group, $\iota : K \to E$ is an injective homomorphism such that $i(K)$ is an open subgroup of E, and $p : E \to G$ a surjective homomorphism whose kernel is $\iota(K)$. This can be visualized by the following short exact sequence

$$1 \longrightarrow K \xrightarrow{\iota} E \xrightarrow{p} G \longrightarrow 1$$

We say that two extensions (E, ι, p) and (E', ι', p') of G by K are isomorphic if there is a homeomorphism $\varphi : E \to E'$ such that $\varphi \circ \iota = \iota'$ and $p = p' \circ \varphi$. That is, the following is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \xrightarrow{\iota} & E & \xrightarrow{p} & G & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle\varphi} & & \| & & \\ 1 & \longrightarrow & K & \xrightarrow{\iota'} & E' & \xrightarrow{p'} & G & \longrightarrow & 1 \end{array}$$

A section of an extension (E, ι, p) of G by K is a map $u : G \to E$ such that $p \circ u = \mathrm{id}_G$. For each $s \in G$, we put $u_s := u(s)$. The map u needs not to be a group isomorphism.

Now assume that we are given an action of G on K. To simplify computations we will write [1] such action using $\beta : G \to \mathrm{Aut}(K)$ and use $\beta_s := \beta(s)$. This action gives both K and Z the structure of G-modules.

Given an extension (E, ι, p), we can get a 2-cocycle α, as long as we assume that we have a section $u : G \to E$ such that

$$u_s \iota(k) u_s^{-1} = \iota(\beta_s(k))$$

for any $(s, k) \in G \times K$. We define a map $\alpha : G^2 \to E$ by putting

$$\alpha(s, t) := u_s u_t (u_{st})^{-1}$$

If we identify K with $\iota(K)$, then $\alpha(s, t)$ belongs to Z, the center of K. Indeed, that $\alpha(s, k)$ belongs to K follows because $\iota(K) = \ker(p)$ and

$$p(\alpha(s, k)) = p(u_s)p(u_t)p(u_{st})^{-1} = st(st)^{-1} = 1_E$$

That $\alpha(s, k)$ belongs to Z is because

$$\iota(k)\alpha(s,t) = \iota(k)u_s u_t (u_{st})^{-1} = u_s \iota(\beta_{s^{-1}}(k)) u_t (u_{st})^{-1} = u_s u_t \iota(\beta_{s^{-1}t^{-1}}(k))(u_{st})^{-1} = u_s u_t (u_{st})^{-1}\iota(k) = \alpha(s,t)\iota(k)$$

(to do the previous computation I assumed that $u_s^{-1} = u_{s^{-1}}$ but that need not to be true, one should be able to prove this without that assumption). We also can check that $\alpha \in Z^2(G, Z)$; that is for any $s, t, r \in G$ we must have

$$\beta_s(\alpha(t, r))\alpha(s, tr) = \alpha(st, r)\alpha(s, t)$$

(still have no idea how to check that the above holds). Further, if u_e is the identity of E, then α is a **normalized** 2-cocycle, that is $\alpha(s, t)$ is the identity element of E provided that at least one of s or t is e.

Conversely, suppose that we have a normalized 2-cycle $\alpha : G^2 \to Z$ in $\mathbb{Z}^2(G, Z)$. Then, we can define an extension of G by K, denoted by $(E(G, K, \alpha), \iota_\alpha, p_\alpha)$, as follows. As a space let $E(G, K, \alpha)$ be $K \times G$ with an operation given by

$$(k, s)(l, t) := \Big(k\beta_s(l)\alpha(s, t), st\Big)$$

This makes $K \times G$ into a locally compact group, with identity given by $(1_K, e)$ and inverse

$$(k, s)^{-1} = \Big(\beta_{s^{-1}}(k^{-1})\alpha(s^{-1}, s)^{-1}, s^{-1}\Big)$$

The only hard part to check is that the operation defined is associative. To do so, notice fist that for any $s_1, s_2, s_3 \in G$

$$\beta_{s_1}(\alpha(s_2, s_3))\alpha(s_1, s_2 s_3) = \alpha(s_1 s_2, s_3)\alpha(s_1, s_2)$$

