# Oral Exam
# Basic of $C^*$-algebras and $K$-theory

Alonso Delfín
Department of Mathematics
University of Oregon

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**Abstract**

The main goal of this document is for me to have some kind of guide for my oral exam. This document contains basic results on $C^*$-algebras and $K$-theory of $C^*$-algebras. The principal references are Murphy and Wegge-Olsen. This is a work in progress, little proofreading has been done and it's possible it contains some typos/mistakes.

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# 1 Basic Definitions

Let $A$ be a $C^*$-algebra. $A_{\text{sa}} := \{a \in A : a^* = a\}$. An element $a \in A$ is normal if $a^*a = aa^*$. A projection is an element $p \in A_{\text{sa}}$ such that $p^2 = p$. We say that $v \in A$ is a partial isometry when $v^*v$ is a projection.

## 1.1 Unital

If $A$ is unital, $\text{Inv}(A)$ is the set of invertible elements in $A$. For $a \in A$, $\sigma(a) := \{\lambda \in \mathbb{C} : \lambda - a \notin \text{Inv}(A)\}$. A unitary is an element $u \in A$ such that $u^*u = uu^* = 1$. An isometry is an element $s \in A$ for which $s^*s = 1$, a coisometry is an element $s \in A$ such that $s^*$ is an isometry.

## 1.2 Unitization

If $A$ is not unital, $\widetilde{A} = A \times \mathbb{C}$ is its unitization which is again a $C^*$-algebra when equipped with the correct norm. When dealing with $K$-theory we will find it useful to define $A^+$ to be $\widetilde{A}$ when $A$ is not unital but $A \oplus \mathbb{C}$ for unital $A$. In any case we get a split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\iota"] & A^+ \arrow[r, "\pi"] & \mathbb{C} \arrow[r] \arrow[l, bend left, "\sigma"] & 0 \end{tikzcd}$$

where $\iota(a) := (a, 0)$; $\pi(a, \lambda) := \lambda$ and $\sigma(\lambda) := (0, \lambda)$. Furthermore, if $\varphi : A \to B$ a $*$-homomorphism there is a unique $*$-homomorphism $\varphi^+ : A^+ \to B^+$ given by $\varphi^*(a, \lambda) = (\varphi(a), \lambda)$.

# 2 Spectral theory and Functional Calculus

**Theorem 2.1.** *Let $A, B$ be $C^*$-algebras and $\varphi : A \to B$ a $*$-homomorphism. Then, $\varphi$ is norm decreasing.*

**Proof.** We assume that $A$ and $B$ are unital (otherwise work with $\widetilde{\varphi} : \widetilde{A} \to \widetilde{B}$). We can further assume that $\varphi(1) = 1$ ($*$-homomorphisms are not assumed to be unital, however if $A$ has a unit, then $\varphi(1)$ is the unit for $\overline{\varphi(A)}$, otherwise $\widetilde{\varphi}(1_{\widetilde{A}}) = 1_{\widetilde{B}}$). Then, it's easy to see that $\sigma(\varphi(a)) \subseteq \sigma(a)$ for any $a \in A$. Thus,

$$\|\varphi(a)\|^2 = \varphi(a^*a)\| = r(\varphi(a^*a)) \leq r(a^*a) = \|a^*a\| = \|a\|^2.$$

That is, $\varphi$ is norm decreasing as wanted. ∎

**Theorem 2.2.** *Let $A, B$ be $C^*$-algebras and $\varphi : A \to B$ an injective $*$-homomorphism. Then, $\varphi$ is isometric.*

**Proof.** We first show this for the particular case that $A$ and $B$ are commutative unital $C^*$-algebras. That is, assume $A = C(X)$ and $B = C(Y)$ for compact Hausdorff spaces $X$ and $Y$. We already know that $X \cong \text{Max}(C(X))$ via $x \mapsto \text{ev}_x$ where $\text{ev}_x(f) = f(x)$. Since $\varphi$ is injective, the induced map $\varphi^* : \text{Max}(C(Y)) \to \text{Max}(C(X))$ given by $\varphi(\text{ev}_y) := \text{ev}_y \circ \varphi$ is surjective. Hence,

$$\|\varphi(f)\|_\infty = \sup_{y \in Y} |\varphi(f)(y)| = \sup_{y \in Y} |(\text{ev}_y \circ \varphi)(f)| = \sup_{x \in X} |\text{ev}_x(f)| = \sup_{x \in X} |f(x)| = \|f\|_\infty$$

For general $C^*$-algebras $A$ and $B$, we only need to show that $\|\varphi(a^*a)\| = \|a^*a\|$. We can do this by working over $\widetilde{A}$, using instead the commutative unital $C^*$-algebra $C^*(a^*a, 1)$ and the map $\widetilde{\varphi} : C^*(a^*a, 1) \to \overline{\widetilde{\varphi}(C^*(a^*a, 1))}$. Thus the general case follows from the particular one above. ∎

The previous result will be useful to show that the image of a $*$-homomorphism is a $C^*$-algebra, but we need to know first that closed ideals of $A$ are also $C^*$-algebras and we need more than just spectral theory to achieve that goal. See Theorem 4.7 below.

# 3 Positive Elements of $C^*$-algebras

**Definition 3.1.** An element $a \in A$ is called **positive**, in symbols $a \geq 0$, if $\sigma(a) \subset \mathbb{R}_{\geq 0}$ and $a = a^*$. The **positive cone** of $A$ is the set

$$A_{\geq 0} := \{a \in A : a \geq 0\} \subset A_{\text{sa}}$$

It follows from functional calculus that each $a \in A_{\geq 0}$ has a unique positive square root, which we denote by $\sqrt{a}$. Of course $(\sqrt{a})^2 = a$. If $b \in A_{\text{sa}}$, then by functional calculus $b^2 \in A_{\geq 0}$, so it makes sense to define $|b| := \sqrt{b^2} \in A_{\geq 0}$.

The following lemma shows that any $C^*$ algebra is spanned by positive elements.

**Lemma 3.2.** *Any element of $A$ can be written as a linear combination of four positive elements. In particular, any $b \in A_{\text{sa}}$ can be uniquely written as $b = b_+ - b_-$ where $b_+, b_- \in A_{\geq 0}$ are such that $b_+b_- = b_-b_+ = 0$.*

**Proof.** We start with the particular case. Define $b_+ := \frac{1}{2}(|b| + b)$ and $b_- := \frac{1}{2}(|b| - b)$. It's clear that $b = b_+ - b_-$ and that $b_+b_- = b_-b_+ = 0$ follows because $b$ commutes with $|b|$ (by functional calculus). To see that $b_+, b_- \in A_{\geq 0}$, suffices to check that their spectrum is in $\mathbb{R}_{\geq 0}$. This is also follows using functional calculus because the functions $\sigma(b) \to \mathbb{R}$ given by $t \mapsto |t| \pm t$ clearly have positive range. Now for a general $a \in A$ we can always write $a = a_1 + ia_2$ for $a_1, a_2 \in A_{\text{sa}}$. The desired result follows from the particular case. ■

The following lemma shows that any unital $C^*$ algebra is spanned by unitary elements.

**Lemma 3.3.** *Any element of $A$ (unital) can be written as a linear combination of unitaries. In particular, any $b \in A_{\text{sa}}$ can be written as the linear combination of two unitaries.*

**Proof.** As in the previous lemma, it's enough to prove the result for the particular case. If $b = 0$ the result is obvious. For $b \neq 0$ put $a := b\|b\|^{-1}$, whence $a \in A_{\text{sa}}$ has norm 1. By spectral theroy $\sigma(a) \subset [-1, 1]$ and therefore $1 - a^2 \in A_{\geq 0}$. Let $u := a + i\sqrt{1 - a^2}$. A direct computation shows that $u$ is unitary and that $u + u^* = 2a$. Therefore, $b = \frac{\|b\|}{2}(u + u^*)$, as wanted. ■

Turns out that $A_{\geq 0} = \{a^*a : a \in A\}$. The inclusion $\subseteq$ follows at once from the existence of a positive square root. The reverse inclusion is the content of the following important theorem.

**Theorem 3.4.** *If $a \in A$, then $a^*a \in A_{\geq 0}$*

**Proof.** The key step is to show that if $-c^*c \in A_{\geq 0}$ for $c \in A$, then $c = 0$. We omit this part. Now since $a^*a \in A_{\text{sa}}$ we get $a^*a = (a^*a)_+ - (a^*a)_-$. Put $c = a(a^*a)_-$ and notice that $-c^*c = (a^*a)_-^3 \in A_{\geq 0}$. Hence, $a^*a = (a^*a)_+ \in A_{\geq 0}$. ■

Using the previous theorem it makes sense to define $|a| := \sqrt{a^*a}$ for any $a \in A$. This agrees with the previous definition of the absolute value for self adjoint elements of $A$.

We make $A_{\text{sa}}$ into a poset by defining $a \leq b$ to mean $b - a \in A_{\geq 0}$. This relation is translation invariant, that is $a \leq b$ implies that $a + c \leq b + c$ for any $a, b, c \in A_{\text{sa}}$. Below, we list the most important properties of this relation

**Proposition 3.5.** *Let $A$ be a $C^*$-algebra and $a, b \in A_{\text{sa}}$*

1. *If $A$ is unital and $A_{\geq 0}$, then $a \leq \|a\|$.*
2. *If $a, b \in A_{\geq 0}$, then $a + b \in A_{\geq 0}$.*
3. *If $a, b \in A_{\geq 0}$ are such that $ab = ba$, then $ab \in A_{\geq 0}$.*
4. *If $a \leq b$, then $ta \leq tb$ for all $t \in \mathbb{R}_{\geq 0}$ and $-b \leq -a$.*
5. *If $a \leq b$, and $c \in A$ then $c^*ac \leq c^*bc$.*
6. *If $0 \leq a \leq b$, then $\|a\| \leq \|b\|$.*
7. *If $A$ is unital and $a, b \in \text{Inv}(A)$ are such that $0 \leq a \leq b$, then $0 \leq b^{-1} \leq a^{-1}$.*
8. *If $0 \leq a \leq b$, then $\sqrt{a} \leq \sqrt{b}$*

**Warning:** It's not true that $0 \leq a \leq b$ implies that $a^2 \leq b^2$. For example, in $A = M_2(\mathbb{C})$ consider the projections

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad q = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Then, $\sigma(q) = \{0, 1\}$, whence $p \leq p + q$. However, since $\sigma(pq + qp + q) = \{\frac{1}{2}(2 + \sqrt{5}), \frac{1}{2}(2 - \sqrt{5})\} \not\subset \mathbb{R}_{\geq 0}$, it follows that $p = p^2 \not\leq (p + q)^2 = p + pq + qp + q$.

It can be shown that if $0 \leq a \leq b$ implies that $a^2 \leq b^2$ for all $a, b \in A$, then $A$ is commutative.

# 4 Approximate Units and ideals

**Definition 4.1.** An **approximate unit** for a $C^*$-algebra $A$ is an increasing net $(e_\lambda)_{\lambda \in \Lambda}$ of positive elements in the closed unit ball of $A$ such that $a = \lim_\lambda ae_\lambda$ for all $a \in A$. Equivalently, $a = \lim_\lambda e_\lambda a$ for all $a \in A$.

For some $C^*$-algebras is easy to find an approximate unit. For example look at $A := \mathcal{K}(\ell^2)$. Let $(\delta_k)_{k=1}^{\infty}$ the usual basis for $\ell^2$. For each $n \in \mathbb{Z}_{>0}$ write

$$p_n\left(\sum_{k=1}^{\infty} \alpha_k \delta_k\right) := \sum_{k=1}^{n} \alpha_k \delta_k$$

For any $x \in \ell^2$ we have $\|p_n(x) - x\|_2 \to 0$. We claim that $(p_n)_{n \in \mathbb{Z}_{>0}}$ is an approximate unit for $A$. It's clear that $p_n^* = p_n^2 = p_n$ and since $p_n$ has finite rank, we must have that $p_n \in A$. Thus, each $p_n$ is a positive element in $A$ with $\|p_n\| \leq 1$. A direct check shows that $p_m - p_n = (p_m - p_n)^2 \geq 0$ for $m \geq n$, whence $p_n \leq p_m$. This gives that $(p_n)_{n=1}^{\infty}$ is indeed an increasing sequence of positive elements in the closed unit ball of $A$. We have to show that $\|p_n u - u\| \to 0$ for any $u \in \mathcal{K}(\ell^2)$. However, since the set of finite rank operators is dense in $\mathcal{K}(\ell^2)$, it suffices to show that $\|p_n u - u\| \to 0$ for a rank one operator $u : \ell^2 \to \ell^2$. Indeed, for each $y, z \in \ell^2$ we consider the rank one operator $u_{y,z}(x) := \langle x, y\rangle z$. Then, $\|u_{y,z}\| = \|y\|_2 \|z\|_2$ and therefore

$$\|p_n u_{y,z} - u_{y,z}\| = \|u_{y,p_n(z)} - u_{y,z}\| = \|u_{y,p_n(z)-z}\| = \|y\|_2 \|p_n(z) - z\|_2 \to 0$$

as wanted.

Turns out that any $C^*$-algebra has an approximate unit. To see this, we will construct a canonical approximate unit. First we need to convince ourselves that the set $\Lambda := \{a \in A_{\geq 0} : \|a\| < 1\} \subset A_{\mathrm{sa}}$ is an upward directed set. That is, we need to show that for any $a, b \in \Lambda$ there is $c \in \Lambda$ such that $a \leq c$ and $b \leq c$. Well, we can see any element $a \in \Lambda$ as an element of $\widetilde{A}$ and since $\|1 - (1 + a)\| = \|a\| < 1$, it follows that $(1 + a)$ is invertible in $\widetilde{A}$. Using functional calculus is easy to see that $\|a(1 + a)^{-1}\| < \frac{1}{4}$ and that $\sigma(a(1 + a)^{-1}) \subset [0, \frac{1}{4})$. Thus, since $A$ sits as an ideal in $\widetilde{A}$, it follows that $a(1 + a)^{-1} \in \Lambda$. Furthermore,

$$a(1 + a)^{-1} = (1 + a)(1 + a)^{-1} - (1 + a)^{-1} = 1 - (1 + a)^{-1}$$

So suppose that $d \in \Lambda$ is such that $0 \leq a \leq d$, then $1 + a \leq 1 + d$ and, by Proposition 3.5, $1 - (1 + a)^{-1} \leq 1 - (1 + d)^{-1}$ which in turn implies that $a(1 + a)^{-1} \leq d(1 + d)^{-1}$. Finally, for any $a, b \in \Lambda$ we use functional calculus to define $a' := a(1 - a)^{-1} \in A_{\geq 0}$ and $b' := b(1 - b)^{-1} \in A_{\geq 0}$. Then, $a' + b' \in A_{\geq 0}$, so if we define $c := (a' + b')(1 + a' + b')^{-1}$ functional calculus shows that $\|c\| < 1$ and therefore $c \in \Lambda$. Clearly $a' \leq a' + b'$ so we must have $a'(1 + a')^{-1} \leq c$ and similarly $b'(1 + b')^{-1} \leq c$. The desired result follows by observing that $a = a'(1 + a')^{-1}$ and $b = b'(1 + b')^{-1}$, which can be easily done with functional calculus

**Theorem 4.2.** *Every $C^*$-algebra $A$ has an approximate unit.*

**Proof.** Let $\Lambda$ be the upwards-directed set from above and define $(e_\lambda)_{\lambda \in \Lambda}$ by putting $e_\lambda := \lambda$. For any $a \in \Lambda$, using the Gelfand representation $C^*(a) \to C_0(\sigma(a))$ and Urysohn's lemma we get that $a = \lim_\lambda a e_\lambda$. By Lemma 3.2 we conclude that $a = \lim_\lambda a e_\lambda$ holds for any $a \in A$. ■

**Corollary 4.3.** *Any separable $C^*$-algebra admits an approximate unit which is a sequence.*

**Proof.** Let $\{a_1, a_n, \ldots\}$ a countable dense subset of $A$ and $(e_\lambda)_\Lambda$ an approximate unit for $A$. Write $F_n := \{a_1, \ldots, a_n\}$ and let choose $\lambda_1 \in \Lambda$ such that $\|a_1 - a e_{\lambda_1}\| < 1$.Now choose $\lambda_2 \geq \lambda_1$ such that $\|a_j - a_j e_{\lambda_2}\| < \frac{1}{2}$ for $j = 1, 2$. We proceed inductively and get an increasing sequence $(\lambda_n)_{n=1}^{\infty}$ such that $\|a - a e_{\lambda_n}\| < \frac{1}{n}$ for any $a \in F_n$. This says that $\|a - a e_{\lambda_n}\| \to 0$ as $n \to \infty$ for $a \in F_n$. We claim that $(e_{\lambda_n})_{n=1}^{\infty}$ is an approximate unit for $A$. Indeed, take any $a \in A$ and let $\varepsilon > 0$. By density there is $j \in \mathbb{Z}_{>0}$ such that $\|a - a_j\| < \frac{\varepsilon}{3}$. In particular $a_j \in F_j$, so there is $N \in \mathbb{Z}_{>0}$ such that $\|a_j - a_j e_{\lambda_n}\| < \frac{\varepsilon}{3}$ for all $n \geq N$. Therefore,

$$\|a - a e_{\lambda_n}\| \leq \|a - a_j\| + \|a_j - a_j e_{\lambda_n}\| + \|a_j - a\| \|e_{\lambda_n}\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \cdot 1 = \varepsilon$$

for all $n \geq N$, proving our claim and finishing the proof. ■

Sometimes we will need to work with approximate units inside the unitization $\widetilde{A}$. The following observation will be useful at least a couple of times: Suppose $e \in A_{\geq 0}$ is such that $\|e\| \leq 1$ then $\|1 - e\| \leq 1$. Indeed, by functional calculus $\sigma(1 - e) \subset (0, 1]$. We will then use that $\|1 - e_\lambda\| \leq 1$ whenever $(e_\lambda)_{\lambda \in \Lambda}$ is an approximate unit.

We now turn our attention to ideals. Having an approximate unit is quite helpful to show that closed two-sided ideals of a $C^*$-algebra are in fact self adjoint. This will imply some other crucial facts. First a lemma.

**Lemma 4.4.** *Let $J$ be a closed left ideal in $A$. Then there is an increasing net $(e_\lambda)_{\lambda \in \Lambda}$ of positive elements in the closed unit ball of $J$ such that $a = \lim_\lambda a e_\lambda$ for all $a \in J$.*

**Proof.** Since $J \cap J^*$ is a $C^*$-algebra it admits an approximate unit. Say $(e_\lambda)_{\lambda\in\Lambda}$ where each $e_\lambda$ is a positive element of the closed unit ball of $J$. Take any $a \in J$. Then $a^*a \in J \cap J^*$ and therefore $\lim_\lambda (aa^*)e_\lambda = aa^*$. Then, working on $\widetilde{A}$ if necessary, we have

$$\|a - ae_\lambda\|^2 = \|(a - ae_\lambda)^*(a - e_\lambda)\| = \|(1 - e_\lambda)a^*a(1 - e_\lambda)\| \leq \|1 - e_\lambda\|\|a^*a - a^*ae_\lambda\| \leq \|a^*a - a^*ae_\lambda\|$$

This proves that $a = \lim_\lambda ae_\lambda$ for any $a \in J$. ∎

**Theorem 4.5.** *If $I$ is a closed ideal in $A$, then $I$ is selfadjoint and therefore a $C^*$-subalgebra of $A$.*

**Proof.** Let $a \in I$ and $(e_\lambda)_{\lambda\in\Lambda}$ be the net in $I$ given by the previous Lemma. Then, $e_\lambda a^* \in I$ for all $\lambda \in \Lambda$, whence

$$a^* = \left(\lim_\lambda ae_\lambda\right)^* = \lim_\lambda (ae_\lambda)^* = \lim_\lambda e_\lambda a^*$$

since $I$ is closed, this proves that $a^* \in I$, so we are done. ∎

If $I$ is a closed ideal in $A$, then $A/I$ is a Banach algebra with the quotient norm. Since $I$ is selfadjoint, $A/I$ is also a $*$-algebra. But is $A/I$ a $C^*$-algebra? The answer is yes but we need to know first a way to easily compute the norm using $(e_\lambda)_{\lambda\in\Lambda}$, an approximate unit for $I$. Well, let $\varepsilon > 0$ and $a \in A$. There is $b \in I$ such that $\|a + b\| < \|a + I\| + \frac{\varepsilon}{2}$. Choose $\lambda_0 \in \Lambda$ such that $\|b - e_\lambda b\| < \frac{\varepsilon}{2}$ for all $\lambda \geq \lambda_0$. Then, working on $\widetilde{A}$ if necessary, we have that for any $\lambda \geq \lambda_0$,

$$\|a - e_\lambda a\| \leq \|a + b - e_\lambda(a + b)\| + \|be_\lambda - b\| = \|(1 - e_\lambda)(a + b)\| + \|be_\lambda - b\| \leq \|(a + b)\| + \|be_\lambda - b\| < \|a + I\| + \varepsilon$$

This gives $\lim_\lambda \|a - e_\lambda a\| \leq \|a + I\|$. Since $e_\lambda a \in I$ for all $\lambda \in \Lambda$, it follows that $\|a + I\| \leq \lim_\lambda \|a - e_\lambda a\|$. All together gives

$$\|a + I\| = \lim_\lambda \|a - e_\lambda a\| = \lim_\lambda \|a^* - a^*e_\lambda\| = \|a^* + I\|$$

This gives a nice way to express the norm $\|a + I\|$ for any $a \in A$, which is used two times to show that $A/I$ is a $C^*$-algebra:

**Theorem 4.6.** *If $I$ is a closed ideal in $A$, then $A/I$ is a $C^*$-algebra.*

**Proof.** The only thing that we don't have yet is the $C^*$-identity. Well, we just proved that $\|a + I\| = \|a^* + I\|$ for all $a \in A$. Hence, it suffices to show that $\|a + I\|^2 \leq \|a^*a + I\|$ for any $a \in I$. Indeed,

$$\|a + I\|^2 = \lim_\lambda \|a - e_\lambda a\|^2 = \lim_\lambda \|(1 - e_\lambda)a^*a(1 - e_\lambda)\| \leq \lim_\lambda \|a^*a - a^*ae_\lambda\| = \|a^*a + I\|$$

as wanted. ∎

**Theorem 4.7.** *Let $A, B$ be $C^*$-algebras and $\varphi : A \to B$ a $*$-homomorphism. Then, $\varphi(A)$ is a $C^*$-subalgebra of $B$.*

**Proof.** Well, now that we know that $A/\ker(\varphi)$ is a $C^*$-algebra, we see that $\varphi$ induces an injective $*$-homomorphism $A/\ker(\varphi) \to B$ via $a + \ker(\varphi) \mapsto \varphi(a)$. This map is clearly inyective so it's isometric thanks to Theorem 2.2. Thus, it's image $\varphi(A)$ is a closed subalgebra of $B$. ∎

## 4.1 Essential Ideals

**Definition 4.8.** We say that a closed ideal $I$ in $A$ is an **essential ideal** if $aI = 0$ implies that $a = 0$. Equivalently, $I$ is essential if $I \cap J \neq \{0\}$ for all non-zero closed ideals $J$ in $A$.

Let $I$ be a closed ideal $I$ and for any $a \in I$ we define $L_a, R_a \in \mathcal{L}(I)$ by $L_a(b) = ab$ and $R_a(b) = ba$ for any $b \in I$. Using the canonical inclusion $I \hookrightarrow M(I)$ sending $a \in I$ to $(L_a, R_a)$ we see that $I$ is an essential ideal of $M(I)$. Further, $I \hookrightarrow M(I)$ extends to $\varphi : A \to M(I)$ where $\varphi(a) := (L_a, R_a)$. It's easy to see that $\varphi$ is the only extension and that $\varphi$ is injective whenever $I$ is essential in $A$.

**Example 4.9.** Below $\mathcal{H}$ is a Hilbert space and $X$ a locally compact Hausdorff space.

1. $\mathcal{K}(\mathcal{H})$ is an essential ideal in $\mathcal{L}(\mathcal{H})$ and the extension $\varphi : \mathcal{L}(\mathcal{H}) \to M(\mathcal{K}(\mathcal{H}))$ is a $*$-isomorphism.
2. $C_0(X))$ is an essential ideal in $C_b(X)$ and the extension $\varphi : C_b(X) \to M(C_0(X))$ is a $*$-isomorphism.

## 4.2 Hereditary Subalgebras

Ideals in $C^*$-algebras are a special case class of $C^*$-subagebras, the hereditary ones:

**Definition 4.10.** We say that a $C^*$-subalgebra $B$ of $A$ is **hereditary** if for $a \in A_{\geq 0}$, $b \in B_{\geq 0}$ the inequality $a \leq b$ implies $a \in B$.

Obviously, the subalgebras $A$ and $\{0\}$ of $A$ are hereditary and any the intersection of hereditary subalgebras is again hereditary. If $S \subset A$, then the hereditary subalgebra generated by $S$ is the smallest hereditary $C^*$-subalgebra of $A$ containing $S$.

**Example 4.11.** Let $p \in A$ be a projection. Then the $C^*$-subalgebra $pAp$ is hereditary. Indeed, suppose $a \in A_{\geq 0}$ is such that $0 \leq a \leq pbp$ for some $b$. Then, from Theorem 3.5 it follows that

$$0 \leq (1-p)a(1-p) \leq (1-p)pbp(1-p) = 0$$

Hence, $(1-p)a(1-p) = 0$ and therefore $\|a^{1/2}(1-p)\|^2 = 0$ by the $C^*$-identity. This gives $a(1-p) = 0$ and therefore $a = ap$ and taking involution gives $a = pa$. Therefore, $a = pap \in pAp$.

