C*-CORRESPONDENCES, HILBERT BIMODULES, AND THEIR L^P VERSIONS

by

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DISSERTATION ABSTRACT

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Title: C*-Correspondences, Hilbert Bimodules, and their L^p versions.

This dissertation initiates the study of L^p-modules, which are modules over L^p-operator algebras inspired by Hilbert modules over C*-algebras. The primary motivation for studying L^p-modules is to explore the possibility of defining L^p analogues of Cuntz-Pimsner algebras.

The first part of this thesis consists of investigating representations of C*-correspondences on pairs of Hilbert spaces. This generalizes the concept of representations of Hilbert bimodules introduced by R. Exel in [10]. We present applications of representing a correspondence on a pair of Hilbert spaces $(\mathcal{H}_0, \mathcal{H}_1)$, such as obtaining induced representations of both $\mathcal{L}_A(\mathsf{X})$ and $\mathcal{K}_A(\mathsf{X})$ on $\mathcal{H}_1$, and giving necessary and sufficient conditions on an (A, B) C*-correspondences to admit a Hilbert A-B-bimodule structure.

The second part is concerned with the theory of L^p-modules. Here we present a thorough treatment of L^p-modules, including morphisms between them and techniques for constructing new L^p-modules. We then use our results on representations for C*-correspondences to motivate and develop the theory of L^p-correspondences, their representations, the L^p-operator algebras they generate, and

present evidence that well-known L^p-operator algebras can be constructed from L^p-correspondences via L^p-Fock representations. Due to the technicality that comes with dealing with direct sums of L^p-correspondences and interior tensor products, we only focus on two particular examples for which a Fock space construction can be carried out. The first example deals with the L^p-module (ℓ^p_d, ℓ^q_d), for which we exhibit a covariant L^p-Fock representation that yields an L^p-operator algebra isometrically isomorphic to $\mathcal{O}^p_d$, the L^p-analogue of the Cuntz-algebra $\mathcal{O}_d$ introduced by N.C. Phillips in [21]. The second example involves a nondegenerate L^p-operator algebra A with a bicontractive approximate identity together with an isometric automorphism $\varphi_A \in \mathrm{Aut}(A)$. In this case, we also present an algebra associated to a covariant L^p-Fock representation, but due to the current lack of knowledge of universality of the L^p-Fock representation, we only show that there is a contractive map from the crossed product $F^p(\mathbb{Z}, A, \varphi_A)$ to this algebra.

This dissertation includes unpublished material.

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TABLE OF CONTENTS

Chapter Page

I. INTRODUCTION 1

C*-component 2

L^p-component 4

Notational Conventions 7

II. PRELIMINARIES ON HILBERT MODULES 9

Hilbert Modules 9

Hilbert Bimodules and C*-correspondences 14

Cuntz Algebras 18

Cuntz-Pimsner Algebras 18

III. REPRESENTING MODULES ON PAIRS OF HILBERT SPACES 38

Concrete Hilbert modules 38

Representations of Hilbert bimodules and Hilbert modules 44

Representations of C*-correspondences 48

Chapter Page

IV. PRELIMINARIES ON L^P OPERATOR ALGEBRAS 61

Multiplier Algebras 61

L^p-operator algebras. 67

Spatial Tensor Product 69

Spatial partial isometries. 70

L^p-Crossed Products 79

V. L^P-MODULES OVER L^P OPERATOR ALGEBRAS 82

L^p-modules and C*-like L^p-modules 82

Finite Direct Sum of L^p-modules 87

Countable Direct Sums of L^p-modules 89

External Tensor Product of L^p-modules 95

Morphisms of L^p-modules 98

VI. L^P CORRESPONDENCES AND THEIR L^P OPERATOR ALGEBRAS 102

L^p-correspondences 102

Tensor Product of L^p-correspondences 104

L^p-Fock representations 106

VII. FUTURE WORK 155

L^p-bimodules and Morita equivalence 155

Chapter | Page

C*-likeness of some L^p-modules 156

L^p-Fock representations and Universality 157

REFERENCES CITED 158

CHAPTER I

INTRODUCTION

Broadly speaking, the principal goal of this work is to initiate the study of modules over L^p-operator algebras, taking Hilbert modules over C*-algebras as a starting point. We refer to these objects as L^p-modules (see Definition 5.1.1). The main motivation for examining such objects was to answer the question of whether there is an analogue of Cuntz-Pimsner algebras (see [24] and [14]) for L^p-operator algebras. To attempt to answer this question, we developed the theory of L^p-correspondences (see Definition 6.1.1), their representations, and the L^p-operator algebras they generate. While more work is needed to fully answer the motivating question, this work presents evidence that some well-known L^p-operator algebras can be constructed from L^p-correspondences. Furthermore, we believe that the use of L^p-modules as a tool to study L^p-operator algebras can be compared to the use of Hilbert modules to study C*-algebras (e.g., Morita equivalence and K-theory). Therefore, part of this work also presents a thorough treatment of L^p-modules, including morphisms between them that generalize the notions of adjointable maps between Hilbert modules, and also presents techniques to get L^p modules from old ones such as direct sums and tensor products.

Given the nature of this work, it can be separated into two main components. The first component is C*-algebraic in nature and is discussed in Chapters II and III. The second component uses some of the theory developed in the C* scenario to motivate definitions for the L^p case, which is discussed in Chapters V and VI. We have also included a chapter which covers preliminary results on L^p operator algebras; see Chapter IV. Most of these results are known and already established, but will be needed throughout the L^p component of this work. In Chapter IV, we have also provided a brief introduction to multiplier algebras for Banach algebras, which will be only used for L^p-operator algebras but was worth presenting in its full generality. Finally, in Chapter VII, we describe some future directions of work related to L^p-modules.

C*-component

Chapter II contains basic definitions and known results about Hilbert modules and bimodules that will be needed throughout this work. We also present a key proposition that will be used later on, but that we did not find in the current literature (see Proposition 2.1.6 and Remark 2.1.7). Similarly, we have recorded in Chapter II basic definitions regarding C*-correspondences, their Fock representations, and both the Toeplitz and Cuntz-Pimnser algebra generated by the universal Fock and covariant Fock representations of a C*-correspondence. We have included detailed proofs of some instances in which Toeplitz and Cuntz-Pimnser algebras are isomorphic to well known C*-algebras. In particular, this chapter also contains a detailed analysis of the Fock correspondence of a C*-algebra A, together with $\varphi \in \mathrm{Aut}(A)$, via the standard Hilbert module $\ell^2(\mathbb{Z}_{\geq 0}) \otimes A$. This approach will provide a framework that will be utilized in our attempts to get an L^p-version of the isomorphism $\mathcal{O}(A, \varphi) \cong C^*(\mathbb{Z}, A, \varphi)$ from Example 2.4.17.

We then turn to Chapter III, which is based on [8]. The main goal of the chapter is to generalize the notion of representations of Hilbert bimodules on pairs of Hilbert spaces to the general setting of C*-correspondences. If A and B are C*-algebras, a representation of a Hilbert A-B-bimodule X on a pair of Hilbert spaces $(\mathcal{H}_0, \mathcal{H}_1)$ was defined by R. Exel in [10] as a triple of maps $(\pi_A, \pi_B, \pi_{\mathsf{X}})$, where

π_A is a representation of A on $\mathcal{H}_1$, π_B is a representation of B on $\mathcal{H}_0$, and π_{X} : $\mathsf{X} \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is a linear map, such that $\pi_{\mathsf{X}}(\mathsf{X})$ has the Hilbert $(\pi_A(A), \pi_B(B))$-bimodule structure where both module actions and both inner products are given by multiplication of operators. See Definition 3.2.1 for more details. That such representations do exist is shown in Propositions 4.7 and 4.8 of [10]. In Theorem 3.3.2, we represent an (A, B) C*-correspondence on a pair of Hilbert spaces in a way that generalizes representations of Hilbert bimodules. The methods we use differ significantly from those used in [10] and [29] for analogous results for Hilbert bimodules and Hilbert modules. In particular, our methods do not rely on the linking algebra but can be easily adapted to work in the Hilbert bimodule setting, making our notion of representations of C*-correspondences more general. Indeed, in Theorem 3.3.3 we adapt our methods from the C*-correspondence case to show the existence of a representation $(\pi_A, \pi_B, \pi_{\mathsf{X}})$ for any Hilbert A-B-bimodule X on a pair of Hilbert spaces $(\mathcal{H}_0, \mathcal{H}_1)$. In contrast with the results from [10], we give an explicit formula for the map π_{X}. This explicit formula is crucial in our proof of Theorem 3.3.7, where we give necessary and sufficient conditions for an (A, B) C*-correspondence to admit a Hilbert A-B-bimodule structure.

A main advantage of having a right Hilbert module represented as a subspace of $\mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is that, assuming some nondegeneracy conditions, the C*-algebras of adjointable maps and compact-module maps of the module can be faithfully represented on $\mathcal{H}_1$. Indeed, this is shown in Propositions 3.1.4, 3.1.5, and 3.2.8. These representations have not been studied in the current literature. However, they play an important role in our definitions for morphisms between L^p-modules, which in turn allow us to define L^p-correspondences.

L^p-component

Chapter IV contains all the necessary background and references for L^p-operator algebras. This includes the main tools needed to define $\mathcal{O}_d^p$, the L^p analogues of the Cuntz algebras introduced by Phillips in [21]. Here, we also present some basic constructions needed to get new L^p-operator algebras from old ones, such as the spatial tensor product $\otimes_p$ and the crossed products $F^p(G, A, \alpha)$, and $F^p_{\mathrm{r}}(G, A, \alpha)$ from [22]. We also provide proofs of analogues of some basic well known results for C*-algebras that also hold when a nondegenerate L^p-operator algebra has a contractive approximate identity. Examples of this include statements about multiplier algebras for Banach algebras (Proposition 4.1.5, Theorem 4.1.6, and Proposition 4.1.8), and also the fact that every nondegenerate representation $C_c(G, A, \alpha) \to \mathcal{L}(L^p(\mu))$ which is L^1-norm contractive is contractive with respect to the universal norm of $F^p(G, A, \alpha)$ (Proposition 4.5.2). All these results play important roles when dealing with L^p-operator algebras generated by L^p-Fock representations of an L^p-operator algebra A.

In Chapters V and VI, we take advantage of our results from Chapter III for representations on Hilbert modules and C*-correspondences to naturally define L^p-modules and L^p-correspondences. The main idea is that we are replacing Hilbert spaces with L^p-spaces. Indeed, roughly speaking, our Definition 5.1.1 for an L^p-module (X, Y) comes by looking at the conditions satisfied by the pair $(\pi_{\mathsf{X}}(\mathsf{X}), \pi_{\mathsf{X}}(\mathsf{X})^*)$ in Definition 3.2.5. A consequence of this definition is that any L^p-module (X, Y) over an L^p-operator algebra A comes equipped with a pairing $\mathsf{Y} \times \mathsf{X} \to A$. Those L^p-modules for which their norm can be recovered using such pairing are called C*-like L^p-modules, so that any Hilbert module over a C*-algebra A is actually a C*-like L^2-module. We then further develop the general theory of

L^p-modules by presenting their finite direct sums, countable direct sums, external tensor products, and finally the notion of morphisms from an L^p-module to itself, which we denote by $\mathcal{L}_A((\mathsf{X}, \mathsf{Y}))$; see (5.5.1). The algebra $\mathcal{L}_A((\mathsf{X}, \mathsf{Y}))$ is also an L^p-operator algebra and is a natural generalization of the adjointable maps for Hilbert modules. Similarly, in (5.5.2) we define $\mathcal{K}_A((\mathsf{X}, \mathsf{Y}))$, a generalization of compact-module maps in the Hilbert module setting. The definition of "adjointable" maps from an L^p-module to itself naturally gives rise to the concept of L^p-correspondence (Definition 6.1.1). Since we have shown that representations of C*-correspondences on pairs of Hilbert spaces are, in some sense, well behaved with respect to the interior tensor product (Theorem 3.3.12), we deduce from there an analogous interior tensor product construction for the L^p case (see Definition 6.2.1). Having all these tools at our disposal while working with L^p-correspondence provides evidence that we should be able to carry an analogue of Fock representations (Definition 2.4.3) and the Fock space construction (see Definition2.4.4) for L^p-correspondences.

