Analysis: Qual. Review

Alonso Delfín
University of Oregon.

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1.1 Theorems and Definitions.

1.1.1 Basic Measure Theory

Definition. Let X be a set. A ***toplogy on*** X is a set $\mathcal{T} \subseteq \mathcal{P}(X)$ such that

(1) $\varnothing, X \in \mathcal{T}$.

(2) If $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$.

(3) If $\{U_\alpha\}_{\alpha \in I} \subseteq \mathcal{T}$, then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$.

Elements of $\mathcal{T}$ are called ***open sets***. ▲

Definition. Let X be a set. A ***σ-algebra on*** X is a set $\mathfrak{M} \subseteq \mathcal{P}(X)$ such that

(1) $\varnothing, X \in \mathfrak{M}$.

(2) If $E \in \mathfrak{M}$, then $X \setminus E \in \mathfrak{M}$.

(3) If $\{E_n\}_{n \in \mathbb{N}} \subseteq \mathfrak{M}$, then $\bigcup_{n \in \mathbb{N}} E_n \in \mathfrak{M}$.

Elements of $\mathfrak{M}$ are called ***measurable sets***. ▲

Definition. Let X be a set and $\mathfrak{M}$ a σ-algebra on X. The pair $(X, \mathfrak{M})$ is called a ***measurable space***. A measure on $(X, \mathfrak{M})$ is a function $\mu : \mathfrak{M} \to [0, \infty]$ such that

(1) $\mu(\varnothing) = 0$.

(2) If $E_1, E_2, \ldots$ are pairwise disjoint elements of $\mathfrak{M}$, then

$$\mu\left(\bigcup_{n \in \mathbb{N}} E_n\right) = \sum_{n \in \mathbb{N}} \mu(E_n).$$

The triple $(X, \mathfrak{M}, \mu)$ is called a ***measure space***. If $\mu(X) < \infty$, the measure μ is said to be ***finite***; and if there is a partition $\{X_n\}_{n \in \mathbb{N}}$ of X such that $\mu(X_n) < \infty$ for all $n \in \mathbb{N}$, the measure μ is said to be ***σ-finite***. ▲

Proposition. *(Properties of measures) Let* $(X, \mathfrak{M}, \mu)$ *be a measure space,* $E, F \in \mathfrak{M}$ *such that* $E \subseteq F$*,* $\{E_n\}_{n\in\mathbb{N}} \subset \mathfrak{M}$ *such that* $E_n \subseteq E_{n+1}$*, and* $\{F_n\}_{n\in\mathbb{N}} \subset \mathfrak{M}$ *such that* $F_n \supseteq F_{n+1}$ *and* $\mu(F_1) < \infty$*. Then,*

(a) $\mu(E) \leq \mu(F)$.

(b) If $\mu(E) < \infty$*, then* $\mu(F \setminus E) = \mu(F) - \mu(E)$.

(c)

$$\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \lim_{n\to\infty} \mu(E_n) \quad \text{and} \quad \mu\left(\bigcap_{n\in\mathbb{N}} F_n\right) = \lim_{n\to\infty} \mu(F_n)$$

Definition. Let $(X, \mathfrak{M})$ and $(Y, \mathfrak{N})$ be measurable spaces. A function $h : X \to Y$ is said to be ***measurable*** if for all $F \in \mathfrak{N}$ it follows that $h^{-1}(F) \in \mathfrak{M}$. If $(Y, \mathcal{T})$ is instead a toplogical space, we only ask that $h^{-1}(V) \in \mathfrak{M}$ for any $V \in \mathcal{T}$. (In fact, any toplogical space is a measurable one with the sigma algebra being the smallest one containing all open sets) ▲

Definition. Let $(X, \mathfrak{M})$ be a measurable space. A ***simple function*** $s : X \to \mathbb{C}$ is a function such that there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ and $E_1, E_2, \ldots E_n \in \mathfrak{M}$ such that $X = \bigsqcup_{k=1}^{n} E_k$ and

$$s = \sum_{k=1}^{n} \alpha_k \chi_{E_k}$$

Furthermore, if μ is a measure on $(X, \mathfrak{M})$, we define ***the integral of*** s ***over*** X ***with respect to*** μ as follows

$$\int_X s \; d\mu := \sum_{k=1}^{n} \alpha_k \mu(E_k)$$

▲

Lemma. *Let* $(X, \mathfrak{M})$ *be a measurable space. If* $f : X \to [0, \infty]$ *is measurable , there is a sequence* $(s_n)_{n=1}^{\infty}$ *of non-negative simple functions such that* $s_n \to f$ *pointwise and* $s_n \leq s_{n+1}$ *for all* $n \in \mathbb{N}$.

Definition. Let $(X, \mathfrak{M}, \mu)$ be a measure space and $f : X \to [0, \infty]$ a non-negative measurable function. Denote by $\mathcal{S}^+(X, \mathfrak{M})$ the set of all non-negative simple functions. We define ***the integral of*** f ***over*** X ***with***

respect to μ as follows

$$\int_X f \; d\mu := \sup\left\{\int_X s \; d\mu : s \in \mathcal{S}^+(X, \mathfrak{M}), \;\; s \leq f\right\}$$

▲

Theorem. *(Monotone Convergence Theorem) Let $(X, \mathfrak{M}, \mu)$ be a measure space and $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable non-negative functions such that*

(i) $f_n \leq f_{n+1}$ *for all* $n \in \mathbb{N}$

(ii) $f(x) := \lim_{n\to\infty} f_n(x)$ *for all* $x \in X$

Then, f is measurable and

$$\int_X f \; d\mu = \lim_{n\to\infty} \int_X f_n \; d\mu$$

Theorem. *(Fatou's Lemma) Let $(X, \mathfrak{M}, \mu)$ be a measure space and $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable non-negative functions. Then,*

$$\int_X \liminf_{n\to\infty} f_n \; d\mu \leq \liminf_{n\to\infty} \int_X f_n \; d\mu$$

Notation. Let X be a set. For a function $g : X \to [-\infty, \infty]$ we define functions $g_+, g_- : X \to [0, \infty]$ by

$$g_+(x) := \max\{g(x), 0\} \qquad \text{and} \qquad g_-(x) := \max\{-g(x), 0\}$$

Observe that $g = g_+ - g_-$. For a function $h : X \to \mathbb{C}$ we denote by $\mathrm{Re}h, \mathrm{Im}h : X \to \mathbb{R}$ the real and imaginary parts of h respectively. Observe that $h = \mathrm{Re}h + i\mathrm{Im}h$. ▼