because α is in $Z^2(G, Z)$. Then

$$\begin{aligned} (k_1, s_1)[(k_2, s_2)(k_3, s_3)] &= (k_1, s_1)(k_2\beta_{s_2}(k_3)\alpha(s_2, s_3), s_2 s_3) \\ &= \Big(k_1\beta_{s_1}\big(k_2\beta_{s_2}(k_3)\alpha(s_2, s_3)\big)\alpha(s_1, s_2 s_3), s_1 s_2 s_3\Big) \\ &= \Big(k_1\beta_{s_1}(k_2)\beta_{s_1 s_2}(k_3)\beta_{s_1}(\alpha(s_2, s_3))\alpha(s_1, s_2 s_3), s_1 s_2 s_3\Big) \\ &= \Big(k_1\beta_{s_1}(k_2)\beta_{s_1 s_2}(k_3)\alpha(s_1 s_2, s_3)\alpha(s_1, s_2), s_1 s_2 s_3\Big) \\ &= \Big(k_1\beta_{s_1}(k_2)\alpha(s_1, s_2)\beta_{s_1 s_2}(k_3)\alpha(s_1 s_2, s_3), s_1 s_2 s_3\Big) \\ &= (k_1\beta_{s_1}(k_2)\alpha(s_1, s_2), s_1 s_2)(k_3, s_3) \\ &= [(k_1, s_1)(k_2, s_2)](k_3, s_3) \end{aligned}$$

[1] On Zeller-Meier's paper the action is simply denoted by $(s, k) \mapsto s \cdot k \in K$ for any $(s, k) \in G \times K$.

The map $\iota_\alpha : K \to K \times G$ is the cannonical inclusion $k \mapsto (k, e)$ and $p_\alpha : K \times G \to G$ is the projection onto G. It's clear that $\iota_\alpha(K) = \ker(p_\alpha)$. Consider the section $u : G \to K \times G$ given by $u(s) := (1_K, s)$. Then, $p \circ u = \mathrm{id}_G$ and

$$u_s\iota_\alpha(k)u_s^{-1} = [(1_K, s)(k, e)](1_K, s)^{-1} = (\beta_s(k), s)(\alpha(s^{-1}, s)^{-1}, s^{-1}) = (\beta_s(k), e) = \iota(\beta_s(k))$$

where we have used that α is a normalized 2-cocycle and that $\beta_s(\alpha(s^{-1}, s)^{-1}) = \alpha(s, s^{-1})^{-1}$ which is also a consequence of $\alpha \in Z^2(G, Z)$. Similarly,

$$u_su_t(u_{st})^{-1} = [(1_K, s)(1_K, t)](1_K, st)^{-1} = (\alpha(s, t), st)(\alpha((st)^{-1}, st)^{-1}, (st)^{-1}) = (\alpha(s, t), e) = \iota(\alpha(s, t))$$

This is saying that any extension of G by K for which the section u with $u_s\iota(k)u_s^{-1} = \iota(\beta_s(k))$ and $u_e = 1_E$ exists, actually looks like $E(K, G, \alpha)$ for a normalized 2-cocycle α.

2.2

Denote by μ_K a left Haar measure for K. Sometimes[2] we write simply $dk := d\mu_K(k)$. We have $\Delta : K \to (0, \infty)$ the modular function for K, that is

$$d(kk_0) = \Delta(k_0)dk$$

Since G acts on K by automorphisms, for a fixed $s \in G$, it's clear that the measure

$$\mu_s(U) := \mu_K(\beta_{s^{-1}}(U))$$

is a left invariant measure on K. Thus, we also have a "modular function" $\delta : G \to (0, \infty)$ for the automorphism $\beta_{s^{-1}}$, that is

$$d(\beta_{s^{-1}}(k)) = \delta(s)dk$$

We will write $\delta_s := \delta(s)$. We have an action[3] of G on $L^1(K)$ given by

$$\gamma_s(f)(k) = f(\beta_{s^{-1}}(k))\delta_s$$

Below, we check that $\gamma_s(f) \in L^1(K)$ provided that $f \in L^1(K)$:

$$\int_K |f(\beta_{s^{-1}}(k))\delta_s|dk = \int_K |f(l)|\delta_s d(\beta_s(l)) = \int_K |f(l)|\delta_s\delta_{s^{-1}}dl = \|f\|_1 < \infty$$