We introduce the concept of L^p-Fock representation and its covariant version in Definition 6.3.1 and Definition 6.3.5. Thus, considering universal representations, we can also make sense, at least in principle, of both Toeplitz and Cuntz-Pimsner L^p-operator algebras for L^p-correspondences. However, given that our current definition of countable direct sums of L^p-modules (Definition 5.3.2) is not easy to deal with for an abstract correspondence, we did not attempt a general analogue of the Fock space construction. Instead, we focused on two particular examples for which the countable direct sum is tractable and show that a Fock space construction can be carried out for those examples. The first example deals with the L^p-module (ℓ_d^p, ℓ_d^q), where $d \in \mathbb{Z}_{\geq 2}$ and q is the Hölder conjugate for p. This module can be made into an L^p-correspondence for which we exhibit a covariant L^p-Fock representation via a Fock space construction. Furthermore, in Theorem 6.3.22 we show that the algebra generated by such a representation is actually isometrically isomorphic to $\mathcal{O}_d^p$. It still remains open whether this comes from the universal L^p-Fock covariant representation as in the C*-case. The second example comes from considering a nondegenerate L^p-operator algebra A with a bicontractive approximate identity. In this case the pair (A, A) is a C*-like L^p-module and can be made into an L^p-correspondence by adding an isometric automorphism $\varphi_A \in \operatorname{Aut}(A)$. We use the fact that, for this correspondence, the direct sum of the tensor correspondences can be computed in a simple fashion as a tensor product (in fact both external and internal constructions coincide for this particular case, as pointed out by Remark 6.3.31) to define a Fock space $(\mathcal{F}^p(A), \mathcal{F}^q(A))$. Moreover, with some care, we can actually mimic the C*-situation (presented in Proposition 2.4.6 and its Corollary 2.4.25) to define a covariant L^p-Fock representation of $((A, A), \varphi_A)$. The main question that arises at this point is whether the L^p-operator algebra generated by this representation is isomorphic to the crossed product $F^p(\mathbb{Z}, A, \varphi_A)$. In order to attempt to answer this question, we have defined a contractive algebra homomorphism from $F^p(\mathbb{Z}, A, \varphi_A)$ to this algebra (see Definition 6.3.48 and the map γ defined after it). Thus, for a positive answer, it still remains to find a contractive algebra homomorphism in the other direction that is inverse to γ. In Proposition 6.3.51, we give a covariant representation of $(\mathbb{Z}, A, \varphi_A)$ on the multiplier algebra of the algebra generated by the L^p-Fock covariant representation of $((A, A), \varphi_A)$. This can be thought as a crucial step required for the construction of the inverse map of γ, but since we have not established the universality of the Fock space construction, it is not an easy task to show that any other nondegenerate covariant representation of $(\mathbb{Z}, A, \varphi_A)$ factors through the one presented in Proposition 6.3.51. It might be possible to do this without universality if we add some extra assumptions to A, and this is discussed with more detail in Remark 6.3.52 at the end of Chapter VI.

Notational Conventions

We end our introduction by establishing some of our notational conventions.

Linear Maps

Let $a\colon V_0 \to V_1$ be a linear map between vector spaces. We follow the common convention of suppressing parentheses for linear maps and write $a\xi$ for the action of a on $\xi \in V_0$. However, if X and Y are vector spaces that are also modules over an algebra A and $t\colon \mathsf{X} \to \mathsf{Y}$ is a linear module map, then we write $t(x)$ for the action of t on $x \in \mathsf{X}$. This is needed to avoid potential confusion when both x and $t(x)$ happen to also be linear maps between vector spaces, which will occur frequently in this work.

If X is a subspace of linear maps between vector spaces V_0 and V_1, the product $\mathsf{X}V_0$ is defined as the linear span of elements in X acting on vectors from V_0, that is

$$\mathsf{X}V_0 = \operatorname{span}\{x\xi\colon x \in \mathsf{X} \text{ and } \xi \in V_0\} \subseteq V_1.$$

The space of bounded linear maps between two Banach spaces E_0 and E_1 is denoted by $\mathcal{L}(E_0, E_1)$ and comes equipped with the usual operator norm $\|a\| = \sup_{\|\xi\|=1} \|a\xi\|$. For a Banach space E, we write $\mathcal{L}(E)$ instead of $\mathcal{L}(E, E)$. Similarly, $\mathcal{K}(E_0, E_1)$ denotes the subspace of $\mathcal{L}(E_0, E_1)$ consisting of compact operators and we write $\mathcal{K}(E)$ instead of $\mathcal{K}(E, E)$.

Hilbert modules

We fix some terminology for Hilbert modules over C*-algebras. The A-valued right inner product for a right Hilbert module will be denoted by $\langle -,-\rangle_A$. The map $(x,y)\mapsto \langle x,y\rangle_A$ is assumed to be linear in the second variable and conjugate linear in the first. Similarly, the A-valued left inner product for a left Hilbert module will be denoted by ${}_A\langle -,-\rangle$. The map $(x,y)\mapsto {}_A\langle x,y\rangle$ is assumed linear in the first variable and conjugate linear in the second. If X is any right Hilbert A-module, we use $\mathcal{L}_A(\mathsf{X})$ to denote adjointable maps from X to itself. For each $x,y\in\mathsf{X}$ we have the "rank one" operator $\theta_{x,y}\in\mathcal{L}_A(\mathsf{X})$, given by $\theta_{x,y}(z)=x\langle y,z\rangle_A$ for any $z\in\mathsf{X}$. We write $\mathcal{K}_A(\mathsf{X})$ for the compact-module maps from X to itself, which are defined as the closed linear span of the "finite rank" operators. That is,

$$\mathcal{K}_A(\mathsf{X})=\overline{\operatorname{span}\{\theta_{x,y}\colon x,y\in\mathsf{X}\}}.$$

We regard Hilbert spaces as right Hilbert $\mathbb{C}$-modules. For this reason, our convention for inner products of Hilbert spaces is the physicist's: they are linear in the second variable and conjugate linear in the first.

L^p-spaces

If $(\Omega,\mathfrak{M},\mu)$ is a measure space, we define $L^0(\Omega,\mathfrak{M},\mu)$ to be the space of complex valued measurable functions modulo functions that vanish a.e $[\mu]$. If $p\in[1,\infty)\cup\{0\}$, most of the time will write $L^p(\mu)$ to mean $L^p(\Omega,\mathfrak{M},\mu)$.

If ν_I is counting measure on a set I, we write $\ell^p(I)$ instead of $L^p(I,2^I,\nu_I)$. In particular, when $I=\{1,\ldots,d\}$ for some $d\in\mathbb{Z}_{\geq 1}$, we simply write ℓ^p_d to mean $\ell^p(\{1,\ldots,d\})$.

CHAPTER II

PRELIMINARIES ON HILBERT MODULES

Hilbert Modules

We start by establishing basic definitions, basic notation, and some key results on Hilbert modules that will be needed throughout this work.

Definition 2.1.1. Let A be a C*-algebra and let X be a complex vector space which is also a right A-module. An *A-valued right inner product on* X is a map

$$\begin{array}{rcl} \mathsf{X} \times \mathsf{X} & \to & A \\ (x, y) & \mapsto & \langle x, y \rangle_A \end{array}$$

such that for any $x, y, y_1, y_2 \in \mathsf{X}$, $a \in A$, and $\alpha \in \mathbb{C}$ we have

1. $\langle x, y_1 + \alpha y_2 \rangle_A = \langle x, y_1 \rangle_A + \alpha \langle x, y_2 \rangle_A$.
2. $\langle x, ya \rangle_A = \langle x, y \rangle_A a$.
3. $\langle x, y \rangle_A^* = \langle y, x \rangle_A$.
4. $\langle x, x \rangle_A \geq 0$ in A.
5. $\langle x, x \rangle_A = 0$ if and only if $x = 0$.

The definition of an A-valued left inner product on X is almost identical, we start with a left A-module instead and modify conditions (1) and (2) above in the obvious way. In this case, the A-valued left inner product will be denoted by ${}_A\langle -, - \rangle$.

Definition 2.1.2. Let A be a C*-algebra. A right *Hilbert A-module* is a complex vector space X which is a right A-module with an A-valued right inner product

$$\begin{aligned} \mathsf{X} \times \mathsf{X} &\to A \\ (x, y) &\mapsto \langle x, y \rangle_A \end{aligned}$$

such that X is complete with the induced norm

$$\|x\| = \|\langle x, x \rangle_A\|^{1/2}.$$

The definition of a left Hilbert A-module is obtained after replacing every instance of the word *right* by the word *left* in the previous definition.

A main difference between Hilbert modules and Hilbert spaces is that not every bounded linear map between Hilbert A-modules has an adjoint.

Definition 2.1.3. Let X and Y be right Hilbert A-modules. A map $t\colon \mathsf{X} \to \mathsf{Y}$ is said to be *adjointable* if there is a map $t^* : \mathsf{Y} \to \mathsf{X}$ such that for any $x \in \mathsf{X}$ and $y \in \mathsf{Y}$,

$$\langle t(x), y \rangle_A = \langle x, t^*(y) \rangle_A.$$

The space of adjointable maps from X to Y is denoted by $\mathcal{L}_A(\mathsf{X}, \mathsf{Y})$ and we write $\mathcal{L}_A(\mathsf{X})$ for $\mathcal{L}_A(\mathsf{X}, \mathsf{X})$.

It standard to verify that adjointable maps between Hilbert modules are automatically linear, bounded and module maps. It is also well known that $\mathcal{L}_A(\mathsf{X})$ is a C*-algebra when equipped with the operator norm. We will have special interest in a particular case of adjointable maps, those of "rank 1":

Definition 2.1.4. Let X and Y be right Hilbert A-modules. For $x \in \mathsf{X}$ and $y \in \mathsf{Y}$, we define a map $\theta_{x,y} \colon \mathsf{Y} \to \mathsf{X}$ by

$$\theta_{x,y}(z) = x\langle y, z\rangle_A.$$

One checks that $\theta_{x,y} \in \mathcal{L}_A(\mathsf{Y}, \mathsf{X})$ with $(\theta_{x,y})^* = \theta_{y,x} \in \mathcal{L}_A(\mathsf{X}, \mathsf{Y})$. The maps $\theta_{x,y}$ give an analogue of the of rank-one operators on Hilbert spaces. So, we define an analogue of the compact operators by letting

$$\mathcal{K}_A(\mathsf{Y}, \mathsf{X}) := \overline{\operatorname{span}\{\theta_{x,y} : x \in \mathsf{X}, y \in \mathsf{Y}\}} \subseteq \mathcal{L}_A(\mathsf{Y}, \mathsf{X}).$$

We call $\mathcal{K}_A(\mathsf{Y}, \mathsf{X})$ the set of compact-module maps from X to Y. Moreover, $\mathcal{K}_A(\mathsf{X}) = \mathcal{K}_A(\mathsf{X}, \mathsf{X})$ is a closed two sided ideal in $\mathcal{L}_A(\mathsf{X})$, whence $\mathcal{K}_A(\mathsf{X})$ is also a C*-algebra.

We record the following Lemma, which is a well known fact that will be used repeatedly throughout this document.

Lemma 2.1.5. *Let A be a C*-algebra and let X be any Hilbert A-module. Then for any $t \in \mathcal{L}_A(\mathsf{X})$ and any $x \in \mathsf{X}$, we have $\langle t(x), t(x)\rangle_A \leq \|t\|^2\langle x, x\rangle_A$.*

Proof. See Proposition 1.2 in [17]. ■

Let A be a C*-algebra, let X be any right Hilbert A-module, and let $n \in \mathbb{Z}_{\geq 1}$. The direct sum X^n is usually regarded as a right Hilbert A-module in an obvious way. However, X^n can also be identified with $M_{1,n}(\mathsf{X})$, the row vectors with n entries in X. This identification makes X^n a right Hilbert $M_n(A)$-module, with the action that comes from the formal matrix multiplication $M_{1,n}(\mathsf{X}) \times M_n(A) \to$

$M_{1,n}(\mathsf{X})$. That is,

$$(x_1, \ldots, x_n) \cdot (a_{i,j})_{i,j} = \Big(\sum_{i=1}^{n} x_i a_{i,1}, \ldots, \sum_{i=1}^{n} x_i a_{i,n}\Big).$$

The $M_n(A)$-valued right inner product comes from the formal matrix multiplication $M_{n,1}(\mathsf{X}) \times M_{1,n}(\mathsf{X}) \to M_n(A)$. That is,

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle_{M_n(A)} = (\langle x_i, y_j \rangle_A)_{i,j}.$$

The following result should be well known. We include a complete proof as we couldn't find one in the current literature.

Proposition 2.1.6. *Let A be a C^*-algebra, let X be any right Hilbert A-module, and let $n \in \mathbb{Z}_{\geq 1}$. For each $t \in \mathcal{L}_A(\mathsf{X})$, we define a map $\kappa(t) \colon \mathsf{X}^n \to \mathsf{X}^n$ by*

$$\kappa(t)(x_1, \ldots, x_n) = \big(t(x_1), \ldots, t(x_n)\big).$$

*Then $\kappa(t) \in \mathcal{L}_{M_n(A)}(\mathsf{X}^n)$, and the map $t \mapsto \kappa(t)$ from $\mathcal{L}_A(\mathsf{X})$ to $\mathcal{L}_{M_n(A)}(\mathsf{X}^n)$ is a *-isomorphism.*

Proof. Firstly we show that $\kappa(t) \in \mathcal{L}_{M_n(A)}(\mathsf{X}^n)$. Indeed, an immediate calculation shows that

$$\langle \kappa(t)(x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle_{M_n(A)} = \langle (x_1, \ldots, x_n), \kappa(t^*)(y_1, \ldots, y_n) \rangle_{M_n(A)}.$$

Therefore, $\kappa(t) \in \mathcal{L}_{M_n(A)}(\mathsf{X}^n)$ and $\kappa(t)^* = \kappa(t^*)$. It is now easily checked that κ is in fact an injective $*$-homomorphism. Thus, to be done, we only need to show that κ is surjective. We establish some notation first. For any $x \in \mathsf{X}$ and any $j \in$

$\{1, \ldots, n\}$, we denote by $\delta_j x$ the element of X^n with x in the j-th coordinate and zero elsewhere. Thus,

$$(x_1, \ldots, x_n) = \sum_{j=1}^{n} \delta_j x_j.$$

Now, take any $s \in \mathcal{L}_{M_n(A)}(\mathsf{X}^n)$. We have linear maps $s_1, \ldots, s_n \colon \mathsf{X}^n \to \mathsf{X}$ such that

$$s(x_1, \ldots, x_n) = \big(s_1(x_1, \ldots, x_n), \ldots, s_n(x_1, \ldots, x_n)\big).$$