Definition. Let $(X, \mathfrak{M}, \mu)$ be a measure space and let R be either $[-\infty, \infty]$ or $\mathbb{C}$. If $f : X \to R$ is measurable, we define ***the integral of*** f ***over*** X ***with respect to*** μ as follows

$$\int_X f \, d\mu := \int_X (\mathrm{Re}f)_+ \; d\mu - \int_X (\mathrm{Re}f)_- \; d\mu + i\left(\int_X (\mathrm{Im}f)_+ \; d\mu - \int_X (\mathrm{Im}f)_- \; d\mu\right)$$

Further, we say that f is ***integrable*** with respect to μ if

$$\int_X |f| \; d\mu < \infty$$

▲

Theorem. *(Lebesgue Dominated Convergence Theorem) Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and* $(f_n)_{n\in\mathbb{N}}$ *be a sequence of measurable functions* $f_n : X \to \mathbb{C}$ *such that*

(i) $f(x) := \lim_{n\to\infty} f_n(x)$ *exists for all* $x \in X$

(ii) There is an integrable function $g : X \to [0, \infty)$ *such that* $|f| \leq g$.

Then, each f_n *and* f *are integrable. Moreover,*

$$\int_X f \; d\mu = \lim_{n\to\infty} \int_X f_n \; d\mu$$

1.1.2 Lebesgue Measure

Definition. Let X be a set. A function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is an ***outer measure*** if

(1) $\mu^*(\varnothing) = 0$.

(2) If $A \subseteq B \subseteq X$, then $\mu^*(A) \leq \mu^*(B)$.

(3) If $A_1, A_2, \ldots \subseteq X$, then

$$\mu^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mu^*(A_n)$$

Furthermore, we say that $E \in \mathcal{P}(X)$ is μ^****-measurable*** if for any $A \subseteq X$ we have

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap (X \setminus E))$$

▲

Remark. Intuitively, since $A \cap E$ and $A \cap (X \setminus E)$ are disjoint, we note that μ^*-measurable sets are those that "break" any subset of X as expected with respect to μ^*. ▼

Theorem. *(Carathéodory construction) Let* X *be a set and* μ^* *an outer measure. The* μ^**-measurable sets form a* σ*-algebra on* X.

Definition. Let $d \in \mathbb{N}$. An ***open box*** in $\mathbb{R}^d$ is a subset of the form

$$B := (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_d, b_d)$$

where for each $1 \leq k \leq d$ we have that $a_k \leq b_k$. We define its volume by

$$\text{vol}(B) := \prod_{k=1}^{d} (b_k - a_k)$$

▲

Definition. Let $d \in \mathbb{N}$. For any $A \subseteq \mathbb{R}^d$ we denote by $\mathcal{B}(A)$ to the set of all countable covers of A by open boxes and we define $m_d^* : \mathcal{P}(\mathbb{R}^d) \to [0, \infty]$ by

$$m_d^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \text{vol}(B_n) : \{B_n\}_{n \in \mathbb{N}} \in \mathcal{B}(A) \right\}$$

▲

Theorem. *The function $m_d^* : \mathcal{P}(\mathbb{R}^d) \to [0, \infty]$ is an outer measure.*

Definition. The ***Lebesgue measurable sets*** on $\mathbb{R}^d$, denoted by $\mathfrak{L}_d$ are those m_d^*-measurable sets and the ***Lebesgue measure*** on $\mathbb{R}^d$, denoted by m_d, is the restriction of m_d^* to $\mathfrak{L}_d$. ▲

Definition. If $(X, \mathcal{T})$ is a toplogical space, the σ-algebra $\mathfrak{B}_X$ generated by all the open sets of X is called the ***Borel σ-algebra of*** X and its elements are called ***Borel sets***. If $\mathfrak{M}$ is another σ-algebra on X and μ a measure on $(X, \mathfrak{M})$, we say that μ is a ***Borel measure*** if $\mathfrak{B}_X \subseteq \mathfrak{M}$. ▲

Theorem. *Let $d \in \mathbb{N}$. Then $\mathfrak{B}_{\mathbb{R}^d} \subset \mathfrak{L}_d$ and therefore m_d is a Borel measure.*

Definition. Let $(X, \mathcal{T})$ be a toplogical space. We say that X is ***locally compact*** if every point of X has an open neighborhood whose closure is compact. ▲

Definition. Let $(X, \mathcal{T})$ be a locally compact Hausdorff space. A Borel measure μ is said to be ***regular*** if

(1) $\mu(K) < \infty$ for all $K \subset X$ compact.

(2) $\mu(E) = \inf\{\mu(U) : U \in \mathcal{T}, E \subseteq U\}$ for all $E \in \mathfrak{B}_X$

(3) If $E \in \mathfrak{B}_X$ with $\mu(E) < \infty$ or if $E \in \mathcal{T}$, then

(3.1) $\mu(E) = \sup\{\mu(K) : K \text{ is compact } , K \subseteq E\}$
(3.2) $\mu(E) = \sup\{\mu(C) : X \setminus C \in \mathcal{T}, C \subseteq E\}$

Condition (2) is known as ***outer regular***, (3.1) as ***strong inner regular*** and (3.2) as ***weak inner regular***. ▲

Theorem. *Lebesgue measure m_d is regular.*

Definition. Let X be a locally compact Hausdorff space. The set of ***continuous functions that vanish at infinity*** is defined as follows

$$C_0(X) := \{f \in C(X) : \forall \varepsilon > 0 , \{x \in X : |f(x)| \geq \varepsilon\} \text{ is compact } \}$$

The set of ***continuous functions with compact support*** is

$$C_c(X) := \{f \in C(X) : \operatorname{supp}(f) \text{ is compact } \}$$

▲

Theorem. *Let X be a locally compact Hausdorff space and equip $C(X)$ with the* sup *norm. Then, $C_0(X)$ is closed in $C(X)$ and $C_c(X)$ is dense in $C_0(X)$.*

Theorem. *(Lusin) Let X be a locally compact Hausdorff space and μ a regular Borel measure. If $f : X \to \mathbb{C}$ is measurable and $A \subset X$ is such that*

(i) $\mu(A) < \infty$

(ii) $f(x) = 0$ *for all* $x \notin A$

Then, for all $\varepsilon > 0$ there is $g \in C_c(X)$ such that

(a) $\mu(\{x \in X : f(x) \neq g(x)\}) < \varepsilon$

(b) $\sup_{x \in X} |g(x)| \leq \sup_{x \in X} |f(x)|$

1.1.3 L^p spaces

Definition. Let $(X, \mathfrak{M}, \mu)$ be a measure space and $p \in [1, \infty)$. The space of p***-integrable functions with respect to*** μ is

$$L^p(X, \mathfrak{M}, \mu) := \frac{\{\text{measurable functions } f : X \to \mathbb{C} : |f|^p \text{ is integrable}\}}{\{\text{measurable functions } f : X \to \mathbb{C} : f = 0 \text{ a.e } [\mu]\}}$$