Hereditary subalgebras are in one-to-one correspondence with closed left ideals. This fact has many useful consequences, some of which we list in the Corollary following the next Theorem

**Theorem 4.12.** *Define $\mathfrak{L} := \{J \subset A : J$ is a closed left ideal $\}$ and $\mathfrak{H} := \{B \subset A : B$ is a hereditary $C^*$-subalgebra $\}$. Then,*

1. *The map $J \mapsto J \cap J^*$ is a bijection $\mathfrak{L} \to \mathfrak{H}$. The inverse map is $B \mapsto \{a \in A : a^*a \in B\}$.*
2. *If $J_1, J_2 \in \mathfrak{L}$, then $J_1 \subset J_2$ if and only if $J_1 \cap J_1^* \subset J_2 \cap J_2^*$.*

**Corollary 4.13.** *Let $A$ be $C^*$-algebra. Then,*

1. *A $C^*$-subalgebra $B$ is hereditary if and only if $bab' \in B$ for all $b, b' \in B$ and all $a \in A$.*
2. *Any closed ideal of $A$ is a hereditary $C^*$-subalgebra.*
3. *For any $a \in A_{\geq 0}$, $\overline{aAa}$ is the hereditary $C^*$-subalgebra generated by $\{a\}$.*
4. *If $B$ is a separable hereditary $C^*$-subalgebra, there is $a \in A_{\geq 0}$ such that $B = \overline{aAa}$.*
5. *If $J$ is a closed ideal of a hereditary $C^*$-subalgebra $B$, there is a closed ideal $I$ in $A$ such that $J = I \cap B$.*
6. *If $A$ is simple and $B$ is a hereditary $C^*$-subalgebra, then $B$ is simple.*

**Example 4.14.** We show that the last assertion is not true for non hereditary $C^*$-subalgebra. It's known that $\mathcal{K}(\mathcal{H})$ is simple (any non-zero ideal of $\mathcal{K}(\mathcal{H})$ is an ideal of $\mathcal{L}(\mathcal{H})$ that contains the finite rank operators), however if $p, q$ are finite rank orthogonal projections then $\mathbb{C}p + \mathbb{C}q$ is a non-simple $C^*$-subalgebra of $\mathcal{K}(\mathcal{H})$ for it contains the non trivial closed ideal $Ap = \mathbb{C}p$.

# 5 Positive Linear Functionals and the GNS representation.

**Definition 5.1.** A linear map $\varphi : A \to B$ between $C^*$-algebras is **positive** if $\varphi(A_{\geq 0}) \subset B_{\geq 0}$.

If $\varphi : A \to B$ is positive, then $\varphi(a_1) \leq \varphi(a_2)$ whenever $a_1 \leq a_2$ and by Lemma 3.2 $\varphi(A_{\text{sa}}) \subset B_{\text{sa}}$.

**Definition 5.2.** A positive linear map $\tau : A \to \mathbb{C}$ is also called a **positive linear functional**. If $\tau$ is a bounded positive linear functional with $\|\tau\| = 1$ we say $\tau$ is a **state** on $A$. We denote by $\mathrm{S}(A)$ the set of states of $A$.

**Definition 5.3.** A positive linear functional $\tau : A \to C$ is called a **trace** if $\tau(ab) = \tau(ba)$ for all $a, b \in A$. A trace which is also a state is called a **tracial state.**

**Example 5.4.** .

1. Ever $*$-homomorphism $\varphi : A \to B$ is positive.
2. Let $X$ be a locally compact group and $\mu$ a regular Borel measure on $X$. The linear functional $C_c(X) \mapsto \mathbb{C}$ given by $f \mapsto \int_X f d\mu$ is positive (and not a homomorphism). This is a trace and it's a tracial state if $\mu(X) = 1$.
3. The linear functional $\mathrm{Tr} : M_n(\mathbb{C}) \to \mathbb{C}$ where $\mathrm{Tr}(a)$ is the usual trace of the matrix $a$ is positive. The normalized trace on $M_n(\mathbb{C})$ is also positive and it's given by $\mathrm{tr}(a) := \frac{1}{n}\mathrm{Tr}(a)$, so tr is a tracial state.

4. Let $\mathcal{H}$ be a Hilbert space and $\xi \in \mathcal{H} \setminus \{0\}$. The map $a \mapsto \langle a\xi, \xi\rangle$ is a positive linear functional on $\mathcal{L}(\mathcal{H})$, but is not a trace in general. This map is a state when $\|\xi\| = 1$.

**Proposition 5.5.** *Let $\tau : A \to \mathbb{C}$ be a positive linear functional on $A$. The map $A \times A \to \mathbb{C}$ given by $(a, b) \mapsto \tau(b^*a)$ is a positive sesquilinear form on $A$.*

**Proof.** Sesquilinerarity follows immediately. Since $\tau$ is positive $\tau(a^*a) \geq 0$ and therefore the form is positive, ■

**Corollary 5.6.** *Let $\tau : A \to \mathbb{C}$ be a positive linear functional on $A$. Then, $\overline{\tau(b^*a)} = \tau(a^*b)$, $|\tau(b^*a)| \leq \tau(a^*a)^{1/2}\tau(b^*b)^{1/2}$ and $a \mapsto \tau(a^*a)^{1/2}$ is a seminorm in $A$.*

**Lemma 5.7.** *Any positive linear functional on $A$ is bounded.*

**Proof.** Let $\tau : A \to \mathbb{C}$ be a positive linear functional. We claim that there is a positive constant $M$ such that $|\varphi(a)| \leq M$ for all $a \in A_{\geq 0}$ with $\|a\| \leq 1$. Assume otherwise that no such $M$ exists. Then, for each $n \in \mathbb{Z}_{>0}$, there exists $a_n \in A_{\geq 0}$ with $\|a_n\| = 1$ such that $\tau(a_n) \geq n$. Consider $b_k = \sum_{n=1}^{k} \frac{a_n}{n^2} \in A_{\geq 0}$ and $a = \lim_{k\to\infty} b_k \in A_{\geq 0}$. Then, $a \geq b_k$ and therefore $\tau(a) \geq \sum_{n=1}^{k} \frac{\tau(a_n)}{n^2} = \sum_{n=1}^{k} \frac{1}{n} \to \infty$, a contradiction. The claim is proved. If $a \in A$ with $\|a\| = 1$ then $a = b + ic$ with $b, c \in A_{\mathrm{sa}}$ and $\|b\|, \|c\| \leq 1$. We now use Theorem 3.2 with our previous claim to get $|\tau(a)| \leq 4M$, whence $\|\tau\| \leq 4M$. ■

**Lemma 5.8.** *Let $\tau : A \to \mathbb{C}$ be a positive linear functional on $A$. Then, $\tau(a^*) = \overline{\tau(a)}$ and $|\tau(a)|^2 \leq \|\tau\|\tau(a^*a)$ for all $a \in A$.*

**Proof.** Let $(e_\lambda)_{\lambda\in A}$ be an approximate unit for $A$. Then, using Corollary 5.6

$$\overline{\tau(a)} = \lim_{\lambda} \overline{\tau(e_\lambda a)} = \lim_{\lambda} \tau(a^*e_\lambda) = \tau(a^*)$$

and

$$|\tau(a)|^2 = \lim_{\lambda} |\tau(e_\lambda a)| \leq \lim_{\lambda} \tau(e_\lambda^2)\tau(a^*a) \leq \lim_{\lambda} \|\tau\|\|e_\lambda^2\|\tau(a^*a) \leq \|\tau\|\tau(a^*a)$$

as desired. ■

**Theorem 5.9.** *Let $\tau \in A^*$. The following are equivalent*

*(a) $\tau$ is positive.*

*(b) For each approximate unit $(e_\lambda)_{\lambda\in\Lambda}$ of $A$, $\|\tau\| = \lim_\lambda \tau(e_\lambda)$.*

*(c) For some approximate unit $(e_\lambda)_{\lambda\in\Lambda}$ of $A$, $\|\tau\| = \lim_\lambda \tau(e_\lambda)$.*

**Corollary 5.10.** *Let $A$ be $C^*$-algebra. Then,*

1. *If $A$ is unital, then $\tau \in A^*$ is positive if and only if $\|\tau\| = \tau(1)$.*
2. *If $\tau, \tau'$ are positive linear functions on $A$, then $\|\tau + \tau'\| = \|\tau\| + \|\tau'\|$.*

**Theorem 5.11.** *If $a$ is a normal element of $A \neq \{0\}$, there is $\tau \in \mathrm{S}(A)$ such that $\|a\| = |\tau(a)|$.*

**Proof.** Suppose $a \neq 0$, otherwise any state works. Look at the commutative unital $C^*$-algebra $B := C^*(1, a)$ in $\widetilde{A}$. Then, since $\mathrm{Max}(B)$ is compact there is $\omega_0 \in \mathrm{Max}(B)$ such that

$$\|a\| = \|\widehat{a}\|_\infty = \sup_{\omega\in\mathrm{Max}(B)} |\omega(a)| = |\omega_0(a)|,$$

and of course $\|\omega_0\| = 1$. By Hahn-Banach, there is $\omega_1 : \widetilde{A} \to \mathbb{C}$ such that $\omega_1|_B = \omega_0$ and $\|\omega_1\| = 1$. We claim that $\tau := \omega_1|_A$ is the state we are looking for. Indeed, since $\omega_1(1) = 1$ the previous Corollary gives that $\omega_1$ is positive and therefore $\tau$ is positive. Since $\tau(a) = \omega_0(a) = \|a\|$, we only need to check that $\tau$ has norm 1. Well, for any $a' \in A$, $|\tau(a')| = |\omega_1(a')| \leq \|a'\|$, whence $\|\tau\| \leq 1$. For the reverse inequality, we have $\|\tau\| \geq |\tau(\frac{a}{\|a\|})| = \frac{|\tau(a)|}{\|a\|} = 1$. ■

**Theorem 5.12.** *Let $\tau$ be a positive linear functional on $A$. Then,*

1. *For any $a \in A$, $\tau(a^*a) = 0$ if and only if $\tau(ba) = 0$ for all $b \in A$.*
2. *$\tau(b^*a^*ab) \leq \|a^*a\|\tau(b^*b)$ for all $a, b \in A$.*

**Proof.** For *1*, the "if" part is obvious. For the "only if" part, we use Corollary 5.6: $|\tau(ba)| \leq \tau(a^*a)^{1/2}\tau(bb^*)^{1/2} = 0$. For *2*, assume that $\tau(b^*b) > 0$ (for if $\tau(b^*b) = 0$ then by *1*, $\tau(b^*a^*ab) = 0$ and the desired result follows) and define

$$\rho(c) := \frac{\tau(b^*cb)}{\tau(b^*b)}$$

It's clear that $\rho$ is a positive linear functional, so if $(e_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for $A$ we have

$$\|\rho\| = \lim_\lambda \rho(e_\lambda) = \lim_\lambda \frac{\tau(b^*cb)}{\tau(b^*b)} = \frac{\tau(b^*b)}{\tau(b^*b)} = 1$$

Hence, $|\rho(a^*a)| \leq \|a^*a\|$, which is precisely $\tau(b^*a^*ab) \leq \|a^*a\|\tau(b^*b)$. ■

## 5.1 GNS construction

**Definition 5.13.** A **representation** of a $C^*$-algebra $A$ is a pair $(\mathcal{H}, \varphi)$ where $\mathcal{H}$ is a Hilbert space and $\varphi : A \to \mathcal{L}(\mathcal{H})$ a $*$-homomorphism. We that say $(\mathcal{H}, \varphi)$

- is **faithful** if $\varphi$ is injective.
- is **cyclic** if there is $\xi \in \mathcal{H}$ such that $\varphi(A)\xi := \operatorname{span}(\{\varphi(a)\xi : a \in A\})$ is dense in $\mathcal{H}$.
- is **non-degenerate** if $\varphi(A)\mathcal{H} := \operatorname{span}(\{\varphi(a)\xi : a \in A, \xi \in \mathcal{H}\})$ is dense in $\mathcal{H}$.

Let $(\mathcal{H}_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$ a family of representations of $A$. Define its direct sum $(\mathcal{H}, \varphi)$ where

$$\mathcal{H} := \bigoplus_{\lambda\in\lambda} \mathcal{H}_\lambda \qquad \text{and} \qquad \varphi(a)(\xi_\lambda)_{\lambda\in\Lambda} := (\varphi_\lambda(a)\xi_\lambda)_{\lambda\in\Lambda}$$

Then, routine verifications show that $(\mathcal{H}, \varphi)$ is a representation of $A$ and that it's faithful if for each non-zero element $a \in A$ there is $\lambda \in A$ such that $\varphi_\lambda(a) \neq 0$.

Given any positive linear functional $\tau : A \to \mathbb{C}$, we get a cyclic representation $(\mathcal{H}_\tau, \varphi_\tau)$ via the Gelfand-Naimark-Segal construction that we sketch below. Recall from Proposition 5.5 that $(a, b) \mapsto \tau(b^*a)$ is a sesquilinear form on $A$. Define

$$N_\tau := \{a \in A : \tau(b^*a) = 0 \ \text{ for all } b \in A\} = \{a \in A : \tau(a^*a) = 0\},$$

where the two sets are equal by part 1 in Theorem 5.12, whereas part 2 shows that $N_\tau$ is a closed left ideal of $A$. Therefore, the sesquiniliear form $(a, b) \mapsto \tau(b^*a)$ descends to a well defined inner-product on the quotient vector space $A/N_\tau$:

$$\langle a + N_\tau, b + N_\tau\rangle_\tau := \tau(b^*a)$$

Thus, with the norm induced by this inner-product, $A/N_\tau$ is a normed vector space. We denote by $\mathcal{H}_\tau$ the Hilbert space completion of $A/N_\tau$ with respect to this inner-product. For any $a \in A$ we define $\varphi_\tau : A \to \mathcal{L}(A/N_\tau)$ by setting

$$\varphi_\tau(a)(b + N_\tau) = ab + N_\tau$$

This map is well define for if $b - c \in N_\tau$, then $ab - ac = a(b - c) \in N_\tau$ for any $a \in A$. Further,

$$\|\varphi_\tau(a)(b + N_\tau)\|^2 = \langle ab + N_\tau, ab + N_\tau\rangle_\tau = \tau((ab)^*ab) = \tau(b^*a^*ab) \leq \|a^*a\|\tau(b^*b) = \|a\|^2\|b + N_\tau\|^2$$

Thus, $\varphi_\tau(a)$ extends to a bounded linear map $\varphi_\tau(a) \in \mathcal{L}(\mathcal{H}_\tau)$ with $\|\varphi_\tau(a)\| \leq \|a\|$. The map $\varphi_\tau : A \to \mathcal{L}(\mathcal{H}_\tau)$ is a $*$-homomorphism:

$$\begin{aligned}
\varphi_\tau(a + \alpha b)(c + N_\tau) = (a + \alpha b)(c + N_\tau) = (ac + \alpha b) + N_\tau = \varphi_\tau(a)(c + N_\tau) + \alpha\varphi_\tau(b)(c + N_\tau)\\
\varphi_\tau(ab)(c + N_\tau) = abc + N_\tau = \varphi_\tau(a)\varphi_\tau(b)(c + N_\tau)\\
\langle b + N_\tau, \varphi(a^*)(c + N_\tau)\rangle_\tau = \tau((a^*c)^*b) = \tau(c^*ab) = \langle\varphi(a)(b + N_\tau), c + N_\tau\rangle_\tau
\end{aligned}$$

Hence, $(\mathcal{H}_\tau, \varphi_\tau)$ is indeed a representation for $A$. We now exhibit a a cyclic vector for $(\mathcal{H}_\tau, \varphi_\tau)$ using an approximate unit $(e_\lambda)_{\lambda\in\Lambda}$ of $A$: we define $\xi_\tau := \lim_\lambda(e_\lambda + N_\tau) \in \mathcal{H}_\tau$. Then,

$$\varphi_\tau(a)\xi_\tau = \lim_\lambda(ae_\lambda + N_\tau) = a + N_\tau$$

and since $A/N_\tau$ is dense in $\mathcal{H}_\tau$ by construction, it follows that $\varphi_\tau(A)\xi_\tau$ is dense in $\mathcal{H}_\tau$. Also, we can recover the positive linear functional $\tau$ using the cyclic vector $\xi_\tau$, indeed

$$\langle\varphi_\tau(a)\xi_\tau, \xi_\tau\rangle_\tau = \langle a + N_\tau, \lim_\lambda(e_\lambda + N_\tau)\rangle_\tau = \lim_\lambda \tau(ae_\lambda) = \tau(a)$$

Hence $\|\xi_\tau\|^2 = \lim_\lambda\langle\varphi_\tau(e_\lambda)\xi_\tau, \xi_\tau\rangle_\tau = \lim_\lambda \tau(e_\lambda) = \|\tau\|$. If $\tau \in \mathrm{S}(A)$, we have $\|\xi_\tau\| = 1$. The vector $\xi_\tau$ is sometimes called the **canonical cyclic vector** for $(\mathcal{H}_\tau, \varphi_\tau)$.

**Definition 5.14.** If $A \neq \{0\}$, we define its **universal representation** by taking the direct sum of all the representation $(\mathcal{H}_\tau, \varphi_\tau)$, where $\tau$ ranges over $\mathrm{S}(A)$.

**Theorem 5.15.** *Any $C^*$-algebra $A$ admits a faithful non-degenerate representation.*

**Proof.** Let $(\mathcal{H}, \varphi)$ the universal representation of $A$, which is non-degenerate because each $(\varphi_\tau, \mathcal{H}_\tau)$ is cyclic. We have to show that $\varphi : A \to \mathcal{L}(\mathcal{H})$ is injective. Assume that $\varphi(a) = 0$. By Theorem 5.11 there is $\tau \in \mathrm{S}(A)$ such that $\tau(a^*a) = \|a^*a\| = \|a\|^2$. Let $b = (a^*a)^{1/4}$ and notice that $\varphi_\tau(b^4) = \varphi_\tau(a^*a) = \varphi_\tau(a^*)\varphi_\tau(a) = 0$ because $\varphi(a) = 0$. Hence, $\varphi_\tau(b) = 0$ and

$$\|a\|^2 = \tau(a^*a) = \tau(b^4) = \langle b^2 + N_\tau, b^2 + N_\tau\rangle_\tau = \|\varphi_\tau(b)(b + N_\tau)\| = 0,$$

Therefore, $a = 0$, whence $\varphi$ is injective. ■

# 6 Representations of $C^*$-algebras.

Given any representation $(\mathcal{H}, \varphi)$ of $A$, if $M$ is a closed subspace of $\mathcal{H}$ such that $\varphi(a)(M) \subset M$ for all $a \in A$ (i.e. $M$ is an **invariant subspace** under $\varphi$) we get a map $\varphi^M : A \to \mathcal{L}(M)$ by restricting to $M$:

$$\varphi^M(a) := \varphi(a)|_M$$

Then $(M, \varphi^M)$ is also a representation of $A$. In particular if we use $M := \overline{\varphi(A)\mathcal{H}}$ we get $\|\varphi(a)\| = \|\varphi^M(a)\|$ for all $a \in A$. Indeed, that $\|\varphi^M(a)\| \leq \|\varphi(a)\|$ is clear, the reverse inequality follows from

$$\|\varphi(a)\|^2 = \|\varphi(a)\varphi(a^*)\| = \|\varphi^M(a)\varphi(a^*)\| \leq \|\varphi^M(a)\|\|\varphi(a^*)\| = \|\varphi^M(a)\|\|\varphi(a)\|$$

Thus, we will often use $(M, \varphi^M)$ instead of $(\mathcal{H}, \varphi)$ to reduce to the case of a non-degenerate representation.

**Lemma 6.1.** *Let $(\mathcal{H}, \varphi)$ be a non-degenerate representation and $(e_\lambda)_{\lambda\in\Lambda}$ an approximate unit for $A$. Then $(\varphi(e_\lambda))_\lambda$ is an approximate unit for $\varphi(A)$ that converges strongly to $\mathrm{id}_{\mathcal{H}}$.*

**Proof.** It's clear thar $(\varphi(e_\lambda))_\lambda$ is an approximate unit for $\varphi(A)$. For $\varphi(a)\xi \in \varphi(A)\mathcal{H}$ we have

$$\|\varphi(e_\lambda)\varphi(a)\xi - \varphi(a)\xi\| = \|\varphi(e_\lambda a) - \varphi(a)\|\|xi\| \to 0$$

By density $\|\varphi(e_\lambda)\xi - \xi\| \to 0$ for any $\xi \in \mathcal{H}$. ■

We now use Zorn's lemma to show that every non-degenerate representation can be written as the direct sum of cyclic representations.

**Theorem 6.2.** *Let $(\mathcal{H}, \varphi)$ be a non-degenerate representation of $A$. Then $(\mathcal{H}, \varphi)$ is a direct sum of cyclic representations.*

**Proof.** For each $\xi \in \mathcal{H}$ put $\mathcal{H}_\xi := \overline{\varphi(A)\xi}$. Clearly $\mathcal{H}_\xi$ is invariant under $\varphi$ and if $(e_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for $A$ we have

$$\xi = \lim_\lambda \varphi(e_\lambda)\xi \in \overline{\varphi(A)\xi} = \mathcal{H}_\xi$$

Thus, $(\mathcal{H}_\xi, \varphi^{\mathcal{H}_\xi}, \xi)$ is a cyclic representation. Let $\mathcal{S} := \{S \subset \mathcal{H} : S \neq \{0\} \text{ and } \mathcal{H}_{\xi_1} \perp \mathcal{H}_{\xi_2}\ \forall\ \xi_1 \neq \xi_2 \text{ in } S\}$ and order it by set inclusion. For any $\xi \in \mathcal{H}$ we clearly have $\{\xi\} \in \mathcal{S}$, whence $\mathcal{S} \neq \varnothing$. Any totally ordered subset of $\mathcal{S}$ has an upper bound in $\mathcal{S}$, namely the union of all the elements in the totally ordered subsets (the totally ordered condition implies that this union is in fact in $\mathcal{S}$). Hence, by Zorn's Lemma, $\mathcal{S}$ has a maximal element, call it $M$. We claim that $\mathcal{H} = \bigoplus_{\xi\in M} \mathcal{H}_\xi$, this will show that $\varphi = \bigoplus_{\xi\in M} \varphi^{\mathcal{H}_\xi}$ and the desired result will follow. To prove the claim, it suffices to show that the linear span of $(\mathcal{H}_\xi)_{\xi\in M}$, denoted as $\sum_{\xi\in M} \mathcal{H}_x$ is dense in $\mathcal{H}$. Take any $\eta \in (\sum_{\xi\in M} \mathcal{H}_x)^\perp$, whence for any $\xi \in M$ and any $a, b \in A$ we have

$$\langle \varphi(a)\eta, \varphi(b)\xi\rangle = \langle \eta, \varphi(a^*b)\xi\rangle = 0$$

This gives $\mathcal{H}_\eta \perp \mathcal{H}_\xi$ for all $\xi \in M$, so $M \cup \{\eta\} \in \mathcal{S}$, but by maximality of $M$ we must have $\eta = 0$. This proves our claim and we are done. ■

**Definition 6.3.** Two representations $(\mathcal{H}_1, \varphi_1)$ and $(\mathcal{H}_2, \varphi_2)$ of $A$ are **unitarily equivalent** if there is a unitary $u : \mathcal{H}_1 \to \mathcal{H}_2$ such that for all $a \in A$

$$u\varphi_1(a) = \varphi_2(a)u$$

We define the set of **intertwining operators** from $\varphi_1$ to $\varphi_2$ by

$$\mathcal{C}(\varphi_1, \varphi_2) := \{v \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) : v\varphi_1(a) = \varphi_2(a)v\ \forall\ a \in A\}$$

Thus, $(\mathcal{H}_1, \varphi_1)$ and $(\mathcal{H}_2, \varphi_2)$ unitarily equivalent whenever $\mathcal{C}(\varphi_1, \varphi_2)$ contains a unitary operator. For a fixed representation $(\mathcal{H}, \varphi)$ of $A$, the set $\mathcal{C}(\varphi) := \mathcal{C}(\varphi, \varphi)$ consist of all the elements of $\mathcal{L}(\mathcal{H})$ that commute with $\varphi(a)$ for all $a \in A$ and it's called the **commutant** of $\varphi$. Sometimes $\mathcal{C}(\varphi)$ is denoted by $\varphi(A)'$.

**Proposition 6.4.** *Let $(\mathcal{H}_1, \varphi_1, \xi_1)$ and $(\mathcal{H}_2, \varphi_2, \xi_2)$ be two cyclic representations. There is a unitary $u \in \mathcal{C}(\varphi_1, \varphi_2)$ with $u(\xi_1) = \xi_2$ if and only if $\langle \varphi_1(a)\xi_1, \xi_1\rangle = \langle \varphi_2(a)\xi_2, \xi_2\rangle$ for all $a \in A$.*

**Proof.** Suppose there is a unitary $u \in \mathcal{C}(\varphi_1, \varphi_2)$ with $u(\xi_1) = \xi_2$. Then,

$$\langle \varphi(a)\xi_1, \xi_1\rangle = \langle u^*\varphi_2(a)u\xi_1, \xi_1\rangle = \langle \varphi_2(a)u\xi_1, u\xi_1\rangle = \langle \varphi_2(a)\xi_2, \xi_2\rangle$$

Conversely, assume that $\langle \varphi_1(a)\xi_1, \xi_1\rangle = \langle \varphi_2(a)\xi_2, \xi_2\rangle$ for all $a \in A$. Define $u : \varphi_1(A)\xi_1 \to \mathcal{H}_2$ de letting $u(\varphi_1(a)\xi_1) = \varphi_2(a)\xi_2$ and extending it linearly to $\varphi_1(A)\xi_1$. We have

$$\|u(\varphi_1(a)\xi_1)\|^2 = \langle \varphi_2(a)\xi_2, \varphi_2(a)\xi_2\rangle = \langle \varphi_1(a)\xi_1, \xi_1\rangle = \|\varphi_1(a)\xi_1\|^2$$

Thus, since $\overline{\varphi_1(A)\xi_1} = \mathcal{H}_1$ and $\overline{\varphi_2(A)\xi_2}$, we have that $u$ extends to a well defined unitary $u : \mathcal{H}_1 \to \mathcal{H}_2$. Clearly $u\varphi_1(a) = \varphi_2(a)u$ on $\varphi_1(A)\xi_1$, so it follows that $u \in \mathcal{C}(\varphi_1, \varphi_2)$. Finally, if $(e_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for $A$, then $\xi_2 = \lim_\lambda \varphi_2(e_\lambda)\xi_2 = u(\lim_\lambda \varphi(e_\lambda)\xi_1) = u(\xi_1)$, as wanted. ■

## 6.1 Irreducible Representations and Pure states

**Definition 6.5.** A representation $(\mathcal{H}, \varphi)$ of $A$ is **irreducible** if it has no non-trivial closed invariant subspaces.