Thus, the action of G on $L^1(K)$ is isometric. Moreover, we can identify $L^1(K)$ and K with their images on the measure algebra $M(K)$ of complex regular measures on K equipped with convolution of measures. Indeed, for each $f \in L^1(K)$ we have a measure μ_f given by $\mu_f(U) := \int_E f(U)dk$, and for every $k \in K$ we have the point mass measure at k, that we denote ν_k. That is,

$$L^1(K) \cong \{\mu_f : f \in L^1(K)\} = \{\mu \in M(K) : \mu \ll \mu_K\}$$

and

$$K \cong \{\nu_k : k \in K\}$$

We can multiply elements of K with elements of $L^1(K)$ by using the convolution of measures in $M(K)$. In particular, for any $k \in K$, $f \in L^1(K)$ and any measurable set U we have

$$(\nu_k * \mu_f)(U) = \int_K\int_K \chi_U(xy)d\nu_k(x)d\mu_f(y) = \int_K \chi_U(ky)d\mu_f(y) = \int_K f(l)\chi_{k^{-1}U}(l)dl = \int_K f(k^{-1}l)\chi_U(l)dl$$

This gives at once that $(\nu_k * \mu_f) \ll \mu_K$, and therefore, $\nu_k * \mu_f$ can be identified with a function in $L^1(K)$, that we call kf. Moreover, we clearly have

$$(kf)(l) = f(k^{-1}l)$$

Suppose know that $k \in Z$, the center of K, and that $fk := \mu_f * \nu_k$. We then have

$$(kf)(l) = f(k^{-1}l) = f(lk^{-1}) = (fk)(l)$$

That is $kf = fk \in L^1(K)$ for any $f \in L^1(K)$ and any $k \in Z$. This will be really useful to "shorten" some formulas below for $k = \alpha(s, t)$ where $\alpha \in Z^2(G, Z)$ is a normalized cocycle.

[2] dk is the notation used in Zeller-Meier, but here we actually need to use μ_K to compare it with other measures.

[3] This action is simply denoted by $s \cdot f$ on Zeller-Meier's paper.

2.3

Let ds be normalized counting measure for G (that is, each $s \in G$ has measure 1). If we are given a normalized 2-cocycle $\alpha \in Z^2(G,Z)$ we can equip the extension $E := E(G,K,\alpha)$ with a left Haar measure

$$d(k,s) := \delta_s dk \otimes ds$$

Then, if $\xi : E \to \mathbb{C}$ is in $L^1(E)$, we have

$$\|\xi\|_1 = \int_E |\xi(k,s)| d(k,s) = \int_{K\times G} |\xi(k,s)|\delta_s dk \otimes ds = \sum_{s\in G} \delta_s \int_K |\xi(k,s)| dk$$

Furtheremore,

$$d\Big((k,s)(k_0,s_0)\Big) = \Big(k\beta_s(k_0)\alpha(s,s_0), ss_0\Big) = \delta_{ss_0} d(k\beta_s(k_0)\alpha(s,s_0)) \otimes d(ss_0) = \delta_{s_0}\Delta(\beta_s(k_0))(\delta_s dk \otimes ds)$$

We claim that $\Delta(\beta_s(k_0)) = \Delta(k_0)$. Indeed, recall that if $\mu_s(U) := \mu_K(\beta_{s^{-1}}(U))$, then $\mu_s(U) = \delta_s \mu_K(U)$. Then, for $k_0 \in K$ we have

$$\begin{aligned}\delta_{s^{-1}}\Delta(k_0)\mu_K(U) &= \delta_{s^{-1}}\mu_K(Uk_0)\\ &= \mu_{s^{-1}}(Uk_0) = \mu_K(\beta_s(Uk_0))\\ &= \mu_K(\beta_s(U)\beta_s(k_0))\\ &= \Delta(\beta_s(k_0))\mu_K(\beta_s(U)) = \Delta(\beta_s(k_0))\delta_{s^{-1}}\mu_K(U)\end{aligned}$$