▲

Theorem. *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and* $p \in [1, \infty)$. *Put*

$$\|[f]\|_p := \left(\int_X |f|^p\right)^{1/p}$$

Then, $\|\cdot\|_p : L^p(X, \mathfrak{M}, \mu) \to [0, \infty)$ *is a well defined norm (so from now on we write* $\|f\|_p := \|[f]\|_p$*) and* $L^p(X, \mathfrak{M}, \mu)$ *is complete with respect to this norm.*

Theorem. *(Hölder) Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and* $p, q \in (1, \infty)$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $f \in L^p(X, \mathfrak{M}, \mu)$ *and* $g \in L^q(X, \mathfrak{M}, \mu)$, *then* $fg \in L^1(X, \mathfrak{M}, \mu)$ *and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Definition. Let $(X, \mathfrak{M}, \mu)$ be a measure space. A measurable function $f : X \to \mathbb{C}$ is said to be ***essentially bounded with respect to*** μ if

$$\|f\|_\infty := \inf\{\alpha > 0 : \mu(\{x \in X : |f(x)| > \alpha\}) = 0\} < \infty$$

The space of essentially bounded functions is denoted by $L^\infty(X, \mathfrak{M}, \mu)$. ▲

Theorem. *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space. Then,* $\|\cdot\|_\infty$ *is a well defined norm and* $L^\infty(X, \mathfrak{M}, \mu)$ *is complete with respect to this norm.*

Theorem. *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space. If* $f \in L^1(X, \mathfrak{M}, \mu)$ *and* $g \in L^\infty(X, \mathfrak{M}, \mu)$, *then* $fg \in L^1(X, \mathfrak{M}, \mu)$ *and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_\infty$$

1.1.4 Complex and Signed measures

Definition. Let $(X, \mathfrak{M})$ be a measurable space. A function $\nu : \mathfrak{M} \to \mathbb{C}$ is a ***complex measure*** if whenever $E_1, E_2, \ldots$ are pairwise disjoint elements in $\mathfrak{M}$, it follows that

$$\nu\left(\bigcup_{n \in \mathbb{N}} E_n\right) = \sum_{n \in \mathbb{N}} \nu(E_n).$$

As consequence of this definition we see that $\nu(\varnothing) = 0$ and that μ may not assume infinite values. ▲

Definition. Let $(X, \mathfrak{M})$ be a measurable space. A function $\nu : \mathfrak{M} \to (-\infty, \infty]$ (or $\to [-\infty, \infty)$), is a ***signed measure*** if whenever $E_1, E_2, \ldots$ are pairwise disjoint elements in $\mathfrak{M}$, it follows that

$$\nu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sum_{n\in\mathbb{N}} \nu(E_n) \quad \text{and} \quad \nu(\varnothing) = 0$$

▲

Definition. Let $(X, \mathfrak{M})$ be a measurable space and ν a complex or signed measure. A measurable set E is said to be ***ν-null*** if whenever $F \subseteq E$ it follows that $\nu(F) = 0$. ▲

Definition. Let $(X, \mathfrak{M})$ be a measurable space and ν a signed measure. A measurable set E is said to be ***ν-positive*** if whenever $F \subseteq E$ it follows that $\nu(F) \geq 0$. A measurable set E is said to be ***ν-negative*** if whenever $F \subseteq E$ it follows that $\nu(F) \leq 0$. ▲

Definition. Let $(X, \mathfrak{M})$ be a measurable space. Two measures μ, λ are ***mutually singular***, denoted by $\mu \perp \lambda$, if there is a measurable set A such that $\mu(A) = 0 = \lambda(X \setminus A)$. If the measures are instead complex or signed measures, we require A to be μ-null and $X \setminus A$ is λ-null. ▲

Definition. Let $(X, \mathfrak{M})$ be a measurable space. A measure μ is ***absolutely continuous with respect to another measure*** λ, denoted by $\mu \ll \lambda$, if whenever $\lambda(E) = 0$ it follows that $\mu(E) = 0$. If the measures are instead complex or signed measures, we require that λ-null implies μ-null. ▲

Proposition. *Let $(X, \mathfrak{M})$ be a measurable space and let μ, λ be finite measures. Then $\mu \ll \lambda$ if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $\lambda(E) < \delta$, then $\mu(E) < \varepsilon$.*

Theorem. *(Hahn Decomposition) Let $(X, \mathfrak{M})$ be a measurable space. If μ is a signed measure, then there exist measurable sets P and N such that P is μ-positive, N is μ-negative and $X = P \sqcup N$. The pair (P, N) is called **a Hahn decomposition of** X. Furthermore, a pair (P_0, N_0) is a Hahn decomposition of X if and only if $P \triangle P_0$ and $N \triangle N_0$ are μ-null.*

Theorem. *(Jordan Decomposition) Let $(X, \mathfrak{M})$ be a measurable space. If μ is a signed measure, then there exist measures μ_+, μ_- such that $\mu = \mu_+ - \mu_-$ and $\mu_+ \perp \mu_-$. Moreover, If $\mu = \lambda_1 - \lambda_2$, then $\mu_+ \leq \lambda_1$ and $\mu_- \leq \lambda_2$, and if also $\lambda_1 \perp \lambda_2$, then $\mu_+ = \lambda_1$ and $\mu_- = \lambda_2$.*

Theorem. *(Radon-Nikodym) Let $(X, \mathfrak{M})$ be a measurable space, μ a σ-finite measure and ν a complex measure such that $\nu \ll \mu$. Then, there exists*

$f : X \to \mathbb{C}$ such that $f \in L^1(\mu)$ and

$$\nu(E) = \int_E f \ d\mu$$

*The function f is called a **Radon-Nikodym derivative of ν with respect to** μ, f is unique a.e $[\mu]$ and it's denoted as*

$$f = \frac{d\nu}{d\mu}$$

Theorem. *(Lebesgue Decomposition) Let $(X, \mathfrak{M})$ be a measurable space, μ a σ-finite measure and ν a complex measure. Then, there exist unique complex measures ν_s and ν_a such that $\nu = \nu_s + \nu_a$, $\nu_s \perp \mu$ and $\nu_a \ll \mu$.*

Proposition. *Let $(X, \mathfrak{M})$ be a measurable space. We denote by $M(X, \mathfrak{M})$ to the set of all complex measures on $(X, \mathfrak{M})$. For $\mu \in M(X, \mathfrak{M})$ and $E \in \mathfrak{M}$ let*

$$|\mu|(E) := \sup \left\{ \sum_{n=1}^{\infty} |\mu(E_n)| : \{E_n\}_{n\in\mathbb{N}} \subset \mathfrak{M} \text{ is a pairwise disjoint cover of } E \right\}$$

Then, $|\mu|$ is a finite measure and $M(X, \mathfrak{M})$ is a Banach space with norm given by $\|\mu\| := |\mu|(X)$.