**Theorem 6.6.** *Let $(\mathcal{H}, \varphi)$ be a non-zero representation of $A$.*

1. *$(\mathcal{H}, \varphi)$ is irreducible if and only if $\mathcal{C}(\varphi) = \mathbb{C}1$ where $1 = \mathrm{id}_\mathcal{H}$.*
2. *If $(\mathcal{H}, \varphi)$ is irreducible, then every non zero vector of $\mathcal{H}$ is cyclic for $(\mathcal{H}, \varphi)$.*

**Proof.**

1. Suppose first that the representation is reducible. Then there is a non-trivial invariant subspace $M \subset \mathcal{H}$. Let $p_M$ be the orthogonal projection onto $M$. Invariance means $\varphi(a)p_M = p_M\varphi(a)$ for all $a \in A$, whence $p_M \in \mathcal{C}(\varphi)$. Since $M$ is non-trivial, $p_M \notin \mathbb{C}1$. For the converse suppose $\mathcal{C}(\varphi) \neq \mathbb{C}1$ and take $v \in \mathcal{C}(\varphi)$ such that $v \notin \mathbb{C}1$. By writing $v = v_1 + iv_2$ with $v_1, v_2$ selfadjoint, we have $v_1, v_2 \in \mathcal{C}(\varphi)$ and we can assume that $v_1 \notin \mathbb{C}1$. We must have at least two different points in $\sigma(v_1)$ say $t_1 \neq t_2$ (otherwise if $\sigma(v_1) = \{t\}$ using functional calculus with the inclusion of $\sigma(v_1)$ in $\mathbb{C}$ we get $v_1 = t1$, which is impossible). Choose $f_1, f_2 \in C(\sigma(v_1))$ with disjoint support such that $f_j(t_k) = \delta_{j,k}$. Then, $f_j(v_1) \in \mathcal{C}(\varphi)$ $(j = 1, 2)$ and $\mathcal{H}_j := \overline{f_j(v_1)\mathcal{H}}$ $(j = 1, 2)$ is a non-zero invariant subspace for $\varphi$. Further, since $f_1$ and $f_2$ have disjoint support it follows that $\mathcal{H}_1$ and $\mathcal{H}_2$ are mutually orthogonal, whence $\mathcal{H}_1$ and $\mathcal{H}_2$ are non-trivial invariant subspaces for $\varphi$.
2. Let $\xi$ be any non zero vector of $\mathcal{H}$. Then, $\overline{\varphi(A)\xi}$ is invariant and non-zero because it contains $\xi = \lim_\lambda \varphi(e_\lambda)\xi$. Irreducibility implies that $\overline{\varphi(A)\xi} = \mathcal{H}$, so $(\mathcal{H}\varphi)$ is indeed cyclic. ■

**Corollary 6.7.** *Let $(\mathcal{H}_1, \varphi_1)$ and $(\mathcal{H}_2, \varphi_2)$ be two irreducible representations of $A$. Then $(\mathcal{H}_1, \varphi_1)$ and $(\mathcal{H}_2, \varphi_2)$ are equivalent if and only if $\mathcal{C}(\varphi_1, \varphi_2)$ is one dimensional, in fact $\mathcal{C}(\varphi_1, \varphi_2) = \{0\}$ whenever $\varphi_1$ and $\varphi_2$ are not equivalent.*

**Proof.** If $(\mathcal{H}_1, \varphi_1)$ and $(\mathcal{H}_2, \varphi_2)$ are equivalent then there is a unitary $u \in \mathcal{C}(\varphi_1, \varphi_2)$ so $\mathcal{C}(\varphi_1, \varphi_2) \neq \{0\}$. Take $v_1, v_2 \in \mathcal{C}(\varphi_1, \varphi_2)$. Since $v_1v_1^* \in \mathcal{C}(\varphi_2)$, the previous theorem gives $v_1v_1^* = \alpha 1$ for some $\alpha \in \mathbb{C}$. Similarly, $v_1^*v_2 \in \mathcal{C}(\varphi_1)$ so there is $\beta \in \mathbb{C}$ such that $v_1^*v_2 = \beta 1$. Then, $v_2 = \alpha\beta v_1$, so $\mathcal{C}(\varphi_1, \varphi_2)$ is one dimensional. Suppose now that $(\mathcal{H}_1, \varphi_1)$ and $(\mathcal{H}_2, \varphi_2)$ are not equivalent but that $\mathcal{C}(\varphi_1, \varphi_2) \neq \{0\}$. As before, for any non-zero $v \in \mathcal{C}(\varphi_1, \varphi_2)$ we have $vv^* = \alpha 1$, for a non-zero $\alpha \in \mathbb{C}$, but this says that $\alpha^{-1/2}v$ is a unitary in $\mathcal{C}(\varphi_1, \varphi_2)$, making the two representation equivalent, a contradiction. ■

**Definition 6.8.** Let $\tau$ and $\rho$ be positive linear functionals on $A$. We write $\rho \leq \tau$ if $\tau - \rho$ is a positive linear functional. A state $\tau \in \mathrm{S}(A)$ is **pure** is whenever $\rho$ is a positive linear functional such that $\rho \leq \tau$, it follows that $\rho = t\tau$ for some $t \in [0, 1]$. The set of pure states on $A$ is denoted by $\mathrm{PS}(A)$.

For thr following results recall that the GNS construction gives a cyclic representation $(\mathcal{H}_\tau, \varphi_\tau, \xi_\tau)$ for any positive linear functional $\tau$ such that

- $\varphi_\tau(a)\xi_\tau = a + N_\tau$
- $\tau(a) = \langle \varphi_\tau(a)\xi_\tau, \xi_\tau\rangle$
- $\|\xi_\tau\|^2 = \|\tau\|$.

**Lemma 6.9.** *Let $\tau \in \mathrm{S}(A)$ and $\rho$ a positive linear functional. Then, $\rho \le \tau$ if and only if there is a unique $v \in \mathcal{C}(\varphi_\tau)$ such that*

$$\rho(a) = \langle \varphi_\tau(a) v \xi_\tau, \xi_\tau \rangle,$$

*and $0 \le v \le 1$.*

**Theorem 6.10.** *Let $\tau \in \mathrm{S}(a)$. Then*

1. *$\tau \in \mathrm{PS}(A)$ if and only if $(\mathcal{H}_\tau, \varphi_\tau)$ is irreducible.*
2. *If $A$ is commutative, then $\tau \in \mathrm{PS}(A)$ if and only if $\tau$ is a character on $A$ (i.e a non-zero homomorphism $A \to \mathbb{C}$).*

**Proof.**

1. Suppose first that $\tau \in \mathrm{PS}(A)$. By the previous lemma, if $v \in \mathcal{C}(\varphi_\tau)$ is such that $0 \le v \le 1$ and we define

$$\rho(a) := \langle \varphi_\tau(a) v \xi_\tau, \xi_\tau \rangle,$$

then $\rho \le \tau$. Hence, there is $t \in [0,1]$ such that $\rho = t\tau$. That is,

$$\langle \varphi_\tau(a) v \xi_\tau, \xi_\tau \rangle = \rho(a) = t\tau(a) = \langle t\varphi_\tau(a)\xi_\tau, \xi_\tau \rangle$$

Then, for any $a, b \in A$

$$\langle v(a + N_\tau), b + N_\tau \rangle = \langle v\varphi_\tau(a)\xi_\tau, \varphi_\tau(b)\xi_\tau \rangle = \langle t\varphi_\tau(b^*a)\xi_\tau, \xi_\tau \rangle = \langle t(a + N_\tau), b + N_\tau \rangle$$

Hence, $v = t1 \in \mathbb{C}1$. By Theorem 3.2 it follows that $\mathcal{C}(\varphi_\tau) = \mathbb{C}1$, whence $(H_\tau, \varphi_\tau)$ is irreducible. Conversely, assume that $(H_\tau, \varphi_\tau)$ is irreducible and that $\rho \le \tau$. We want to show that there is $t \in [0,1]$ such that $\rho = t\tau$. By the previous lemma, there is $v \in \mathcal{C}(\varphi_\tau) = \mathbb{C}1$ such that $0 \le v \le 1$ and $\rho(a) = \langle \varphi_\tau(a) v\xi_\tau, \xi_\tau \rangle$. So $v = \alpha 1$ for some $\alpha \in \mathbb{C}$ and therefore $\rho = \alpha\tau$. But since $0 \le v \le 1$ this means $\{\alpha\} = \sigma(v) \subset [0,1]$.

2. Suppose $A$ is commutative. If $\tau \in \mathrm{PS}(A)$, then $(\mathcal{H}_\tau, \varphi_\tau)$ is irreducible by part 1 and therefore $\mathcal{C}(\varphi_\tau) = \mathbb{C}1$. Since $A$ is commutative, it's clear that $\varphi_\tau(A) \subset \mathcal{C}(\varphi)$ and therefore for each $a \in A$ there is $\alpha(a) \in \mathbb{C}$ such that $\varphi_\tau(a) = \alpha(a)1$. Thus, since $1 = \|\xi_\tau\|^2 = \langle \xi_\tau, \xi_\tau \rangle$,

$$\begin{aligned} \tau(ab) &= \langle \varphi_\tau(ab)\xi_\tau, \xi_\tau \rangle \\ &= \langle \varphi_\tau(a)\varphi_\tau(b)\xi_\tau, \xi_\tau \rangle \\ &= \alpha(a)\alpha(b)\langle \xi_\tau, \xi_\tau \rangle \\ &= \alpha(a)\langle \xi_\tau, \xi_\tau \rangle \alpha(b) \langle \xi_\tau, \xi_\tau \rangle \\ &= \langle \varphi_\tau(a)\xi_\tau, \xi_\tau \rangle \langle \varphi_\tau(b)\xi_\tau, \xi_\tau \rangle = \tau(a)\tau(b) \end{aligned}$$

So $\tau$ is a character on $A$. Conversely, assume that $\tau$ is a character on $A$ and that $\rho$ is a positive linear functional such that $\rho \le \tau$. We want to show that there is $t \in [0,1]$ such that $\rho = t\tau$. Notice first that $\ker(\tau) \subset \ker(\rho)$ for if $\tau(a) = 0$, then using Lemma 5.8 and that $\tau$ is a character

$$|\rho(a)|^2 \le \|\rho\|\rho(a^*a) \le \|\rho\|\tau(a^*a) = \|\rho\|\tau(a^*)\tau(a) = 0$$

Now, since $\tau$ is non zero, there is $a_0 \in A$ with $\tau(a_0) = 1$; and clearly for any $a \in A$ we have $a - \tau(a)a_0 \in \ker(\tau)$. Thus, $a - \tau(a)a_0 \in \ker(\rho)$, whence $\rho(a) = \rho(a_0)\tau(a)$. It now suffices to show that $\rho(a_0) \in [0,1]$. Well, $0 \le \rho(a_0^*a_0) = \rho(a_0)\tau(a_0^*a_0) = \rho(a_0)$ and $\rho(a_0) = \frac{\rho(a_0^*a_0)}{\tau(a_0^*a_0)} \le 1$.

■

The next result shows that any cyclic representation actually comes from as state as in the GNS construction.

**Theorem 6.11.** *Let $(\mathcal{H}, \varphi, \xi)$ be a cyclic representation of $A$ with $\|\xi\| = 1$. Then, the function $\tau : A \to \mathbb{C}$ given by*

$$\tau(a) := \langle \varphi(a)\xi, \xi \rangle$$

*is a state of $A$ and $(\mathcal{H}, \varphi)$ is equivalent to $(\mathcal{H}_\tau, \varphi_\tau)$. Moreover, if $(\mathcal{H}, \varphi)$ is irreducible, then $\tau$ is pure.*

**Proof.** If $a \ge 0$, then $\tau(a) = \|\varphi(a^{1/2})\xi\|^2 \ge 0$, whence $\tau$ is a positive linear functional on $A$. Then, for $(e_\lambda)$, an approcimate unit for $A$, we know $\|\tau\| = \lim_\lambda \tau(e_\lambda) = \lim_\lambda \langle \varphi(e_\lambda)\xi, \xi \rangle = \langle \xi, \xi \rangle = 1$. Hence, $\tau \in \mathrm{S}(a)$. The equivalence part follows from the following fact

$$\langle \varphi_\tau(a)\xi_\tau, \xi_\tau \rangle = \tau(a) = \langle \varphi(a)\xi, \xi \rangle,$$

together with Proposition 6.4. Finally, that $\tau \in \mathrm{PS}(A)$ whenerver $(\mathcal{H}, \tau)$ is irreducible follows at once from the previous Theorem. ■

**Example 6.12.** For a Hilbert space $\mathcal{H}$, the pure states of $\mathcal{K}(\mathcal{H})$ are given by the positive linear functionals

$$\tau_\xi(a) := \langle a\xi, \xi\rangle$$

where $\|\xi\| = 1$. A consequence of this is that any non-zero irreducible representation of $\mathcal{K}(\mathcal{H})$ is equivalent to $(\iota, \mathcal{H})$ where $\iota(a) = a \in \mathcal{L}(\mathcal{H})$ for any $a \in \mathcal{K}(\mathcal{H})$. If we look at $\mathcal{L}(\mathcal{H})$, any $\tau_\xi$ as above is also a pure state. However, if $\mathcal{H}$ is infinite dimensional and separable, not all the pure states on $\mathcal{L}(\mathcal{H})$ are of the form $\tau_\xi$.

**Definition 6.13.** If $K$ is a convex set, $k \in K$ is an **extreme point of** $K$ if whenever $k = tk_1 + (1-t)k_2$ for $t \in (0,1)$ and $k_1, k_2 \in K$, then $k = k_1 = k_2$. We denote by $\mathrm{Ext}(K)$ to the set of extreme points of $K$.

**Theorem 6.14.** *Let $K(A)$ be the set of norm-decreasing positive linear functionals on $A$. Then $K(A)$ is a weak-$*$ compact set and it's convex. Moreover,* $\mathrm{Ext}(K(A)) = \mathrm{PS}(A) \cup \{0\}$.

**Proof.** It's clear that any converging net of positive linear functionals converges to a positive linear function. Thus, $K(A)$ is a weak-$*$ closed subset of $\overline{B_1(0)}$ in $A^*$. By Banach-Alaoglu it follows that $K(A)$ is weak-$*$ compact. Let $\tau_1, \tau_2 \in K(A)$ and $t \in [0,1]$. Then, $t\tau_1 + (1-t)\tau_2$ is also a norm-decresing positive linear functional, whence $K(A)$ is a convex set. Suppose that $0 = t\tau_1 + (1-t)\tau_2$ with $t \in (0,1)$ and $\tau_1, \tau_2 \in K(A)$. For any $a \in A_{\geq 0}$ we have $0 \geq -t\tau_1(a) = (1-t)\tau_2(a) \geq 0$; so $\tau_1 = \tau_2 = 0$ and therefore $0 \in \mathrm{Ext}(K(A))$. Now suppose that $\tau \in \mathrm{PS}(A)$ is such that $\tau = t\tau_1 + (1-t)\tau_2$ with $t \in (0,1)$ and $\tau_1, \tau_2 \in K(A)$. Clearly 1 is an extreme point of $[0,1]$ and since $1 = \|\tau\| = 1\|\tau_1\| + (1-t)\|\tau_2\|$, we must have $\|\tau_1\| = \|\tau_2\| = 1$. Also, $t\tau_1 \leq \tau$ and therefore there is $t' \in [0,1]$ so that $t\tau_1 = t'\tau$ because $\tau$ is pure. But, $t = \|t\tau_1\| = \|t'\tau\| = t'$, whence $\tau = \tau_1$ and from here it's easy to see that $\tau = \tau_2$, which gives $\tau \in \mathrm{Ext}(K(A))$.

So far, we have shown $\mathrm{PS}(A) \cup \{0\} \subset \mathrm{Ext}(K(A))$. For the reverse inclusion, take $\tau$ any non-zero element of $\mathrm{Ext}(K(A))$. We have to show that $\tau$ is a pure state. Since $\tau \in \mathrm{Ext}(K(A))$,

$$\tau = \|\tau\| \cdot \frac{\tau}{\|\tau\|} + (1 - \|\tau\|) \cdot 0$$

and $\frac{\tau}{\|\tau\|}, 0 \in K(A)$, it follows that $\|\tau\| = 1$; so $\tau$ is a state. Assume now that $\rho \leq \tau$ but that $\rho \neq \tau$ and $\rho \neq 0$. Then, $\|\rho\| \in (0,1)$ and $\|\tau - \rho\| = \lim_\lambda(\tau - \rho)(e_\lambda) = 1 - \|\rho\|$. Therefore,

$$\tau = \|\rho\| \cdot \frac{\rho}{\|\rho\|} + (1 - \|\rho\|) \cdot \frac{\tau - \rho}{\|\tau - \rho\|}$$

Since $\tau \in \mathrm{Ext}(K(A))$ and $\frac{\rho}{\|\rho\|}, \frac{\tau-\rho}{\|\tau-\rho\|} \in K(A)$, it follows that $\rho = \|\rho\|\tau$, so $\tau \in \mathrm{PS}(A)$. ■

**Remark 6.15.** A fact that we won't prove is that a representation on $A$ is algebraically irreducible if and only if it is topologically irreducible. An important consequence of this fact is that whenever $\tau$ is a pure state, then $A/N_\tau = \mathcal{H}_\tau$ simply because $A/N_\tau$ is an invariant (not necessarily closed a priori) non-zero subspace of $\mathcal{H}_\tau$ so irreducibility of $(\mathcal{H}_\tau, \varphi_\tau)$ (see Theorem 6.10) implies that $A/N_\tau = \mathcal{H}_\tau$.

## 6.2 Modular and Primitive ideals

**Definition 6.16.** An ideal $I$ in $A$ is **modular** if there is $u \in A$ such that $a - au \in I$ and $a - ua \in I$ for all $a \in A$. Similarly, a left ideal $J$ in $A$ is **modular** if there is $u \in A$ such that $a - au \in J$ for all $a \in A$.

**Lemma 6.17.** *If $\tau \in \mathrm{PS}(A)$, the left ideal $N_\tau$ is modular.*

**Proof.** From Remark 6.15 we know that there is $u \in A$ such that $\xi_\tau = u + N_\tau \in \mathcal{H}_\tau$. Also, for any $a \in A$, we have $a + N_\tau = \varphi_\tau(a)\xi_\tau = au + N_\tau$, whence $a - au \in N_\tau$. ■

**Theorem 6.18.** *The correspondence $\tau \mapsto N_\tau$ is a bijection from* $\mathrm{PS}(A)$ *onto the set of all modular maximal left ideals of $A$.*

**Definition 6.19.** Given a closed modular maximal left modular ideal $J$ in $A$, the ideal

$$I := \{a \in A : aA \subset J\}$$

is a largest ideal of $A$ contained in $J$. We call $I$ the **primitive ideal** of $A$ associated with $J$. We denote by $\mathrm{Prim}(A)$ to the set of all primitive ideals in $A$.

If $\tau \in \mathrm{PS}(A)$, It's easy to see that the primitive ideal associated with $N_\tau$ is $\ker(\varphi_\tau)$. This particular case can be seen as a general characterization of primitive ideals:

**Proposition 6.20.** *An ideal $I$ is in* $\mathrm{Prim}(A)$ *if and only if there is an irreducible representation* $(\mathcal{H}, \varphi)$ *of $A$ such that* $I = \ker(\varphi)$.

**Remark 6.21.** A primitive ideal gives an irreducible representation. An irreducible representation has to come from a pure state using the GNS construction. We already saw that any non-zero vector of an irreducible representation is cyclic and that we can use norm one vectors define a state whose GNS representation is equivalent to the original one. Then, for each primitive ideal $I = \ker(\varphi)$ we get a lot of pure states associated with $(\mathcal{H}, \varphi)$. Also it worth keeping in mind that equivalent representation might have different kernels.

**Definition 6.22.** For $S \subset A$ we let $\mathrm{hull}(S) := \{I \in \mathrm{Prim}(A) : S \subset I\}$. If $\varnothing \neq R \subset \mathrm{Prim}(A)$, we put $\ker(R) = \bigcap_{I \in R} I$ and $\ker(\varnothing) = A$.

**Theorem 6.23.** *If $A$ is a proper modular ideal in $A$, then* $\mathrm{hull}(I) \neq \varnothing$*. Moreover, if $I$ is also closed, then*

$$\ker(\mathrm{hull}(I)) = I$$

**Remark 6.24.** The previous theorem gives that any modular maximal ideal of $A$ is primitive. If $A$ is commutative the converse is true, for $\mathrm{Prim}(A)$ is identified with it's character space, which coincides with the modular maximal ideals of $A$.

**Definition 6.25.** There is a unique topology on $\mathrm{Prim}(A)$ such that $R \subset \mathrm{Prim}(A)$ is closed if and only if $\mathrm{hull}(\ker(R)) = R$. This is known as the **hull-kernel** topology.

**Definition 6.26.** If $A$ is non-zero, we denote by $\widehat{A}$ to the set of unitary equivalence classes of non-zero irreducible representations of $A$, that is

$$\widehat{A} := \{[\mathcal{H}, \varphi] : (\mathcal{H}, \varphi) \text{ is irreducible }\}$$

We topologize $\widehat{A}$ by considereing the weakest topology making the surjective map $\widehat{A} \ni [\mathcal{H}, \varphi] \mapsto \ker(\varphi) \in \mathrm{Prim}(A)$ continuous.

**Proposition 6.27.** *The canonical map* $\widehat{A} \mapsto \mathrm{Prim}(A)$ *is a homeomorphism if and only if any two non-zero irreducible representations of $A$ with the same kernel are unitarily equivalent.*

## 6.3 Liminal and Postliminal $C^*$-algebras

**Lemma 6.28.** *Let $B$ be a $C^*$-subalegebra of $\mathcal{L}(\mathcal{H})$ such that $B$ has no non-trivial invariant subspaces and $B \cap \mathcal{K}(\mathcal{H}) \neq \{0\}$. Then $\mathcal{K}(\mathcal{H}) \subset B$.*

**Definition 6.29.** $A$ is said to be **liminal** (also called a CCR $C^*$-algebra) if for every non-zero irreducible representation $(\mathcal{H}, \varphi)$ of $A$ we have $\varphi(A) = \mathcal{K}(\mathcal{H})$. It's enough to ask $\varphi(A) \subset \mathcal{K}(\mathcal{H})$, because the inclusion $\mathcal{K}(\mathcal{H}) \subset \varphi(A)$ will then automatically hold by the above Lemma, being $(\mathcal{H}, \varphi)$ an irreducible representation.

**Example 6.30.** .

1. Any commutative $C^*$-algebra is liminal. Indeed, Let $(\mathcal{H}, \varphi)$ be a non-zero irreducible representation. Then $\mathcal{C}(\varphi) = \mathbb{C}1$, but since $A$ is commutative, $\varphi(A) \subset \mathbb{C}1$. One checks that this implies that $\mathcal{L}(\mathcal{H}_1) = \mathbb{C}1$, so $\mathcal{H}$ is one-dimensional and therefore $\varphi(A) \subset \mathcal{L}(\mathcal{H}) = \mathcal{K}(\mathcal{H})$.

2. Any finite dimensional $C^*$-algebra is liminal, for if $(\mathcal{H}, \varphi)$ is irreducible, then any non-zero $\xi\ n\mathcal{H}$ is cyclic and by finite dimentionality $\varphi(A)\xi = \overline{\varphi(A)\xi} = \mathcal{H}$, whence $\mathcal{H}$ is finite dimensional. Thus, $\varphi(A) \subset \mathcal{L}(\mathcal{H}) = \mathcal{K}(\mathcal{H})$.

3. Recall from Example 6.12 that any non-zero irreducible representation of $\mathcal{K}(\mathcal{H})$ is equivalent to the inclusion $(\iota, \mathcal{H})$. Thus, $\mathcal{K}(\mathcal{H})$ is liminal.

4. The algebra $\mathcal{L}(\mathcal{H})$ is **not** liminal when $\mathcal{H}$ is infinite dimensional; indeed $\mathcal{B}(\mathcal{H}) \neq \mathcal{K}(\mathcal{H})$, so the identity representation, which is irreducible because $\mathcal{L}(\mathcal{H})' = \mathbb{C}1$, fails the requirement for liminality.

**Theorem 6.31.** *If $A$ is liminal, then its $C^*$-subalgebras and its quotient $C^*$-algebras are liminal also.*

The converse to the above theorem is not true. One can check that if $\mathcal{H}$ is infinite dimensional, then $\widetilde{\mathcal{K}(\mathcal{H})}$ can't be liminal (because the identity on $\mathcal{H}$ is an infinite dimensional irreducible representation and liminal algebras only have finite dimensional irreducible representations). However, both $\mathcal{K}(\mathcal{H})$ and $\widetilde{\mathcal{K}(\mathcal{H})}/\mathcal{K}(\mathcal{H}) = \mathbb{C}1$ are liminal.