For each $i, j \in \{1, \ldots, n\}$, we define a linear map $s_{i,j} \colon \mathsf{X} \to \mathsf{X}$ by letting $s_{i,j}(x) = s_i(\delta_j x)$ for any $x \in \mathsf{X}$. Therefore,

$$s(x_1, \ldots, x_n) = \sum_{j=1}^{n} \big(s_{1,j}(x_j), \ldots, s_{n,j}(x_j)\big).$$

Since s is adjointable, we have a map $s^* \colon \mathsf{X}^n \to \mathsf{X}^n$, which in turn gives, for each $i, j \in \{1, \ldots, n\}$, a linear map $(s^*)_{i,j} \colon \mathsf{X} \to \mathsf{X}$. The equation

$$\langle s(x_1, \ldots, x_n), (y_1, \ldots, y_n)\rangle_{M_n(A)} = \langle (x_1, \ldots, x_n), s^*(y_1, \ldots, y_n)\rangle_{M_n(A)},$$

becomes

$$\left(\left\langle \sum_{k=1}^{n} s_{i,k}(x_k), y_j \right\rangle_A\right)_{i,j} = \left(\left\langle x_i, \sum_{k=1}^{n} (s^*)_{j,k}(y_k) \right\rangle_A\right)_{i,j}. \tag{2.1.1}$$

In particular, let $l, m \in \{1, \ldots, n\}$ be distinct. Take any $x, y \in \mathsf{X}$ and notice that the (l, l) entry in (2.1.1), applied to the elements $\delta_m x \in \mathsf{X}^n$ and $\delta_l y \in \mathsf{X}^n$, becomes the equation

$$\langle s_{l,m}(x), y\rangle_A = \langle 0, (s^*)_{l,l}(y)\rangle_A = 0.$$

Thus, for each $l, m \in \{1, \ldots, n\}$ with $l \neq m$, we have shown that $s_{l,m} = 0$. An analogous computation also shows that $(s^*)_{l,m} = 0$ when $l \neq m$. Then (2.1.1) implies

$$\langle s_{i,i}(x_i), y_j\rangle_A = \langle x_i, (s^*)_{j,j}(y_j)\rangle_A$$

for all $i, j \in \{1, \ldots, n\}$. It now follows at once that, for all $i \in \{1, \ldots, n\}$, $s_{i,i} \in \mathcal{L}_A(\mathsf{X})$ with $(s_{i,i})^* = (s^*)_{i,i}$. Furthermore, this also proves that $s_{i,i} = s_{j,j}$ for all $i, j \in \{1, \ldots, n\}$. It is now clear that $\kappa(s_{i,i}) = s$ for any $i \in \{1, \ldots, n\}$, which finishes the proof. ■

Remark 2.1.7. On page 39 of [17] it is claimed, with no proof, that $\mathcal{L}_A(\mathsf{X}^n) \cong \mathcal{L}_{M_n(A)}(\mathsf{X}^n)$. Proposition 2.1.6 shows that the claim is false in general. Indeed, if $A = \mathsf{X} = \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 2}$, then it is clear that $\mathcal{L}_{\mathbb{C}}(\mathbb{C}^n) \cong M_n(\mathbb{C})$. However, by Proposition 2.1.6 we have $\mathcal{L}_{M_n(\mathbb{C})}(\mathbb{C}^n) \cong \mathcal{L}_{\mathbb{C}}(\mathbb{C}) \cong \mathbb{C}$.

Hilbert Bimodules and C*-correspondences

Definition 2.2.1. Let A and B be C*-algebras. A Hilbert A-B-bimodule is a complex vector space X that is a left Hilbert A-module and a right Hilbert B-module (see Definitions 2.1.1 and 2.1.2) such that for all $x, y, z \in \mathsf{X}$,

$$_A\langle x, y\rangle z = x\langle y, z\rangle_B. \tag{2.2.1}$$

Example 2.2.2. Let A be a C*-algebra and X any right Hilbert A-module. Then X is also a left Hilbert $\mathcal{K}_A(\mathsf{X})$-module with the obvious action and with left inner product given by

$$_{\mathcal{K}_A(\mathsf{X})}\langle x, y\rangle = \theta_{x,y}.$$

Remark 2.2.3. For the definition of Hilbert A-B-bimodule, some authors also require that A acts on X via $\langle -, -\rangle_B$-adjointable operators and B acts on X via

${}_A\langle -,-\rangle$-adjointable operators. That is, for all $a \in A$, $b \in B$, and $x, y \in \mathsf{X}$ the following holds $\langle ax, y\rangle_B = \langle x, a^*y\rangle_B$ and ${}_A\langle x, yb\rangle = {}_A\langle xb^*, y\rangle$. However, this is redundant as it already follows from (2.2.1); see comments after Remark 1.9 in [2].

Definition 2.2.4. Let A and B be C*-algebras. An (A, B) *C*-correspondence* is a pair (X, φ), where X is a right Hilbert B-module and $\varphi\colon A \to \mathcal{L}_B(\mathsf{X})$ is a $*$-homomorphism. We say that A acts *nondegenerately* on X whenever $\varphi(A)\mathsf{X}$ is dense in X. Whenever $A = B$, we say (X, φ) is a *C*-correspondence over A.*

We observe that Remark 2.2.3 implies that any Hilbert A-B-bimodule, as in Definition 2.2.1, is in fact an (A, B) C*-correspondence with φ given by the left action of the bimodule. In fact, if X is Hilbert A-B-bimodule, then it is well known that A always acts nondegenerately on X. However, not every C*-correspondence is a Hilbert bimodule. Thus, C*-correspondences are a generalization of Hilbert bimodules.

We will need the interior tensor product of C*-correspondences. This is a well known construction. We only list some of the basic properties that will be needed below. We refer the reader to Proposition 4.5 in [17] and the afterwards discussion for more details. Let A, B, and C be C*-algebras, let $(\mathsf{X}, \varphi_{\mathsf{X}})$ be an (A, B) C*-correspondence and let $(\mathsf{Y}, \varphi_{\mathsf{Y}})$ be a (B, C) C*-correspondence. We consider the algebraic B-balanced tensor product of modules $\mathsf{X} \odot_B \mathsf{Y}$ which has a C-valued right pre-inner product given on elementary tensors by

$$\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle_C = \langle y_1, \varphi_{\mathsf{Y}}(\langle x_1, x_2\rangle_B) y_2\rangle_C. \tag{2.2.2}$$

The completion of $\mathsf{X} \odot_B \mathsf{Y}$ under the norm induced by the C-valued right pre-inner product from equation (2.2.2) is a right Hilbert C-module, which we denote by

$\mathsf{X} \otimes_{\varphi_\mathsf{Y}} \mathsf{Y}$. It is useful to keep in mind that, by construction, if $x \in \mathsf{X}$, $b \in B$, and $y \in \mathsf{Y}$, then

$$xb \otimes y = x \otimes \varphi_\mathsf{Y}(b)y. \tag{2.2.3}$$

Furthermore, A acts on $\mathsf{X} \otimes_{\varphi_\mathsf{Y}} \mathsf{Y}$ via $\langle -, - \rangle_C$-adjointable operators and the action $\widetilde{\varphi_\mathsf{X}}\colon A \to \mathcal{L}_C(\mathsf{X} \otimes_{\varphi_\mathsf{Y}} \mathsf{Y})$ is determined by φ_X as follows:

$$\widetilde{\varphi_\mathsf{X}}(a)(x \otimes y) = \varphi_\mathsf{X}(a)x \otimes y, \tag{2.2.4}$$

for $a \in A$, $x \in \mathsf{X}$, and $y \in \mathsf{Y}$. All this makes $(\mathsf{X} \otimes_{\varphi_\mathsf{Y}} \mathsf{Y}, \widetilde{\varphi_\mathsf{X}})$ into an (A, C) C*-correspondence, called the *interior tensor product* of $(\mathsf{X}, \varphi_\mathsf{X})$ with $(\mathsf{Y}, \varphi_\mathsf{Y})$. Whenever $\mathsf{X} = \mathsf{Y}$ and $A = B = C$, we will find it convenient to write $\varphi_\mathsf{X}^{(2)}$ instead of $\widetilde{\varphi_\mathsf{X}}$.

We now present the notion of morphisms between C*-correspondences.

Definition 2.2.5. Let A, B, C, and D be C*-algebras, let $(\mathsf{X}, \varphi_\mathsf{X})$ be an (A, B) C*-correspondence, and let $(\mathsf{Y}, \varphi_\mathsf{Y})$ be a (C, D) C*-correspondence. A *morphism* from $(\mathsf{X}, \varphi_\mathsf{X})$ to $(\mathsf{Y}, \varphi_\mathsf{Y})$ consist of a triple (π_l, π_r, π) where the maps $\pi_l\colon A \to C$, $\pi_r\colon B \to D$ are *-homomorphisms, and $\pi\colon \mathsf{X} \to \mathsf{Y}$ is a linear map such that for all $a \in A$, and $x, x_1, x_2 \in \mathsf{X}$, the following two conditions hold

1. $\pi(\varphi_\mathsf{X}(a)x) = \varphi_Y(\pi_l(a))\pi(x)$,

2. $\pi_r(\langle x_1, x_2 \rangle_B) = \langle \pi(x_1), \pi(x_2) \rangle_D$.

We will sometimes write $(\pi_l, \pi_r, \pi)\colon (\mathsf{X}, \varphi_\mathsf{X}) \to (\mathsf{Y}, \varphi_\mathsf{Y})$. If the maps π_l, π_r are *-isomorphisms and the map π is invertible, then we say $(\mathsf{X}, \varphi_\mathsf{X})$ is *isomorphic to* $(\mathsf{Y}, \varphi_\mathsf{Y})$ and write $(\mathsf{X}, \varphi_\mathsf{X}) \cong (\mathsf{Y}, \varphi_\mathsf{Y})$ or simply $\mathsf{X} \cong \mathsf{Y}$ when the left actions are understood by context.

Remark 2.2.6. When using morphisms between C*-correspondences, we might encounter cases in which $A = C$ and $B = D$. In these cases, the maps π_l and π_r will usually be id_A and id_B.

Remark 2.2.7. It is worth mentioning that any morphism $(\pi_l, \pi_r, \pi)\colon (\mathsf{X}, \varphi_{\mathsf{X}}) \to (\mathsf{Y}, \varphi_{\mathsf{Y}})$ between C*-correspondences will automatically satisfy a third condition:

3. $\pi(xb) = \pi(x)\pi_r(b)$ for any $x \in \mathsf{X}$, $b \in B$.

Indeed, let $y = \pi(xb) - \pi(x)\pi_r(b) \in \mathsf{Y}$. Now using condition 2 in 2.2.5 at the third step and that $\pi_r\colon B \to D$ is a *-homomorphism at the fifth one yields

$$\begin{aligned}
\|y\|^2 &= \|\langle \pi(xb) - \pi(x)\pi_r(b), \pi(xb) - \pi(x)\pi_r(b)\rangle_D\| \\
&= \|\langle \pi(xb), \pi(xb)\rangle_D - \langle \pi(xb), \pi(x)\pi_r(b)\rangle_D - \langle \pi(x)\pi_r(b), \pi(xb)\rangle_D + \langle \pi(x)\pi_r(b), \pi(x)\pi_r(b)\rangle_D\| \\
&= \|\pi_r(\langle xb, xb\rangle_B) - \pi_r(\langle xb, x\rangle_B)\pi_r(b) - \pi_r(b^*)\pi_r(\langle x, xb\rangle_B) + \pi_r(b^*)\pi_r(\langle x, x\rangle_B)\pi_r(b)\| \\
&= \|\pi_r(b^*\langle x, x\rangle_B b) - \pi_r(b^*\langle x, x\rangle_B)\pi_r(b) - \pi_r(b^*)\pi_r(\langle x, x\rangle_B b) + \pi_r(b^*)\pi_r(\langle x, x\rangle_B)\pi_r(b)\| \\
&= \|\pi_r(b^*\langle x, x\rangle_B b) - \pi_r(b^*\langle x, x\rangle_B)b) - \pi_r(b^*\langle x, x\rangle_B b) + \pi_r(b^*\langle x, x\rangle_B b)\| \\
&= 0,
\end{aligned}$$

proving the desired third condition.

Remark 2.2.8. Observe that if $(\pi_l, \pi_r, \pi)\colon (\mathsf{X}, \varphi_{\mathsf{X}}) \to (\mathsf{Y}, \varphi_{\mathsf{Y}})$ is a morphism between C*-correspondences, then the map π is automatically bounded. Indeed, if $x \in \mathsf{X}$, then using condition 2 on Definition 2.2.5 at the second step yields

$$\|\pi(x)\|^2 = \|\langle \pi(x), \pi(x)\rangle_D\| = \|\pi_r(\langle x, x\rangle_A)\| \leq \|\langle x, x\rangle_A\| = \|x\|^2$$

Cuntz Algebras

Let $d \in \mathbb{Z}_{\geq 2}$. Recall that the Cuntz algebra $\mathcal{O}_d$, defined in [5], is the universal unital C*-algebra generated by d isometries $s_1, \ldots, s_d$ satisfying

$$\sum_{j=1}^{d} s_j s_j^* = 1 \tag{2.3.1}$$

It is well known that $\mathcal{O}_d$ is a simple C*-algebra with the following universal property: If A is a unital C*-algebra containing elements $a_1, \ldots, a_d$ satisfying

$$a_j^* a_j = 1 \quad \text{and} \quad \sum_{j=1}^{d} a_j a_j^* = 1,$$

then there is a unique $*$-homomorphism $\varphi\colon \mathcal{O}_d \to A$ such that $\varphi(s_j) = a_j$.