Theorem. *Let $(X, \mathfrak{M})$ be a measurable space and $\mu \in M(X, \mathfrak{M})$. Then there is a function $h : X \to \mathbb{C}$ such that $|h| = 1$ a.e $[|\mu|]$ and*

$$\mu(E) = \int_E h \ d|\mu|$$

Definition. Let X be a topological space and $\mu \in M(X, \mathfrak{M})$. We say that μ is regular if $|\mu|$ is regular. ▲

Theorem. *(Riesz Representation Theorems) Let X be a locally compact Hausdorff space.*

(a) If $\omega \in C_0(X)^*$, then there exist a unique regular measure $\mu \in M(X, \mathfrak{B}_X)$ such that

$$\omega(f) = \int_X f \ d\mu \quad \text{and} \quad \|\omega\| = \|\mu\|$$

(b) If $\omega \in C_c(X)^*$ is positive (i.e. $\omega(f) \geq 0$ whenever $f \geq 0$), then there exist a unique regular positive measure μ such that

$$\omega(f) = \int_X f \ d\mu$$

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2.1 Theorems and Definitions.

2.1.1 Fubini's Theorem

Definition. Let $(X, \mathfrak{M})$ and $(Y, \mathfrak{N})$ be measurable spaces. A ***measurable rectangle*** is a set of the form $E \times F$ where $E \in \mathfrak{M}$ and $F \in \mathfrak{N}$. ▲

Definition. Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be σ-finite measure spaces. Then, we denote by $\mathfrak{M} \otimes \mathfrak{N}$ to the σ-algebra on $X \times Y$ geberated by the measurable rectangles. ▲

Theorem. *Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be σ-finite measure spaces. Then, there exist a unique measure $\mu \otimes \nu$ on $\mathfrak{M} \otimes \mathfrak{N}$ such that*

$$(\mu \otimes \nu)(E \times F) = \mu(E)\nu(F) \ \forall \ (E \in \mathfrak{M}, \ F \in \mathfrak{N}).$$

Theorem. *(Tonelli [Fubini]) Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be σ-finite measure spaces, $f : X \times Y \to \mathbb{C}$ a $\mathfrak{M} \otimes \mathfrak{N}$-measurable function. Suppose that f is non-negative [f is $\mu \otimes \nu$-integrable]. Then,*

(a) For all $y \in Y$, $x \mapsto f(x, y)$ is $\mathfrak{M}$-measurable [μ-integrable for a.e. $y \in Y$].

(b) The function

$$y \mapsto \int_X f(x, y) \ d\mu(x)$$

is defined a.e $[\nu]$ and it's $\mathfrak{N}$-measurable [ν-integrable].

(c)

$$\int_Y \left(\int_X f(x, y) \ d\mu(x) \right) d\nu(y) = \int_{X \times Y} f \ d(\mu \otimes \nu)$$

Furthermore, items (a), (b) and (c) above also hold when interchanging $y \leftrightarrow x$, $Y \leftrightarrow X$, $\mathfrak{M} \leftrightarrow \mathfrak{N}$ and $\mu \leftrightarrow \nu$.

2.1.2 Differentiation

Definition. Let $d \in \mathbb{N}$ and let μ be a regular Borel measure on $\mathbb{R}^d$. For $r > 0$ and $x \in \mathbb{R}^d$, we define

$$(Q_r\mu)(x) := \frac{\mu(B_r(x))}{m(B_r(x))}$$

where m is the Lebesgue measure in $\mathbb{R}^d$. For $x \in \mathbb{R}^d$ set

$$D_\mu(x) := \lim_{r \to 0^+} (Q_r\mu)(x)$$

▲

Definition. Let $d \in \mathbb{N}$ and let μ be a complex regular Borel measure on $\mathbb{R}^d$. For $x \in \mathbb{R}^d$ we put

$$M_\mu(x) := \sup_{r>0} (Q_r|\mu|)(x)$$

▲

Lemma. *Let $d \in \mathbb{N}$ and let μ be a complex regular Borel measure on $\mathbb{R}^d$. The function M_μ is lower semicontinuous, that is $M_\mu^{-1}(\ (t, \infty)\ )$ is open for every $t \in \mathbb{R}$.*

Definition. Let $d \in \mathbb{N}$. A ***locally integrable functions*** is a Lebesgue measurable function $f : R^d \to \mathbb{C}$ such that $\int_K |f|\ dm < \infty$ for all compact subsets $K \subset \mathbb{R}^d$. We denote the set of locally integrable functions by $L^1_{\text{loc}}(\mathbb{R}^d)$. ▲

Definition. For $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, define

$$Mf(x) := \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f|\ dm$$

▲

Definition. Let $d \in \mathbb{N}$ and $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. A point $x_0 \in \mathbb{R}^k$ is a ***Lebesgue point of*** f if

$$\lim_{r \to 0^+} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f - f(x_0)|\ dm = 0$$

▲

Theorem. *Let $d \in \mathbb{N}$ and $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. Then, the complement of the set of Lebesgue points of f has measure 0. That is, a.e. $[m]$ $x \in \mathbb{R}^d$ is a Lebesgue point.*

Theorem. *Let $d \in \mathbb{N}$ and let μ be a complex regular Borel measure on $\mathbb{R}^d$ such that $\mu \ll m$. If $f = \frac{d\mu}{dm}$ a.e. $[m]$, it follows that $f(x) = D_\mu(x)$ for all Lebesgue points of f.*

Definition. Let $d \in \mathbb{N}$ and $x \in \mathbb{R}^d$. We say that a sequence of subsets of $\mathbb{R}^d$, $(E_n)_{n\in\mathbb{N}}$, ***shrinks nicely to*** x is there is a sequence $(r_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ and $\alpha > 0$ such that

(1) $r_n \to 0$

(2) $E_n \subset B_{r_n}(x)$

(3) $m(E_n) \geq \alpha m(B_{r_n}(x))$

▲

Theorem. *If $d \in \mathbb{N}$, $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ is a Lebesgue point of f and $(E_n)_{n\in\mathbb{N}}$ shrinks nicely to x, then*

$$\lim_{n\to\infty} \frac{1}{m(E_n)} \int_{E_n} f \; fm = f(x)$$

Definition. A function $f : [a,b] \to \mathbb{C}$ is ***absolutely continuous*** if for every $\varepsilon > 0$, there is $\delta > 0$ such that if $(a_1,b_1),\ldots,(a_n,b_n)$ are disjoint intervals in $[a,b]$ and $\sum_{k=1}^n (b_k - a_k) < \delta$, it follows that $\sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon$. ▲

Theorem. *If $f : [a,b] \to \mathbb{C}$ is absolutely continuous, it's derivative f' exist a.e $[m]$, $f' \in L^1([a,b],m)$ and*

$$f(x) = f(a) + \int_a^x f' \; dm$$

Theorem. *Suppose that $f : [a,b] \to \mathbb{R}$ is non decreasing. The following are equivalent.*

(i) f is absolutely continuous.