**Definition 6.32.** $A$ is said to be **postliminal** (most commonly called type I $C^*$-algebras) if for every non-zero irreducible representation $(\mathcal{H}, \varphi)$ of $A$ we have $\mathcal{K}(\mathcal{H}) \subset \varphi(A)$. By a Lemma above, this is equivalent to ask $\varphi(A) \cap \mathcal{K}(\mathcal{H}) \neq \{0\}$, being $(\mathcal{H}, \varphi)$ an irreducible representation.

**Theorem 6.33.** *Let $I$ be a closed ideal in $A$. Then, $A$ is postliminal if and only if $I$ and $A/I$ are postliminal.*

**Example 6.34.** Any liminal $C^*$-algebra is also postliminal. $\mathcal{T}$, the Toeplitz algebra which is the $C^*$-algebra generated by the unilateral shift in $\mathcal{L}(\ell^2)$, is postliminal but not liminal. To see this, we need to know that $\mathcal{T}$ can be represented in the Hardy space $H^2 := \{f \in L^2(S^1) : \widehat{f}(n) = 0, n < 0\}$. One checks that $\mathcal{K}(H^2)$ is an ideal in $\mathcal{T}$ and that $\mathcal{T}/\mathcal{K}(H^2) \cong C(S^1)$. Now apply the previous theorem.

**Theorem 6.35.** *If $(\mathcal{H}_1, \varphi_1)$ and $(\mathcal{H}_2, \varphi_2)$ are two non-zero irreducible representations of a postliminal $C^*$-algebra $A$, then $[\mathcal{H}_1, \varphi_1] = [\mathcal{H}_2, \varphi_2]$ in $\widehat{A}$ if and only if* $\ker(\varphi_1) = \ker(\varphi_2)$.

**Corollary 6.36.** *If $A$ is a non-zero postliminal $C^*$-algebra, the canonical map $\widehat{A} \to \mathrm{Prim}(A)$ is an isomorphism.*

# 7 Direct Limits of $C^*$-algebras

## 7.1 Direct Limit of Groups

Let $(G_i)_{i\in\Lambda}$ be a directed family of groups (i.e. $\Lambda$ is a directed set: a proset such that for every $i, j \in \Lambda$ there is $k \in \Lambda$ with $i \leq k$ and $j \leq k$). Suppose that for each $i \leq j$ in $\Lambda$ there is a group homorphism $\varphi_{j,i} : G_i \to G_j$ such that

- $\varphi_{i,i} := \mathrm{id}_{G_i}$
- $\varphi_{k,1} = \varphi_{k,j} \circ \varphi_{j,i}$ for $i \leq j \leq k$.

The pair $((G_i)_{i\in\Lambda}, (\varphi_{j,i})_{i\leq j})$ is called a **directed system of groups**. Given such a directed system of groups we will now define its direct limit $\varinjlim((G_i)_{i\in\Lambda}, (\varphi_{j,i})_{i\leq j})$. First consider the set

$$G_\infty := \left\{(g_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} G_i : \exists\ i_0 \in \Lambda \ \text{ s.t. } \ g_j = \varphi_{j,i_0}(g_{i_0})\ \forall\ j \geq i_0 \ \right\}$$

It's easily seen that $G_\infty$ is a subgroup of $\prod_{i\in\Lambda} G_i$ with pointwise multiplication. Consider the set

$$F := \left\{(g_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} G_i : \exists\ i_0 \in \Lambda \ \text{ s.t. } \ g_j = 1_{G_j} \forall\ j \geq i_0 \ \right\}$$

Observe that $F$ is a normal subgroup of $G_\infty$.

**Definition 7.1.** Put $\varinjlim G_i := \varinjlim((G_i)_{i\in\Lambda}, (\varphi_{j,i})_{i\leq j}) := G_\infty/F$. Write $[(g_i)_{i\in\Lambda}]_F$ for the image of $(g_i)_{i\in\Lambda}$ in $\varinjlim G_i$.

For each $j \in \Lambda$ we get a map $\varphi_{\infty,j} : G_j \to G_\infty/N$ defined by

$$\varphi_{\infty,j}(g) := [(g_i)_{i\in\Lambda}]_F \text{ where } g_i := \begin{cases} \varphi_{i,j}(g) & \text{if } i \geq j \\ 1_{G_i} & \text{otherwise} \end{cases}$$

**Lemma 7.2.** *The group $\varinjlim G_i$ is such that*

$$\varinjlim G_i = \bigcup_{i\in\Lambda} \varphi_{\infty,i}(G_i)$$

**Proof.** Take any $x := [(g_i)_{i\in\Lambda}]_F \in \varinjlim G_i$, we have to show that $x \in \varphi_{\infty,i}(G_i)$ for some $i \in \Lambda$. Well, since $(g_i)_{i\in\Lambda} \in G_\infty$, there is $i_0$ such that $\varphi_{j,i_0}(g_{i_0}) = g_j$ for all $j \geq i_0$, whence $x = \varphi_{\infty,i_0}(g_{i_0}) \in \varphi_{\infty,i_0}(G_{i_0})$. ■

Furthermore, it's clear that the group $\varinjlim G_i$ together with the maps $(\varphi_{\infty,i})_i$ make the following diagram commute

$$\begin{tikzcd} & \varinjlim G_i & \\ G_i \arrow[ur, "\varphi_{\infty,i}"] \arrow[rr, "\varphi_{j,i}"'] & & G_j \arrow[ul, "\varphi_{\infty,j}"'] \end{tikzcd}$$

That is, $\varphi_{\infty,j} \circ \varphi_{j,i} = \varphi_{\infty,i}$ whenever $i \leq j$. In fact, $(\varinjlim G_i, \varphi_{i,\infty})_{i\in\Lambda}$ is the categorical universal object making this diagram commute:

**Theorem 7.3.** *Let $(G_i)_{i\in\Lambda}, (\varphi_{j,i})_{i\leq j}$ be a directed system of groups. Suppose $H$ is a group with group homomorphisms $\psi_i : G_i \to H$ such that $\psi_j \circ \varphi_{j,i} = \psi_i$ whenever $i \leq j$. Then, there is a unique group homomorphism $\Psi : \varinjlim G_i \to H$ such that $\psi_i = \Psi \circ \varphi_{\infty,i}$ for all $i \in \Lambda$:*

$$\begin{tikzcd} & H \\ & \varinjlim G_i \arrow[u, dotted, "\Psi"'] \\ G_i \arrow[uur, bend left, "\psi_i"] \arrow[ur, "\varphi_{\infty,i}"'] & \end{tikzcd}$$

**Proof.** For existence, using that $\varinjlim G_i$ is the union of $\varphi_{\infty,i}(G_i)$ by Lemma 7.2, we only need to check that the map $\Psi(\varphi_{\infty,i}(g)) := \psi_i(g)$ is well defined. Suppose that there are $g_i \in G_i$ and $g_j \in G_j$ such that $\varphi_{\infty,i}(g_i) = \varphi_{\infty,j}(g_j)$ in $\varinjlim G_i$. Then, there is $k \in \Lambda$ with $k \geq i$ and $k \geq j$ such that $\varphi_{k,i}(g_i) = \varphi_{k,j}(g_j)$, whence

$$\psi_i(g_i) = (\psi_k \circ \varphi_{k,i})(g_i) = (\psi_k \circ \varphi_{k,j})(g_j) = \psi_j(g_j).$$

For uniqueness assume $\Phi : \varinjlim G_i \to H$ is another such map. Then, for any $x \in \varinjlim G_i$ we know from Lemma 7.2 that there is $i \in \Lambda$ such that $x = \varphi_{\infty,i}(g)$ for some $g \in G_i$, whence

$$\Phi(x) = (\Phi \circ \varphi_{\infty,i})(g) = \psi_i(g) = (\Psi \circ \varphi_{\infty,i})(g) = \Psi(x)$$

This finishes the proof. ■

## 7.2 Direct Limit of $C^*$-algebras

The direct limit construction for $C^*$-algebras is very similar to the one for groups. The advantage is that we will not need to kill things on the analogue of $G_\infty$ because $C^*$-algebras have an additive identity. The disadvantage is that the $C^*$-seminorm we will put on the analogue of $G_\infty$ needs not to be a norm and needs not to be complete after killing the null space of the seminorm. Thus, a completion process will be necessary.

Let $((A_i)_{i\in\Lambda}, (\varphi_{j,i})_{i\leq j})$ be a directed system of $C^*$-algebras. Define

$$A_\infty := \left\{ (a_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} A_i : \exists\ i_0 \in \Lambda \ \text{ s.t. } \ a_j = \varphi_{j,i_0}(a_{i_0}) \ \forall\ j \geq i_0 \ \right\}$$

When equipped with pointwise operations, $A_\infty$ is a $*$-algebra. For each $j \in \Lambda$ we have a map $\varphi^0_{\infty,j} : A_j \to A_\infty$ given by

$$\varphi^0_{\infty,j}(a) := (a_i)_{i\in\Lambda} \text{ where } a_i := \begin{cases} \varphi_{i,j}(a) & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases}$$

We then get an analogue of Lemma 7.2

$$A_\infty = \bigcup_{i\in\Lambda} \varphi^0_{\infty,i}(A_i)$$

This allows us to define a map $\alpha : A_\infty \to \mathbb{R}^+$ as

$$\alpha(\varphi^0_{\infty,i}(a)) := \limsup_{j\geq i} \|\varphi_{j,i}(a)\| = \lim_{j\geq i} \sup_{k\geq j} \|\varphi_{k,i}(a)\|$$

Since $\|\varphi_{k,i}(a)\| \leq \|a\|$, $\alpha(\varphi^0_{\infty,i}(a))$ is a finite number. We still have to check $\alpha$ is well defined. Indeed, if $\varphi^0_{\infty,i}(a_i) = \varphi^0_{\infty,j}(a_j)$ for some $i, j \in \Lambda$, then for any $k \in \Lambda$ with $k \geq i$ and $k \geq j$ we must have $\varphi_{k,i}(a_i) = \varphi_{k,j}(a_j)$. Notice that $\alpha$ is a seminorm. Also, it's easy to check that for any $x, y \in A_\infty$ we have

- $\alpha(xy) \leq \alpha(x)\alpha(y)$
- $\alpha(x^*) = \alpha(x)$
- $\alpha(x^*x) = \alpha(x)^*$

Thus $\alpha$ is infact a $C^*$-seminorm (it's a $C^*$-norm whenerver all the maps $\varphi_{i,j}$ are injective, whence isometric by Theorem 2.2). Thus, if $N := \alpha^{-1}(\{0\})$, we have that $\alpha$ descends to a $C^*$-norm on $A_\infty/N$ by letting

$$\|x + N\|_\alpha := \alpha(x)$$

**Definition 7.4.** We define $\varinjlim A_i := \varinjlim((A_i)_{i\in\Lambda}, (\varphi_{j,i})_{i\leq j})$ to be the completion of $A_\infty/N$ with respect to the $C^*$-norm induced by $\alpha$.

Notice that, by construction, $A_\infty/N$ is a dense subalgebra of $\varinjlim A_i$. Furthermore, for each $j \in \Lambda$ we get maps $\varphi_{\infty,j} : A_j \to \varinjlim A_i$ by letting

$$\varphi_{\infty,j}(a) := \varphi^0_{\infty,j}(a) + N$$

Then, $\bigcup_{i\in\Lambda} \varphi_{\infty,i}(A_i) = A_\infty/N$ and therefore $\varinjlim A_i = \overline{\bigcup_{i\in\Lambda} \varphi_{\infty,i}(A_i)}$. Moreover, a direct check shows that whenever $i \leq j$, then $(\varphi_{\infty,j} \circ \varphi^0_{j,i})(a) - \varphi^0_{\infty,i}(a) \in N$ for $a \in A_i$. Thus, we also get

$$\begin{tikzcd} & \varinjlim A_i & \\ A_i \arrow[ur, "\varphi_{\infty,i}"] \arrow[rr, "\varphi_{j,i}"'] & & A_j \arrow[ul, "\varphi_{\infty,j}"'] \end{tikzcd}$$

As it was the case for groups, the pair $(\varinjlim A_i, \varphi_{\infty,i})_{i\in\Lambda}$ is the universal object satisfying the above commutative diagram.

**Theorem 7.5.** *Let $((A_i)_{i\in\Lambda}, (\varphi_{j,i})_{i\leq j})$ be a directed system of $C^*$-algebras. Suppose $B$ is a $C^*$-algebra with $*$-homomorphisms $\psi_i : A_i \to B$ such that $\psi_j \circ \varphi_{j,i} = \psi_i$ whenever $i \leq j$. Then, there is a unique $*$-homomorphism $\Psi : \varinjlim A_i \to B$ such that $\psi_i = \Psi \circ \varphi_{\infty,i}$ for all $i \in \Lambda$:*

$$\begin{tikzcd} & B \\ & \varinjlim A_i \arrow[u, dotted, "\Psi"'] \\ A_i \arrow[uur, bend left, "\psi_i"] \arrow[ur, "\varphi_{\infty,i}"'] & \end{tikzcd}$$

**Proof.** Define $\Psi^0 : \bigcup_{i\in\Lambda} \varphi_{\infty,i}(A_i) \to B$ by letting

$$\Psi^0(\varphi_{\infty,i}(a)) := \psi_i(a)$$

We have to check $\Psi^0$ is well defined, so assume $\varphi_{\infty,i}(a_i) = \varphi_{\infty,j}(a_j)$ for some $i, j \in \Lambda$. Then, let $\varepsilon > 0$, since $\varphi^0_{\infty,i}(a_i) - \varphi^0_{\infty,j}(a_j) \in N$, there is $k$ such that $k \geq i$ and $k \geq j$ such that

$$\|\varphi_{k,i}(a_i) - \varphi_{k,j}(a_j)\| < \varepsilon.$$

Then,

$$\|\psi_i(a_i) - \psi_j(a_j)\| = \|(\psi_k \circ \varphi_{k,i})(a_i) - (\psi_k \circ \varphi_{k,j})(a_j)\| \leq \|\varphi_{k,i}(a_i) - \varphi_{k,j}(a_j)\| < \varepsilon.$$

Letting $\varepsilon \to 0$ yields $\psi_i(a_i) = \psi_j(a_j)$, so $\Psi^0$ is well defined. Furthermore, for any $j \geq i$

$$\|\Psi^0(\varphi_{\infty,i}(a))\| = \|\psi_i(a)\| = \|\psi_j(\varphi_{j,i}(a))\| \leq \|\varphi_{j,i}(a)\|$$

Hence, $\|\Psi^0(\varphi_{\infty,i}(a))\| \leq \|\varphi_{\infty,i}(a)\|$. Since $\Psi^0$ is linear, multiplicative and it preserves involution, we use density to extend it to a well defined $*$-homomorphism $\Psi : \varinjlim A_i \to B$. This proves existence. For uniqueness, notice that any two such maps will agree on the dense subset $\bigcup_{i\in\Lambda} \varphi_{\infty,i}(A_i)$. ■

**Corollary 7.6.** *Let $(A_i)_{i\in\Lambda}$ be a directed family of $C^*$-subalgebras of $A$ such that $A_i \subset A_j$ whenever $i \leq j$ and such that $\bigcup_{i\in\Lambda} A_i$ is dense in $A$. Then, $A \cong \varinjlim(A_i, \iota_{j,i})$, where $\iota_{j,i} : A_i \hookrightarrow A_j$ is the inclusion map.*

**Theorem 7.7.** *Let $((A_i)_{i\in\Lambda}, (\varphi_{j,i})_{i\leq j})$ be a directed system of simple $C^*$-algebras. Then $\varinjlim A_i$ is simple.*

**Proof.** Consider the set $\mathcal{S} := \{\varphi_{\infty,i}(A_i) : i \in \Lambda\}$. Then $\mathcal{S}$ is an upward directed of simple $C^*$-subalgebras of $\varinjlim A_i$ whose union is dense in $\varinjlim A_i$. Notice that to show that $\varinjlim A_i$ is simple suffices to prove that if $B$ is any $C^*$ algebra, then any surjective $*$-homomorphism $\pi : \varinjlim A_i \to B$ is also injective. For any $S \in \mathcal{S}$, simplicity of $S$ gives that $\pi|_S : S \to B$ is either the zero map or injective and therefore isometric. However, since $\bigcup_{S\in\mathcal{S}} S$ is dense in $\varinjlim A_i$, $\pi$ can't be the zero map when restricted to any non zero $S \in \mathcal{S}$, otherwise the restriction to any $T$ with $S \subset T$ will not be injective. Then, $\pi$ is isometric when restricted to $\bigcup_{S\in\mathcal{S}} S$ and by density $\pi$ is isometric on $A$, whence injective. ■

**Lemma 7.8.** *Let $a \in A_{\mathrm{sa}}$ such that $\|a^2 - a\| < \frac{1}{4}$. Then there is a projection $p \in A$ such that $\|a - p\| < \frac{1}{2}$.*

The previous lemma says that if a selfadjoint element is "almost idempotent", then it's close to a projection. This is an important fact for $K$-theory but also to lift projections in the next important result.

**Theorem 7.9.** *Let $A := \varinjlim((A_i)_{i\in\Lambda}, (\varphi_{j,i})_{i\leq j})$. Let $\varepsilon > 0$ and $x \in A$. Then, there is $i \in \Lambda$ and $a_i \in A_i$ such that*

$$\|x - \varphi_{\infty,i}(a_i)\| < \varepsilon$$

*Moreover, if $x$ is self-adjoint, positive, positive of norm less than one, or a projection, then $a_i$ may be chosen to be of the same kind as $x$. If all the $A_i$'s are unital and the connecting maps $\varphi_{i,j}$ are unital, then $a_i$ may be chosen to be invertible of unitary if $x$ is so.*

## 7.3 UHF and AF Algebras

UHF and AF algebras are a special case of direct limits of $C^*$-algebras. Before defining them, we need a couple of lemmas.

**Lemma 7.10.** *Let $p, q$ be projections in a unital $C^*$-algebra $A$ such that $\|q - p\| < 1$. Then there is a unitary $u \in A$ such that $q = upu^*$ and $\|1 - u\| \leq \sqrt{2}\|q - p\|$.*

The previous lemma implies that "sufficiently close" projections are unitarily equivalent (see Definition K).

**Lemma 7.11.** *If $A$ is a non-zero finite-dimensional $C^*$-algebra, then $A$ is simple if and only if it's of the form $M_n(\mathbb{C})$ for some $n$.*

**Proof.** An algebraic argument shows that $M_n(\mathbb{C})$ is simple. Now suppose $A$ is simple and finite-dimensional. Let $(\mathcal{H}, \varphi)$ be any non-zero irreducible representation of $A$. Since $A$ is finite dimensional, $A$ is liminal and therefore $\varphi(A) = \mathcal{K}(\mathcal{H})$ and $\mathcal{H}$ is finite dimensional. Since $A$ is simple, $\ker(\varphi) = \{0\}$, whence $\varphi : A \to \mathcal{L}(\mathcal{H}) = \mathcal{K}(\mathcal{H})$ is a $*$-isomorphism. ■

**Lemma 7.12.** *$M_n(\mathbb{C})$ has a unique tracial state.*

**Proof.** We already saw (Example 5.4) that $a \mapsto \mathrm{tr}(a)$ is a tracial state. Suppose $\tau : A \to \mathbb{C}$ is another one. Notice that any two rank-one projections $p, q$ in $M_n(\mathbb{C})$ are unitarily equivalent. Indeed, if $p = v_{\xi,\xi}$ and $q = v_{\eta,\eta}$ for unit vectors $\xi, \eta \in \mathbb{C}^n$ (where $v_{\xi,\eta}(\zeta) := \langle \zeta, \xi \rangle \eta)$. Find a unitary $u \in M_n(\mathbb{C})$ such that $u(\xi) = \eta$. Then

$$q(\zeta) = \langle \zeta, \eta \rangle \eta = \langle \zeta, u(\xi) \rangle u(\xi) = u(\langle u^*(\zeta), \xi \rangle \xi) = upu^*(\zeta)$$

That is, $q = upu^*$. Then, $\tau(q) = \tau(p)$ for all rank-one projections; say their common value is $r$. In particular, notice that if $\xi_1, \ldots, \xi_1$ is the canonical orthonormal basis for $\mathbb{C}^n$, then

$$1 = \tau(\mathrm{id}) = \sum_{k=1}^{n} \tau(v_{\xi_k,\xi_k}) = nr$$

Hence, for any rank-one projection $p$ we must have $\tau(p) = \frac{1}{n} = \mathrm{tr}(p)$. Since the rank-one projections span $M_n(\mathbb{C})$, we have $\tau = \mathrm{tr}$. ■

**Remark 7.13.** .

- The same argument as the one given in the previous lemma shows that if $\mathcal{H}$ is infinite dimensional, then $\mathcal{K}(\mathcal{H})$ does not admit a tracial state. Indeed, if $\tau : \mathcal{K}(\mathcal{H}) \to \mathbb{C}$ happened to be a tracial state and $E$ is an orthonotmal basis for $\mathcal{H}$, then for each $n$ take $\xi_1, \ldots, \xi_n \in E$ and get

$$\tau\Big(\sum_{k=1}^{n} v_{\xi_k,\xi_k}\Big) = nr,$$

  where as before, $r$ is the common value of the rank one-projections. But $\tau(\sum_{k=1}^n v_{\xi_k,\xi_k}) \leq 1$, because $\sum_{k=1}^n v_{\xi_k,\xi_k}$ is a projection. This gives, $n \leq \frac{1}{r}$ for all $n$, a contradiction.

- $\mathcal{L}(\mathcal{H})$ does not have a tracial state for infinite dimensional $\mathcal{H}$. To prove this one needs more machinery. $\mathcal{L}(\mathcal{H})$ is a **purely infinite** algebra, which means there are isometries $s_1, s_2 \in \mathcal{L}(\mathcal{H})$ such that $s_1^* s_2 = 0$.

**Definition 7.14.** A **uniformly hyperfinite algebra** or **UHF algebra** is a unital $C^*$-algebra $A$ which has an increasing sequence $(A_n)_{n=1}^{\infty}$ of finite dimensional simple $C^*$-subalgebras each containing the unit of $A$ and whose union is dense in $A$.

**Proposition 7.15.** *If $A$ is a UHF algebra, then it has a unique tracial state.*

**Proof.** By Lemma 7.12, each $A_n$ has a unique tracial state, call it $\tau_n$. Since $1 \in A_n \subset A_{n+1}$, the restriction $\tau_{n+1}|_{A_n}$ is also a tracial state on $A_n$ (it's clearly a trace and has norm$\tau_{n+1}|_{A_n}(1) = \tau_{n+1}(1) = 1$). Thus, by uniqueness of the trace on $A_n$ we must have $\tau_{n+1}|_{A_n} = \tau_n$. This allows us to define $\tau : \bigcup_{n=1}^n A_n \to \mathbb{C}$ by letting $\tau(a) := \tau_n(a)$ for $a \in A_n$. Extending $\tau$ by density to all of $A$ gives a tracial state on $A$. Uniqueness follows from uniqueness on each $A_n$. ■

For any $m, n \in \mathbb{Z}_{>0}$ we have a map $\iota_{m,n} : M_n(\mathbb{C}) \to M_{mn}(\mathbb{C})$ sending $a$ to $\iota_{m,n}(a) = \mathrm{diag}(a, \ldots, a)$ (that is the matrix who has $m$ blocks of $a$ down the mail diagonal and zeros elsewhere). Denote by $\mathbb{S}$ to the set of all functions $s : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$. A Cantor diagonal argument shows that $\mathbb{S}$ is uncountable. For each $s \in \mathbb{S}$ define $s! \in \mathbb{S}$ by $s!(n) := s(1) \ldots s(n)$. For each $s \in \mathbb{S}$ and $n \leq m$, we define $\varphi_{m,n} : M_{s!(n)}(\mathbb{C}) \to M_{s!(m)}(\mathbb{C})$ by $\varphi_{m,n} := \iota_{s(n+1)\cdots s(m), s!(n)}$. Then, put

$$M_s := \varinjlim (M_{s!(n)}(\mathbb{C}), (\varphi_{m,n})_{n \leq m})$$

Since each $M_{s!(n)}(\mathbb{C})$ is simple, $M_s$ is a UHF algebra and therefore it has a unique tracial state. Let $\mathbb{P}$ be the set of prime numbers. Define $E_s : \mathbb{P} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ by

$$E_s(r) := \sup\{m \in \mathbb{Z}_{\geq 0} : r^m \mid s!(n) \text{ for some } n \in \mathbb{Z}_{>0}\}$$

**Theorem 7.16.** *Let $s, s' \in \mathbb{S}$ and suppose that $M_s$ and $M_{s'}$ are $*$-isomorphic. Then $E_s = E_{s'}$*

**Proof.** By symmetry suffices to show that $E_s \leq E_{s'}$, so it's enough to show that for each $n \in \mathbb{Z}_{>0}$, there is $m \in \mathbb{Z}_{>0}$ such that $s!(n) \mid s!(m)$. Well, let $\pi : M_s \to M_{s'}$ be a $*$-isomorphism, let $\tau$ and $\tau'$ be their unique tracial sates. Also let $\varphi_{\infty,n} : M_{s!(n)}(\mathbb{C}) \to M_s$ and $\psi_{\infty,n} : M_{s'!(n)}(\mathbb{C}) \to M_{s'}$ be the direct limit maps. Since $\tau' \circ \pi$ is a tracial state on $M_s$, we have $\tau = \tau' \circ \pi$. Now, let $p$ be a rank-one projection on $M_{s!(n)}(\mathbb{C})$ and since $\tau \circ \varphi_{\infty,n}$ ought to be the only tracial state of $M_{s!(n)}(\mathbb{C})$, we must have $\tau(\varphi_{\infty,n}(p)) = \frac{1}{s!(n)}$. Since $\pi(\varphi_{\infty,n}(p))$ is a projection in $M_{s'}$, by Theorem 7.9 there is $m \in \mathbb{Z}_{>0}$ and a projection $q \in M_{s'!(m)}(\mathbb{C})$ such that

$$\|\pi(\varphi_{\infty,n}(p)) - \psi_{\infty,m}(q)\| < 1$$

So, by Lemma 7.10, $\pi(\varphi_{\infty,n}(p))$ and $\psi_{\infty,m}(q)$ are unitarily equivalent projections in $M_{s!(n)}(\mathbb{C})$. Hence, $\tau'(\psi_{\infty,m}(q)) = \tau'(\pi(\varphi_{\infty,n}(p))) = \tau(\varphi_{\infty,n}(p)) = \frac{1}{s!(n)}$. But, since $\tau' \circ \psi_{\infty,m}$ is the unique tracial state on $M_{s'!(m)}(S)$, we must have $\tau'(\psi_{\infty,m}(q)) = \frac{d}{s'!(m)}$ for some $d \in \mathbb{Z}_{>0}$. That is $s'!(m) = ds!(n)$, as we needed to prove. ■

**Corollary 7.17.** *There uncountably many non isomorphic UHF algebras.*

**Proof.** Enumerate the prime numbers $\mathcal{P} = \{r_1, r_2, \ldots\}$. For each $s \in \mathbb{S}$ define $\overline{s}(n) := r_n^{s(n)}$. Then $\overline{s} \in \mathbb{S}$ and $E_{\overline{s}}(r_n) = s(n)$. Therefore $s = s'$ if and only if $E_{\overline{s}} = E_{\overline{s'}}$. Thus, the previous Theorem implies that $(M_{\overline{s}})_{s\in\mathbb{S}}$ is a family of non isomorphic UHF algebras. The desired result follows because $\mathbb{S}$ is uncountable. ■

We finish this section by presenting AF algebras.