Definition 2.3.1. For $d \in \mathbb{Z}_{\geq 2}$, look at the generating isometries $s_1, \ldots, s_{d+1} \in \mathcal{O}_{d+1}$ and let $\mathcal{E}_d$ be the C*-algebra in $\mathcal{O}_{d+1}$ generated by $s_1, \ldots, s_d$. That is, $\mathcal{E}_d$ is the universal unital C*-algebra generated by d isometries, whose orthogonal ranges do not add up to 1.

Since $\mathcal{O}_d$ has elements satisfying the relations of $\mathcal{E}_d$, there is a surjective map $\mathcal{E}_d \to \mathcal{O}_d$. The kernel of this map is the ideal in $\mathcal{E}_d$ generated by $s_{d+1}s_{d+1}^* = 1 - \sum_{j=1}^{d} s_j s_j^*$, which we denote by $\mathcal{J}_d$. Then $\mathcal{E}_d/\mathcal{J}_d \cong \mathcal{O}_d$.

Cuntz-Pimsner Algebras

The following is a minor modification of Definition 2.1 in [14].

Definition 2.4.1. Let (X, φ) be a C*-correspondence over a C*-algebra A. A *Fock representation* for (X, φ) consist of a triple $(B, \pi_A, \pi_{\mathsf{X}})$ where B is a C*-algebra, $\pi_A\colon A \to B$ a *-homomorphism, and $\pi_{\mathsf{X}}\colon \mathsf{X} \to B$ a linear map satisfying

1. $\pi_X(\varphi(a)x) = \pi_A(a)\pi_{\mathsf{X}}(x)$ for all $a \in A$, $x \in \mathsf{X}$,

2. $\pi_A(\langle x, y\rangle_A) = \pi_X(x)^*\pi_X(y)$ for all $x, y \in \mathsf{X}$.

We denote by $C^*(B, \pi_A, \pi_{\mathsf{X}})$ to the C*-subalgebra in B generated by $\pi_A(A)$ and $\pi_{\mathsf{X}}(\mathsf{X})$.

Remark 2.4.2. Below Definition 2.1 in [14], it is shown that whenever $(B, \pi_A, \pi_{\mathsf{X}})$ is a Fock representation for (X, φ), then a third condition is automatically satisfied:

3. $\pi_{\mathsf{X}}(xa) = \pi(x)\pi_A(a)$ for all $x \in \mathsf{X}$, $a \in A$.

This is a particular case of situation already addressed in Remark 2.2.7 for morphisms of C*-correspondences.

Definition 2.4.3. Let A be a C*-algebra and let (X, φ) be a C^*-correspondence over A. We define $\mathcal{T}(\mathsf{X}, \varphi)$, the Toeplitz algebra of (X, φ), as the universal C*-algebra algebra generated by Fock representations. That is, there exists the universal Fock representation $(C, \rho_A, \rho_{\mathsf{X}})$ such that $\mathcal{T}(\mathsf{X}, \varphi) = C^*(C, \rho_A, \rho_{\mathsf{X}})$ and for any other Fock representation $(B, \pi_A, \pi_{\mathsf{X}})$ there is a surjective $*$-homomorphism $\sigma \colon \mathcal{T}(\mathsf{X}, \varphi) \to C * (B, \pi_A, \pi_{\mathsf{X}})$ satisfying $\pi_A = \sigma \circ \rho_A$ and $\pi_{\mathsf{X}} = \sigma \circ \rho_{\mathsf{X}}$.

We now give an explicit construction of the universal Fock representation for a C*-correspondence (X, φ) over A.

Definition 2.4.4. Given (X, φ) a C*-correspondence over A, the *Fock space of* X is the Hilbert A-module given by

$$\mathcal{F}(\mathsf{X}) = \bigoplus_{n \geq 0} \mathsf{X}^{\otimes n},$$

where $\mathsf{X}^{\otimes 0} = A$ and $\mathsf{X}^{\otimes n} = \underbrace{\mathsf{X} \otimes_\varphi \ldots \otimes_\varphi \mathsf{X}}_{n \text{ times}}$.

An arbitrary element of $\mathcal{F}(\mathsf{X})$ is a tuple $(\kappa_n)_{n\geq 0}$ where each κ_n is an element of the n-th degree tensor product of X. For each $n \in \mathbb{Z}_{\geq 0}$, $(\mathsf{X}^{\otimes n}, \varphi^{(n)})$ is a C*-correspondence over A where $\varphi^{(n)}\colon A \to \mathcal{L}_A(\mathsf{X}^{\otimes n})$ is given by

$$\varphi^{(0)}(a)(b) = ab,\ \forall a, b \in A,$$

and determined by

$$\varphi^{(n)}(a)(x \otimes z) = \varphi(a)x \otimes z,\ \forall x \in \mathsf{X}, z \in \mathsf{X}^{\otimes(n-1)},$$

when $n \geq 1$. Thus, we can define $\varphi^{\infty}\colon A \to \mathcal{L}(\mathcal{F}(\mathsf{X}))$ by letting

$$\varphi^{\infty}(a)\Big((\kappa_n)_{n\geq 0}\Big) = \big(\varphi^{(n)}(a)\kappa_n\big)_{n\geq 0}.$$

This makes $(\mathcal{F}(\mathsf{X}), \varphi^{\infty})$ into a C*-correspondence over A. For a fixed $x \in \mathsf{X}$ and any $n \in \mathbb{Z}_{\geq 1}$ we have a *creation operator* $c(x)\colon \mathsf{X}^{\otimes n} \to \mathsf{X}^{\otimes(n+1)}$ given by

$$c(x)(y) = x \otimes y,\ \forall y \in \mathsf{X}^{\otimes n}.$$

If $n = 0$ we set $c(x) : A \to \mathsf{X}$

$$c(x)(a) = xa,\ \forall a \in A$$

Each $c(x)$ is an adjointable map. In fact, if $n \in \mathbb{Z}_{\geq 1}$, $c(x)^*\colon \mathsf{X}^{\otimes(n+1)} \to \mathsf{X}^{\otimes n}$ is the *annihilation operator*, satisfying

$$c(x)^*(y \otimes z) = \varphi^{(n)}(\langle x, y\rangle_A)z,\ \forall y \in \mathsf{X}, z \in \mathsf{X}^{\otimes n},$$

and $c(x)^* : \mathsf{X} \to A$ is simply

$$c(x)^*(y) = \langle x, y\rangle_A, \ \forall y \in \mathsf{X}.$$

Notice that $c(x)$ increases the degree by one, whereas $c(x)^*$ decreases the degree by one. For any $x, y \in \mathsf{X}$ and $n \geq 0$, we record the following important properties for the map $c(x)^*c(y) : \mathsf{X}^{\otimes n} \to \mathsf{X}^{\otimes n}$ and $c(x)c(y)^* : \mathsf{X}^{\otimes(n+1)} \to \mathsf{X}^{\otimes(n+1)}$,

$$c(x)^*c(y) = \varphi^{(n)}(\langle x, y\rangle_A) \in \mathcal{L}_A(\mathsf{X}^{\otimes n}),$$

$$c(x)c(y)^* = \theta^{(n+1)}_{x,y} \in \mathcal{L}_A(\mathsf{X}^{\otimes(n+1)}).$$

For any $x, y \in \mathsf{X}$, we abuse notation and consider the elements $c(x), c(y)^*$ as elements of $\mathcal{L}_A(\mathcal{F}(\mathsf{X}))$ acting coordinate-wise. That is,

$$c(x)\big((\kappa_n)_{n\geq 0}\big) = \Big(0, \big(c(x)(\kappa_n)\big)_{n\geq 0}\Big),$$

and

$$c(y)^*\big((\kappa_n)_{n\geq 0}\big) = \Big(c(y)^*(\kappa_n)\Big)_{n\geq 1}.$$

The following important result was shown by Pimsner (see Theorem 3.4 in [24]) for the particular case when φ is injective and extended to the general case by Katsura (see Theorem 6.2 and Proposition 6.5 in [14]).

Theorem 2.4.5. *(Pimsner-Katsura) Let A be a C^*-algebra and let (X, φ) be a C^*-correspondence over A. The triple $(\mathcal{L}_A(\mathcal{F}(\mathsf{X})), \varphi^\infty, c)$ is the universal Fock space representation of (X, φ). That is, $\mathcal{T}(\mathsf{X}, \varphi)$, the Toeplitz algebra of (X, φ), is the C^*-subalgebra in $\mathcal{L}_A(\mathcal{F}(\mathsf{X}))$ generated by the creation operators $c(\mathsf{X})$ and by $\varphi^\infty(A)$.*

Remark 2.4.6. It is clear that the map $\varphi^{(0)}\colon A \to \mathcal{L}_A(A)$ is injective, whence we may identify A with its image in $\mathcal{T}(\mathsf{X},\varphi)$ given by $a \mapsto \varphi^{\infty}(a)$. Notice that if X is full as Hilbert A-module, then $\mathcal{T}(\mathsf{X},\varphi)$ is solely generated by the creation operators. Indeed, since $\varphi^{\infty}(\langle x,y\rangle_A) = c(x)^*c(y) \in \mathcal{T}(\mathsf{X},\varphi)$ for any $x,y \in \mathsf{X}$, fullness implies that $\varphi^{\infty}(A)$ is already in the C*-algebra generated by the creation operators.

The following is an easy consequence of the universal property for $\mathcal{T}(\mathsf{X},\varphi)$

Proposition 2.4.7. *Let $(B,\pi_A,\pi_{\mathsf{X}})$ be any Fock representation for (X,φ). Then $\pi_A\colon A \to B$ has a unique extension $\widetilde{\pi}\colon \mathcal{T}(\mathsf{X},\varphi) \to B$ that sends $c(x)$ to $\pi_{\mathsf{X}}(x)$.*

Proof. We claim that $\sigma\colon \mathcal{T}(\mathsf{X},\varphi) \to C^*(B,\pi_A,\pi_X)$, the natural surjection given by universality, is the desired extension. Clearly $\sigma(c(x)) = (\sigma \circ c)(x) = \pi_{\mathsf{X}}(x)$ and also $\sigma(\varphi^{\infty}(a)) = (\sigma \circ \varphi^{\infty})(a) = \pi_A(a)$, which shows that σ extends π_A. ■

The following is Lemma in 2.2 in [13]

Lemma 2.4.8. *Let $(B,\pi_A,\pi_{\mathsf{X}})$ be any Fock representation for (X,φ). Then there is a *-homomorphism $\pi_{\mathcal{K}}\colon \mathcal{K}_A(\mathsf{X}) \to B$ that satisfies $\pi_{\mathcal{K}}(\theta_{x,y}) = \pi_{\mathsf{X}}(x)\pi_{\mathsf{X}}(y)^*$ for all $x,y \in \mathsf{X}$.*

Definition 2.4.9. For a C* correspondence (X,φ) over A, we define Katsura's ideal J_{X} to be

$$J_{\mathsf{X}} = \varphi^{-1}(\mathcal{K}_A(\mathsf{X})) \cap (\ker(\varphi))^{\perp} = \{a \in A : \varphi(a) \in \mathcal{K}_A(\mathsf{X}) \text{ and } ab = 0 \text{ for all } b \in \ker(\varphi)\}.$$

Definition 2.4.10. A Fock representation $(B,\pi_A,\pi_{\mathsf{X}})$ for (X,φ) is said to be covariant if

$$\pi_{\mathcal{K}}(\varphi(a)) = \pi_A(a)\ \forall a \in J_{\mathsf{X}}.$$

Definition 2.4.11. Let A be a C*-algebra and let (X, φ) be a C^*-correspondence over A. We define $\mathcal{O}(\mathsf{X}, \varphi)$, the Cuntz-Pimsner algebra of (X, φ), as the universal C*-algebra algebra generated by covariant Fock representations. That is, there exists the universal covariant Fock representation $(D, \tau_A, \tau_{\mathsf{X}})$ such that $\mathcal{O}(\mathsf{X}, \varphi) = C * (D, \tau_A, \tau_{\mathsf{X}})$ and for any other covariant Fock representation $(B, \pi_A, \pi_{\mathsf{X}})$ there is a surjective $*$-homomorphism $\sigma\colon \mathcal{O}(\mathsf{X}, \varphi) \to C^*(B, \pi_A, \pi_{\mathsf{X}})$ satisfying $\pi_A = \sigma \circ \tau_A$ and $\pi_{\mathsf{X}} = \sigma \circ \tau_{\mathsf{X}}$.

We now give an explicit construction of the universal covariant Fock representation for a C*-correspondence (X, φ) over A.

Lemma 2.4.12. *Let (X, φ) be a C^* correspondence over A. Then J_{X} is indeed a closed ideal in A, $\mathcal{F}(\mathsf{X})J_{\mathsf{X}}$ is a Hilbert J_{X}-module, and*

$$\mathcal{K}_{J_{\mathsf{X}}}(\mathcal{F}(\mathsf{X})J_{\mathsf{X}}) = \overline{\operatorname{span}}\{\theta_{\kappa a, \tau}\colon \kappa, \tau \in \mathcal{F}(\mathsf{X}), a \in J_{\mathsf{X}}\} \trianglelefteq \mathcal{L}_A(\mathcal{F}(\mathsf{X}))$$

Proof. Follows from Corollary 1.4 in [15]. ∎

We will also need the quotient C*-algebra $\mathcal{Q}_A(\mathsf{X}) = \mathcal{L}_A(\mathcal{F}(\mathsf{X}))/\mathcal{K}_{J_{\mathsf{X}}}(\mathcal{F}(\mathsf{X})J_{\mathsf{X}})$ together with the quotient map $q\colon \mathcal{L}_A(\mathcal{F}(\mathsf{X})) \to \mathcal{Q}_A(\mathsf{X})$.