(ii) If $m(E) = 0$, then $m(f(E)) = 0$.

(iii) f' exist a.e $[m]$, $f' \in L^1([a,b],m)$ and

$$f(x) = f(a) + \int_a^x f' \; dm$$

2.1.3 Basic Functional Analysis

Theorem. *(Hahn-Banach) Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. Let E be a normed vector space over $\mathbb{K}$ and let M be a subspace of E. If $\omega_0 : M \to \mathbb{K}$ is a bounded linear functional, there exist a linear functional $\omega : E \to \mathbb{K}$ such that*

(i) $\omega|_M = \omega_0$

(ii) $\|\omega\| = \|\omega_0\|$

Corollary I. *If E is a normed vector space and $\xi_0 \in E \setminus \{0\}$, then there exist $\omega : E \to \mathbb{C}$ such that*

(i) $\omega(\xi_0) = \|\xi_0\|$

(ii) $\|\omega\| = 1$

Corollary II. *If E is a normed vector space and $\Phi : E \to E^{**}$ is given by*

$$[\Phi(\xi)](\omega) := \omega(\xi) \quad \text{for all } \xi \in E \text{ and } \omega \in E^* \quad ,$$

then Φ is an injective isometry. We may, and do, identify E as a subset of E^{**}.

Theorem. *(Baire Category Theorem) Let X be a complete metric space. Let $U_1, U_2, \ldots,$ be open dense subsets of X. Then,*

$$\bigcap_{n=1}^{\infty} U_n \quad \text{is dense in} \quad X$$

Theorem. *(Uniform Boundedness Principle) Let E be a Banach space, F a normed vector space and $S \subseteq L(E, F)$. Assume there is a dense G_δ set $B \subseteq E$ (i.e. B is a countable intersection of open sets) such that for any $\xi \in B$,*

$$\sup_{a \in S} \|a(\xi)\| < \infty$$

Then,

$$\sup_{a \in S} \|a\| < \infty$$

Theorem. *(Open Mapping Theorem) Let E, F be Banach spaces and $a \in L(E, F)$ surjective. Then,*

(i) There is $\delta > 0$ such that $B_\delta^F(0) \subseteq a(B_1^E(0))$.

(ii) If $U \subset E$ is open, then $a(U)$ is open.

(iii) If a is also injective, then $a^{-1} \in L(F, E)$.

Theorem. *(Closed Graph) If $a : E \to F$ is linear, then a is bounded if and only if the graph of a is a closed subset of $E \times F$.*

2.1.4 Hilbert Spaces

Definition. Let $\mathcal{H}$ be a vector space over $\mathbb{C}$. A ***scalar product*** on $\mathcal{H}$ is a function $(\xi, \eta) \mapsto \langle \xi, \eta \rangle$ from $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that

(1) It's linear in ξ for each fixed η.

(2) $\overline{\langle \xi, \eta \rangle} = \langle \eta, \xi \rangle$.

(3) $\langle \xi, \xi \rangle \geq 0$.

(4) If $\langle \xi, \xi \rangle = 0$, then $\xi = 0$.

▲

Theorem. *(Cauchy-Schwarz)*

$$|\langle \xi, \eta \rangle|^2 \leq \langle \xi, \xi \rangle \langle \eta, \eta \rangle$$

Remark. A scalar product on $\mathcal{H}$ induces a norm in $\mathcal{H}$ by letting $\|\xi\| := (\langle \xi, \xi \rangle)^{1/2}$, so Cauchy-Schwarz gives $|\langle \xi, \eta \rangle| \leq \|\xi\| \|\eta\|$. ▼

Definition. If $\mathcal{H}$ is equipped with a scalar product, we say that $\mathcal{H}$ is a ***Hilbert space*** if it is complete with respect to the induced norm. ▲

Theorem. *(Parallelogram Law) In any scalar product space we have*

$$\|\xi + \eta\|^2 + \|\xi - \eta\|^2 = 2(\|\xi\|^2 + \|\eta\|^2)$$

Definition. Let $\mathcal{H}$ be a scalar product space. We say that $\xi, \eta \in \mathcal{H}$ are ***orthogonal***, denoted by $\xi \perp \eta$, if $\langle \xi, \eta \rangle = 0$. For $S, T \subseteq \mathcal{H}$, we say that S ***is orthogonal to*** T, denoted by $S \perp T$, if $\xi \perp \eta$ for all $\xi \in S$ and all $\eta \in T$. Further, we put

$$S^{\perp} := \{\eta \in \mathcal{H} : \eta \perp \xi, \ \forall \ \xi \in S\}$$

▲

Theorem. *If $\mathcal{H}$ is a Hilbert space and $S \subseteq \mathcal{H}$, then $S^{\perp}$ is a closed subset of $\mathcal{H}$.*

Theorem. *If $\mathcal{H}$ is a Hilbert space and K is a closed convex subset of $\mathcal{H}$, then there is a unique $\xi_0 \in K$ such that*

$$\operatorname{dist}(0, K) = \|\xi_0\|$$

Theorem. *Let $\mathcal{H}$ be is a Hilbert space and $M \subseteq \mathcal{H}$ a closed subspace. Then, for every $\xi \in \mathcal{H}$ there are unique $p(\xi) \in M$ and $q(\xi) \in M^{\perp}$ such that*

(1) $\xi = p(\xi) + q(\xi)$.

(2) $p : \mathcal{H} \to M$, $q : \mathcal{H} \to M^{\perp}$ *are linear with* $\|p\|, \|q\| \leq 1$ *and* $p^2 = p$, $q^2 = q$.