**Definition 7.18.** An **approximately finite algebra** or **AF algebra** is $C^*$-algebra $A$ which has an increasing sequence $(A_n)_{n=1}^{\infty}$ of finite dimensional $C^*$-subalgebras whose union is dense in $A$.

Of course $A$ is an AF algebra if and only if it's isomorphic to the direct limit of a sequence of finite dimensional $C^*$-algebras. Any UHF algebra is an AF algebra. But there are AF algebras that are not UHF algebras. This is the case of $\mathcal{K}(\mathcal{H})$ when $\mathcal{H}$ is an infinite dimensional. Indeed, $\mathcal{K}(\mathcal{H})$ is not a UHF algebra because we already saw it doesn't admit a tracial state. To see that $\mathcal{K}(\mathcal{H})$ is an AF algebra, take $\{\xi_1, \xi_2, \ldots\}$ an orthonormal basis for $\mathcal{H}$ and $p_n$ the projection onto $\operatorname{span}(\xi, \ldots, \xi_n)$. We already saw that $(p_n)_{n=1}^{\infty}$ is an approximate unit for $\mathcal{K}(\mathcal{H})$, so if $A_n := p_n\mathcal{K}(\mathcal{H})p_n$ it follows that $(A_n)_{n=1}^{\infty}$ is an incresing sequence of $C^*$-subalgebras and that $\bigcup_{n=1}^{\infty} A_n$ is dense in $\mathcal{K}(\mathcal{H})$. We only need to show that $A_n$ is finite dimensional. Well, notice that $A_n$ is spanned by the rank one operators $(v_{\xi_j,\xi_k})_{j,k=1}^{n}$:

$$p_n u p_n = \sum_{k,j=1}^{n} v_{\xi_k,\xi_k} u v_{\xi_j,\xi_j} = \sum_{k,j=1}^{n} \langle u(\xi_j), \xi_k\rangle v_{\xi_j,\xi_k}$$

so each $A_n$ has dimension at most $n^2$.

**Theorem 7.19.** *If $I$ is a closed ideal in an AF-algebra $A$, then $I$ and $A/I$ are AF algebras.*

# 8 Tensor Products of $C^*$-alegebras.

We briefly recall the tensor product of Hilbert spaces. Let $Hi_1, \mathcal{H}_2$ be two Hilbert spaces. Denote by $\mathcal{H}_1 \odot \mathcal{H}_2$ to the algebraic tensor product. Then, there is a unique inner product on $\mathcal{H}_1 \odot \mathcal{H}_2$ such that

$$\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2\rangle = \langle \xi_1, \eta_1\rangle\langle \xi_2, \eta_2\rangle$$

The Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is the completion of $\mathcal{H}_1 \odot \mathcal{H}_2$ under the above inner product.

**Lemma 8.1.** *Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{G}_1, \mathcal{G}_2$ be Hilbert Spaces, $a \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $b \in \mathcal{L}(\mathcal{G}_1, \mathcal{G}_2)$. Then, there exists a unique element of $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{G}_1, \mathcal{H}_2 \otimes \mathcal{G}_2)$ which extends $a \otimes b : \mathcal{H}_1 \odot \mathcal{G}_1 \to \mathcal{H}_2 \odot \mathcal{G}_2$ (recall that $a \otimes b$ is the unique linear map for which $(a \otimes b)(\xi_1 \otimes \eta_1) = a(\xi_1) \otimes b(\eta_1)$.) This map is also called $a \otimes b$ and we have $\|a \otimes b\| = \|a\|\|b\|$.*

With suitable domains, $(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1a_2) \otimes (b_1b_2)$ and $(a \otimes b)^* = a^* \otimes b^*$. To prove the second assertion, we need to use dense subspaces and elementary tensors.

Let now $A$ and $B$ be $C^*$-algebras. Their algebraic tensor product $A \odot B$ is easily seen to be an algebra with the unique multiplication given by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1a_2) \otimes (b_1b_2)$$

for $a_k \in A$, $b_k \in B$. Further, we can make $A \odot B$ into a $*$-algebra with the well defined involution

$$(a \otimes b)^* = a^* \otimes b^*$$

Further, if $A'$ and $B'$ are also $C^*$-algebras and $\varphi : A \to A'$, $\psi : B \to B'$ are $*$-homomorphisms, there is a unique $*$-algebra homomorphism $\varphi \otimes \psi : A \odot A' \to B \odot B'$ such that

$$(\varphi \otimes \psi)(a \otimes b) = \varphi(a) \otimes \psi(b)$$

**Theorem 8.2.** *Suppose $(\mathcal{H}_1, \varphi)$ and $(\mathcal{H}_2, \psi)$ are representations of $A$ and $B$ respectively. Then, there is a unique $*$-homomorphism $\pi : A \odot B \to \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ such that*

$$\pi(a \otimes b) = \varphi(a) \otimes \psi(b),$$

*where $\varphi(a) \otimes \psi(b) \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is the map from Lemma 8.1. Moreover, if both $\varphi$ and $\psi$ are injective, then so is $\pi$. We usually denote $\pi = \varphi \otimes \psi$.*

## 8.1 Spatial Norm

**Definition 8.3.** Let $A$ and $B$ be $C^*$-algebras with universal representations given by $(\mathcal{H}_1, \varphi)$ and $(\mathcal{H}_2, \psi)$ respectively. Then, we use the map $\pi = \varphi \otimes \psi$ from the previous theorem to define the **spatial norm** on $A \odot B$ by letting

$$\|c\|_* := \|\pi(c)\|$$

for any $c \in A \odot B$. Clearly $\|\cdot\|_*$ is a $C^*$-norm on $A \odot B$, the completion of $A \odot B$ with respect to this norm is called the **spatial tensor product of** $A$ **and** $B$ and we denote it by $A \otimes_* B$.

**Remark 8.4.** Let $A$ and $B$ be $C^*$-algebras with universal representations given by $(\mathcal{H}_1, \varphi)$ and $(\mathcal{H}_2, \psi)$ respectively.

- $\|a \otimes b\|_* = \|(\varphi \otimes \psi)(a \otimes b)\| = \|\varphi(a) \otimes \psi(b)\| = \|\varphi(a)\|\|\psi(b)\| = \|a\|\|b\|$.
- There might be more than one $C^*$-norm on $A \odot B$. If $\gamma$ is any $C^*$-norm on $A \otimes B$, we denote its $C^*$-completion with respect to $\gamma$ by $A \otimes_\gamma B$.
- Recall that the universal representation comes from states, one checks that

  $$\|c\|_* = \sup_{\tau \in S(A), \rho \in S(B)} \|(\varphi_\tau \otimes \varphi_\rho)(c)\|$$

  for any $c \in A \odot B$.

**Theorem 8.5.** *Let $A$ and $B$ be non-zero $C^*$-algebras and suppose that $\gamma$ is a $C^*$-norm on $A \odot B$. Let $(\mathcal{H}_\pi)$ be a non-degenerate representation of $A \otimes_\gamma B$. Then, there exist unique $*$-homomorphisms $\pi_A : A \to \mathcal{L}(\mathcal{H})$ and $\pi_B : B \to \mathcal{L}(\mathcal{H})$ such that*

$$\pi(a \otimes b) = \pi_A(a)\pi_B(b) = \pi_B(b)\pi_A(a)$$

*for all $a \in A, b \in B$. Moreover, the representations $(\pi_A, \mathcal{H})$ and $(\pi_B, \mathcal{H})$ are non-degenerate.*

**Sketch of Proof.** Let $(u_\alpha)_{\alpha \in I}$ and $(v_\beta)_{\beta \in J}$ be approximate identities for $A$ and $B$ respectively and define

$$\pi_A(a)\xi := \lim_\beta \pi(a \otimes v_\beta)\xi \qquad \pi_B(b)\xi := \lim_\alpha \pi(u_\alpha \otimes b)\xi$$

for $\xi \in (A \odot B)\mathcal{H}$. One checks both $\pi_A$ and $\pi_B$ are well defined. Since $\pi$ is non-degenerate, we extend to all $\mathcal{H}$ and get $*$-homomorphisms $\pi_A : A \to \mathcal{L}(\mathcal{H})$ and $\pi_B : B \to \mathcal{L}(\mathcal{H})$. Non-degeneracy of $\pi$ implies that both $\pi_A$ and $\pi_B$ are non degenerate. The commuting assertion follows because both $\pi(a \otimes v_\beta)\pi(u_\alpha \otimes b)$ and $\pi(u_\alpha \otimes b)\pi(a \otimes v_\beta)$ are strongly convergent to $\pi(a \otimes b)$. Finally, uniqueness follows because $\pi(a \otimes v_\beta)$ is strongly convergent to any other such $\pi_A$ and $\pi(u_\alpha \otimes b)$ to any other such $\pi_B$. "□"

**Corollary 8.6.** *Let $A$ and $B$ be non-zero $C^*$-algebras and suppose that $\gamma$ is a $C^*$-**seminorm** on $A \odot B$. Then $\gamma(a \otimes b) \leq \|a\|\|b\|$.*

**Proof.** Consider the $C^*$-norm $\delta := \max\{\gamma, \|\cdot\|_*\}$. Then, let $(\pi, \mathcal{H})$ be the universal representation of the $C^*$-algebra $A \otimes_\delta B$, which is injective by Theorem 5.15 and non-degenerate because its cyclic. We get non-degenerate representations $\pi_A$ and $\pi_B$ from the previous theorem. Finally,

$$\gamma(a \otimes b) \leq \delta(a \otimes b) = \|\pi(a \otimes b)\| = \|\pi_A(a) \otimes \pi_B(b)\| = \|\pi_A(a)\|\|\pi_B(b)\| \leq \|a\|\|b\|$$

as wanted. ■

## 8.2 Maximal Norm

Let $A$ and $B$ be $C^*$-algebras and denote by $\Gamma$ the set of all $C^*$-norms $\gamma$ on $A \odot B$. We define

$$\|c\|_{\max} := \sup_{\gamma \in \Gamma} \gamma(c)$$

for each $c \in A \odot B$. As a consequence of the previous corollary, $\|c\|_{\max} < \infty$ and therefore we get a $C^*$-norm on $A \odot B$. We call $A \otimes_{\mathrm{m}} B$ the **maximal tensor product** of $A$ and $B$. The maximal tensor product has a useful universal property

**Theorem 8.7.** *Let $A, B$ and $C$ be $C*$-algebras. Suppose $\varphi : A \to C$ and $\psi : B \to C$ are $*$-homomorphisms such that $\varphi(a)\psi(b) = \psi(b)\varphi(a)$ for all $a \in A, b \in B$. Then, there is a unique $*$-homomorphism $\pi : A \otimes_{\mathrm{m}} B \to C$ such that $\pi(a \otimes b) = \varphi(a)\psi(b)$.*

### 8.3 Nuclear $C^*$-Algebras

**Definition 8.8.** A $C^*$-algebra $A$ is said to be **nuclear** if for any $C^*$-algebra $B$, there is only one $C^*$-norm on $A \odot B$.

**Lemma 8.9.** *If a $*$-algebra admits a complete $C^*$-norm, then it is the only $C^*$-norm on $A$.*

**Proof.** Let $\|\cdot\|$ be a complete $C^*$-norm on $A$. Assume $\gamma$ is a (potentially not compete) $C^*$-norm on $A$. Let $A_\gamma$ be the completion of $A$ with respect to $\gamma$. The inclusion $\varphi : A \to A_\gamma$ is clearly an inyective $*$-homomorphism and therefore isometric by Theorem 2.2. Hence, $\gamma(a) = \|a\|$ for all $a \in A$. ■

**Example 8.10.** .

1. The $C^*$-algebra $M_n(\mathbb{C})$ is nuclear. Indeed, if $A$ is any other $C^*$-algebra, we have $M_n(\mathbb{C}) \otimes A \cong M_n(A)$ via the map $e_{j,k} \otimes a \mapsto (\delta_{j,k} a)_{l,m}$. Since $M_n(A)$ admits a complete $C^*$-norm (represent $A$ on $\mathcal{H}$ and see $M_n(A)$ in $\mathcal{L}(\mathcal{H}^n)$), nuclearity follows from the previous lemma.
2. In fact any finite-dimensional algebra is nuclear. A finite dimensional algebra looks like $A = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$. Let $B$ be any $C^*$-algebra. Then $A \otimes B \cong M_{n_1}(B) \oplus \cdots \oplus M_{n_k}(B)$ admits only one $C^*$-norm.
3. Direct limit of nuclear $C^*$-algebras is nuclear. Indeed, if $B$ is any $C^*$-algebra then $\bigcup_{i \in \Lambda} (\varphi_{\infty,i}(A_i) \odot B)$ is dense in $(\varinjlim A_i) \otimes_\gamma B$ for any $C^*$-norm $\gamma$ on $(\varinjlim A_i) \odot B$. Nuclearity will follow because the restriction of $\gamma$ to $\varphi_{\infty,i}(A_i) \odot B$ is unique as each $\varphi_{\infty,i}(A_i)$ is nuclear.
4. If $\mathcal{H}$ is an infinite dimensional Hilbert space, then $\mathcal{K}(\mathcal{H})$ is nuclear. This follows from 1 and 3 above.
5. Any commutative $C^*$-algebra is nuclear. This needs a lot more of work. In particular one gets this in the road to show that the spacial norm is the least $C^*$-norm on the tensor product of two $C^*$-algebras.
6. Suppose that $0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$ with $I$ and $B$ nuclear. Then $A$ is nuclear.

# 9 Projections and $K_0$

**Definition 9.1.** Two projections $p, q \in A$ are **orthogonal** when $pq = 0$. In such case the sum $p+q$ is also a projection which we will denote by $p \oplus q$.

**Remark 9.2.** Let $p \in A$ be a projection and consider $1 - p \in \widetilde{A}$. Then, $1-p$ is a projection which is orthogonal to $p$:

$$p(1-p) = (p,0)(-p,1) = (-p+p,0) = 0$$

More generally, if $p, q$ are projections and $p \leq q$, then $q - p$ is a projection orthogonal to $p$. Indeed, if we represent $A$ on $\mathcal{L}(\mathcal{H})$, $p \leq q$ implies that $p(\mathcal{H}) \subset q(\mathcal{H})$ and from this it follows that $qp = pq = p$, whence $(q-p)^2 = q - p$ and $p(q-p) = 0$.

**Lemma 9.3.** *Let $v$ be a partial isometry (that is $v^*v$ is a projection). We call the projection $p := v^*v$ the **support projection**. Then, $q := vv^*$ is also projection, called **range projection**. Moreover,*

$$v = vv^*v = vp = qv = qvp \quad \textit{and} \quad v^* = v^*vv^* = v^*q = pv^* = pv^*q$$

**Proof.** First we show that $v = vv^*v$. Indeed, let $z := v - vv^*v$. Then,

$$z^*z = (v - vv^*v)^*(v - vv^*v) = (v^* - v^*vv^*)(v - vv^*v) = p - p^2 - p^2 + p^3 = p - p - p + p = 0$$

Thus $\|z\|^2 = \|z^*z\| = 0$, whence $z = 0$. That is, $v = vv^*v$ and the first chain of equialities follows. This also proves that $q$ is a projection:

$$q^2 = vv^*vv^* = (vv^*v)v^* = vv^* = q$$

The second chain of equalities comes from taking involution on the first chain. ■

**Remark 9.4.** The previous lemma, although an easy result, is extremely important and used all the time. A particular consequence is that when $v$ is a partial isometty on $\mathcal{L}(\mathcal{H})$, then $v^*(\mathcal{H})$ and $v(\mathcal{H})$ are isometrically isomorphic via $v$.

**Definition 9.5.** Projections $p, q$ in $A$ are said to be

- **Murray-von Neumann equivalent** (or simply **equivalent**), denoted $p \sim q$, if there is a partial isometry $v \in A$ such that $p = v^*v$ and $q = vv^*$.
- **unitarily equivalent**, denoted $p \sim_u q$, if there is a unitary $u \in \widetilde{A}$ such that $p = u^*qu$.

- **homotopic**, denoted $p \sim_h q$, if $p$ and $q$ are connected by a norm continuous path of projections.

**Lemma 9.6.** *If $p_1, p_2, q_1$ and $q_2$ are projections in $A$ such that $p_1 \sim q_1$, $p_2 \sim q_2$, $p_1 \perp p_2$ and $q_1 \perp q_2$, then $p_1 \oplus p_2 \sim q_1 \oplus q_2$.*

**Proof.** We have partial isometries $v_1$ and $v_2$ with $p_1 = v_1^* v_1$, $q_1 = v_1 v_1^*$, $p_2 = v_2^* v_2$ and $q_2 = v_2 v_2^*$. Orthogonality and Lemma 9.3 imply that $v_j^* v_k = 0 = v_j v_k^*$ for $j \neq k$. Thus, if $v = v_1 + v_2$, it follows that $p_1 \oplus p_2 = v^* v \sim vv^* = q_1 \oplus q_2$. ∎

**Proposition 9.7.** *Let $p$ and $q$ be projections in $A$. Then*

1. $p \sim_h q \Longrightarrow p \sim_u q \Longrightarrow p \sim q$.
2. $p \sim q \Longrightarrow \mathrm{diag}(p, 0) \sim_u \mathrm{diag}(q, 0)$ *in* $M_2(A)$.
3. $p \sim_u q \Longrightarrow \mathrm{diag}(p, 0) \sim_h \mathrm{diag}(q, 0)$ *in* $M_2(A)$.

**Proof.**

1. That $p \sim_h q \Longrightarrow p \sim_u q$ requires some technical work and we omit it. If $p \sim_u q$, there is a unitary $u \in \widetilde{A}$ such that $p = u^* q u$. Let $v = qu$ and notice that $v^* v = u^* q u = v$ and $vv^* = q$, whence $p \sim q$.

2. Suppose $p \sim q$. We have a partial isometry $v$ with $p = v^* v$ and $q = vv^*$. Let $u = \begin{pmatrix} v^* & 1-p \\ 1-q & v \end{pmatrix} \in M_2(\widetilde{A})$. Then, using Lemma 9.3
$$u^* u = \begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix} \begin{pmatrix} v^* & 1-p \\ 1-q & v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
Similarly $uu^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so $u$ is a unitary in $M_2(\widetilde{A})$. We compute
$$u^* \mathrm{diag}(p, 0) u = \begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v^* & 1-p \\ 1-q & v \end{pmatrix} = \begin{pmatrix} vp & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v^* & 1-p \\ 1-q & v \end{pmatrix} \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix},$$
so indeed $\mathrm{diag}(p, 0) \sim_u \mathrm{diag}(q, 0)$ in $M_2(A)$.

3. Now assume that $p \sim_u q$. We have a unitary $u \in \widetilde{A}$ such that $q = upu^*$. We omit the proof that $w_0 := \mathrm{diag}(u, u^*) \sim_h 1_{M_2} =: w_1$ via a continuous path of unitaries $(w_t)_{t \in [0,1]}$ in $M_2(\widetilde{A})$. Then, each $p_t := w_t \mathrm{diag}(p, 0) w_t^*$ is a projection in $M_2(A)$ with $p_0 = \mathrm{diag}(q, 0)$ and $p_1 = \mathrm{diag}(p, 0)$. ∎

## 9.1 The monoid $V(A)$

**Definition 9.8.** Define $M_\infty(A) := \bigcup_{n=1}^\infty M_n(A)$. Two projections $p, q$ in $M_\infty(A)$ are **equivalent**, dented $p \sim q$, if there is $v \in M_\infty(A)$ with $p = v^* v$ and $q = vv^*$. The equivalence class of a projection $p \in M_\infty(A)$ is denoted by $[p]$. We define
$$V(A) := \{[p] : p^2 = p^* = p \in M_\infty(A)\}$$
Addition in $V(A)$ is defined by
$$[p] + [q] = [\mathrm{diag}(p, q)] = [p' \oplus q']$$
where $p' \sim p$, $q' \sim q$ and $p' \perp q'$.

**Remark 9.9.** Addition in $V(A)$ is well defined and by Proposition 9.7, all notions of equivalence of projections agree on $M_\infty(A)$.

**Proposition 9.10.** *$V(A)$ is an Abelian semigroup with additive identity $[0]$. If $\varphi : A \to B$ is a $*$-homomoprhism, then the induced map $\varphi_* : V(A) \to V(B)$ given by*
$$\varphi_*([(a_{i,j})]) = [(\varphi(a_{i,j}))]$$
*is a well defined homomorphism of semigroups. The correspondence $A \mapsto V(A)$ together with $\varphi \mapsto \varphi_*$ is a covariant functor from the category of $C^*$-algebras to the one of abelian semigroups.*

**Proof.** That $V(A)$ is an Abelian semigroup with additive identity $[0]$ is easy. Let $p = (a_{i,j})$ be a projection in $M_\infty(A)$. Then, $\varphi(p) := (\varphi(a_{i,j})) \in M_\infty(B)$ is a projection because $\varphi(p)^2 = \varphi(p^2) = \varphi(p) = \varphi(p^*) = \varphi(p)^*$. Now suppose $p \sim p'$ in $M_\infty(A)$, so there is $v \in M_\infty(A)$ so that $p = v * v$ and $p' = vv^*$. Then $\varphi(v)$ implements the equivalence between $\varphi(p)$ and $\varphi(p')$. This gives that $\varphi_*$ is well defined. Since
$$\varphi_*([p] + [q]) = \varphi_*(\mathrm{diag}(p, q)) = \mathrm{diag}(\varphi(p), \varphi(q)) = [\mathrm{diag}(\varphi(p), 0)] + [\mathrm{diag}(0, \varphi(p))] = \varphi_*([p]) + \varphi_*([q]),$$
it follows that $\varphi_*$ is a homomorphism. Finally, if we consider $\mathrm{id_A} : A \to A$, it's clear that $(\mathrm{id}_A)_* = \mathrm{id}_{V(A)}$ and if $\psi : B \to C$, clearly $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$. ∎

**Example 9.11.** Let $\mathcal{H}$ be separable infinite dimensional Hilbert space. We compute $V(\mathbb{C})$, $V(M_n(\mathbb{C}))$, $V(\mathcal{K}(\mathcal{H}))$, $V(\mathcal{L}(\mathcal{H}))$ and $V(\mathcal{Q}(\mathcal{H}))$, where $\mathcal{Q}(\mathcal{H}) := \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the Calkin algebra.

1. To compute $V(\mathbb{C})$ we need to look at projections in $M_\infty(\mathbb{C})$. Let $p \sim q$ be equivalent projections. Choose $n \in \mathbb{Z}_{>0}$ and $p', q' \in M_n(\mathbb{C})$ so that $p' \sim p$ and $q' \sim q$. There is $v \in M_n(\mathbb{C})$ such that $p' = v^*v$ and $q' = v^*v$. Then, we know from Remark 9.4 that $v^*(\mathbb{C}^n)$ and $v(\mathbb{C}^n)$ are isometrically isomorphic subspaces of $\mathbb{C}^n$. Further, notice that $p'(\mathbb{C}^n) = v^*(\mathbb{C}^n)$: Indeed $p'\xi = v^*v\xi \in v^*(\mathbb{C}^n)$ and $v^*\eta = (vp')^*\eta = p'v^*\eta \in p'(\mathbb{C}^n)$. Similarly, $q'(\mathbb{C}^n) = v(\mathbb{C}^n)$ and therefore $\operatorname{rank}(p') = \operatorname{rank}(q')$. We have just shown that equivalent projections in $M_\infty(\mathbb{C})$ have equal rank. The converse is true. Take any two projections in $M_\infty(\mathbb{C})$ with equal rank, say $k \in Z_{\geq 0}$. Since $p \sim \operatorname{diag}(p, 0)$ and the rank is not affected, with no loss of generality we may assume $p, q \in M_n(\mathbb{C})$ for some $n \in \mathbb{Z}_{>0}$. Let $\xi_1, \dots, \xi_k$ be an orthonormal basis for $p(\mathbb{C}^n)$ and $\eta_1, \dots, \eta_k$ an orthonormal basis for $q(\mathbb{C}^n)$. Define $v : \mathbb{C}^n \to \mathbb{C}^n$ by letting $v(\xi_j) := \eta_j$ on $p(\mathbb{C}^n)$ and $v = 0$ on $(1-p)(\mathbb{C}^n)$. It's clear that $v$ is a linear map, so $v \in M_n(\mathbb{C})$ and that $v^* : \mathbb{C}^n \to \mathbb{C}^n$ is such that $v^*(\eta_j) = \xi_j$. Now for any $\xi \in \mathbb{C}^n$ we have
$$p\xi = \sum_{j=1}^k a_j \xi_j = \sum_{j=1}^k a_j v^*(\eta_j) = \sum_{j=1}^k a_j v^*v(\xi_j) = v^*v(p\xi)$$
This proves $p = v^*v$ and similarly we get $q = vv^*$, whence $p \sim q$. Putting all together we conclude that
$$V(\mathbb{C}) \cong \mathbb{Z}_{\geq 0}$$
via $[p] \mapsto \operatorname{rank}(p)$.