Our claim now follows from comparing both ends on the previous equation. Thus, the modular function for E is $\Delta_E(k_0,s_0) := \delta_{s_0}\Delta(k_0)$. We now make $L^1(E)$ into an $*$-Banach algebra by letting

$$(\xi * \eta)(k,s) := \int_E \xi(l,t)\eta((l,t)^{-1}(k,s))d(l,t) = \sum_{t\in G}\delta_t \int_K \xi(l,t)\eta(\beta_{t^{-1}}(l^{-1}k)\alpha(t^{-1},t)^{-1}\alpha(t^{-1},s), t^{-1}s)dl$$

and

$$\xi^*(k,s) := \overline{\xi((k,s)^{-1})}\Delta_E((k,s)^{-1}) = \overline{\xi(\beta_{s^{-1}}(k^{-1})\alpha(s^{-1},s)^{-1}, s^{-1})}\Delta(k^{-1})\delta_{s^{-1}}$$

Turns out that, as Banach spaces, $L^1(E)$ is isometrically isomorphic to $\ell^1(G, L^1(K))$. To see this, we recall for an element $f \in \ell^1(G, L^1(K))$ we put $f_s := f(s) \in L^1(K)$. Now, define a map $\Phi : L^1(E) \to \ell^1(G, L^1(K))$ as follows

$$\Phi(\xi)_s(k) := \delta_s \xi(k,s)$$

It's immediate to check that Φ is linear and that $\|\Phi(\xi)\|_{\ell^1} = \|\xi\|_1$. To check that Φ is surjective, take any $f \in \ell^1(G, L^1(K))$ and define $\xi_f : E \to \mathbb{C}$ by

$$\xi_f(k,s) := \delta_{s^{-1}} f_s(k)$$

Then,

$$\|\xi_f\|_1 = \sum_{s\in G}\|f_s\|_1 = \|f\| < \infty$$

so $\xi_f \in L^1(E)$ and clearly $\Phi(\xi_f) = f$. We can then use the convolution and involution on $L^1(E)$ to make $\ell^1(G, L^1(K))$ into a $*$-Banach algebra, which we will denote $\ell^1(G, L^1(K), \alpha, \gamma)$. Indeed, for $f, g \in \ell^1(G, L^1(K))$ set

$$(f*g)_s(k) := \sum_{t\in G}\delta_t \int_K f_t(l) g_{t^{-1}s}(\beta_{t^{-1}}(l^{-1}k)\alpha(t^{-1},t)^{-1}\alpha(t^{-1},s)dl$$

Since $\alpha \in Z^2(G,Z)$, we have $\alpha(t^{-1},s) = \alpha(t^{-1},t)\beta_{t^{-1}}(\alpha(t,t^{-1}s)^{-1})$, so ww have

$$(f*g)_s(k) = \sum_{t\in G}\delta_t \int_K f_t(l) g_{t^{-1}s}(\beta_{t^{-1}}(l^{-1}k\alpha(t,t^{-1}s)^{-1}))dl = \sum_{t\in G}\int_K f_t(l)\gamma_t(g_{t^{-1}s})(l^{-1}k\alpha(t,t^{-1}s)^{-1})dl$$

Now recall that, if working over the measure algebra $M(K)$, we can multiply elements in Z by elements in $L^1(K)$ and get back an element of $L^1(K)$ (as we did in 2.2). We then actually have

$$(f*g)_s = \sum_{t\in G} f_t\gamma_t(g_{t^{-1}s})\alpha(t,t^{-1}s)$$

For the involution we get

$$f_s^*(k) := \overline{f_{s^{-1}}(\beta_{s^{-1}}(k^{-1})\alpha(s^{-1},s)^{-1})}\Delta(k^{-1})\delta_{s^{-1}}$$

Again, since α is a 2-cocycle, it follows that $\alpha(s^{-1},s)^{-1} = \beta_{s^{-1}}(\alpha(s,s^{-1})^{-1})$, so that

$$f_s^*(k) = \overline{f_{s^{-1}}(\beta_{s^{-1}}(k^{-1}\alpha(s,s^{-1})^{-1})}\Delta(k^{-1})\delta_{s^{-1}} = \overline{\gamma_s(f_{s^{-1}})(k^{-1}\alpha(s,s^{-1})^{-1})}\Delta(k^{-1})$$