(3) $\|\xi\|^2 = \|p(\xi)\|^2 + \|q(\xi)\|^2$

(4) $p(\xi)$ *is the nearest point in* M *to* ξ, $q(\xi)$ *is the nearest point in* $M^{\perp}$ *to* ξ,

Corollary. *Let $\mathcal{H}$ be is a Hilbert space and $M \subseteq \mathcal{H}$ a closed subspace. Then,*

$$\mathcal{H} = M \oplus M^{\perp}$$

Theorem. *(Riesz) Let $\mathcal{H}$ be is a Hilbert space and $\omega \in \mathcal{H}^*$. Then, there is a unique $\eta \in \mathcal{H}$ such that*

$$\omega(\xi) = \langle \xi, \eta \rangle \ \forall \ \xi \in \mathcal{H}$$

Moreover, $\|\omega\| = \|\eta\|$.

Theorem. *Let $\mathcal{H}$ be is a Hilbert space and $a \in L(\mathcal{H})$. Then there is a unique $a^* \in L(\mathcal{H})$ such that*

$$\langle a\xi, \eta \rangle = \langle \xi, a^*\eta \rangle \ \forall \ \xi, \eta \in \mathcal{H}$$

Furthermore, for any $a, b \in L(\mathcal{H})$ and $\lambda \in \mathbb{C}$

(1) $\|a^*\| = \|a\|$.

(2) $(a + \lambda b)^* = a^* + \overline{\lambda} b^*$.

(3) $(ab)^* = b^* a^*$.

(4) $\mathrm{id}^* = \mathrm{id}$.

(5) $(a^*)^* = a$.

(6) $\|a^*a\| = \|a\|^2$.

Definition. Let $\mathcal{H}$ be is a Hilbert space. A family $(\xi_i)_{i\in I}$ of elements in $\mathcal{H}$ is: ***orthogonal*** if $\xi_i \perp \xi_j$ for all $i \neq j$ in I; ***orthonormal*** if in also $\|xi_i\| = 1$ for all $i \in I$; and a ***Hilbert basis*** if in addition $\mathrm{span}(\{\xi_i : i \in I\})$ is dense in $\mathcal{H}$. ▲

Definition. Let $\mathcal{H}$ be is a Hilbert space and $(\xi_i)_{i\in I}$ a family of elements in $\mathcal{H}$. We say that

$$\sum_{i\in I} \xi_i \to \xi \in \mathcal{H}$$

if for all $\varepsilon > 0$ there is a finite set $F \subset I$, such that for every finite set $S \subset I$ with $F \subset S$, we have

$$\Big\| \sum_{i\in S} \xi_i - \xi \Big\| < \varepsilon$$

▲

Theorem. *Let $\mathcal{H}$ be any Hilbert space.*

(a) $\mathcal{H}$ has an orthonormal basis.

(b) Any other orthonormal basis of $\mathcal{H}$ has the same cardinality.

(c) For an orthonormal basis $(\xi_i)_{i\in I}$, the map $\ell^2(I) \to \mathcal{H}$ given by

$$(x_i)_{i\in I} \mapsto \sum_{i\in I} x_i \xi_i$$

is an isometric isomorphisim with inverse given by

$$\xi \mapsto (\langle \xi, \xi_i \rangle)_{i\in I}$$

Theorem. *Let $\mathcal{H}$ be is a Hilbert space and $(\xi_i)_{i\in I}$ a family of elements in $\mathcal{H}$ such that $\overline{\mathrm{span}(\{\xi_i : i \in I\})} = M$. Then,*

(a) For all $\xi \in \mathcal{H}$,

$$\sum_{i\in I} |\langle \xi, \xi_i \rangle|^2 \leq \|\xi\|^2$$

(b) The orthogonal projection $p : \mathcal{H} \to M$ is

$$p\xi = \sum_{i\in I} \langle \xi, \xi_i \rangle \xi_i$$

2.1.5 Basic Fourier Analysis

Definition. For $f, g \in L^1(\mathbb{R}, m)$ we define the ***convolution*** $f * g$ by

$$(f * g)(x) := \int_{\mathbb{R}} f(y)g(x - y)\ dm(y)$$

▲

Theorem. *If $f, g \in L^1(\mathbb{R}, m)$ are Lebesgue measurable, then*

(a) $(x, y) \mapsto f(y)g(x - y)$ *is Lebesgue measurable*

(b) $y \mapsto f(y)g(x - y)$ *is integrable for a.e.* x

(c) $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$

Definition. Put $\overline{m} := (2\pi)^{-1/2} m$, and let $f \in L^1(\mathbb{R}, \overline{m})$. We define the ***Fourier Transform of*** f, denoted by $\widehat{f}$, by

$$\widehat{f}(t) := \int_{\mathbb{R}} e^{-itx} f(x)\ d\overline{m}(x)$$

▲

Proposition. *(Properties of $\widehat{f}$) For $f \in L^1(\mathbb{R}, \overline{m})$, $\alpha \in \mathbb{R}$ and $\beta \in (0, \infty)$ we have*

(a) If $g(x) = e^{i\alpha x} f(x)$*, then* $\widehat{g}(t) = \widehat{f}(t - \alpha)$

(b) If $g(x) = f(x - \alpha)$*, then* $\widehat{g}(t) = e^{-i\alpha t}\widehat{f}(t)$

(c) If $g \in L^1(\mathbb{R}, \overline{m})$*, then* $\widehat{(f * g)} = \widehat{f} \cdot \widehat{g}$

(d) If $g(x) = \overline{f(-x)}$*, then* $\widehat{g}(t) = \overline{\widehat{f}(t)}$

(e) If $g(x) = f(\beta^{-1}x)$*, then* $\widehat{g}(t) = \beta \widehat{f}(t)$

(f) If $g(x) = -ixf(x)$ *is such that* g *is in* $L^1(\mathbb{R}, \overline{m})$*, then* $(\widehat{f})'$ *exists and is equal to* $\widehat{g}$*.*

Theorem. *The map* $f \mapsto \widehat{f}$ *is a contractive linear map from* $L^1(\mathbb{R}, \overline{m})$ *to* $C_0(\mathbb{R})$*; that is* $\widehat{f} \in C_0(\mathbb{R})$ *and* $\|\widehat{f}\|_\infty \leq \|f\|_1$.