2. Since $M_m(M_n(\mathbb{C})) \cong M_{mn}(\mathbb{C})$, it follows as in the case of $\mathbb{C}$ that $V(M_n(\mathbb{C})) \cong \mathbb{Z}_{\geq 0}$.

3. For $\mathcal{K}(\mathcal{H})$, notice that $M_n(\mathcal{K}(\mathcal{H})) = \mathcal{K}(\mathcal{H}^n) \cong \mathcal{K}(\mathcal{H})$. We claim that a projection in $\mathcal{L}(\mathcal{H})$ is in $\mathcal{K}(\mathcal{H})$ if and only if it has finite rank. The *if* part is clear. For the *only if*, assume $p \in \mathcal{K}(\mathcal{H})$ is a projection. Since $p$ is idempotent, we have $p(\mathcal{H}) = \ker(1-p)$, but since $p$ is compact, it follows that $1-p$ is Fredholm and therefore has finite dimensional kernel. The claim is proved. Thus, we also have that two preojections in $M_\infty(\mathcal{K}(\mathcal{H}))$ are equivalent if and only if they have the same rank, and only finite ranks are possible. This gives $V(\mathcal{K}(\mathcal{H})) = \mathbb{Z}_{\geq 0}$.

4. For $\mathcal{L}(\mathcal{H})$ we also have $M_n(\mathcal{L}(\mathcal{H})) = \mathcal{L}(\mathcal{H}^n) \cong \mathcal{L}(\mathcal{H})$. Using orthonormal basis, the argument used for finite dimensions shows that projections in $M_\infty(\mathcal{L}(\mathcal{H}))$ are equivalent if and only if they have equal rank. However, we noe have projections with infinite rank, like the identity. In fact any infinite rank projection is equivalent to the identity. Thus, $V(\mathcal{L}(\mathcal{H})) = \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

5. For $\mathcal{Q}(\mathcal{H})$, we have again $M_n(\mathcal{Q}(\mathcal{H})) \cong \mathcal{Q}(\mathcal{H})$. Thus, it suffices to look at equivalence classes of projections in $\mathcal{Q}(\mathcal{H})$. Clearly any two finite rank projections in $\mathcal{L}(\mathcal{H})$, descend to 0 in $\mathcal{Q}(\mathcal{H})$. Turns out that any non-zero projection in $\mathcal{Q}(\mathcal{H})$ comes from an infinite rank projection in $\mathcal{L}(\mathcal{H})$ (sketch: if $u + \mathcal{K}$ is a non-zero projection, $u$ can be chosen to be self adjoint and therefore $u - u^2$ is compact, now use the spectral theorem decomposition to "perturb" $u$ and get the desired projection). Finally, any two non-zero projections in $\mathcal{Q}(\mathcal{H})$ are equivalent, as they come from equivalent projections in $\mathcal{L}(\mathcal{H})$. This proves that there are only two classes of projections in the Calkin algebra, the zero projection and the non-zero ones, that is $V(\mathcal{Q}(\mathcal{H})) \cong \{0, \infty\}$.

## 9.2 The group $K_0(A)$

Suppose $(V, +)$ is a commutative semigroup with identity $0_+$. For $(a,b), (c,d) \in V \times V$ we identify $(a,b) \sim (c,d)$ if there is $e \in V$ such that $a + d + e = c + b + e$. This is an equivalence relation and we write $[(a,b)] := a - b$ and denote the set of equivalence classes by $G(V) := \{a - b : a, b \in V\}$. We endow $G(V)$ with an binary operation
$$(a-b) + (c-d) := (a+c) - (b-d)$$
It's easily seen that this operation is well defined and makes $G(V)$ into an Abelian group with identity $0 := 0_V - 0_V = a - a$ and inverse $-(a-b) = b - a$. This is called the Groethendieck group (this construction works even when only have a commutative semigroup). We get a map $\iota_V := V \mapsto G(V)$ given by $\iota_V(a) = a - 0_V$ (actually this could be defined using any $b \in V$ in place of $0_V$: $\iota_V(a) = (a+b) - b$). The pair $(G, \iota_V)$ is universal in the sense that any additive map from $V$ to another Abelian group $H$ factors through $\iota_V$. We make this more precise in the following theorem:

**Theorem 9.12.** *Let $(V, +)$ is a commutative semigroup with identity $0_V$, $G(V)$ its Grothendieck semigroup and $\iota_V : V \to G(V)$ the canonical map. Then*

1. *(Universal Property) If $(H, +)$ is an Abelian group and $\varphi : V \to H$ an additive map, then there is a unique group homomorphism $\psi : G(V) \to H$ such that $\psi \circ \varphi = \iota_V$*

$$\begin{tikzcd} V \arrow[rr, "\varphi"] \arrow[dr, "\iota_V"'] & & H \\ & G(V) \arrow[ur, dotted, "\psi"'] & \end{tikzcd}$$

2. *(Functoriality) If $V, W$ are semigroups and $\varphi; V \to W$ an additive map, then there is a unique group homomorphism $G(\varphi) : G(V) \to G(W)$ such that*

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\iota_V"'] & W \arrow[d, "\iota_W"'] \\ G(V) \arrow[r, dotted, "G(\varphi)"'] & G(W) \end{tikzcd}$$

*commutes.*

3. $G(V) := \{\iota_V(a) - \iota_V(b) : a, b \in V\}$.

4. *$a, b \in V$, then $\iota_V(a) = \iota_V(b)$ if and only if there is $e \in V$ such that $a + e = b + e$.*

5. *$\iota_V$ is injective if anf only if $V$ has the cancellation property.*

**Definition 9.13.** For any $C^*$-algebra $A$ we put $K_{00}(A) := G(V(A))$. We have the canonical map $\iota_A := \iota_{V(A)} : V(A) \to K_{00}(A)$ given by $\iota_A([p]) = [p] - [0]$.

**Example 9.14.** We already know that $V(\mathbb{C}) = \mathbb{Z}_{\geq 0}$. It's now a standard exercise to verify that $K_{00}(\mathbb{C}) = \mathbb{Z}$.

**Example 9.15.** $K_{00}$ is not that interesting when $A$ is not unital. Recall that the projections of $C_0(X)$ are the indicator functions of subsets of $X$ which are both compact and open. Then, $V(C_0(\mathbb{R}^2)) = \{0\}$ and therefore $K_{00}(C_0(\mathbb{R}^2)) = \{0\}$. However, if we adjoint a unit, we get $C(S^2)$, adding only one new projections, the identtity function, never the less we get more non-trivial projections in $M_2(C(S^2))$. One actually gets $K_{00}(C(S^2)) = \mathbb{Z} \oplus \mathbb{Z}$.

For a $*$-homomorphism $\varphi : A \to B$, we already got an additive map $\varphi_* : V(A) \to V(A)$. We denote again by $\varphi_*$ to the map $G(\varphi_*) : K_{00}(A) \to K_{00}(B)$ gotten from functoriality of the Grothendieck group.

Recall that $A^+$ means $\widetilde{A}$ when $A$ is not unital and $A \oplus \mathbb{C}$ when $A$ is uinital. In any case we always have an exact sequence

$$0 \longrightarrow A \longrightarrow A^+ \xrightarrow{\pi} \mathbb{C} \longrightarrow 0$$

This gives a group homomorphism $\pi_* : K_{00}(A^+) \to \mathbb{Z}$. We use the $K_{00}$ group of $A^+$ to define the $K_0$ group of $A$.

**Definition 9.16.** For any $C^*$-algebra $A$ we define $K_0(A) := \ker(\pi_*) \subset K_{00}(A^+)$.

We still have a canonical map from $V(A)$ to $K_0(A)$ that we denote again by $\iota_A : V(A) \to K_0(A)$ and it's given also by $\iota_A([p]) := [p] - [0]$. We have to be careful here. A priori, $[p] - [0] \in K_{00}(A^+)$; we have to check that it is actually in $K_0(A)$. Indeed, since $p$ is a projection in $A$ and $\ker(\pi) = A$, we have

$$\pi_*([p] - [0]) = [\pi(p)] - [\pi(0)] = 0$$

**Theorem 9.17.** *For a $*$-homomorphism $\varphi : A \to B$, there is a well defined group homomorphism $\varphi_* : K_0(A) \to K_0(B)$ given by*

$$\varphi_*([(a_{i,j})] - [(b_{i,j})]) := [(\varphi^+(a_{i,j}))] - [(\varphi^+(b_{i,j}))],$$

*where $(a_{i,j}), (b_{i,j}) \in M_\infty(A^+)$ are projections. This makes $K_0$ is a covariant functor from the category of $C^*$-algebras to the one of Abelian groups.*

**Proof.** The only non-obvious part is that $[(\varphi^+(a_{i,j}))] - [(\varphi^+(b_{i,j}))]$ is actually an element of $K_0(B)$. Well, since $[(a_{i,j})] - [(b_{i,j})] \in K_0(A)$, this means that $[\pi(a_{ij})] - [\pi(b_{i,j})] = 0 \in \mathbb{Z}$. Since $V(\mathbb{C})$ has cancellation, the canonical map is injective and thereofore $\pi(a_{i,j}) = \pi(b_{i,j})$. which implies that the matrices $(a_{i,j})$ and $(b_{i,j})$ have the same scalar part. Hence, the matrices $(\varphi^+(a_{i,j}))$ and $(\varphi^+(b_{i,j}))$ also have the same scalar part and therefore $\pi_*\big([(\varphi^+(a_{i,j}))] - [(\varphi^+(b_{i,j}))]\big) = 0$. ∎

The following result is important and as an immediate consequence one gets that $K_0(A) = K_{00}(A)$ when $A$ is unital.

**Proposition 9.18.** *Let $A_1, A_2$ be $C^*$-algebras and put $A := A_1 \oplus A_2$. For $k = 1, 2$, let $\pi_k : A \to A_k$ be the projection onto $A_k$. The induced map $(\pi_k)_*$ has three different interpretations (either on $V(A)$, on $K_{00}(A)$ or $K_0(A)$). Then, the maps $(\pi_1)_* \oplus (\pi_2)_*$ are isomorphisms between $V(A) \to V(A_1) \oplus V(A_2)$, $K_{00}(A) \to K_{00}(A_1) \oplus K_{00}(A_2)$, $K_0(A) \to K_0(A_1) \oplus K_0(A_2)$.*

**Proof.** If $p_1 \in M_\infty(A_1)$ and $p_2 \in M_\infty(A_2)$ are projections, we have

$$(\pi_1)_* \oplus (\pi_2)_*([(p_1, p_2)]) = \big((\pi_1)_*([(p_1, p_2)]), (\pi_2)_*([(p_1, p_2)])\big) = \big([\pi_1(p_1, p_2)], [\pi_2(p_1, p_2) = (p_1, p_2)]\big) = ([p_1], [p_2])$$

The result now easily follows. ∎

**Corollary 9.19.** *For every $C^*$-algebra $A$, whether unital or not, the split exact sequence*

$$0 \longrightarrow A \longrightarrow A^+ \underset{\sigma}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0$$

*induces a split exact sequence*

$$0 \longrightarrow K_0(A) \longrightarrow K_{00}(A^+) \underset{\sigma_*}{\overset{\pi_*}{\rightleftarrows}} \mathbb{Z} \longrightarrow 0$$

*Thus, $K_{00}(A^+) = K_0(A) \oplus \mathbb{Z}$. In particuar, $K_{00}(A) \cong K_0(A)$ when $A$ is unital.*

**Proof.** Exactness at $K_0(A)$ is because $K_0(A)$ is a subgroup of $K_{00}(A^+)$. By definition $K_0(A) = \ker(\pi_*)$, so this gives exactness at the middle. Exactness at $\mathbb{Z}$ and splitness come from functoriality. Thus, $K_{00}(A^+) = K_0(A) \oplus \mathbb{Z}$. If $A$ is unital, then $A^+ = A \oplus \mathbb{C}$, so the previous Proposition gives $K_{00}(A^+) = K_{00}(A) \oplus \mathbb{Z}$; from where we extract that $K_{00}(A) \cong K_0(A)$. ■

**Example 9.20.** Let $\mathcal{H}$ be separable infinite dimensional Hilbert space. We compute $K_0(\mathbb{C})$, $K_0(M_n(\mathbb{C}))$, $K_{00}(\mathcal{K}(\mathcal{H}))$, $K_0(\mathcal{L}(\mathcal{H}))$ and $K_0(\mathcal{Q}(\mathcal{H}))$, where $\mathcal{Q}(\mathcal{H}) := \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the Calkin algebra.

1. $\mathbb{C}$ is unital and therefore $K_0(\mathbb{C}) = K_{00}(\mathbb{C}) = \mathbb{Z}$.
2. $M_n(\mathbb{C})$ is unital and therefore $K_0(M_n(\mathbb{C})) = K_{00}(M_n(\mathbb{C})) = \mathbb{Z}$.
3. $\mathcal{K}(\mathcal{H})$ is not unital, at this point we can only say that $K_{00}(\mathcal{K}(\mathcal{H})) = \mathbb{Z}$. We will use stability to show that $K(\mathcal{K}(\mathcal{H})) = \mathbb{Z}$.
4. $\mathcal{L}(\mathcal{H})$ is unital so $K_0(\mathcal{L}(\mathcal{H})) = K_{00}(\mathcal{L}(\mathcal{H})) = G(V(\mathcal{L}(\mathcal{H}))) = G(\mathbb{Z}_{\geq 0} \cup \{\infty\})$. In the semigroup $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ we have $\infty + \infty = \infty$ and $n + \infty = \infty$ for any $n \in {}_{\_}\geq 0$. Thus, $a - b = c - d$ for any $a, b, c, d \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. This gives $G(\mathbb{Z}_{\geq 0} \cup \{\infty\}) = \{0\}$.
5. $\mathcal{Q}(\mathcal{H})$ is unital so $K_0(\mathcal{Q}(\mathcal{H})) = K_{00}(\mathcal{Q}(\mathcal{H})) = G(V(\mathcal{Q}(\mathcal{H}))) = G(\{0, \infty\})$. Again we have $G(\{0, \infty\}) = \{0\}$.

**Theorem 9.21.** *(A picture of $K_0(A)$)*

1. *$K_0(A)$ is an Abelian group.*
2. *Any element in $K_0(A)$ can be seen as a formal difference $[p] - [q]$ where $p, q$ are projections in $M_k(A^+)$ for some $k \in \mathbb{Z} > 0$ and $p - q \in M_k(A)$ (that is $p$ and $q$ have the same scalar part). If $A$ is unital, then $p$ and $q$ may be chosen to be in $M_k(A)$.*
3. *Actually, each element of $K_0(A)$ can be written as $[p] - [p_n]$ where $p$ is a projection in $M_k(A^+)$ for some $k \in \mathbb{Z}_{>0}$, $p_n := \mathrm{diag}(\underbrace{1, \ldots, 1}_{n}, 0, \ldots, 0) \in M_k(A^+)$ with $n \leq k$, and $p - p_n \in M_k(A)$.*

**Proposition 9.22.** *Let $((A_i)_{i \in \Lambda}, (\varphi_{j,i})_{i \leq j})$ be a directed system of $C^*$-algebras. Then $((K_0(A_i))_{i \in \Lambda}, ((\varphi_{j,i})_*)_{i \leq j})$ is a directed system of groups and*

$$K_0(\varinjlim A_i) \cong \varinjlim K_0(A_i)$$

**Sketch of Proof.** For $i \leq j$ in $\Lambda$ we get $\psi_{j,i} := (\varphi_{j,i})_* : V(A_i) \to V(A_j)$. It's easy to check that $((V(A_i))_{i \in \Lambda}, (\psi_{j,i})_{i \leq j})$ is a directed system of semigroups. Just as we did for groups, we can construct the direct limit of a direct system of semigroups. We get a semigroup $\varinjlim V(A_i)$ together with the canonical maps $\psi_{\infty,i} : V(A_i) \to \varinjlim V(A_i)$. The semigroup $V(\varinjlim(A_i))$ together with the maps $(\varphi_{\infty,i})_* : V(A_i) \to V(\varinjlim A_i)$ are such that $(\varphi_{\infty,i})_* = (\varphi_{\infty,j})_* \circ \psi_{j,i}$ when $i \leq j$. Then, there is a unique additive map $\Psi : \varinjlim V(A_i) \to V(\varinjlim(A_i))$ such that

$$\begin{tikzcd}
 & V(\varinjlim(A_i)) \\
 & \varinjlim V(A_i) \arrow[u, dotted, "\Psi"'] \\
V(A_i) \arrow[uur, bend left, "(\varphi_{\infty,i})_*"] \arrow[ur, "\psi_{\infty,i}"'] &
\end{tikzcd}$$

Now use the lifting results from Theorem 7.9 to show that $\varinjlim V(A_i) \cong V(\varinjlim(A_i))$ via $\Psi$. This clearly implies $\varinjlim K_0(A_i) \cong K_0(\varinjlim(A_i))$. "□"

**Lemma 9.23.** *For $n \in \mathbb{Z}_{>0}$, consider the embedding $\varphi_{n,1} : A \hookrightarrow M_n(A)$ given by $\varphi_{n,1}(a) := \operatorname{diag}(a, \mathbf{0})$. Then, $(\varphi_{n,1})_* : K_0(A) \to K_0(M_n(A))$ is an isomorphism.*

**Proof.** It suffices to show that $(\varphi_{n,1})_* : V(A) \to V(M_n(A))$ is an isomorphism. To do so we need to play in $M_k(A)$ for different sizes. First, it's clear that $(\varphi_{n,1})_*$ is additive. If $p, q \in M_\infty(A)$ are projections such that $\varphi_{n,1}(p) \sim \varphi_{n,1}(q)$, then $p \sim q$, so $(\varphi_{n,1})_*$ is injective. If $p \in M_\infty(M_n(A))$ is a projection, then $p \in M_{nk}(A)$ for some $k$, so $p \in M_\infty(A)$ and clearly $\varphi_{n,1}(p) \sim p$, whence $(\varphi_{n,1})_*$ is surjective. ■

**Corollary 9.24.** *Let $\mathcal{H}$ be an infinite dimensional speparable Hilbert space and put $\mathcal{K} := \mathcal{K}(\mathcal{H})$. Let $v_{1,1}$ be a rank one projection in $\mathcal{K}$. The morphisim $a \mapsto a \otimes v_{1,1}$ from $A \to A \otimes \mathcal{K}$ induces an isomorphism $K_0(A) \cong K_0(A \otimes \mathcal{K})$.*

**Proof.** For $m \leq n$, let $\varphi_{n,m} : M_m(A) \hookrightarrow M_n(A)$ be given by $\varphi_{n,m}(a) := \operatorname{diag}(a, \mathbf{0})$. We saw that $\mathcal{K}$ is an AF algebra and that it's actually given by $\mathcal{K} = \varinjlim M_n(\mathbb{C})$. Since $M_n(A) = A \otimes M_n(\mathbb{C})$, we have

$$A \otimes \mathcal{K} = \varinjlim M_n(A)$$

Thus, by continuity of $K_0$, we have $\varinjlim K_0(M_n(A)) \cong K_0(A \otimes \mathcal{K})$. By the previous Lemma, for $m \leq n$, the following diagram is commutative

$$\begin{tikzcd} & K_0(A) & \\ K_0(M_m(A)) \arrow[ur, "(\varphi_{m,1})_*^{-1}"] \arrow[rr, "(\varphi_{n,m})_*"'] & & K_0(M_n(A)) \arrow[ul, "(\varphi_{n,1})_*^{-1}"'] \end{tikzcd}$$

By universality of direct limit of groups, this gives a unique homomorphism $\Psi : K_0(A \otimes \mathcal{K}) \to K_0(A)$ such that

$$\begin{tikzcd} & K_0(A) \\ & K_0(A \otimes \mathcal{K}) \arrow[u, dotted, "\Psi"'] \\ K_0(M_n(A)) \arrow[uur, bend left, "(\varphi_{n,1})_*^{-1}"] \arrow[ur, "\psi_{\infty,n}"'] & \end{tikzcd}$$

where $\psi_{\infty,n} : K_0(M_n(A)) \to \varinjlim K_0(M_n(A)) \cong K_0(A \otimes \mathcal{K})$. Using that each $(\varphi_{n,1})_*^{-1}$ is an isomorphism, it follows that $\Psi$ is also an isomorphism, whence $K_0(A) \cong K_0(A \otimes \mathcal{K})$. The fact that the isomorphism is implemented by $a \mapsto a \otimes v_{1,1}$ follows from the uniqueness of $\Psi$ and that $\Psi^{-1} = \psi_{\infty,n} \circ (\varphi_{n,1})_* = (\psi_{\infty,1})_*$. ■

**Corollary 9.25.** $K_0(\mathcal{K}) = K_0(\mathbb{C}) = \mathbb{Z}$

**Theorem 9.26.** *A short exact sequence of $C^*$-algebras*

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$$

*induces an exact sequence of groups.*

$$K_0(I) \xrightarrow{\varphi_*} K_0(A) \xrightarrow{\psi_*} K_0(B)$$

The previous theorem is saying that the functor $K_0$ is half exact. It's not exact as injectivity fails on

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \xrightarrow{\iota} \mathcal{L}(\mathcal{H}) \xrightarrow{\pi} \mathcal{Q}(\mathcal{H}) \longrightarrow 0$$

and surjectivity fails on

$$0 \longrightarrow C_0((0,1)) \xrightarrow{\iota} C([0,1]) \xrightarrow{\varphi} \mathbb{C} \oplus \mathbb{C} \longrightarrow 0$$

where $\varphi(f) := (f(0), f(1))$. It's not true that functor $K_{00}$ is half exact. Consider the short exact sequence

$$0 \longrightarrow C_0(\mathbb{R}^2) \xrightarrow{\iota} C(S^2) \xrightarrow{\pi} \mathbb{C} \longrightarrow 0$$

The induced $K_{00}$ sequence is

$$0 \longrightarrow 0 \xrightarrow{\iota_*} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\pi_*} \mathbb{Z} \longrightarrow 0$$

## 9.3 Homotopy invariance of $K_0$

**Definition 9.27.** Let $A$ and $B$ be $C^*$-algebras.

1. Two $*$-homomorphism $\varphi, \psi : A \to B$ are **homotopic**, denoted by $\varphi \sim \psi$, if there is a path $(\gamma_t)_{t\in[0,1]}$ of $*$-homomorphism such that $t \mapsto \gamma_t(a)$ is continuous for every fixed $a \in A$ and such that $\gamma_0 = \varphi$, $\gamma_1 = \psi$.

2. $A$ is **homotopically equivalent** to $B$ if there are maps $\varphi : A \to B$ and $\psi : B \to A$ such that $\varphi \circ \psi \sim \mathrm{id}_B$ and $\psi \circ \varphi \sim \mathrm{id}_A$.

3. $B$ is a **deformation retract of** $A$ if there $\varphi : A \to B$ and $\psi : B \to A$ such that $\varphi \circ \psi = \mathrm{id}_B$ and $\psi \circ \varphi \sim \mathrm{id}_A$. In this case $\varphi$ is a **a deformation retraction**.

4. We say $A$ is **contractible** when the identity map $\mathrm{id}_A : A \to A$ is homotopic to the zero map $0 : A \to A$.

**Theorem 9.28.** *If $\varphi, \psi : A \to B$ are homotopic, then $\varphi_* = \psi_* : K_0(A) \to K_0(B)$*

**Proof.** There is $(\gamma_t)_{t\in[0,1]}$ with $t \mapsto \gamma_t(a)$ continuous for every fixed $a \in A$ and such that $\gamma_0 = \varphi$, $\gamma_1 = \psi$. Take any projection $p \in M_k(A^+)$, $t \mapsto \gamma_t^+(p)$ is a continuous path of projections from $\varphi^+(p)$ to $\psi^+(p)$; whence $\varphi^+(p) \sim_h \psi^+(p)$. Therefore $[\varphi^+(p)] \sim [\psi^+(q)]$. That $\psi_* = \phi_*$ now follows because any element in $K_0(A)$ looks like $[p] - [q]$. ∎

**Corollary 9.29.** *Let $A$ and $B$ be $C^*$-algebras.*

1. *If $A$ is homotopically equivalent to $B$, then $K_0(A) \cong K_0(B)$.*

2. *If $B$ is a deformation retract of $A$, then $K_0(A) \cong K_0(B)$*

3. *If $A$ is contractible, then $K_0(A) \cong \{0\}$.*

**Example 9.30.** If $X$ is a compact Hausdorff space that is contractible, then $C(X)$ is homotopically equivalent to $\mathbb{C}$. Therefore, $K_0(C(X)) \cong \mathbb{Z}$.

**Definition 9.31.** Let $A$ and $B$ be a $C^*$-algebra.

1. The **cone** of $A$ is $CA := \{f \in C([0,1], A) : f(0) = 0\}$; this is a $C^*$-algebra with pointwise operations and sup norm.