Thus, going up again to the measure algebra $M(K)$ (involution here is $\mu^*(U) := \overline{\mu(U^{-1})}$ and therefore $\nu_k^* = \nu_{k^{-1}}$ for any $k \in K$) we actually have

$$f_s^* = \gamma_s(f_{s^{-1}})^\star \alpha(s,s^{-1})^*$$

The Banach $*$-algebra $\ell^1(G, A, \alpha, \gamma)$

2.4

We give the analog of the previous section when we take a C^*-algebra A in place of $L^1(K)$. As before G is a discrete group, where $e \in G$ denotes the identity element for G. We also suppose that we are given an action of G on A; that is a homomorphism $\gamma : G \to \mathrm{Aut}(A)$. Moreover, we can regard A^{**} as the enveloping Von-Neumann algebra of A. Indeed, A sits inside of A^{**} via $i : A \hookrightarrow A^{**}$, where $i(a)(\varphi) = \varphi(a)$ for any $\varphi \in A^*$. It's known that $i(A)$ is weakly-$*$ dense in A^{**}. Then, since the C^*-algebraic operations are continuous, they extend to A^{**}. These extensions turn A^{**} into a Banach algebra; the C^* identity also extends, making A^{**} into a unital C^*-algebra. Let Z be the center of A^{**} and define

$$C := \{\omega \in Z : i(a)\omega \in i(A) \ \forall \ a \in A\}$$

It's clear that C is a sub C^*-algebra of A^{**}. We set C_u to be the subgroup of C consisting of unitary elements. Since for any $u \in C_u$ and $a \in A$, we have that $i(a)u = ui(a) \in i(A)$, we see the product $i(a)u = ui(a)$ as an element of A and simply write $ua = au \in A$. Moreover, we regard C_u as a G-module using the dual action induced by γ. We write the action of $s \in G$ on $u \in C_u$ by $\beta_s(u)$. This action is compatible with the given action in the following sense

$$\gamma_s(ua) = \beta_s(a)\gamma_s(a) = \gamma_s(a)\beta_s(u) = \gamma_s(au)$$

Let $\alpha \in Z^2(G, C_u)$ be normalized. We now define $\ell^1(G, A, \alpha, \gamma)$ as the set $\ell^1(G, A)$ with the following multiplication and involution: For $f, g \in \ell^1(G, A)$ we set

$$(fg)_s := \sum_{t \in G} f_t \gamma_t(g_{t^{-1}s})\alpha(t, t^{-1}s)$$

and

$$f^*_s := \gamma_s(f_{s^{-1}})^\star \alpha(s, s^{-1})^*$$

These two operations are motivated from the ones we already had in $\ell^1(G, L^1(K), \alpha, \gamma)$. Furthermore, since each $\alpha(s,t)$ is a unitary operator and each γ_s an automorphism of A, we have

$$\|fg\|_1 = \sum_{s \in G} \left\| \sum_{t \in G} f_t \gamma_t(g_{t^{-1}s})\alpha(t, t^{-1}s) \right\| \leq \sum_{s \in G}\sum_{t \in G} \|f_t\|\|g_{t^{-1}s}\| = \|f\|_1\|g\|_1$$

and

$$\|f^*\|_1 = \sum_{s \in G} \|\gamma_s(f_{s^{-1}})\alpha(s, s^{-1})^*\| = \sum_{s \in G} \|f_{s^{-1}}\| = \|f\|_1$$

Thus the product and involution are well defined. To prove that we actually get a Banach $*$-algebra, notice that the dense subset $k(G, A) = k(G) \otimes A$, of finitely supported functions, is closed under the given multiplication

$$(\chi_s \otimes a)(\chi_t \otimes b) = \chi_{st} \otimes (a\gamma_s(b)\alpha(s, t))$$

A direct check also gives

$$(\chi_s \otimes a)^* = \chi_{s^{-1}} \otimes (\gamma_{s^{-1}}(a^*)\alpha(s^{-1}, s)^*)$$

from where we get $(\chi_s \otimes a)^{**} = (\chi_s \otimes a)$ and $[(\chi_s \otimes a)(\chi_s \otimes b)]^* = (\chi_s \otimes b)^*(\chi_s \otimes a)^*$. As a consequence one gets that $\ell^1(G, A, \alpha, \gamma)$ is indeed a Banach $*$-algebra. Moreover, $\ell^1(G, A, \alpha, \gamma)$ separable whenever G is countable and A separable.