Theorem. *(Fourier Inversion) Suppose that* $f \in L^1(\mathbb{R}, \overline{m})$ *and that* $\widehat{f} \in L^1(\mathbb{R}, \overline{m})$*. If*

$$g(x) := \int_{\mathbb{R}} e^{itx} \widehat{f}(t)\ d\overline{m}(t),$$

then $g \in C_0(\mathbb{R})$ *and* $g = f$ *a.e.*

Corollary. *If* $f \in L^1(\mathbb{R}, \overline{m})$ *and* $\widehat{f} = 0$*, then* $f = 0$ *a.e.*

Theorem. *(Plancherel) There is a mapping* $\mathcal{F} : L^2(\mathbb{R}, \overline{m}) \to L^2(\mathbb{R}, \overline{m})$ *such that*

(1) $\mathcal{F}(f) = \widehat{f}$ *for all* $f \in L^1(\mathbb{R}, \overline{m}) \cap L^2(\mathbb{R}, \overline{m})$*.*

(2) $\|\mathcal{F}(f)\|_2 = \|f\|_2$ *for all* $f \in L^2(\mathbb{R}, \overline{m})$

(3) $\mathcal{F}$ *is a Hilbert space isomorphism of* $L^2(\mathbb{R}, \overline{m})$ *onto itself.*

(4) If

$$\varphi_A(t) := \int_{-A}^{A} e^{-itx} f(x)\ d\overline{m}(x) \quad \text{and} \quad \psi_A(x) := \int_{-A}^{A} e^{itx} \mathcal{F}(f)(t)\ d\overline{m}(t),$$

Then

$$\lim_{A\to\infty} \|\varphi_A - \mathcal{F}(f)\|_2 = 0 \quad \text{and} \quad \lim_{A\to\infty} \|\psi_A - f\|_2 = 0$$

Corollary. *If* $f \in L^2(\mathbb{R}, \overline{m})$ *and* $\mathcal{F}(f) \in L^1(\mathbb{R}, \overline{m})$*, then*

$$f(x) = \int_{\mathbb{R}} e^{itx} \mathcal{F}(f)(t)\ d\overline{m}(t) \ \text{a.e}$$

Remark. $L^1(\mathbb{R}, \overline{m})$ is a Banach algebra with multiplication given by convolution. ▼

Theorem. *To every non-zero complex homomorphism* $\omega : L^1(\mathbb{R}, \overline{m}) \to \mathbb{C}$ *(i.e. Banach algebra homomorphism) corresponds a unique* $t \in \mathbb{R}$ *such that* $\omega(f) = \widehat{f}(t)$ *for all* $f \in L^1(\mathbb{R}, \overline{m})$

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3.1 Theorems and Definitions.

3.1.1 Basic Complex Analysis

Definition. For Ω an open subset of $\mathbb{C}$, $a \in \Omega$, we say that $f : \Omega \to \mathbb{C}$ is ***complex differentiable at*** a is the following limit exits

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

In such case, the limit is denoted by $f'(a)$. Equivalently, if $f := u + iv$, $z = x + iy$, then f is complex differentiable at $a := x_0 + iy_0$ if u and v satisfy the Cauchy Riemann equations:

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \quad \text{and} \quad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$$

Furthermore,

$$f'(a) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$$

▲

Lemma. *If $f'(a)$ exists, then f is continuous at a.*

Definition. Let $\Omega \subseteq \mathbb{C}$ open and $f : \Omega \to \mathbb{C}$. We say that f is ***representable by power series on*** Ω if for all $a \in \Omega$ and for all $r > 0$ with $B_r(a) \subseteq \Omega$, there is a sequence $(c_n)_{n \in \mathbb{N}_0}$ in $\mathbb{C}$ such that

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n \ \forall\ z \in B_r(a)$$

▲

Definition. Let $\Omega \subseteq \mathbb{C}$ open and $f : \Omega \to \mathbb{C}$. We say that f is ***weakly representable by power series on*** Ω if for all $z_0 \in \Omega$, there is $a \in \Omega$ such that f is given by a power series about a on some neighborhood of z_0. ▲

Theorem. *For a sequence $(c_n)_{n \in \mathbb{N}_0}$ in $\mathbb{C}$, we define $R \in [0, \infty]$ by*

$$\frac{1}{R} := \limsup_{n \to \infty} |c_n|^{1/n}$$

Then,

(a) *If* $|z-a| > R$, *then* $c_n(z-a)^n \not\to 0$ *as* $n \to \infty$.

(b) *For all* $s \in (0, R)$, *for all* $a \in \mathbb{C}$, *the series* $\sum_{n\in\mathbb{N}_0} c_n(z-a)^n$ *converges uniformly and absolutely on* $\overline{B_s(a)}$

Theorem. *Let* $\Omega \subseteq \mathbb{C}$ *open . If* $f : \Omega \to \mathbb{C}$ *is representable by power series on* Ω, *then* f *is complex differentiable on* Ω *and* f' *is also representable by power series on* Ω. *Further,*

$$\text{if } f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n, \text{ then } f'(z) = \sum_{n=1}^{\infty} n c_n (z-a)^{n-1}.$$

Corollary. *If* $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$, *then* $f^{(k)}(a) = k! c_k$ *for all* $n \in \mathbb{N}_0$.

Lemma. *Let* (X, μ) *be a complex measure space,* $\Omega \subseteq \mathbb{C}$ *open, and* $\varphi : X \to \mathbb{C}$ *measurable with* $\varphi(X) \cap \Omega = \varnothing$. *Then,*

$$f(z) := \int_X \frac{1}{\varphi(x) - z} d\mu(x)$$

is representable by power series on Ω *with coefficients around* $a \in \Omega$ *given by*

$$c_n = \int_X \frac{1}{(\varphi(x) - a)^{n+1}} d\mu(x)$$

Definition. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a C^1 curve and f a complex valued function defined on $\text{Ran}(\gamma)$. We define the ***line integral of*** f ***along*** γ by

$$\int_\gamma f(\zeta)\, d\zeta := \int_\alpha^\beta f(\gamma(t))\, \gamma'(t)\, dt$$

▲

Definition. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a piecewise C^1 closed curve, and let $\Omega := \mathbb{C} \setminus \text{Ran}(\gamma)$. The ***winding number of*** γ ***around*** $z \in \Omega$ is defined by

$$\text{Ind}_\gamma(z) := \int_\gamma \frac{1}{\zeta - z} d\zeta$$

▲

Theorem. *Let* $\gamma : [\alpha, \beta] \to \mathbb{C}$ *be a piecewise* C^1 *closed curve, and let* $\Omega := \mathbb{C} \setminus \text{Ran}(\gamma)$. *The function* $\text{Ind}_\gamma : \Omega \to \mathbb{Z}$ *is continuous and vanishes on the unbounded component of* Ω.