The following important result was shown by Pimsner (see Theorem 3.12 in [24]) for the particular case when φ is injective and extended to the general case by Katsura (see Theorem 6.4 and Proposition 6.5 in [14]).

Theorem 2.4.13. *(Pimsner-Katsura) Let A be a C^*-algebra and let (X, φ) be a C^*-correspondence over A. Then $(\mathcal{Q}_A(\mathsf{X}), q \circ \varphi^\infty, q \circ c)$ is the universal covariant Fock representation for (X, φ). That is, $\mathcal{O}(\mathsf{X}, \varphi)$, the Cuntz-Pimsner algebra of X, is the C^*-subalgebra in $\mathcal{Q}_A(\mathsf{X})$ generated by $q(c(\mathsf{X}))$ and by $q(\varphi^\infty(A))$.*

Remark 2.4.14. Just as before, we identify A as a subset of $\mathcal{O}(\mathsf{X},\varphi)$ via $a \mapsto q(\varphi^\infty(a))$. Similarly, if X is full, then $\mathcal{O}(\mathsf{X},\varphi)$ is generated solely by $q(c(\mathsf{X}))$.

The following is an easy consequence of the universal property for the Cuntz-Pimsner algebra $\mathcal{O}(\mathsf{X},\varphi)$.

Proposition 2.4.15. *Let $(B,\pi_A,\pi_{\mathsf{X}})$ be any covariant Fock representation for (X,φ). Then $\pi_A\colon A\to B$ has is a unique extension $\widetilde{\pi_A}\colon \mathcal{O}(\mathsf{X},\varphi)\to B$ that sends $q(c(x))$ to $\pi_{\mathsf{X}}(x)$.*

Proof. Just as in the Toeplitz case above, the natural surjection given by universality is the required extension. ■

We now present some examples of C*-correspondences (X,φ) together with their Toeplitz and Cuntz-Pimsner algebras.

Example 2.4.16. Let $d\in\mathbb{Z}_{\geq 2}$ and regard $\mathbb{C}^d$ as a Hilbert $\mathbb{C}$-module. Let $\varphi_{\mathbb{C}}:\mathbb{C}\to\mathcal{L}_{\mathbb{C}}(\mathbb{C}^d)$ by given by

$$\varphi_{\mathbb{C}}(z)(\zeta_1,\dots,\zeta_d)=(z\zeta_1,\dots,z\zeta_d).$$

Then $(\mathbb{C}^d,\varphi_{\mathbb{C}})$ is a C*-correspondence. We claim that $\mathcal{T}(\mathbb{C}^d,\varphi_{\mathbb{C}}) \cong \mathcal{E}_d$. For simplicity we only show this when $d=2$ as the proof is essentially the same for $d>2$. We start by showing that $\mathcal{T}(\mathbb{C}^2,\varphi_{\mathbb{C}})$ has elements satisfying the relations of $\mathcal{E}_2$. Indeed, consider $v_1=c((1,0))$ and $v_2=c((0,1))$. We have to check that $v_1^*v_1=v_2^*v_2=1$ in $\mathcal{L}_{\mathbb{C}}(\mathcal{F}(\mathbb{C}^2))$. We only do $v_1^*v_1=1$, the other one being analogous. For $n=0$, take $z\in\mathbb{C}$

$$(v_1^*v_1)(z)=c((1,0))^*(z,0)=\langle(1,0),(z,0)\rangle_{\mathbb{C}}=z.$$

For $n \geq 1$,

$$(v_1^* v_1) = c((1,0))^* c((1,0)) = \varphi_{\mathbb{C}}(\langle (1,0),(1,0)\rangle_{\mathbb{C}}) = \varphi(1) = 1 \in \mathcal{L}_{\mathbb{C}}(\mathcal{F}(\mathbb{C}^2)).$$

By universality of $\mathcal{E}_2$, there is a unique $*$-homomorphism $\psi : \mathcal{E}_2 \to \mathcal{T}(\mathbb{C}^2, \varphi_{\mathbb{C}})$, which sends the s_j to v_j for $j = 1, 2$. Notice that ψ is surjective because v_1, v_2 generate $\mathcal{T}(\mathbb{C}^2, \varphi)$. We still need to show that it is injective. Let $\pi_{\mathbb{C}} : \mathbb{C} \to \mathcal{E}_2$ be given by $\pi_{\mathbb{C}}(z) = z \cdot 1$ and $\pi_{\mathbb{C}^2} : \mathbb{C}^2 \to \mathcal{E}_2$ be given by

$$\pi_{\mathbb{C}^2}(\zeta_1, \zeta_2) = \zeta_1 s_1 + \zeta_2 s_2.$$

It is obvious that $\pi_{\mathbb{C}}$ is a $*$-homomorphism and that $\pi_{\mathbb{C}^2}$ is a linear map. Moreover, using that $s_j^* s_j = 1$ we get

$$\pi_{\mathbb{C}^2}(\zeta_1, \zeta_2)^* \pi_{\mathbb{C}^2}(\eta_1, \eta_2) = (\overline{\zeta_1}\eta_1 + \overline{\zeta_2}\eta_2)1 = \pi_{\mathbb{C}}(\langle (\zeta_1, \zeta_2), (\eta_1, \eta_2)\rangle_{\mathbb{C}})$$

Finally,

$$\pi_{\mathbb{C}}(z)\pi_{\mathbb{C}^2}(\zeta_1, \zeta_2) = z1(\zeta_1 s_1 + \zeta_2 s_2) = z\zeta_1 s_1 + z\zeta_2 s_2 = \pi_{\mathbb{C}^2}(\varphi(z)(\zeta_1, \zeta_2))$$

Hence, $(\mathcal{E}_2, \pi_{\mathbb{C}}, \pi_{\mathbb{C}^2})$ is a Fock representation of $(\mathbb{C}^2, \varphi)$ and therefore Proposition 2.4.7 gives a $*$-homomorphism $\widetilde{\pi} : \mathcal{T}(\mathbb{C}^2, \varphi) \to \mathcal{E}_2$ sending $c(x)$ to $\pi_{\mathbb{C}^2}(x)$ for any $x \in \mathbb{C}^2$. Since $\pi_{\mathbb{C}^2}(1,0) = s_1$ and $\pi_{\mathbb{C}^2}(0,1) = s_2$, it follows $\widetilde{\pi}$ is a left inverse for ψ, whence ψ is injective. We also claim that $\mathcal{O}(\mathbb{C}^d, \varphi_{\mathbb{C}}) \cong \mathcal{O}_d$. As before, we only prove this for $d = 2$. We first show that $\mathcal{O}(\mathbb{C}^2, \varphi_{\mathbb{C}})$ fits the universal property for

$\mathcal{O}_2$. Let $w_1 = q(c((1,0)))$ and $w_2 = q(c((0,1)))$. As before, we have

$$w_1^* w_1 = w_2^* w_2 = q(1) = 1 \in \mathcal{Q}_{\mathbb{C}}(\mathbb{C}^2).$$

Also, notice that since $\varphi_{\mathbb{C}}$ is injective and $\mathcal{L}_{\mathbb{C}}(\mathbb{C}^2) = \mathcal{K}_{\mathbb{C}}(\mathbb{C}^2)$. Hence, in this case the Katsura ideal is $J_{\mathbb{C}^2} = \mathbb{C}$. Therefore $\theta_{(1,0,0,\ldots),(1,0,0,\ldots)} \in \mathcal{K}_{J_{\mathbb{C}^2}}(\mathcal{F}(\mathbb{C}^2)J_{\mathbb{C}^2})$. Thus, we find

$$w_1 w_1^* + w_2 w_2^* = q(1 - \theta_{(1,0,0,\ldots),(1,0,0,\ldots)}) = q(1) = 1 \in \mathcal{Q}_{\mathbb{C}}(\mathbb{C}^2).$$

Hence, universality of $\mathcal{O}_2$ gives a surjective $*$-homomorphism $\mathcal{O}_2 \to \mathcal{O}(\mathbb{C}^2, \varphi_{\mathbb{C}})$ sending s_j to w_j, for $j = 1, 2$. Since $\mathcal{O}_2$ is simple, such homomorphism has to be injective and we are done. More generally, let $d \in \mathbb{Z}_{\geq 2}$, let A be a C*-algebra, and think of A^d as a Hilbert A-module. Let $\varphi_A \colon A \to \mathcal{L}_A(A^d)$ by given by

$$\varphi_A(a)(a_1, \ldots, a_d) = (aa_1, \ldots, aa_d).$$

Then (A^d, φ_A) is a C*-correspondence over A. Similar arguments as those presented above show that $\mathcal{T}(A^d, \varphi_A) \cong A \otimes \mathcal{E}_d$ and $\mathcal{O}(A^d, \varphi_A) \cong A \otimes \mathcal{O}_d$.

Example 2.4.17. Let A be any C*-algebra and let $\varphi \in \operatorname{Aut}(A)$ be an automorphism of A. Then A is a right Hilbert A-module. Notice that A acts via adjointable maps on A by left multiplication by φ. Indeed, for any $a, b, c \in A$

$$\langle \varphi(a)b, c\rangle_A = (\varphi(a)b)^* c = b^* \varphi(a^*) c = \langle b, \varphi(a^*)c\rangle_A.$$

That is, $\varphi(a) \in \mathcal{L}_A(A)$ for any $a \in A$, whence (A, φ) is a C*-correspondence. In fact, for any $n \in \mathbb{Z}$, (A, φ^n) is a C*-correspondence, where by convention $\varphi^0 = \mathrm{id}_A$.

Lemma 2.4.18. *Let A be a C^*-algebra. Then $(A \otimes_\varphi A, \varphi^{(2)}) \cong (A, \varphi^2)$ in the sense of Definition 2.2.5.*

Proof. Let $n \in \mathbb{Z}_{\geq}1$ and $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$. Then using equation (2.2.2) at the second step gives

$$\begin{aligned}\Big\| \sum_{j=1}^{n} \varphi(a_j) b_j \Big\|^2 &= \Big\| \sum_{j,k=1}^{n} b_j^* \varphi(a_j^* a_k) b_k \Big\| \\ &= \Big\| \sum_{j,k=1}^{n} \langle a_j \otimes b_j, a_k \otimes b_k \rangle_A \Big\| \\ &= \Big\| \sum_{j=1}^{n} a_j \otimes b_j \Big\|^2.\end{aligned}$$

Furthermore, since the assignment $\tau(a \otimes b) = \varphi(a)b \in A$ clearly satisfies $\tau(a \otimes \varphi(a_1)b) = \tau(aa_1 \otimes b)$, it follows that it can be extended to an isometry $\tau : A \otimes_\varphi A \to A$. Since τ is clearly surjective, it is in fact an invertible linear map. Finally, we claim that $(\mathrm{id}_A, \mathrm{id}_A, \tau)\colon (A \otimes_\varphi A, \varphi^{(2)}) \to (A, \varphi^2)$ is an isomorphism as in Definition 2.2.5. We only need to check conditions 1-2 in Definition 2.2.5 and it suffices to do so for elementary tensors. Let $a, a_1, a_2, b_1, b_2 \in A$ and notice that

$$\tau(\varphi^{(2)}(a)(a_1 \otimes a_2)) = \tau(\varphi(a)a_1 \otimes a_2) = \varphi(\varphi(a)a_1)a_2 = \varphi^2(a)\tau(a_1 \otimes a_2),$$

proving condition 1. Finally, since

$$\langle a_1 \otimes a_2, b_1 \otimes b_2 \rangle_A = a_2^* \varphi(a_1^* a_2) b_2 = \tau(a_1 \otimes a_2)^* \tau(b_1 \otimes b_2),$$

condition 2 also holds, proving the claim. We are done. ■

Corollary 2.4.19. *Let A be a C^*-algebra and let $n \in \mathbb{Z}_{\geq 0}$. Then $(A^{\otimes n}, \varphi^{(n)}) \cong (A, \varphi^n)$ in the sense of Definition 2.2.5.*

Therefore, the Fock space of A is

$$\mathcal{F}(A) = \bigoplus_{n\geq 0} A = \left\{ (a_n)_{n=0}^{\infty} \colon a_n \in A,\ \sum_{n=0}^{\infty} a_n^* a_n \text{ converges in norm} \right\}.$$

This makes $\mathcal{F}(A)$ into a C*-correspondence over A with right action given pointwise and left action given by

$$\varphi^{\infty}(a)(a_n)_{n=0}^{\infty} = (\varphi^n(a)a_n)_{n=0}^{\infty}.$$

For each $a \in A$, the creation operator $c(a) \in \mathcal{L}_A(\mathcal{F}(A))$ is given by

$$c(a)(a_n)_{n=0}^{\infty} = (0, (\varphi^n(a)a_n)_{n=0}^{\infty}). \tag{2.4.1}$$

Similarly, its adjoint $c(a)^* \in \mathcal{L}_A(\mathcal{F}(A))$ is

$$c(a)^*(a_n)_{n=0}^{\infty} = (\varphi^n(a^*)a_{n+1})_{n=0}^{\infty}. \tag{2.4.2}$$

Since φ is injective and $A \cong \mathcal{K}_A(A)$, the Katsura Ideal for (A, φ) is $J_A = A$, and since A is full as a Hilbert A-module, we only need to look at images of creation operators in $\mathcal{L}_A(\mathcal{F}(A))/\mathcal{K}_A(\mathcal{F}(A))$ to get $\mathcal{O}(A, \varphi)$ (see Remark 2.4.14). That is, if $q \colon \mathcal{L}_A(\mathcal{F}(A)) \to \mathcal{L}_A(\mathcal{F}(A))/\mathcal{K}_A(\mathcal{F}(A))$ is the quotient map, then $\mathcal{O}(A, \varphi)$ is the C* subalgebra of $\mathcal{L}_A(\mathcal{F}(A))/\mathcal{K}_A(\mathcal{F}(A))$ generated by the set $q(c(A))$. The following lemma will help us do computations on $\mathcal{O}(A, \varphi)$.