Notice that A sits as a subalgebra of $\ell^1(G, A, \alpha, \gamma)$ via the map $a \mapsto (\chi_e \otimes a)$. Assume $(a_\lambda)_{\lambda \in \Lambda}$ is an approximate unit for A. Then, $(\chi_1 \otimes a_\lambda)_{\lambda \in \Lambda}$ is an approximate unit for $\ell^1(G, A, \alpha, \gamma)$.

2.5

We now show that $\ell^1(G, A, \alpha, \gamma)$ only depends of the class of α in $H^2(G, C_u)$. Indeed, assume that $\alpha' \in Z^2(G, C_u)$ is such that $[\alpha] = [\alpha'] \in H^2(G, C_u)$. Then, there is a normalized $\sigma \in C^1(G, C_u)$ (that is a map $\sigma : G \to C_u$ with $\sigma(e) = 1_{C_u}$) such that

$$\alpha'(s, t)\alpha(s, t)^* = (d^2\sigma)(s, t)$$

for ant $s, t \in G$. That is,

$$\alpha'(s, t) = \alpha(s, t)\beta_s(\sigma(t))\sigma(st)^*\sigma(s)$$

We claim that $\ell^1(G, A, \alpha, \gamma)$ and $\ell^1(G, A, \alpha', \gamma)$ are isomorphic as Banach $*$-algebras. To prove this claim, we consider the map $\Phi : \ell^1(G, A, \alpha, \gamma) \to \ell^1(G, A, \alpha', \gamma)$ given by

$$\Phi(f)_s := \sigma(s)^* f_s$$

Since $\sigma(s)$ is unitary, it's clear that Φ is an ismorphism of $\ell^1(G,A)$ into itself. To show that it is a Banach $*$-algebra isomorphism from $\ell^1(G,A,\alpha,\gamma)$ to $\ell^1(G,A,\alpha',\gamma)$, it suffices to show that multiplication and involution are preserved when restricting to elements in the dense subspace $k(G,A)$. Well, for any $s,t\in G$, $a,b\in A$ we have

$$\begin{aligned}\Phi((\chi_s\otimes a)(\chi_t\otimes b)) &= \Phi\big(\chi_{st}\otimes(a\gamma_s(b)\alpha(s,t))\big)\\ &= \chi_{st}\otimes(a\gamma_s(b)\beta_s(\sigma(t))^*\sigma(s)^*\alpha'(s,t))\\ &= \chi_{st}\otimes(\sigma(s)^*a\gamma_s(\sigma(t)^*b)\alpha'(s,t))\\ &= \Phi((\chi_s\otimes a))\Phi((\chi_t\otimes b))\end{aligned}$$

and

$$\begin{aligned}\Phi((\chi_s\otimes a)^*) &= \Phi\big(\chi_{s^{-1}}\otimes(\gamma_{s^{-1}}(a^*)\alpha(s^{-1},s)^*)\big)\\ &= \chi_{s^{-1}}\otimes\big(\sigma(s^{-1})^*\gamma_{s^{-1}}(a^*)\alpha(s^{-1},s)^*)\big)\\ &= \chi_{s^{-1}}\otimes\big(\sigma(s^{-1})^*\sigma(s)\beta_{s^{-1}}(\gamma(s))\gamma_{s^{-1}}(a^*)\alpha'(s^{-1},s)^*)\big)\\ &= \chi_{s^{-1}}\otimes\big(\gamma_{s^{-1}}((\sigma(s)^*a)^*)\alpha'(s^{-1},s)^*\big)\\ &= \Phi(\chi_s\otimes a)^*\end{aligned}$$

Therefore, Φ is indeed a Banach $*$-algebra isomorphism, as claimed.