2. The **suspension** of $A$ is $SA := \{f \in CA : f(1) = 0\}$; this is a $C^*$-subalgebra of $CA$.

3. If $\varphi : A \to B$ is a $*$-homomorphism, the **mapping cone for** $\varphi$ is $C_\varphi := \{(a, f) \in A \oplus CB : f(1) = \varphi(a)\}$

**Proposition 9.32.** *$CA$ is a contractible $C^*$-algebra. $SA$ is contractible if $A$ is contractible.*

**Proof.** First we show that $\mathrm{id}_{CA} \sim 0$. Indeed, for each $t \in [0,1]$ define $\gamma_t : CA \to CA$ by $(\gamma_t(f))(s) := f(ts)$ for any $s \in [0,1]$. Clearly $t \mapsto \gamma_t(s)$ is continuous and of course $\gamma_0 = 0$ and $\gamma_1 = \mathrm{id}_{CA}$. Now suppose that $A$ is contractible. That is there are maps $\alpha_t : A \to A$ such that $\alpha_0 = \mathrm{id_A}$ and $\alpha_1 = 0$. Then, define $\beta_t : SA \to SA$ by $(\beta_t(f))(s) := \alpha_t(f(s))$ for any $s \in [0,1]$. Continuity of $\beta_t$ follows from continuity of $\alpha_t$, $\beta_0 = \mathrm{id}_{SA}$ and $\beta_1 = 0$. ∎

# 10 Unitaries and $K_1$

As we've been doing so far, if $\pi : A^+ \to \mathbb{C}$ is the map $(a, \lambda) \mapsto \lambda$, we also denote by $\pi$ to the induced entry-wise map $M_n(A^+) \to M_n(\mathbb{C})$. We denote by $1_n$ to the identity matrix in $M_n(\mathbb{C})$.

**Definition 10.1.**

$$\mathrm{GL}_n^+(A) := \{a \in M_n(A^+) : a \text{ is invertible, } \pi(a) = 1_n\} \subset \mathrm{GL}_n(A^+) \qquad \mathrm{GL}_\infty^+(A) := \bigcup_{n=1}^{\infty} \mathrm{GL}_n^+(A)$$

$$\mathcal{U}_n^+(A) := \{u \in M_n(A^+) : u^*u = uu^* = 1_{M_n(A^+)},\ pi(u) = 1_n\} \subset \mathcal{U}_n(A^+) \qquad \mathcal{U}_\infty^+(A) := \bigcup_{n=1}^{\infty} \mathcal{U}_n^+(A)$$

Of course $\mathrm{GL}_n^+(A) \subset \mathrm{GL}_{n+1}^+(A)$ via $a \mapsto \mathrm{diag}(a, 1)$, whence if $a \in \mathrm{GL}_n^+(A)$, we regard it as an element of $\mathrm{GL}_\infty^+(A)$ written as $\mathrm{diag}(a, 1_\infty)$ (similarly for $\mathcal{U}_n^+(A)$). These are all topological groups, the topology on the "$\infty$" ones comes from the direct limit topology. For each of these groups, we denote the connected component of 1 by adding a $_0$ subscript. . Moreover, if $A$ is unital and $n \in \mathbb{Z}_{>0} \cup \infty$, then $\mathrm{GL}_n^+(A) \cong \mathrm{GL}_n(A)$ and $\mathcal{U}_n^+(A) \cong \mathcal{U}_n(A)$.

**Lemma 10.2.** *If $u$ and $v$ are unitaries in a $C^*$-algebra with $\|u - v\| \le 2$, then $u \sim_h v$. In particular $\mathcal{U}_n^+(A)$ is locally path connected and connected components coincide with path compotents.*

**Sketch of Proof.** That $\|u - v\| < 2$ implies that $\sigma(uv^*)$ has a gap arround 2. Then we can use continuous functional calculus to define

$$u_t := (\exp(t \log(uv^*)))v$$

Since $u_t^* u_t = u_t^* u_t = 1$, $u_0 = v$ and $u_1 = u$ we've found a path of unitaries connecting $u$ and $v$. In particular if both $u$ and $v$ are normalized matrices (that is $\pi(u) = \pi(v) = 1_n$) it also follows that $\pi(u_t) = 1_n$ "□"

**Remark 10.3.** Since we have Borel functional calculus in $\mathcal{L}(\mathcal{H})$, then **any** two unitaries are homotopicaly equivalent in $\mathcal{L}(\mathcal{H})$ (in fact this is the case for any von-Neumann algebra). An result we'll use is that $\mathcal{U}_n(\mathbb{C})$ is connected (same for $\mathrm{GL}_n(\mathbb{C})$).

**Proposition 10.4.** *For $n \in \mathbb{Z}_{>0} \cup \{\infty\}$,*

$$\mathrm{GL}_n^+(A)/\mathrm{GL}_n^+(A)_0 \cong \mathcal{U}_n^+(A)/\mathcal{U}_n(A)_0 \cong \mathrm{GL}_n(A^+)/\mathrm{GL}_n(A^+)_0 \cong \mathcal{U}_n(A^+)/\mathcal{U}(A+)_0$$

**Sketch of Proof.** That $\mathrm{GL}_n^+(A)/\mathrm{GL}_n^+(A)_0 \cong \mathcal{U}_n^+(A)/\mathcal{U}_n(A)_0$ comes from the map induced by the map $\mathrm{GL}_n^+(A) \to \mathcal{U}_n^+(A)$ given by $a \mapsto a|a|^{-1}$. That $\mathrm{GL}_n(A^+)/\mathrm{GL}_n(A^+)_0 \cong \mathrm{GL}_n^+(A)/\mathrm{GL}_n^+(A)_0$ comes from the map $a \mapsto a\pi(a^{-1})$ from $\mathrm{GL}_n(A^+) \to \mathrm{GL}_n^+(A)$. "□"

**Definition 10.5.** We define $K_1(A)$ to be any of the isomorphic groups of the previous Proposition with $n = \infty$. For $u \in \mathcal{U}_n^+(A)$, we denote by $[u] \in K_1(A)$ to the element in $\mathcal{U}_\infty^+(A)/\mathcal{U}_\infty(A)_0$ given by the connected component that contains $\mathrm{diag}(u, 1_\infty)$.

**Theorem 10.6.** *$K_1(A)$ is an Abelian group with multiplication $[u][v] := [uv] = [\mathrm{diag}(u, v)]$.*

**Proof.** First, we show that multiplication is well defined. Suppose $[u_0] = [u_1]$ and $[v_0] = [v_1]$ and assume with no loss on generality that all matrices lie on $M_k(A^+)$ for a suffciently large $k$. Then we have homotopies $u_t$ and $v_t$ in $\mathcal{U}_k^+(A)$, $t \mapsto u_t v_t$ is the path connecting $u_0 v_0$ with $u_1 v_1$. One also checks that $\mathrm{diag}(uv, 1) \sim_h \mathrm{diag}(u, v) \sim_h \mathrm{diag}(v, u) \sim_h \mathrm{diag}(vu, 1)$. This gives $[uv] = [\mathrm{diag}(u, v)]$ and that the group is Abelian. ■

We now list several properties of $K_1$.

**Proposition 10.7.** *Let $A$ and $B$ be $C^*$-algebras.*

1. *If $\varphi : A \to B$ is a $*$-isomorphism, then there is a well defined group homomorphism $\varphi_* : K_1(A) \to K_1(B)$ given by*
$$\varphi_*([u]) := [\varphi^+(u)],$$
*where $u \in M_\infty(A^+)$ is either an invertible element or an unitary. This makes $K_1$ into covariant functor from the category of $C^*$-algebras to the one of Abelian groups.*
2. *$K_1$ is half exact.*
3. *$K_1(A \oplus B) = K_1(A) \oplus K_1(B)$.*
4. *$K_1$ is a homotopy invariant functor. In particular $K_1(A) = 0$ if $A$ is contractible.*
5. *Let $((A_i)_{i\in\Lambda}, (\varphi_{j,i})_{i\leq j})$ be a directed system of $C^*$-algebras. Then $((K_1(A_i))_{i\in\Lambda}, ((\varphi_{j,i})_*)_{i\leq j})$ is a directed system of groups and*
$$K_1(\varinjlim A_i) \cong \varinjlim K_1(A_i)$$
6. *Let $\mathcal{K} := \mathcal{K}(\mathcal{H})$ for a separable Hilbert space $\mathcal{H}$. Then*
   (a) *$K_1(A) \cong K_1(A \otimes \mathcal{K})$.*
   (b) *$K_1(A) \cong \mathcal{U}_1^+(A \otimes \mathcal{K})/\mathcal{U}_1^+(A \otimes \mathcal{K})_0 \cong \mathrm{GL}_1^+(A \otimes \mathcal{K})/\mathrm{GL}_1^+(A \otimes \mathcal{K})_0$.*

**Example 10.8.** Below we say why all groups $K_1(\mathbb{C})$, $K_1(\mathcal{L}(\mathcal{H})$, and $K_1(\mathcal{K}(\mathcal{H}))$ are the trivial group.

1. $\mathrm{GL}_n(\mathbb{C})$ is connected so all unitaries are equivalent. The same happens for $\mathcal{L}(\mathcal{H})$ because the Borel functional calculus on $\log : S^1 \to \mathbb{C}$ can be used to connect any two unitaries in $\mathcal{L}(\mathcal{H}^n) \cong \mathcal{L}(\mathcal{H})$. Hence $K_1(\mathbb{C}) \cong K_1(\mathcal{L}(\mathcal{H}) \cong \{0\}$.
2. $K_1(\mathcal{K}(\mathcal{H})) \cong \varinjlim K_1(M_n(\mathbb{C})) \cong \varinjlim K_1(\mathcal{L}(\mathbb{C}^n)) = \{0\}$.

## 10.1 Suspended $C^*$-algebras

The suspension of $A$ was defined as $SA := \{f \in C([0,1], A) : f(0) = f(1) = 0\}$. Equivalently, we have

$$SA \cong A \otimes C_0(\mathbb{R}) \cong C_0(\mathbb{R}, A) \cong C_0((0,1), A) \cong \{f \in C(S^1, A) : f(1) = 0\}.$$

We omit the proof of the following technical result

**Theorem 10.9.** *There is a natural isomorphism $K_1(A) \cong K_0(SA)$.*

**Example 10.10.** Notice that $(S\mathbb{C})^+ = C(S^1)$. Then, $K_1(C(S^1)) \cong K_0(\mathbb{C}) = \mathbb{Z}$.

A much deep result, known as Bott periodicity, says that $K_0(A) \cong K_1(SA)$.

## 10.2 The index map

Given a short exact sequence of $C^*$-algebras,

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0 \tag{10.11}$$

we get two half exact sequences of $K$ groups:

$$\begin{tikzcd} K_0(I) \arrow[r, "\varphi_*"] & K_0(A) \arrow[r, "\psi_*"] & K_0(B) \\ K_1(B) \arrow[u, dotted, "\delta_1"] & K_1(A) \arrow[l, "\psi_*"] & K_1(I) \arrow[l, "\varphi_*"] \end{tikzcd} \tag{10.12}$$

We wish to connect them by constructing a map $\delta_1 : K_1(B) \to K_0(I)$. This map will come from the universal property of $K_1$:

Since $K_1$ is defined as the quotient $\mathcal{U}_\infty^+(A)/\mathcal{U}_\infty(A)_0$, then it has a universal property. Indeed, assume that $H$ is an (additive) Abeliean group and that $\varphi$ is a group homomorphism such that $\varphi(u) = \varphi(v)$ whenever $[u] = [v]$ and $\varphi(1) = 0$. Then, there is a unique group homomorphism $\psi$ such that

$$\begin{tikzcd} \mathcal{U}_\infty^+(A) \arrow[rr, "\varphi"] \arrow[dr, "{[\cdot]}"'] & & H \\ & K_1(A) \arrow[ur, dotted, "\psi"'] & \end{tikzcd}$$

We also need to recall that any $C^*$-algebra induces the split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\iota"] & A^+ \arrow[r, "\pi"] & \mathbb{C} \arrow[l, bend left, "\sigma"] \arrow[r] & 0 \end{tikzcd}$$

We then get a scalar map $s : A^+ \to A^+$ given by $s = \sigma \circ \pi$. This scalar map has the following properties

- $s(a, \lambda) = (0, \lambda) = \lambda 1$.
- $\pi(s(x)) = \pi(x)$ for all $x \in A^+$
- $x - s(x) \in A \subset A^+$ for all $x \in A^+$

We need the following somewhat technical lemma

**Lemma 10.13.** *We have the short exact sequence in 10.11 and $u \in \mathcal{U}_n(B^+)$.*

1. *There is $v \in \mathcal{U}_{2n}(A^+)$ and a projection $p \in M_{2n}(I^+)$ such that $\psi^+(v) = \mathrm{diag}(u, u^*)$, $\varphi^+(p) = v\mathrm{diag}(1_n, 0)v^*$ and $s(p) = \mathrm{diag}(1_n, 0)$.*
2. *If there is $w \in \mathcal{U}_{2n}(A^+)$ and a projection $q \in M_{2n}(I^+)$ such that $\psi^+(w) = \mathrm{diag}(u, u^*)$ and $\varphi^+(p) = w^*\mathrm{diag}(1_n, 0)w^*$, then $s(q) = \mathrm{diag}(1_n, 0)$ and $q \sim_u q$.*

**Definition 10.14.** The previous lemma guarantees that the map $\alpha : \mathcal{U}_\infty(B^+) \to K_0(I)$ given by

$$\alpha(u) := [p] - [s(p)]$$

is well defined.

Furtermore, we have that

1. $\alpha(u_1 u_2) = \alpha(u_1) + \alpha(u_2)$.
2. $\alpha(1) = 0$.
3. $\alpha(u_1) = \alpha(u_2)$ whenever $[u_1] = [u_2]$.
4. $\alpha(\psi^+(v)) = 0$ for all $v \in \mathcal{U}_\infty(A^+)$.
5. $\varphi_*(\alpha(u)) = 0$ for all $u\mathcal{U}_\infty(B^+)$

**Definition 10.15.** Using the universal property of $K_1$, there is a unique group homomorphism $\delta_1 : K_1(B) \to K_0(I)$ such that

$$\delta_1([u]) = \alpha(u)$$

This map is called the **index map** for 10.11.

By construction $\mathrm{im}(\psi_*) \subset \ker(\delta_1)$ and $\mathrm{im}(\delta_1) \subset \ker(\varphi_*)$. With some work we can show the reverse inclusions. This is exactness of the sequence 10.12 at $K_1(B)$ and $K_0(I)$.

**Example 10.16.** Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and write $\mathcal{K} := \mathcal{K}(\mathcal{H})$, $\mathcal{L} := \mathcal{L}(\mathcal{H})$ and $\mathcal{Q} := \mathcal{L}/\mathcal{K}$. We get a short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{L} \longrightarrow \mathcal{Q} \longrightarrow 0$$

We already know that $K_0(\mathcal{K}) = \mathbb{Z}$, $K_0(\mathcal{L}) = K_0(\mathcal{Q}) = \{0\}$ and that $K_1(\mathcal{K}) = K_1(\mathcal{L}) = \{0\}$. Thus, we have the following exact sequence

$$\begin{tikzcd} \mathbb{Z} \arrow[r] & \{0\} \arrow[r] & \{0\} \\ K_1(\mathcal{Q}) \arrow[u, "\delta_1"] & \{0\} \arrow[l] & \{0\} \arrow[l] \end{tikzcd}$$

This implies that the index map, whatever it looks like, is an isomorphism. Hence, $K_1(\mathcal{Q}) \cong \mathbb{Z}$.

**Example 10.17.** Let $\mathcal{H}$ be a separable infinite dimensional Hilbert. Recall that a map $u \in \mathcal{L}(\mathcal{H})$ is **Fredholm** if $u(\mathcal{H})$ is closed, and both $\ker(u)$ and $\ker(u^*)$ are finite dimensional. Its index is

$$\mathrm{ind}(u) := \dim(\ker(u)) - \dim(\ker(u^*))$$

The following are equivalent definitions for $u$ to be Fredholm

- There is $v \in \mathcal{L}(\mathcal{H})$ such that $1 - vu$, $1 - uv$ are compact.
- If $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{Q}(\mathcal{H})$ is the quotient map, then $\pi(u)$ is invertible in $\mathcal{Q}(\mathcal{H})$.

Recall that $K_0(\mathcal{K}(\mathcal{H})) \cong \mathbb{Z}$ via the map induced by $p \mapsto \dim(p(\mathcal{H})) = \mathrm{Tr}(p)$. The Fredholm index is in fact a disguised version of the index map:

$$\mathrm{ind}(u) = (\mathrm{Tr}_* \circ \delta_1)([\pi(u)])$$

# 11 $K$-theory for some examples

## 11.1 AF-Algebras

**Definition 11.1.** A **partially ordered group** is a pair $(G, \leq)$ consisting of an Abelian group $G$ and a partial order $\leq$ on $G$ such that if $G_+ := \{g \in G : 0 \leq g\}$ then $G = G_+ - G_+$ and if $g_1 \leq g_2$ then $g_1 + g \leq g_2 + g$ for all $g \in G$.

**Definition 11.2.** If $G$ is an abelian group and $N$ is a subset of $G$ such that $N + N \subset N$, $G = N - N$ and $N \cap (-N) = \{0\}$, we call $N$ a **cone on** $G$. Given a cone we get a partial order on $G$ be letting $g_1 \leq g_2$ if and only if $g_2 - g_1 \in N$. In this case $G_+ = N$.

**Theorem 11.3.** *Let $A$ be an AF-algebra. Then $V(A)$ is a cone in $K_0(A)$.*

**Definition 11.4.** If $\varphi : G_1 \to G_2$ is a group homomorphism between partially ordered groups, we say $\varphi$ is **positive** if $\varphi(G_{1+}) \subset G_{2+}$. If in adition $\varphi$ is an isomorphism for which $\varphi^{-1}$ is also positive, we say $\varphi$ is an **order isomorphism**.

**Definition 11.5.** If $A$ and $B$ are unital $C^*$-algebras and $\tau : K_0(A) \to K_0(B)$ is a homomorphism, we say $\tau$ is **unital** if $\tau([1_A]) = [1_B]$.

**Theorem 11.6.** *Let $A$ and $B$ be unital AF-algebras and $\tau : K_0(A) \to K_0(B)$ a unital order isomorphism. Then there is a $*$-isomorphism $\varphi : A \to B$ such that $\varphi_* = \tau$.*

**Corollary 11.7.** *Two unital AF-algebras are isomorphic if and only if there is a unital order isomorphism between their $K_0$ groups.*

## 11.2 The Toeplitz Algebra

**Definition 11.8.** Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{\xi_1, \xi_2, \ldots\}$. The unilateral shift on $\mathcal{L}(\mathcal{H})$ is the operator $s(\xi_n) = \xi_{n+1}$. It's easy to check that $s^*s = 1$. The **Toeplitz algebra** is the unital $C^*$-alegbra $\mathcal{T}$ in $\mathcal{L}(\mathcal{H})$ generated by $s$.

The closed two sided ideal in $\mathcal{T}$ generated by $1 - ss^*$ is $\mathcal{K} := \mathcal{K}(\mathcal{H})$. One can show that $\mathcal{T}/\mathcal{K} \cong C(S^1)$. Thus, the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \longrightarrow C(S^1) \longrightarrow 0$$

Induces an exact sequence (we haven't talk at all about the map $\delta_0 : K_0(B) \to K_1(A)$ though)

$$\begin{tikzcd} \mathbb{Z} \arrow[r] & K_0(\mathcal{T}) \arrow[r] & \mathbb{Z} \arrow[d, "\delta_0"] \\ \mathbb{Z} \arrow[u, "\delta_1"] & K_1(\mathcal{T}) \arrow[l] & \{0\} \arrow[l] \end{tikzcd}$$

which can be used to deduce that $K_0(\mathcal{T}) = \mathbb{Z}$ and $K_1(\mathcal{T}) = 0$.

## 11.3 Cuntz Algebras

Let $n \geq 2$ be an integer and $\mathcal{H}$ an infinite dimensional separable Hilbert space. Then, there are elements $s_1, s_2, \ldots, s_n \in \mathcal{L}(\mathcal{H})$ such that

$$s_j^* s_j = 1 \quad \text{and} \quad \sum_{j=1}^{n} s_j s_j^* = 1 \tag{11.9}$$

**Definition 11.10.** We define $\mathcal{O}_n$, the Cuntz algebra of order $n$, as $C^*(s_1, \ldots, s_n)$. In fact, the construction of $\mathcal{O}_n$ is independent of the Hilbert space $\mathcal{H}$ and the choice of isometries as long as the relations 11.9 are satisfied.

The algebra $\mathcal{O}_n$ is a simple $C^*$-algebra and has the following universal property: If $A$ is a unital $C^*$-algebra containing elements $a_1, \ldots, a_n$ such that

$$a_j^* a_j = 1 \quad \text{and} \quad \sum_{j=1}^{n} a_j a_j^* = 1,$$

then there is a unique $*$-homomorphism $\varphi : \mathcal{O}_n \to A$ such that $\varphi(s_j) = a_j$.

**Remark 11.11.** The projections $s_j s_j^*$ are muttually orthogonal, therefore

$$[1] = \sum_{j=1}^{n} [s_j s_j^*] = \sum_{j=1}^{n} [s_j^* s_j] = n[1]$$

This gives $(n-1)[1] = 0$. So $K_0(\mathcal{O}_n)$ has torsion.

**Theorem 11.12.** *$K_0(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$ and $K_1(\mathcal{O}_n) \cong \{0\}$.*

**Sketch of Proof.** We will not show that $K_1(\mathcal{O}_n) \cong \{0\}$, although this fact will be used. Consider $v_1, \ldots, v_{n+1}$ to be $n+1$ isometries whose range projections add up to 1. Then $\mathcal{E}_n := C^*(v_1, \ldots, v_n) \neq \mathcal{O}_n$. Let $J_n$ be the ideal generated by $v_{n+1}v_{n+1}^*$ in $\mathcal{E}_n$. Then $J_n \cong \mathcal{K}(\mathcal{H})$ and $\mathcal{E}_n / J_n \cong \mathcal{O}_n$. Cuntz proved that $K_0(\mathcal{E}_n) = \mathbb{Z}$. Thus, the exact sequence

$$0 \longrightarrow J_n \longrightarrow \mathcal{E}_n \longrightarrow \mathcal{O}_n \longrightarrow 0$$

Induces an exact sequence

$$\begin{tikzcd} (n-1)\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & K_0(\mathcal{O}_n) \arrow[d, "\delta_0"] \\ \{0\} \arrow[u, "\delta_1"] & K_1(\mathcal{E}_n) \arrow[l] & \{0\} \arrow[l] \end{tikzcd}$$

Hence, $K_0(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$. "□"

### 11.4 Rotation Algebras

For $\theta \in \mathbb{R}$ we consider the homeomorphism $\varphi_\theta : S^1 \to S^1$ given by rotation by the angle $2\pi\theta$:

$$\varphi_\theta(z) = e^{i2\pi\theta}z$$

This gives a homeomorphism $h_\theta : C(S^1) \to C(S^1)$ given by

$$h_\theta(f)(z) = f(\varphi_\theta(z))$$

and therefore we get an action of $\mathbb{Z}$ on $C(S^1)$. We extend $h_\theta : C(S^1) \to C(S^1)$ to $h_\theta : \mathcal{L}(L^2(S^1)) \to \mathcal{L}(L^2(S^1))$, with $h_\theta^* = h_{-\theta}$. This makes $h_\theta$ into a unitary on $L^2(S^1)$. Also $C(S^1)$ can be faithfully represented on $L^2(S^1)$as multiplication operators via $f \mapsto m_f$. Let $\mathrm{id} : S^1 \to \mathbb{C}$ be the identity map, then $m_{\mathrm{id}}$ is a unitary on $L^2(S^1)$. Notice that for any $g \in L^2(S^1)$

$$[h_\theta(m_{\mathrm{id}}g)](z) = e^{i2\pi\theta}zg(e^{i2\pi\theta}z) = e^{i2\pi\theta}[m_{\mathrm{id}}(h_\theta(g))](z)$$

That is, $h_\theta \circ m_{\mathrm{id}} = e^{i2\pi\theta}m_{\mathrm{id}} \circ h_\theta$.

**Definition 11.13.** We define $A_\theta$ as the unital $C^*$-algebra generated by two unitaries $u, v$ satysfying $vu = e^{i2\pi\theta}uv$. Equivalently

- $A_\theta = C^*(\mathbb{Z}, C(S^1), h_\theta)$.
- If $\theta$ is irrational, $A_\theta$ is isomorphic to the $C^*$-subalgebra of $\mathcal{L}(L^2(S^1))$ generated by $m_{\mathrm{id}}$ and $h_\theta$.

When $\theta$ is irrational, $h_\theta$ and 1 are not homotopic unitaries in $A_\theta$, but $h_{\theta*} = 1$ as a map on $K$-theory. This fact, together with the Pimsner-Voiculescu sequence (we did not discussed this) regarding $A_\theta$ as a corssed product, yields

$$K_0(A_\theta) \cong K_1(A_\theta) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

## 12 Hilbert Modules

**Definition 12.1.** A Hilbert $A$-module $E$ is a right $A$-module together with a pairing $\langle \cdot, \cdot \rangle : E \times E \to A$ such that

1. For each $\eta \in E$, the map $\langle \xi, \cdot \rangle : E \to A$ is linear,
2. $\langle \xi, \eta a \rangle = \langle \xi, \eta \rangle a$ for any $\xi, \eta \in E$ and $a \in A$,
3. $\langle \xi, \eta \rangle = \langle \eta, \xi \rangle^*$ for any $\xi, \eta \in E$,
4. $\langle \xi, \xi \rangle \geq 0$ in $A$ for any $\xi \in E$ and if $\langle \xi, \xi \rangle = 0$, then $\xi = 0$.
5. $E$ is complete with the norm $\|\xi\| := \|\langle \xi, \xi \rangle\|^{1/2}$.

The pairing $\langle \cdot, \cdot \rangle : E \times E \to A$ satisfying 1-4 above is referred to as an "$A$-valued inner product".