Lemma 2.4.20. *Let $a, b \in A$. Then*

1. $q(\varphi^{\infty}(a^*b)) = q(c(a)^*c(b))$,
2. $q(\varphi^{\infty}(\varphi^{-1}(ab^*))) = q(c(a)c(b)^*)$.

Proof. A routine computation shows that $\varphi^\infty(a^*b) = c(a)^*c(b)$, proving part 1. For part 2, let $\kappa = (\varphi^{-1}(a), 0, 0, \ldots) \in \mathcal{F}(A)$ and $\tau = (\varphi^{-1}(b), 0, 0, \ldots) \in \mathcal{F}(A)$ and observe that

$$\varphi^\infty(\varphi^{-1}(ab^*)) - c(a)c(b)^* = \theta_{\kappa,\tau} \in \mathcal{K}_A(\mathcal{F}(A)).$$

Applying q to the previous equation finishes the proof. ■

It has been pointed out that $\mathcal{O}(A, \varphi)$ is isomorphic to $C^*(\mathbb{Z}, A, \varphi)$, see for instance Example (4) of [24]. However, we have not found complete details for this. Below we construct a map from $\mathcal{O}(A, \varphi) \to C^*(\mathbb{Z}, A, \varphi)$ and show that it is actually an isomorphism. It is important to keep in mind that, since A is a full Hilbert A-module, $\mathcal{O}(A, \varphi)$ is solely generated by $q(c(A))$. Also, recall that we regard A as a subset of $\mathcal{O}(A, \varphi)$ via $a \mapsto q(\varphi^\infty(a))$.

We now introduce some notation first. If $a \in A$ and $j \in \mathbb{Z}$, we define the function $au_j \colon \mathbb{Z} \to A$ by letting $au_j(j) = a$ and $au_j(k) = 0$ for any $k \neq j$. It is clear that $au_j \in C_c(\mathbb{Z}, A, \varphi)$. Furthermore, notice that for any $a, b \in A$ and any $j, k \in \mathbb{Z}$

$$(au_j)(bu_k) = a\varphi^j(b)u_{k+j}, \tag{2.4.3}$$

(multiplication on LHS is twisted convolution) and

$$(au_j)^* = \varphi^{-j}(a^*)u_{-j}. \tag{2.4.4}$$

We define $\pi_0 \colon A \to C^*(\mathbb{Z}, A, \varphi)$ by

$$\pi_0(a) = au_0,$$

and $\pi_1 \colon A \to C^*(\mathbb{Z}, A, \varphi)$ by

$$\pi_1(a) = \varphi^{-1}(a)u_{-1}.$$

It is easily checked that π_0 is a $*$-homomorphism, that π_1 is a linear map, and that for any $a, b \in A$, the following holds

1. $\pi_1(\varphi(a)b) = \pi_0(a)\pi_1(b)$.

2. $\pi_1(ab) = \pi_1(a)\pi_0(b)$.

3. $\pi_0(\langle a, b\rangle_A) = \pi_1(a)^*\pi_1(b)$.

4. $\pi_0(\varphi^{-1}(ab^*)) = \pi_1(a)\pi_1(b)^*$

Then since $J_A = A$, conditions 1, 3, and 4 above show that $(C^*(\mathbb{Z}, A, \varphi), \pi_0, \pi_1)$ is a covariant Fock representation for (A, φ) (see Definitions 2.4.1 and 2.4.10). Thus, by Proposition 2.4.15, the map π_0 extends to a $*$-homomorphism $\widetilde{\pi_0} \colon \mathcal{O}(A, \varphi) \to C^*(\mathbb{Z}, A, \varphi)$ such that $q(c(a))$ gets mapped to $\pi_a(a)$ and whose range contains both $\pi_0(A)$ and $\pi_1(A)$. Since $\pi_0(A)$ and $\pi_1(A)$ generate $C_c(\mathbb{Z}, A, \varphi)$, a dense subalgebra of $C^*(\mathbb{Z}, A, \varphi)$, it follows that $\widetilde{\pi_0}$ is surjective. We still need to show that $\widetilde{\pi_0}$ is injective. To do so, it suffices to show that $\widetilde{\pi_0}$ admits a left inverse. First we need a useful lemma:

Lemma 2.4.21. *Let $n \in \mathbb{Z}_{\geq 1}$ and $a_1, \ldots, a_n \in A$. Then*

$$\|c(a_1) \cdots c(a_n)\| = \|\varphi^{n-1}(a_1) \cdots \varphi(a_{n-1})a_n\|$$

Proof. Notice that for any $(b_k)_{k=0}^{\infty} \in \mathcal{F}(A)$,

$$c(a_1)\cdots c(a_n)(b_k)_{k=0}^{\infty} = \Big(\underbrace{0,\ldots,0}_{n}, \big(\varphi^k(\varphi^{n-1}(a_1)\cdots\varphi(a_{n-1})a_n)b_k\big)_{k=0}^{\infty}\Big).$$

The desired norm equality now follows immediately. ∎

We are now ready to define a map $\gamma\colon C_c(\mathbb{Z}, A, \varphi)$. First of all, for each $a \in A$, we set

$$\gamma(au_{-1}) = q(c(\varphi(a))). \tag{2.4.5}$$

We claim that this is enough to uniquely determine a $*$-homomorphism γ : $C_c(\mathbb{Z}, A, \varphi) \to \mathcal{O}(A, \varphi)$. Indeed, looking at (2.4.4) and taking adjoins in both sides of equation (2.4.5) forces $\gamma(au_1) = q(c(a^*)^*)$ for any $a \in A$. Therefore, following the multiplication (2.4.3), we must put $\gamma(au_0) = q(\varphi^{\infty}(a))$ for $a \in A$. All this, together with (2.4.3), allows us to define $\gamma(au_j)$ for any $j \in \mathbb{Z}$. Indeed, let $a \in A$, let $j \in \mathbb{Z}_{>0}$, and use the Cohen-Hewitt factorization theorem (Theorem 1 in [4] is enough) to find $a_1, \ldots, a_j$ such that

$$a = a_1^*\varphi(a_2^*)\ldots\varphi^{n-1}(a_n^*)$$

Then since we want γ to be an algebra homomorphisms, (2.4.3) forces us to set

$$\gamma_n(au_j) = q(c(a_1)^*\cdots c(a_j)^*).$$

Furthermore, Lemma 2.4.21 implies that the definition of $\gamma_n(au_j)$ is independent of the factorization used. Finally, taking adjoints will also give a formula for each $\gamma_n(au_{-j})$ that is independent of the factorization. Thus, for any finite subset $J \subset$

$\mathbb{Z}$ and any finite sequence $(a_j)_{j\in J}$ of elements in A, we define $\gamma(\sum_{j\in J} a_j u_j)$ to be $\sum_{j\in J}\gamma(a_j u_j)$.

Lemma 2.4.22. *Let J be a finite subset of $\mathbb{Z}$ and let $(a_j)_{j\in J}$ a finite sequence of elements in A. Then*

$$\Big\|\gamma\Big(\sum_{j\in J} a_j u_j\Big)\Big\| \le \sum_{j\in J}\|a_j\|.$$

Proof. This will follow if $\|\gamma(a_j u_j)\| \le \|a_j\|$ for any $j \in \mathbb{Z}$. This is in fact an immediate consequence of the definition of γ and Lemma 2.4.21. Indeed, for instance when $j \in \mathbb{Z}_{\geq 0}$ and $a = a_1^*\varphi(a_2^*)\ldots\varphi^{n-1}(a_n^*)$,

$$\begin{aligned}\|\gamma(a_j u_j)\| &= \|q(c(a_1)^*\cdots c(a_j)^*)\|\\ &\le \|(c(a_j)\cdots c(a_1))^*\|\\ &= \|\varphi^{n-1}(a_j)\cdots\varphi(a_2)a_1\|\\ &= \|a_1^*\varphi(a_2^*)\cdots\varphi^{n-1}(a_j^*)\| = \|a\|.\end{aligned}$$

An analogous calculation (or an argument via adjoints) shows that $\|\gamma(a_j u_{-j})\| \le \|a\|$, so we are done. ■

It follows from Corollary 2.46. in [28] that a direct consequence of Lemma 2.4.22 is that for any $f \in C_c(\mathbb{Z}, A, \varphi)$

$$\|\gamma(f)\| \le \|f\|_{C^*(\mathbb{Z},A,\varphi)}.$$

Therefore γ extends to a $*$-homomorphism $\widetilde{\gamma} : C^*(\mathbb{Z}, A, \varphi) \to \mathcal{O}(A, \varphi)$. Now notice that $\widetilde{\gamma}\circ\widetilde{\pi}_0$ acts as the identity on $q(c(A))$. Indeed, let $a \in A$,

$$\widetilde{\gamma}\Big(\widetilde{\pi}_0\big(q(c(a))\big)\Big) = \widetilde{\gamma}(\pi_1(a)) = \widetilde{\gamma}(\varphi^{-1}(a)u_{-1}) = q(c(a)).$$

Therefore, since $q(c(A))$ generates $\mathcal{O}(A,\varphi)$, it follows that $\widetilde{\gamma}\circ\widetilde{\pi}_0 = \mathrm{id}_{\mathcal{O}(A,\varphi)}$, which shows that $\widetilde{\pi}_0$ is injective as we wanted to check.

Remark 2.4.23. One of the purposes of this example is to carefully look at a proof of the fact $\mathcal{O}(A,\varphi)\cong C^*(\mathbb{Z},A,\varphi)$ in order to generalize it in later chapters to the L^p setting. However, in that setting it is more convenient to exploit the fact that $(\mathcal{F}(A),\varphi^\infty)$ is isomorphic to $(\ell^2(\mathbb{Z}_{\geq 0})\otimes_{\mathbb{C}} A,\widehat{\varphi})$ as C*-correspondences over A. This statement is made more precise in the following proposition.

Proposition 2.4.24. *Let A be any C^*-algebra and let $\varphi \in \mathrm{Aut}(A)$ be an automorphism of A. Let $\ell^2(\mathbb{Z}_{\geq}0)\otimes_{\mathbb{C}} A$ be the interior tensor product of $\ell^2(\mathbb{Z}_{\geq}0)$, regarded as a C^*-correspondence over $\mathbb{C}$, with A, regarded as a $(\mathbb{C},A)$ C^*-correspondence in the obvious way. Then there is a $*$-homomorphisms $\widehat{\varphi}\colon A\to \mathcal{L}_A(\ell^2(\mathbb{Z}_{\geq}0)\otimes_{\mathbb{C}} A)$ that satisfies, for any $\xi\in\ell^2(\mathbb{Z}_{\geq 0})$, $a,b\in A$,*

$$\widehat{\varphi}(a)(\xi\otimes b)=\sum_{n=0}^{\infty}\xi(n)\delta_n\otimes\varphi^n(a)b, \tag{2.4.6}$$

where $\{\delta_n\colon n\in\mathbb{Z}_{\geq 0}\}$ is the canonical orthonormal basis for $\ell^2(\mathbb{Z}_{\geq}0)$. Furthermore, $(\ell^2(\mathbb{Z}_{\geq}0)\otimes_{\mathbb{C}} A,\widehat{\varphi})$ is a C^*-correspondence over A that is isomorphic (in the sense of Definition 2.2.5) to $(\mathcal{F}(A),\varphi^\infty)$, the C^*-correspondence over A defined below Corollary 2.4.19, via the map determined by $\xi\otimes a\mapsto(\xi(n)a)_{n=1}^{\infty}$.