Representations of $\ell^1(G,A,\alpha,\gamma)$

2.6

Let (G,A,α,γ) be as above. A **representation** of (G,A,α,γ) is a pair (u,ρ) such that $\rho: A\to\mathcal{L}(\mathcal{H}_\rho)$ is a non degenerate representation of A and a map $u: G\to\mathcal{U}(\mathcal{H}_\rho)$ such that

$$u_s\rho(a)u_s^* = \rho(\gamma_s(a))$$

and

$$u_su_t = \widetilde{\rho}(\alpha(s,t))u_{st}$$

where $\widetilde{\rho}$ is the extension of ρ to A^{**}. We observe that, since α is normalized, then it follows that $u_e = \mathrm{id}_{\mathcal{H}_\rho}$. When α is trivial, the second condition is saying that u is a unitary representation of G and the pair (u,ρ) is known as a covariant representation of (G,A,γ).

Given a representation (u,ρ) of (G,A,α,γ), we define[4] $\pi:\ell^1(G,A,\alpha,\gamma)\to\mathcal{L}(\mathcal{H}_\rho)$ by

$$\pi(f) := \sum_{s\in G}\rho(f_s)u_s$$

We claim that π is a non-degenerate representation of $\ell^1(G,A,\alpha,\gamma)$ on $\mathcal{H}_\rho$. To see this, notice first that

$$\|\pi(f)\|\le\sum_{s\in G}\|\rho(f_s)u_s\|\le\sum_{s\in G}\|\rho(f_s)\|\le\sum_{s\in G}\|f_s\| = \|f\|_1$$

Thus, π is a well defined continuous linear map. To show that π is indeed a representation, suffices to show that is multiplicative and preserves the involution on elements of $k(G,A)$. Well, for any $s,t\in G$, $a,b\in A$ we have

$$\begin{aligned}\pi((\chi_s\otimes a)(\chi_t\otimes b)) &= \pi\big(\chi_{st}\otimes(a\gamma_s(b)\alpha(s,t))\big)\\ &= \rho(a\gamma_s(b)\alpha(s,t))u_{st}\\ &= \rho(a)\rho(\gamma_s(b))\widetilde{\rho}(\alpha(s,t))u_{st}\\ &= \rho(a)\rho(\gamma_s(b))u_su_t\\ &= \rho(a)u_s\rho(b)u_t\\ &= \pi(\chi_s\otimes a)\pi(\chi_t\otimes b)\end{aligned}$$

and

$$\begin{aligned}\pi((\chi_s\otimes a)^*) &= \rho(\gamma_{s^{-1}}(a^*)\alpha(s^{-1},s)^*)u_{s^{-1}}\\ &= \widetilde{\rho}(\alpha(s^{-1},s)^*)\rho(\gamma_{s^{-1}}(a^*))u_{s^{-1}}\\ &= \widetilde{\rho}(\alpha(s^{-1},s)^*)u_{s^{-1}}\rho(a^*)\\ &= (u_{s^{-1}}^*\widetilde{\rho}(\alpha(s^{-1},s))^*\rho(a^*)\\ &= u_s^*\rho(a^*)\\ &= (\rho(a)u_s)^*\\ &= \pi(\chi_s\otimes a)^*\end{aligned}$$

[4]A modern notation for π is $\pi = u\ltimes\rho$

We still need to check that π is non degenerate. Since ρ is non degenerate, if $(a_\lambda)_{\lambda\in\Lambda}$ is an approximate identity for A, we have that

$$\|\rho(a_\lambda)\xi - \xi\| \to 0$$

for any $\xi \in \mathcal{H}_\rho$. Then,

$$\|\pi(\chi_e \otimes a_\lambda)\xi - \xi\| = \|\rho(a_\lambda)u_e\xi - \xi\| = \|\rho(a_\lambda)\xi - \xi\| \to 0$$

for any $\xi \in \mathcal{H}_\rho$. Therefore, π is also non degenerate.