We have to check that $\|\xi\| := \|\langle \xi, \xi \rangle\|^{1/2}$ actually gives a norm. To get the triangle inequality one needs a version of Cauchy-Schwarz:

**Proposition 12.2.** *Let $E$ be a Hilbert $A$-module and $\xi, \eta \in E$. Then,*

$$\langle \eta, \xi \rangle \langle \xi, \eta \rangle \leq \|\langle \xi, \xi \rangle\| \langle \eta, \eta \rangle$$

*In particular $\|\langle \xi, \eta \rangle\|^2 = \|\langle \eta, \xi \rangle \langle \xi, \eta \rangle\| \leq \|\xi\|^2 \|\eta\|^2$ and therefore*

$$\|\langle \xi, \eta \rangle\| \leq \|\xi\| \|\eta\|$$

**Proof.** If $\xi = 0$, the result is clear. For $\xi \neq 0$, assume wlog that $\|\langle \xi, \xi \rangle\| = 1$. Then, for any $a \in A$, by Proposition 3.5 we have $a^*\langle \xi, \xi \rangle a \leq a^*a$. Hence,

$$0 \leq \langle \xi a - \eta, \xi a - \eta \rangle = a^*\langle \xi, \xi \rangle a - a^*\langle \xi, \eta \rangle - \langle \eta, \xi \rangle a + \langle \eta, \eta \rangle \leq a^*a - a^*\langle \xi, \eta \rangle - \langle \eta, \xi \rangle a + \langle \eta, \eta \rangle$$

If $a := \langle \xi, \eta \rangle$, this implies $\langle \eta, \xi \rangle \langle \xi, \eta \rangle \leq \langle \eta, \eta \rangle$ and we are done. ■

**Lemma 12.3.** *Let $E$ be a Hilbert $A$-module. Then $EA = \{\xi a : \xi \in E, a \in A\}$ is dense in $E$.*

**Proof.** Let $(u_\lambda)_{\lambda \in \Lambda}$ be an approximate unit for $A$. For any $\xi \in E$ we have

$$\|\xi u_\lambda - \xi\|^2 = \langle \xi u_\lambda - \xi, \xi u_\lambda - \xi \rangle = u_\lambda \langle \xi, \xi \rangle u_\lambda - \langle \xi, \xi \rangle u_\lambda - u_\lambda \langle \xi, \xi \rangle + \langle \xi, \xi \rangle$$

Thus, $\lim_\lambda \|\xi u_\lambda - \xi\|^2 = 0$. ■

**Definition 12.4.** A Hilbert $A$-module $E$ is **full** if $\langle E, E\rangle := \operatorname{span}\{\langle \xi, \eta\rangle : \xi, \eta \in E\}$ is dense in $A$.

**Remark 12.5.** Even if $E$ is not full, we always have $E\langle E, E\rangle$ is dense in $E$. The same proof given in the previous lemma works using an approcimate identity for the closure of the two sided ideal $\langle E, E\rangle$.

**Example 12.6.** Fix a $C^*$-algebra $A$.

1. Hilbert spaces are precisely Hilbert $\mathbb{C}$-modules, with the Physicist's convention of linearity in second coordinate for the inner product.

2. $A$ is itself a full Hilbert $A$-module when equipped with right multiplication as action and $\langle a, b\rangle := a^*b$ as $A$-valued inner product. Any closed right ideal of $A$ is a sub-$A$-module of $A$.

3. if $E_1, \dots, E_n$ are Hilbert $A$-modules, the direct sum
$$\bigoplus_{n=1}^{k} E_k := \{\xi = (\xi_1, \dots, \xi_n) : \xi_k \in E_k\}$$
is again a Hilbert $A$-module with the component-wise right action of $A$ and $A$-valued inner product
$$\langle \xi, \eta\rangle := \sum_{k=1}^{n} \langle \xi_k, \eta_k\rangle$$

4. If $(E_\lambda)_{\lambda\in\Lambda}$ is an arbitrary family of Hilbert $A$-modules, we can form their direct sum
$$\bigoplus_{\lambda\in\Lambda} E_\lambda := \left\{\xi = (\xi_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} E_\lambda : \sum_{\lambda\in\Lambda} \langle \xi_\lambda, \xi_\lambda\rangle \text{ converges in } A\right\}$$
which is a right $A$-module with coordinate-wise action and it becomes a Hilbert $A$-module when equipped with the well defined $A$-valued inner product
$$\langle \xi, \eta\rangle := \sum_{\lambda\in\Lambda} \langle \xi_\lambda, \eta_\lambda\rangle$$

5. A particular case of the above one is when $\Lambda = \mathbb{Z}_{>0}$ and each $E_\lambda := A$. This is called the standard Hilbert $A$-module and denoted by $\mathcal{H}_A$
$$\mathcal{H}_A := \left\{a = (a_j)_{j=1}^{\infty} \in \prod_{j=1}^{\infty} A : \sum_{j=1}^{\infty} a_j^* a_j \text{ converges in } A\right\}$$

6. Let $X$ be a compact Hausdorff space and $\pi : E \to X$ a complex vector bundle over $X$. Let $\Gamma(E)$ be the space of continuous sections of $E$ equipped with a Riemannian metric. Then, $\Gamma(E)$ is a Hilbert $C(X)$-module with inner product $\langle \sigma_1, \sigma_2\rangle(x) := g(\sigma_1(x), \sigma_2(x))$.

For an arbitrary $C^*$-algebra $A$, the Hilbert $A$-modules are a good generalization of Hilbert spaces. However, many nice properties of Hilbert spaces, such as complementability of subspaces, are not guaranteed for general Hilbert $A$-modules.

**Example 12.7.** Let $A := C(X)$ for a compact Hausdorff space $X$. Regard $A$ as a Hilbert $A$-module. Let $Y$ be a closed subset of $X$ such that $X \setminus Y$ is dense in $X$. Let $E := \{f \in A : f(Y) = \{0\}\}$. Then $E$ is a proper sub-$A$-module of $A$. Notice that $E^\perp := \{g \in A : \overline{g}f = 0 \;\forall\; f \in E\} = \{0\}$ and therefore $E \oplus E^\perp \neq A$. Further, $E \neq E^{\perp\perp} = A$. A similar thing happens with the submodule $C_0((0,1))$ of $C([0,1])$. Turns out that a closed submodule of a Hilbert $A$-module is complementable precesely when it's the range of an adjointable map.

Nevertheless, Hilbert-$A$ modules provide a good tool to study the $C^*$-algebra $A$. For example, one can visualize the multiplier algebra of $A$ using some kind of operators between Hilbert $A$-modules, the adjointable ones (see the definition below). Also, there are alternate descriptions of $K_0(A)$ using isomorphism classes of finitely generated projective $A$-modules or some particular kind of projections in $\mathcal{H}_A$.

**Definition 12.8.** Let $E$ and $F$ be a Hilbert $A$-modules. A map $t : E \to F$ is said to be **adjointable** if there is a map $t^* : F \to E$ such that for any $\xi \in E$, and $\eta \in F$
$$\langle t(\xi), \eta\rangle = \langle \xi, t^*(\eta)\rangle$$
The space of adjointable maps from $E$ to $F$ is denoted by $\mathcal{L}(E, F)$ and $\mathcal{L}(E) := \mathcal{L}(E, E)$.

It's easy to check that adjointable maps are module maps that are actually bounded linear maps with the usual operator norm and that $\mathcal{L}(E)$ is a $C^*$-algebra.

**Example 12.9.** Not every bounded linear map between Hilbert modules is adjointable. Indeed, let $E := C(X)$ for a compact Hausdorff space $X$ and $F := \{f \in C(X) : f(Y) = \{0\}\}$ where $Y$ is a closed non-empty subset of $X$ such that $X \setminus Y$ is dense in $X$. Consider the inclusion $\iota : F \to E$, which is clearly a bounded linear map. However, it's not adjointable. Assume on the contrary that $\iota$ is adjointable. Then, $\overline{f}g = \overline{f}\iota^*(g)$ for all $f \in F$ and $g \in E$, in particular if $g = \mathbf{1}$ we would have $\overline{f} = \overline{f}\iota^*(\mathbf{1})$ for all $f \in F$ and this implies $\iota(\mathbf{1})(x) = 1$ for all $x \notin Y$. By density of $X \setminus Y$ we get that $\iota^*(\mathbf{1}) = \mathbf{1}$ but $\mathbf{1} \notin F$.

**Lemma 12.10.** *An element $t \in \mathcal{L}(E)$ is positive if and only if $\langle t\xi, \xi\rangle \geq 0$ for all $\xi \in E$.*

**Proof.** The usual proof on Hilbert spaces uses $\ker(t)^\perp = \overline{\operatorname{im}(t^*)}$ but this fails for general Hilbert modules $E$. One direction is clear: If $t$ is positive then $t = s^*s$ for some $s \in \mathcal{L}(E)$, whence $\langle t\xi, \xi\rangle = \langle s\xi, s\xi\rangle \geq 0$. Conversely, assume that $\langle t\xi, \xi\rangle \geq 0$ for all $\xi \in E$. Then, $\langle t\xi, \xi\rangle = \langle t\xi, \xi\rangle^* = \langle \xi, t\xi\rangle = \langle t^*\xi, \xi\rangle$ for any $\xi \in E$. Since the polarization identity is valid for the $A$-valued inner product we have $t^* = t$, so $t$ is self adjoint. By Lemma 3.2 we write $t = t_+ - t_-$ where $t_+, t_- \geq 0$ and $t_+t_- = 0 = t_-t_+$. Suffices to prove that $t_- = 0$. Well, for any $\eta \in E$ we have

$$0 \leq \langle (t_+ - t_-)\eta, \eta\rangle = \langle t_+\eta, \eta\rangle - \langle -t_-\eta, \eta\rangle$$

Thus, $\langle t_-\eta, \eta\rangle \leq \langle t_+\eta, \eta\rangle$. Hence, since $t_- \geq 0$, clearly $t_+^3 \geq 0$, whence

$$0 \leq \langle t_-^3\xi, \xi\rangle = \langle t_-^2\xi, t_-\xi\rangle \leq \langle t_+t_-\xi, t_-\xi\rangle = 0$$

This says $t_-^3 = 0$ (again by polarization) but of course this means that $t_- = 0$. ∎

**Definition 12.11.** Let $E$ and $F$ be a Hilbert $A$-modules. For $\xi \in E$ and $\eta \in F$, we define a map $\theta_{\xi,\eta} : F \to E$ by

$$\theta_{\xi,\eta}(\zeta) := \xi\langle \eta, \zeta\rangle$$

One easily checks that $\theta_{\xi,\eta} \in \mathcal{L}(E, F)$, that $(\theta_{\xi,\eta})^* = \theta_{\eta,\xi} \in \mathcal{L}(F, E)$ and that $\|\theta_{\xi,\eta}\| \leq \|\xi\|\|\eta\|$. This gives an analogous of the class of rank-one operators on Hilbert spaces. So, we define an analogous of the compact operators by letting

$$\mathcal{K}(E, F) := \overline{\operatorname{span}\{\theta_{\xi,\eta} : \xi \in E, \eta \in F\}}$$

It's also not hard to verify that if $E, F, G$ are Hilbert $A$-modules, $u \in \mathcal{L}(E, G)$ and $v \in \mathcal{L}(G, F)$ then

- $u\theta_{\xi,\eta} = \theta_{u\xi,\eta}$
- $\theta_{\xi,\eta}v = \theta_{\xi,v^*\eta}$

In particular $\mathcal{K}(E) := \mathcal{K}(E, E)$ is a closed two sided ideal in $\mathcal{L}(E)$, whence $\mathcal{K}(E)$ is also a $C^*$-algebra. We have to be careful and not call these maps compact operators, in fact they do not have to be compact as maps between the two Banach spaces $E$ and $F$. For example, if $A$ is an infinite dimensional unital $C^*$ algebra and $E = F = A$ with inner product given by $a^*b$, then $\mathrm{id}_A = \theta_{1,1} \in \mathcal{K}(A)$ is not a compact operator.

**Theorem 12.12.** *If we regard $A$ as a Hilbert $A$-module, then $A \cong \mathcal{K}(A)$ as $C^*$-algebras.*

**Proof.** Define a map $\Phi : \mathcal{K}(A) \to A$ by letting $\Phi(\theta_{a,b}) = ab^*$ and extending to all of $\mathcal{K}(A)$. We check that $\Phi$ is a $*$-homomorphism. Indeed, multiplicativity follows because $\theta_{a,b}\theta_{c,d} = \theta_{ab^*c,d}$. Since $\theta_{a,b}^* = \theta_{b,a}$ and $(ab^*)^* = ba^*$, $\Phi$ preserves the involution. Also $\theta_{a,b} = 0$ if and only if $ab^* = 0$, whence $\Phi$ is injective. For any $a \in A$, if $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for $A$, then $(\theta_{u_\lambda,a^*})_{\lambda\in\Lambda}$ is a Cauchy net in $\mathcal{K}(A)$ and clearly $\Phi(\lim_\lambda \theta_{u_\lambda,a^*}) = a$, so surjectivity follows. ∎

Recall from Example 4.9 that $\mathcal{K}(\mathcal{H})$ is an essential ideal in $\mathcal{L}(\mathcal{H})$ and that $M(\mathcal{K}(\mathcal{H})) = \mathcal{L}(\mathcal{H})$. This is also the case for Hilbert modules. We only deal with the particular case $E = A$.

**Theorem 12.13.** *$\mathcal{K}(A)$ is an essential ideal in $\mathcal{L}(E)$.*

**Proof.** Suppose $t \in \mathcal{L}(E)$ is such that $t\mathcal{K}(A) = \{0\}$. We have to show $t = 0$. Well, for any $a, b \in A$ we have $0 = t\theta_{a,b}c = \theta_{ta,b}c = t(a)b^*c = t(ab^*c)$. In particular if $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for $A$, we have $t(a) = t(\lim_\lambda au_\lambda) = \lim_\lambda t(au_\lambda^{1/2}u_\lambda^{1/2}) = 0$. So $t = 0$ as wanted. ∎

We know from the discussion previous to Example 4.9 that the multiplier algebra $M(A)$ is the largest unital algebra containing $A$ as an essential ideal. Since we've already shown that $A \cong \mathcal{K}(A)$ sits in $\mathcal{L}(A)$ as an essential ideal, to prove that $M(A) \cong \mathcal{L}(A)$ it suffices to prove maximality. For this we first need to talk about representations of $A$ on Hilber modules and a couple of lemmas.

**Definition 12.14.** Let $A$, $C$ be $C^*$-algebras and $E$ a Hilbert $C$-module. A rrepresentation of $A$ on $E$ is a $*$-homomotphism $\varphi : A \to \mathcal{L}(E)$. As for Hilbert spaces, we say that $\varphi$ is **non-degenerate** when $\varphi(A)E$ is dense in $E$.

**Lemma 12.15.** *Identify $A \cong \mathcal{K}(A)$ and look at the inclusion $\iota : A \to \mathcal{L}(E)$. This is a non-degenerate representation.*

**Proof.** It's obviously a representation. Recall that $A\langle A, A\rangle$ is dense in $A$, so prove that $\iota(A)A$ is dense in $A$ it suffices to prove $\iota(A)A\langle A, A\rangle$ is dense in $A\langle A, A\rangle$. Let $(v_\lambda)_\lambda$ an approximate unit for $\iota(A) = \mathcal{K}(A)$. Then, for $a\langle b, c\rangle \in A\langle A, A\rangle$

$$\lim_\lambda v_\lambda(a\langle b, c\rangle) = \lim_\lambda v_\lambda \theta_{a,b}(c) = \theta_{a,b}(c) = a\langle b, c\rangle$$

as wanted. ■

**Lemma 12.16.** *Let $A$, $B$ and $C$ be $C^*$-algebras such that $A$ is an ideal in $B$ and $E$ is a Hilbert $C$-module. Suppose that $\varphi : A \to \mathcal{L}(E)$ is a non-degenerate representation. Then, $\varphi$ extends uniquely to a unique $*$-homomorphism $\varphi' : B \to \mathcal{L}(C)$, Furthermore, if $\varphi$ is injective and $A$ is essential, then $\varphi'$ is injective.*

**Proof.** For $b \in B$ we define $\varphi'(b) : \varphi(A)E \to E$ by

$$\varphi'(b)\left(\sum_{k=1}^n \varphi(a_j)\xi_j\right) := \sum_{k=1}^n \varphi(ba_j)\xi_j$$

Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for $A$. Then,

$$\Big\|\sum_{k=1}^n \varphi(ba_j)\xi_j\Big\| = \lim_\lambda \Big\|\sum_{k=1}^n \varphi(bu_\lambda a_j)\xi_j\Big\| = \lim_\lambda \Big\|\varphi(bu_\lambda)\sum_{k=1}^n \varphi(a_j)\xi_j\Big\| \le \lim_\lambda \|\varphi(bu_\lambda)\|\Big\|\sum_{k=1}^n \varphi(a_j)\xi_j\Big\| \le \|b\|\Big\|\sum_{k=1}^n \varphi(a_j)\xi_j\Big\|$$

Thus, $\varphi'(b)$ extends by density to a unique well defined map $\varphi'(b) : E \to E$. In a similar way (using $(u_\lambda)_\lambda$) we can check that $\varphi'(b) \in \mathcal{L}(E)$ with $\varphi'(b^*) = \varphi'(b)^*$. Thus, $\varphi'$ is indeed a $*$-homomorphism extending $\varphi$.

Finally, if in addition $\varphi$ is injective and $A$ an essential ideal, the ideal $\ker(\varphi')$ intersected with $A$ is the $\ker(\varphi) = \{0\}$. But essential ideals have non-zero intersection with any non-zero ideal of $B$. Thus, $\ker(\varphi') = \{0\}$, whence $\varphi'$ is injective. ■

**Theorem 12.17.** $M(A) \cong \mathcal{L}(A)$.

**Proof.** As we already pointed out, suffices to show that if $B$ is any other $C^*$-algebra containing $A$ as an essential ideal is contained in $\mathcal{L}(A)$. Indeed, consider the inclusion $\iota : A \to \mathcal{L}(A)$, combining the previous two lemmas we get a unique injective extension $\iota' : B \to \mathcal{L}(A)$. ■

**Remark 12.18.** A direct proof of the previous theorem (that requires to see $M(A)$ as double centralizers) is to check that the map $t \mapsto (t, \widetilde{t})$ is a $*$-isomorphism from $\mathcal{L}(A)$ to $M(A)$, where $\widetilde{t}(a) := t^*(a^*)^*$.

**Remark 12.19.** One needs more work to show a more general result $M(\mathcal{K}(E)) \cong \mathcal{L}(E)$. If $\mathcal{K} := \mathcal{K}(\mathcal{H})$ for a separable infinite dimensional Hilbert space $\mathcal{H}$, one can show that $\mathcal{K}(\mathcal{H}_A) \cong \mathcal{K} \otimes A$. Then, one gets at once $M(\mathcal{K} \otimes A) \cong \mathcal{L}(\mathcal{H}_A)$.

## 12.1 Morita Equivalence

Given a Hilbert $A$-module $E$, there is a connection between the $C^*$-algebras $A$ and $\mathcal{K}(E)$. Observe that $E$ is a left $\mathcal{K}(E)$-module when equipped with the obvious left action $v \cdot \xi := v(\xi)$. Further, there is a $\mathcal{K}(E)$-valued left inner product on $E$ defined by

$$(\xi, \eta) := \theta_{\xi,\eta}$$

for any $\xi, \eta \in E$. Indeed:

- $(\xi_1 + \alpha\xi_2, \eta) = \theta_{\xi_1+\alpha\xi_2,\eta} = \theta_{\xi_1,\eta} + \alpha\theta_{\xi_2,\eta}$.
- $(v\xi, \eta) = \theta_{v\xi,\eta} = v\theta_{\xi,\eta} = v(\xi, \eta)$.
- $(\xi, \eta)^* = \theta^*_{\xi,\eta} = \theta_{\eta,\xi} = (\eta, \xi)$.
- $\langle(\xi, \xi)\eta, \eta\rangle = \langle\xi\langle\xi, \eta\rangle, \eta\rangle = \langle\xi, \eta\rangle^*\langle\xi, \eta\rangle \ge 0$, whence $(\xi, \xi) \ge 0$ by Lemma 12.10.
- If $(\xi, \xi) = 0$, then $\langle\xi, \xi\rangle = 0$ and therefore $\xi = 0$.
- Since $\|(\xi, \xi)\| = \|\langle\xi, \xi\rangle\|$ ($\le$ is immediate and $\ge$ requires some play with functional calculus), it follows that $E$ is complete with the norm induced by $(\cdot, \cdot)$.

Hence $E$ is also a left Hilbert $\mathcal{K}(E)$-module. Even better, the right action of $A$ on $E$ is compatible with the left action of $\mathcal{K}(E)$ on $E$. Indeed, for $v \in \mathcal{K}(E)$, $\xi \in E$ and $a \in A$

$$(v \cdot \xi)a = v(\xi)a = v(\xi a) = v \cdot (\xi a)$$

The correct terminology is to say that $E$ is a Hilbert $(\mathcal{K}(E), A)$-bimodule.

**Definition 12.20.** Two $C^*$-algebras $A$ and $B$ are said to be **Morita equivalent** if there is a Hilbert $(A, B)$-bimodule $E$ (we use ${}_A(\cdot, \cdot)$ for $A$-valued inner product and $\langle \cdot, \cdot \rangle_B$ for the $B$-valued one) such that

1. $E$ is a full left Hilbert $A$-module, $E$ is a full right Hilbert $B$-module.
2. For all $\xi, \eta, \zeta \in E$, $a \in A$ and $b \in B$
   (2.1) $\langle a\xi, \eta \rangle_B = \langle \xi, a^*\eta \rangle_B$.
   (2.2) ${}_A(\xi b, \eta) = {}_A(\xi, \eta b^*)$.
   (2.3) ${}_A(\xi, \eta) \cdot \zeta = \xi \cdot \langle \eta, \zeta \rangle_B$.

If $A$ and $B$ are Morita equivalent $C^*$-algebras, then the module $E$ implementing the equivalence is called an ***A*-*B* imprimitivity bimodule**.

**Example 12.21.** Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Then $\mathbb{C}$ and $\mathcal{K}(\mathcal{H})$ are Morita equivalent $C^*$-algebras via the $\mathcal{K}(\mathcal{H})$-$\mathbb{C}$ imprimitivity bimodule $\mathcal{H}$.

If $A$ and $B$ are Morita equivalent, there is an equivalence between the categories of representations of $A$ and representations of $B$. To see this, we need to discuss first inner tensor products of Hilbert modules.

## 12.2 Inner Tensor product

Let $A$ and $B$ be $C^*$-algebras. Suppose $E$ is a Hilbert $B$-module, that $F$ is a Hilbert $A$-module and that there is a $*$-homomorphism $\phi : B \to \mathcal{L}(F)$. This naturally makes $F$ a left $B$-module with the action induced by $\phi$. We can then form the algebraic tensor product of $E$ and $F$ over $B$, denoted by $E \odot_B F$. To do so, we start with the algebraic tensor product $E \odot F$ and take the quotient by the subspace generated by

$$\{\xi b \otimes \eta - \xi \otimes \phi(b)\eta : \xi \in E, \eta \in F, b \in B\}$$

This quotient is $E \odot_B F$. We abuse notation and call the image of $\xi \otimes \eta$ in $E \odot_B F$ also by $\xi \otimes \eta$. Then, $E \odot_B F$ is a right $A$-module with an action defined by

$$(\xi \otimes \eta)a = \xi \otimes (\eta a)$$

We now define an $A$-valued inner product on $E \odot_B F$. First we put

$$\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle := \langle \eta, \phi(\langle \xi, \xi' \rangle)\eta' \rangle$$

for any $\xi, \xi' \in E$ and $\eta, \eta' \in F$. One checks that this is indeed a well defined $A$-valued inner product on $E \odot_B F$, so to get a Hilbert $A$-module we complete $E \odot_B F$ with respect to the norm induced by this inner product. We denote the completion $E \otimes_\phi F$ and we call it the interior tensor product of $E$ and $F$ by $\phi$.

**Theorem 12.22.** *If $A$ and $B$ are Morita equivalent $C^*$-algebras, then the category of representations of $A$ is equivalent to the one on $B$.*

**Sketch of Proof.** Let $E$ be the $A$-$B$ imprimitivity bimodule implementing the equivalence and $\pi : B \to \mathcal{L}(\mathcal{H}_\pi)$ be a representation of $B$. Write $\langle \cdot, \cdot \rangle_B$ for the $B$-valued right inner product on $E$. Then, regarding $\mathcal{H}_\pi$ as a right $\mathbb{C}$-module, we can form the Hilbert space $E \otimes_\pi \mathcal{H}_\pi$ whose inner product on elementary tensors looks like

$$\langle \xi_1 \otimes \upsilon_1, \xi_2 \otimes \upsilon_2 \rangle = \langle \upsilon_1, \pi(\langle \xi_1, \xi_2 \rangle_B)\upsilon_2 \rangle)$$

for $\xi_k \in E$ and $\upsilon_k \in \mathcal{H}_B$. We define $\mathrm{Ind}\pi : A \to \mathcal{L}(E \otimes_\pi \mathcal{H}_\pi)$ by first letting

$$[\mathrm{Ind}\pi(a)](\xi \otimes \upsilon) = (a\xi) \otimes \upsilon$$

and then extending to all $E \otimes_\pi \mathcal{H}_\pi$. Using that $A$ is Morita equivalent to $B$, this gives a $*$-homomorphism and therefore $\mathrm{Ind}\pi$ is a representation of $A$. One checks that $\pi$ is irreducible if and only if $\mathrm{Ind}\pi$ is irreducible and every irreducible representation of $A$ is of this form. The Functor Ind from the category of representations of $A$ to the one of representations of $B$ is the one implementing the equivalence. "□"

Department of Mathematics, University of Oregon, Eugene OR 97403-1222, USA.
*E-mail address*: `alonsod@uoregon.edu`