Proof. We first establish the existence of $\widehat{\varphi}$. Take $a,b\in A$, $\xi\in\ell^2(\mathbb{Z}_{\geq 0})$ and $m,k\in\mathbb{Z}_{\geq 0}$ with $m>k$. Then

$$\Big\|\sum_{n=k}^{m}\xi(n)\delta_n\otimes\varphi^n(a)b\Big\|^2=\Big\|\sum_{n=k}^{m}|\xi(n)|^2(\varphi^n(a)b)^*\varphi^n(a)b\Big\|\leq\|a\|^2\|b\|^2\sum_{n=k}^{m}|\xi(n)|^2.$$

Thus, since $\xi \in \ell^2(\mathbb{Z}_{\geq}0)$, the right hand side of equation (2.4.6) defines an element of $\ell^2(\mathbb{Z}_{\geq}0) \otimes_{\mathbb{C}} A$. Now fix $k \in \mathbb{Z}_{>0}$, let $a, b_1, \ldots, b_k \in A$, and let $\xi_1, \ldots, \xi_k \in \ell^2(\mathbb{Z}_{\geq 0})$. We claim that for any $n \in \mathbb{Z}_{\geq 0}$,

$$\sum_{j=1}^{k}\sum_{i=1}^{k}\langle \varphi^n(a)\xi_j(n)b_j, \varphi^n(a)\xi_i(n)b_i\rangle_A \leq \|a\|^2 \sum_{j=1}^{k}\sum_{i=1}^{k}\langle \xi_j(n)b_j, \xi_i(n)b_i\rangle_A.$$

To prove the claim, first set $\mathbf{b} = (b_1, \ldots, b_k) \in A^k$. We consider the isomorphism $\kappa : \mathcal{L}_A(A) \to \mathcal{L}_{M_k(A)}(A^k)$ from Proposition 2.1.6 and we use Lemma 2.1.5 to get

$$\langle \kappa(\varphi^n(a))\mathbf{b}, \kappa(\varphi^n(a))\mathbf{b}\rangle_{M_k(A)} \leq \|\varphi^n(a)\|^2\langle \mathbf{b}, \mathbf{b}\rangle_{M_k(A)} \leq \|a\|^2\langle \mathbf{b}, \mathbf{b}\rangle_{M_k(A)},$$

which becomes $(\langle \varphi^n(a)b_i, \varphi^n(a)b_j\rangle_A)_{i,j} \leq \|a\|^2(\langle b_i, b_j\rangle_A)_{i,j}$ in $M_k(A)$. Now let $(e_\lambda)_{\lambda\in\lambda}$ be an approximate identity for A and define $\boldsymbol{\xi}_\lambda = (\xi_1(n)e_\lambda^{1/2}, \ldots, \xi_k(n)e_\lambda^{1/2}) \in A^k$. Then working with the obvious action of $M_k(A) = \mathcal{K}_A(A^k)$ on A^k, we have

$$\langle((\langle \varphi^n(a)b_i, \varphi^n(a)b_j\rangle_A)_{i,j}\boldsymbol{\xi}_\lambda, \boldsymbol{\xi}_\lambda\rangle_A \leq \|a\|^2\langle((\langle b_i, b_j\rangle_A)_{i,j}\boldsymbol{\xi}_\lambda, \boldsymbol{\xi}_\lambda\rangle_A,$$

which becomes

$$\sum_{j=1}^{k}\sum_{i=1}^{k}\langle \varphi^n(a)\xi_j(n)b_j, \varphi^n(a)\xi_i(n)b_i\rangle_A e_\lambda \leq \|a\|^2 \sum_{j=1}^{k}\sum_{i=1}^{k}\langle \xi_j(n)b_j, \xi_i(n)b_i\rangle_A e_\lambda.$$

The claim now follows after taking $\lim_\lambda$ on both sides of the previous equation. Thus, using the inequality from the claim at the second step we find

$$\Big\|\sum_{j=1}^{k}\sum_{n=0}^{\infty}\xi_j(n)\delta_n \otimes \varphi^n(a)b_j\Big\|^2 = \Big\|\sum_{n=0}^{\infty}\sum_{j=1}^{k}\sum_{i=1}^{k}\langle \varphi^n(a)\xi_j(n)b_j, \varphi^n(a)\xi_i(n)b_i\rangle_A\Big\|$$

$$
\begin{aligned}
&\leq \|a\|^2 \Big\| \sum_{n=0}^{\infty} \sum_{j=1}^{k} \sum_{i=1}^{k} \langle \xi_j(n) b_j, \xi_i(n) b_i \rangle_A \Big\| \\
&= \|a\|^2 \Big\| \sum_{j=1}^{k} \sum_{i=1}^{k} \langle \xi_j, \xi_i \rangle \langle b_j, b_i \rangle_A \Big\| \\
&= \|a\|^2 \Big\| \sum_{j=1}^{k} \xi_j \otimes b_j \Big\|^2.
\end{aligned}
$$

This implies that, for each $a \in A$, the formula (2.4.6) extends to a well defined linear map $\widehat{\varphi}(a) : \ell^2(\mathbb{Z}_{\geq 0}) \otimes_{\mathbb{C}} A \to \ell^2(\mathbb{Z}_{\geq 0}) \otimes_{\mathbb{C}} A$ satisfying $\|\widehat{\varphi}(a)\| \leq \|a\|$. Let $a \in A$, to check that $\widehat{\varphi}(a)$ is indeed an element of $\mathcal{L}_A(\ell^2(\mathbb{Z}_{\geq 0}) \otimes_{\mathbb{C}} A)$, it suffices to check adjointability on elementary tensors. To do so, let $\xi_1, \xi_2 \in \ell^2(\mathbb{Z}_{\geq 0})$ and $b_1, b_2 \in A$ and observe

$$
\begin{aligned}
\langle \widehat{\varphi}(a)(\xi_1 \otimes b_1), \xi_2 \otimes b_2 \rangle_A &= \sum_{n=0}^{\infty} \langle \xi_1(n) \delta_n, \xi_2 \rangle \langle \varphi^n(a) b_1, b_2 \rangle_A \\
&= \sum_{n=0}^{\infty} \langle \xi_1, \xi_2 \delta_n \rangle \langle b_1, \varphi^n(a^*) b_2 \rangle_A \\
&= \langle \xi_1 \otimes b_1, \widehat{\varphi}(a^*)(\xi_2 \otimes b_2) \rangle_A.
\end{aligned}
$$

Hence, $\widehat{\varphi} : A \to \mathcal{L}_A(\ell^2(\mathbb{Z}_{\geq 0}) \otimes_{\mathbb{C}} A)$ is a *-homomorphism which makes $(\ell^2(\mathbb{Z}_{\geq} 0) \otimes_{\mathbb{C}} A, \widehat{\varphi})$ into a C*-correspondence over A. It only remains to show that it is isomorphic to $(\mathcal{F}(A), \varphi^{\infty})$. For each $\xi \in \ell^2$ and $a \in A$ we define $\tau(\xi \otimes a) = (\xi(n)a)_{n=1}^{\infty}$. Notice that

$$
\sum_{n=0}^{\infty} (\xi(n)a)^* \xi(n) a = a^* a \|\xi\|_2,
$$

and therefore $\tau(\xi \otimes a) \in \mathcal{F}(A)$. Furthermore, if $k \in \mathbb{Z}_{>0}$, $\xi_1, \ldots, \xi_k \in \ell^2(\mathbb{Z}_{\geq 0})$, and $a_1, \ldots, a_k \in A$, then

$$\Big\| \sum_{j=1}^{k} (\xi_j(n) a_j)_{n=1}^{\infty} \Big\|^2 = \Big\| \sum_{j=1}^{k} \sum_{i=1}^{k} \langle \xi_j, \xi_i \rangle a^* a \Big\| = \Big\| \sum_{j=1}^{k} \xi_j \otimes a_j \Big\|^2.$$

Thus, τ extends to an isometric linear map $\tau\colon \ell^2(\mathbb{Z}_{\geq 0}) \otimes_{\mathbb{C}} A \to \mathcal{F}(A)$. To show that τ is also surjective, take any $(a_n)_{n=0}^{\infty} \in \mathcal{F}(A)$ and observe that for $k, m \in \mathbb{Z}_{\geq 0}$ with $k < m$ we have

$$\Big\| \sum_{n=k}^{m} \delta_n \otimes a_n \Big\|^2 = \Big\| \sum_{n=k}^{m} a_n^* a_m \Big\|.$$

Thus, $\sum_{n=0}^{\infty} \delta_n \otimes a_n \in \ell^2(\mathbb{Z}_{\geq 0}) \otimes_{\mathbb{C}} A$ and gets mapped to $(a_n)_{n=0}^{\infty}$ via τ, showing that τ is onto. Finally, we claim that $(\mathrm{id}_A, \mathrm{id}_A, \tau)\colon (\ell^2(\mathbb{Z}_{\geq 0}) \otimes_{\mathbb{C}} A, \widehat{\varphi}) \to (\mathcal{F}(A), \varphi^{\infty})$ is an isomorphism as in Definition 2.2.5. We only need to check conditions 1-2 in Definition 2.2.5 and it suffices to do so for elementary tensors. Let $a, b \in A$ and $\xi \in \ell^2(\mathbb{Z}_{\geq 0})$, and notice

$$\tau(\widehat{\varphi}(a)(\xi \otimes b)) = \sum_{n=0}^{\infty} (\xi(n)\delta_n(k)\varphi^n(a)b)_{k=0}^{\infty} = (\varphi^n(a)\xi(n)b)_{n=0}^{\infty} = \varphi^{\infty}(a)\tau(\xi \otimes b),$$

proving condition 1. Finally, if $\xi_1, \xi_2 \in \ell^2(\mathbb{Z}_{\geq}0)$ and $b_1, b_2 \in A$ we compute

$$\langle \tau(\xi_1 \otimes b_1), \tau(\xi_2 \otimes b_2) \rangle = \sum_{n=0}^{\infty} (\xi_1(n) b_1)^* \xi_2(n) b_2 = \langle \xi_1, \xi_2 \rangle b_1^* b_2 = \langle \xi_1 \otimes b_1, \xi_2 \otimes b_2 \rangle.$$

proving condition 2 and thus finishing the proof. $\blacksquare$

Corollary 2.4.25. *Under the isomorphism from Proposition 2.4.24, the creation operator (see equation (2.4.1)) on* $\ell^2(\mathbb{Z}_{\geq 0} \otimes_{\mathbb{C}} A)$ *is determined by*

$$c(a)(\xi \otimes b) = \sum_{n=1}^{\infty} (s\xi)(n)\delta_n \otimes \varphi^{n-1}(a)b \tag{2.4.7}$$

where $s \in \mathcal{L}(\ell^2(\mathbb{Z}_{\geq 0})$ *is right translation. Similarly, its adjoint (see equation (2.4.2)) is determined by*

$$c(a)^*(\xi \otimes b) = \sum_{n=0}^{\infty} (t\xi)(n)\delta_n \otimes \varphi^{n}(a^*)b \tag{2.4.8}$$

where $t \in \mathcal{L}(\ell^2(\mathbb{Z}_{\geq 0})$ *is left translation.*

Proof. Let τ be as in the proof of Proposition 2.4.24. The first formula follows by appropriately composing the formula in (2.4.1) with τ and τ^{-1} Form there, checking that everything is well defined needs analogous arguments and computations as the ones used in Proposition 2.4.24 for the map $\widehat{\varphi}$. ∎

CHAPTER III

REPRESENTING MODULES ON PAIRS OF HILBERT SPACES

Concrete Hilbert modules

In this section, we describe a concrete example of a right Hilbert module X over a concrete C*-algebra $A \subseteq \mathcal{L}(\mathcal{H}_0)$ for a Hilbert space $\mathcal{H}_0$. We provide useful representations for the C*-algebras $\mathcal{L}_A(\mathsf{X})$ and $\mathcal{K}_A(\mathsf{X})$. In section 3.2 we will see that any right Hilbert A-module can be represented in this fashion.

Example 3.1.1. Let $\mathcal{H}_0, \mathcal{H}_1$ be Hilbert spaces and let $A \subseteq \mathcal{L}(\mathcal{H}_0)$ be a concrete C*-algebra. Suppose that $\mathsf{X} \subseteq \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is a closed subspace such that $xa \in \mathsf{X}$ for all $x \in \mathsf{X}$ and all $a \in A$, and such that $x^*y \in A$ for all $x, y \in \mathsf{X}$. For each $x, y \in \mathsf{X}$ we put

$$\langle x, y\rangle_A = x^*y \in A. \tag{3.1.1}$$

Proposition 3.1.2. *Let X be as in Example 3.1.1. Then X is a right Hilbert A-module with A-valued inner product given by equation (3.1.1) and $\|\langle x, x\rangle_A\|^{1/2} = \|x\|$.*

Proof. It is clear that X is a right A-module. It is easily checked that $(x, y) \mapsto \langle x, y\rangle_A$ satisfies all the axioms of an A-valued inner product on X. We claim that X is complete with the induced norm $\|x\|_A = \|\langle x, x\rangle_A\|^{1/2}$. Indeed, elements of the C*-algebra $\mathcal{L}(\mathcal{H}_0 \oplus \mathcal{H}_1)$ can be written as 2×2 operator valued matrices and $\mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is isometrically isomorphic to the lower left corner of $\mathcal{L}(\mathcal{H}_0 \oplus \mathcal{H}_1)$, while $\mathcal{L}(\mathcal{H}_0)$ is isomorphic to the upper left corner. Hence, if $x \in \mathsf{X}$, the C*-equation at the second step yields

$$\|x\|^2 = \left\| \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} x^*x & 0 \\ 0 & 0 \end{pmatrix} \right\| = \|x^*x\| = \|x\|_A^2.$$

The claim now follows because X is closed in $\mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$. Thus, X is indeed a right Hilbert A-module. ■

Remark 3.1.3. Thanks to Proposition 3.1.2 above, when X is as in Example 3.1.1, we are free to not make any distinction between the norm $x \in X$ has as an element of $\mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ and the module norm $\|x\|_A$. Thus, from now on we drop the subscript A and simply write $\|x\|$.

Next, we will show that the compact-module maps and adjointable maps of the Hilbert module in Example 3.1.1 above can be realized as closed C*-subalgebras of $\mathcal{L}(\mathcal{H}_1)$, provided that some nondegeneracy conditions hold.