2.7

Turns out that any non degenerate representation of $\ell^1(G, A, \alpha, \gamma)$ arises uniquely from a representation (u, ρ) of (G, A, α, γ) in the above fashion. Indeed, if $\pi : \ell^1(G, A, \alpha, \gamma) \to \mathcal{L}(\mathcal{H}_\pi)$ is non degenerate, since A sits inside of $\ell^1(G, A, \alpha, \gamma)$ we can define

$$\rho := \pi|_A : A \to \mathcal{L}(\mathcal{H}_\pi)$$

Once checks that ρ is a non degenerate representation of A such that $u_s\rho(a)u_s^* = \rho(\gamma_s(a))$ for any $s \in G$, $a \in A$. If $(a_\lambda)_{\lambda\in\Lambda}$ is an approximate identity for A, we define $u : G \to \mathcal{L}(\mathcal{H}_\pi)$ by letting

$$u_s\xi := \lim_\lambda \pi(\chi_s \otimes a_\lambda)\xi$$

for any $s \in G$, $\xi \in \mathcal{H}_\rho$. One checks that $u_s \in \mathcal{U}(\mathcal{H}_\pi)$ for any $s \in G$ and that $u_su_t = \widetilde{\rho}(\alpha(s,t))u_{st}$ for any $s, t \in G$. Moreover, this is independent of the approximate identity chosen. This gives that $\pi = u \ltimes \rho$, as wanted.

As a consequence of this we find that if $\pi = u \ltimes \rho$ is injective on $k(G, A)$, then ρ is injective. This follows at once from the following estimate

$$\|\pi(\chi_s \otimes a)\| = \|\rho(a)u_s\| \leq \|\rho(a)\|$$

Crossed Products of A by G

2.8

Let (G, A, α, γ) be as above and define Π as the collection of all non degenerate representations of $\ell^1(G, A, \alpha, \gamma)$. In what follows, we will ignore the set theoretic problem that Π might not be a set. For $f \in \ell^1(G, A, \alpha, \gamma)$ define

$$N(f) := \sup_{\pi\in\Pi} \|\pi(f)\|$$

This gives a sub multiplicative seminorm on $\ell^1(G, A, \alpha, \gamma)$. Moreover, note that $N(f^*) = N(f)$ and $N(f^*f) = N(f)^2$, so we actually have a C^*-seminorm. We define **the crossed product of A by G**, denoted $C^*(G, A, \alpha, \gamma)$, as the enveloping C^*-algebra of $(\ell^1(G, A, \alpha, \gamma), N)$.

We get an isometric copy of A inside of $C^*(G, A, \alpha, \gamma)$ via the map $a \mapsto \chi_e \otimes a$

We saw in 2.5 that $\ell^1(G, A, \alpha, \gamma)$ is only depends (up to isomorphism) of the chohomology class of α in $H^2(G, C_u)$. Therefore, $C^*(G, A, \alpha, \gamma)$ only depends (up to isomorphism) of the chohomology class of α in $H^2(G, C_u)$.

2.9

As a particular case we take $A = \mathbb{C}$. The action of G on $\mathbb{C}$ is the trivial action; that is $\gamma_s(a) = a$ for all $s \in G$. We can then omit γ from our notation. Further, here $C_u = S^1 = \{a \in \mathbb{C} : |a| = 1\}$. For a 2-cocycle $\alpha \in Z^2(G, S^1)$, we say that an α-representation of $(G, \mathbb{C})$ is a map $u : G \to \mathcal{U}(\mathcal{H})$, for a Hilbert space $\mathcal{H}$, such that

$$u_su_t = \alpha(s,t)u_{st}$$

for all $s, t \in G$. Then, $C^*(G, \mathbb{C}, \alpha)$ is the universal C^*-algebra for α-representations of G. If α is the trivial 2-cocycle, then α-representations are simply unitary representations of G. Then, $C^*(G, \mathbb{C}, \alpha)$ is the universal C^*-algebra for unitary representations of G; which is commonly known as the **group C^*-algebra of G**, denoted simply by $C^*(G)$.

References

[1] G. Zeller-Meier. *Produits croisés d'une C*-algèbre par un groupe d'automorphismes. J. Math. Pures Appl. (9)*, vol. 47: pp. 101–239, 1968.

Department of Mathematics, University of Oregon, Eugene OR 97403-1222, USA.
E-mail address: alonsod@uoregon.edu