Theorem. *Let $\Omega \subseteq \mathbb{C}$ open, f holomorphic on Ω and $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a piecewise C^1 closed curve such that $\operatorname{Ind}_\gamma = 0$ on $\mathbb{C} \setminus \Omega$. Then,*

(a) (Cauchy's Theorem)

$$\int_\gamma f(\zeta)\ d\zeta = 0$$

(b) (Cauchy's Formula) If $a \in \Omega \setminus \operatorname{Ran}(\gamma)$

$$\operatorname{Ind}_\gamma(z) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - a} d\zeta$$

Theorem. *Let $\Omega \subset \mathbb{C}$ be a region (i.e. open and connected), f holomorphic on Ω, and $Z(f) := \{z \in \Omega : f(z) = 0\}$. Then,*

(a) $Z(f) = \Omega$ or $Z(f)$ has no limit point in Ω

(b) If $Z(f)$ has no limit point in Ω, then $Z(f)$ is at most countable and for all $a \in \mathbb{Z}(f)$ there is a unique $n \in \mathbb{N}$ and a holomorphic function g on Ω with $g(a) \neq 0$, such that

$$f(z) = (z - a)^n g(z)\ \forall\ z \in \Omega$$

The number n is known as ***the order of the zero*** *a.*

Corollary. *Let $\Omega \subset \mathbb{C}$ be a region, f, g holomorphic on Ω and $A \subset \Omega$ with limit points in ω. If $f|_A = g|_A$, then $f = g$.*

Definition. If $\Omega \subseteq \mathbb{C}$ is open and $f : \Omega \setminus \{a\} \to \mathbb{C}$ holomorphic, we say that f has ***a singularity at*** a. Further,

(1) a is ***removable*** if there is a holomorphic function g on Ω such that $g|_{\Omega \setminus \{a\}} = f$.

(2) a is ***a pole*** if there is $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{C}$, with $c_n \neq 0$ and a holomorphic function g on Ω, such that

$$f(z) = \sum_{k=1}^{n} \frac{c_z}{(z - a)^k} + g(z) \ \ \forall\ z \in \Omega \setminus \{a\}$$

The number n is known as ***the order of the pole*** a.

(3) a is ***an essential singularity*** if for all $r > 0$

$$f((B_r(a) \cap \Omega) \setminus \{a\})$$

is dense in $\mathbb{C}$.

▲

Theorem. *(Cauchy Estimates) Let $a \in \mathbb{C}$, $r > 0$. If f is holomorphic on $B_r(a)$ and $|f(z)| \leq M$ for all $z \in B_r(a)$, then*

$$|f^{(n)}(a)| \leq \frac{n!M}{r^n} \ \forall n \in \mathbb{N}$$

Theorem. *(Liouville's Theorem) Any bounded entire function is constant.*

Theorem. *(Maximum Modulus) Let $\Omega \subset \mathbb{C}$ be a region, f holomorphic on Ω and $\overline{B_r(a)} \subset \Omega$ for some $a \in \Omega$, $r > 0$.Then*

$$|f(a)| \leq \sup_\theta |f(a + re^{i\theta})|$$

Equality occurs if and only if f is constant in Ω.

Corollary. *Let $\Omega \subset \mathbb{C}$ be a region. If f is holomorphic on Ω and $|f|$ has a local maximum on Ω, then f is constant.*

Theorem. *(Open Mapping) Let $\Omega \subset \mathbb{C}$ be open, f holomorphic on Ω and $a \in \Omega$ such that $f(a) \neq 0$. Then, there is an open neighborhood $V \subset \Omega$ of a such that*

(a) $f|_V$ is injective.

(b) $W := f(V)$ is open.

(c) $(f|_V)^{-1} : W \to V$ is holomorphic.

Corollary I. *Let $\Omega \subset \mathbb{C}$ be open, f holomorphic on Ω. If f is non constant, then f is an open map.*

Corollary II. *Let $\Omega \subset \mathbb{C}$ be open, f holomorphic on Ω. If f is injective, then $f(\Omega)$ is open, f^{-1} is holomorphic and $f'(z) \neq 0$ for all $z \in \Omega$.*

Theorem. *(Morera) Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ continuous. Suppose that for every closed triangle $\Delta \subset \Omega$ we have*

$$\int_{\partial\Delta} f(\zeta)\, d\zeta = 0$$

Then, f is holomorphic on Ω.

Definition. Let $\Omega \subset \mathbb{C}$ be open. A complex valued function f is said to be ***meromorphic*** *[**a function with isolated singularities**]* ***on*** Ω if there is a subset $A \subseteq \Omega$ with no limit points in Ω such that f is holomorphic on $\Omega \setminus A$ and at each $a \in A$ f as a pole [a pole or an essential singularity]. If $a \in A$, we set

$$\mathrm{Res}(f, a) := \frac{1}{2\pi i} \int_{\gamma} f(\zeta)\ d\zeta,$$

where $\gamma(t) := a + re^{it}$ for $t \in [0, 2\pi]$ and $r > 0$ is such that $\overline{B_r(a)} \cap A = \{a\}$ and $\overline{B_r(a)} \subseteq \Omega$. One checks that $\mathrm{Res}(f, a)$ is independent of the r chosen. ▲

Theorem. *(Residue) Let $\Omega \subset \mathbb{C}$ be open and f a meromophic (or a a function with isolated singularities) with set of singularities given by A. If Γ is a cycle in $\Omega \setminus A$ such that $\mathrm{Ind}_\Gamma(z) = 0$ for all $z \in \mathbb{C} \setminus A$, then*

$$\frac{1}{2\pi i} \int_{\Gamma} f(\zeta)\ d\zeta = \sum_{a \in A} \mathrm{Ind}_\Gamma(a) \mathrm{Res}(f, a)$$

Theorem. *Let $\Omega \subset \mathbb{C}$ be open and Γ is a cycle in Ω such that*

(1) $\mathrm{Ind}_\Gamma(z) = 0$ *for all* $z \in \mathbb{C} \setminus \Omega$

(2) $\mathrm{Ind}_\Gamma(z) \in \{0, 1\}$ *for* $z \in \Omega \setminus \mathrm{Ran}(\Gamma)$.

Let $U := \{z : \mathrm{Ind}_\Gamma = 1\}$ and f a holomorphic function on Ω that is not zero on any unbounded component and with no zeros in $\mathrm{Ran}(\Gamma)$. *Define*

$$N_f := \#\text{number of zeros of } f \text{ (counting multiplicity) in } U$$

Then,

(a) (Argument Principle)

$$N_f = \frac{1}{2\pi i} \int_{\Gamma} \frac{f'(\zeta)}{f(\zeta)}\ d\zeta$$

(b) (Rouché) If g is holomorphic on Ω and $|f(z) - g(z)| < |f(z)|$ on $\mathrm{Ran}(\Gamma)$, *then $N_f = N_g$.*