Proposition 3.1.4. *Let X be the right Hilbert A-module described in Example 3.1.1 above. Suppose that $\mathsf{X}\mathcal{H}_0$ is dense in $\mathcal{H}_1$. Then there is a $*$-isomorphism from $\mathcal{K}_A(\mathsf{X})$ to*

$$\overline{\operatorname{span}\{xy^*\colon x, y \in \mathsf{X}\}} \subseteq \mathcal{L}(\mathcal{H}_1)$$

which sends $\theta_{x,y}$ to xy^* for $x, y \in \mathsf{X}$.

Proof. Let $K_1 = \operatorname{span}\{xy^*\colon x, y \in \mathsf{X}\} \subseteq \mathcal{L}(\mathcal{H}_1)$ and let $K_2 = \operatorname{span}\{\theta_{x,y}\colon x, y \in \mathsf{X}\} \subseteq \mathcal{K}_A(\mathsf{X})$. Recall that K_2 is dense in $\mathcal{K}_A(\mathsf{X})$. Let $n \in \mathbb{Z}_{\geq 1}$ and let $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathsf{X}$. Then for any $z \in \mathsf{X}$

$$\Big\|\sum_{j=1}^{n} \theta_{x_j,y_j}(z)\Big\|_{\mathsf{X}} = \Big\|\Big(\sum_{j=1}^{n} x_j y_j^*\Big) z\Big\|_{\mathcal{L}(\mathcal{H}_0,\mathcal{H}_1)} \leq \Big\|\sum_{j=1}^{n} x_j y_j^*\Big\|_{\mathcal{L}(\mathcal{H}_1)} \|z\|.$$

This implies

$$\Big\|\sum_{j=1}^{n}\theta_{x_j,y_j}\Big\|_{\mathcal{K}_A(\mathsf{X})} \leq \Big\|\sum_{j=1}^{n}x_jy_j^*\Big\|_{\mathcal{L}(\mathcal{H}_1)}. \tag{3.1.2}$$

Let $\iota\colon K_1 \to \mathcal{K}_A(\mathsf{X})$ be the linear extension of the map which sends xy^* to $\theta_{x,y}$ for $x, y \in \mathsf{X}$. That is,

$$\iota\Big(\sum_{j=1}^{n}x_jy_j^*\Big) = \sum_{j=1}^{n}\theta_{x_j,y_j}.$$

That ι is well defined follows from (3.1.2). In fact, (3.1.2) gives $\|\iota(k)\| \leq \|k\|$ for all $k \in K_1$. Thus, we can extend ι by continuity to a map $\tilde{\iota}\colon \overline{K_1} \to \mathcal{K}_A(\mathsf{X})$ such that $\|\tilde{\iota}(s)\|_{\mathcal{K}_A(\mathsf{X})} \leq \|s\|_{\mathcal{L}(\mathcal{H}_1)}$ for all $s \in \overline{K_1}$. Our goal is to show that $\tilde{\iota}$ is a $*$-isomorphism from $\overline{K_1}$ to $\mathcal{K}_A(\mathsf{X})$. Notice that $\tilde{\iota}$ is already a $*$-homomorphism between C*-algebras. We will show that $\tilde{\iota}$ is injective, which in turn will make $\tilde{\iota}$ an isometry. Since $\tilde{\iota}$ maps K_1 onto K_2, a dense subset of $\mathcal{K}_A(\mathsf{X})$, proving injectivity will automatically show that $\tilde{\iota}$ is a $*$-isomorphism and this will finish the proof.

Take any $s \in \overline{K_1}$ and fix $x \in \mathsf{X}$. We claim that the element $sx \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is actually in X and that it is equal to $\tilde{\iota}(s)(x) \in \mathsf{X}$. Indeed, for any $k \in K_1$, the element $kx \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is an element of X (because $x_1x_2^*x \in \mathsf{X}A \subseteq \mathsf{X}$ for all $x_1, x_2 \in \mathsf{X}$) and it coincides with $\iota(k)(x) \in \mathsf{X}$. Thus, by continuity it follows that $sx = \tilde{\iota}(s)(x)$, as claimed.

Finally, to prove that $\tilde{\iota}$ is injective, let $s \in \overline{K_1}$ satisfy $\tilde{\iota}(s) = 0$ in $\mathcal{K}_A(\mathsf{X})$. We have to show that $s = 0$ in $\mathcal{L}(\mathcal{H}_1)$, but since $\mathsf{X}\mathcal{H}_0$ is dense in $\mathcal{H}_1$, it is enough to prove that $s(x\xi) = 0$ for all $x \in \mathsf{X}$ and $\xi \in \mathcal{H}_0$. Indeed, thanks to our last claim, we have

$$s(x\xi) = sx(\xi) = [\tilde{\iota}(s)(x)]\xi = 0.$$

This finishes the proof. ■

Proposition 3.1.5. *Let X be the right Hilbert A-module described in Example 3.1.1 above. Suppose that $\mathsf{X}\mathcal{H}_0$ is dense in $\mathcal{H}_1$. Define $B \subseteq \mathcal{L}(\mathcal{H}_1)$ by*

$$B = \{b \in \mathcal{L}(\mathcal{H}_1) \colon bx, b^*x \in \mathsf{X} \textit{ for all } x \in \mathsf{X}\}.$$

For each $b \in B$ we get a map $\tau(b) \colon \mathsf{X} \to \mathsf{X}$, given by $\tau(b)(x) = bx$. Then B is $*$-isomorphic to $\mathcal{L}_A(\mathsf{X})$, via the map that sends $b \in B$ to $\tau(b)$.

Proof. For any $b \in B$ and any $x, y \in \mathsf{X}$, we have

$$\langle bx, y\rangle_A = (bx)^*y = x^*(b^*y) = \langle x, b^*y\rangle_A.$$

Thus, $\tau(b) \in \mathcal{L}_A(\mathsf{X})$ and $\tau(b)^* = \tau(b^*)$. It is also easily checked that τ is $*$-homomorphism. Furthermore, it follows from density of $\mathsf{X}\mathcal{H}_0$ in $\mathcal{H}_1$ that τ is also injective.

We will finish the proof if we show that τ is surjective. Take any $t \in \mathcal{L}_A(\mathsf{X})$. We have maps $t \colon \mathsf{X} \to \mathsf{X}$ and $t^* \colon \mathsf{X} \to \mathsf{X}$ satisfying

$$t(x)^*y = x^*t^*(y) \tag{3.1.3}$$

for all $x, y \in \mathsf{X}$. Thus, if $n \in \mathbb{Z}_{\geq 1}$, $x_1, \ldots, x_n \in \mathsf{X}$, and $\xi_1, \ldots, \xi_n \in \mathcal{H}_0$, then we find, using (3.1.3) at the final step,

$$\Big\|\sum_{j=1}^{n} t(x_j)\xi_j\Big\|^2 = \sum_{j=1}^{n}\sum_{i=1}^{n}\big\langle t(x_j)\xi_j, t(x_i)\xi_i\big\rangle = \sum_{j=1}^{n}\sum_{i=1}^{n}\big\langle x_i^*(t^*t)(x_j)\xi_j, \xi_i\big\rangle. \tag{3.1.4}$$

Recall from Proposition 2.1.6 that X^n can be viewed as a Hilbert $M_n(A)$-module and that $\mathcal{L}_A(\mathsf{X}) \cong \mathcal{L}_{M_n(A)}(\mathsf{X}^n)$ via the map $t \mapsto \kappa(t)$. Applying Lemma 2.1.5 to

$\kappa(t) \in \mathcal{L}_{M_n(A)}(\mathsf{X}^n)$, we get

$$(x_i^*(t^*t)(x_j))_{i,j} = (\langle t(x_i), t(x_j)\rangle_A)_{i,j} \leq \|\kappa(t)\|^2(\langle x_i, x_j\rangle_A)_{i,j} = \|t\|^2(x_i^*x_j)_{i,j}. \quad (3.1.5)$$

Therefore, if we let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in \mathcal{H}_0^n$, and consider the obvious action of $M_n(A)$ on $\mathcal{H}_0^n$, then we get, using (3.1.5) at the second step,

$$\begin{aligned}
\sum_{j=1}^n \sum_{i=1}^n \big\langle x_i^*(t^*t)(x_j)\xi_j, \xi_i\big\rangle &= \big\langle (x_i^*(t^*t)(x_j))_{i,j}\boldsymbol{\xi}, \boldsymbol{\xi}\big\rangle \\
&\leq \|t\|^2\big\langle (x_i^*x_j)_{i,j}\boldsymbol{\xi}, \boldsymbol{\xi}\big\rangle \\
&= \|t\|^2 \sum_{j=1}^n \sum_{i=1}^n \langle x_j\xi_j, x_i\xi_i\rangle \\
&= \|t\|^2 \Big\|\sum_{j=1}^n x_j\xi_j\Big\|^2.
\end{aligned}$$

This, together with (3.1.4), shows that

$$\Big\|\sum_{j=1}^n t(x_j)\xi_j\Big\| \leq \|t\|\Big\|\sum_{j=1}^n x_j\xi_j\Big\|. \quad (3.1.6)$$

We can now define $b_t\colon \mathsf{X}\mathcal{H}_0 \to \mathcal{H}_1$ by letting $b_t(x\xi) = t(x)\xi \in \mathcal{H}_1$, and extending linearly to all of $\mathsf{X}\mathcal{H}_0$. That is,

$$b_t\Big(\sum_{j=1}^n x_j\xi_j\Big) = \sum_{j=1}^n t(x_j)\xi_j.$$

Notice that (3.1.6) shows that b_t is well defined and that $\|b_t(\eta)\| \leq \|t\|\|\eta\|$, for all $\eta \in \mathsf{X}\mathcal{H}_0 = \operatorname{span}\{x\xi\colon x \in \mathsf{X} \text{ and } \xi \in \mathcal{H}_0\}$. Thus, we extend b_t by continuity to all of $\mathcal{H}_1$, and get a well defined map $b_t \in \mathcal{L}(\mathcal{H}_1)$ such that $\|b_t(\eta)\| \leq \|t\|\|\eta\|$ for all $\eta \in \mathcal{H}_1$. Let $x \in \mathsf{X}$. Since for all $\xi \in \mathcal{H}_0$, we have $(b_tx)\xi = b_t(x\xi) = t(x)\xi$, it follows that $b_t x = t(x) \in \mathsf{X}$. Similarly, for any $x, y \in \mathsf{X}$, we have $x^* t(y) = x^* b_t y = (b_t^* x)^* y$ and therefore

$$\langle t^*(x), y \rangle_A = \langle x, t(y) \rangle_A = x^* t(y) = (b_t^* x)^* y = \langle b_t^* x, y \rangle_A.$$

Hence, $b_t^* x = t^*(x) \in \mathsf{X}$. Thus, $b_t \in B$ and since $\tau(b_t)(x) = b_t x = t(x)$, surjectivity of τ now follows, finishing the proof. ∎

We end this section with a lemma that only needs familiarity with Example 3.1.1 and Proposition 3.1.4, but that will be needed later to show that our definition of infinite direct sums of L^p modules agrees with the known definition of Hilbert modules when $p = 2$; see Theorem 5.3.3.

Lemma 3.1.6. *Let $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$ be Hilbert spaces. Let $\mathcal{H} = \bigoplus_{j=1}^n \mathcal{H}_j$ and for each $j \in \{1, \ldots, n\}$, let $x_j \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_j)$. Then the norm of the matrix $(x_j x_k^*)_{j,k=1}^n \in \mathcal{L}(\mathcal{H})$ is given by*

$$\|(x_j x_k^*)_{j,k=1}^n\| = \Big\| \sum_{j=1}^n x_j^* x_j \Big\|_{\mathcal{L}(\mathcal{H}_0)}.$$

Proof. For each $j \in \{1, \ldots, n\}$, define $\mathsf{X}_j = \mathcal{L}(\mathcal{H}_0, \mathcal{H}_j)$ and think of it as a right Hilbert $\mathcal{L}(\mathcal{H}_0)$-module as in Example 3.1.1. With no loss of generality, we may replace $\mathcal{H}_j$ by $\overline{\mathsf{X}_j \mathcal{H}_0}$. Hence, $x_j \in \mathsf{X}_j$ for each $j \in \{1, \ldots, n\}$. Consider the direct sum Hilbert module $\mathsf{X} = \mathsf{X}_1 \oplus \cdots \oplus \mathsf{X}_n$. Then X is a right Hilbert $\mathcal{L}(\mathcal{H}_0)$-module and it is a closed subspace of $\mathcal{L}(\mathcal{H}_0, \mathcal{H})$ satisfying the assumptions of Example 3.1.1. Furthermore, $\mathsf{X}\mathcal{H}_0$ is dense in $\mathcal{H}$ (because $\mathsf{X}_j \mathcal{H}_0$ is dense in $\mathcal{H}_j$ for each j) and therefore the hypothesis of Proposition 3.1.4 is met. Thus, for any $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathsf{X}$, we now have $(x_j x_k^*)_{j,k=1}^n = \theta_{\boldsymbol{x},\boldsymbol{x}} = \boldsymbol{x}\boldsymbol{x}^* \in \mathcal{K}_A(\mathsf{X}) \subseteq \mathcal{L}(\mathcal{H})$.

Hence,

$$\|(x_j x_k^*)_{j,k=1}^n\| = \|\theta_{\boldsymbol{x},\boldsymbol{x}}\| = \|\boldsymbol{x}\|^2 = \Big\| \sum_{j=1}^n x_j^* x_j \Big\|_A,$$

as we wanted to show. ∎

Representations of Hilbert bimodules and Hilbert modules

The main purpose of this section is to state known results for representations of Hilbert modules and bimodules. Our main contribution here is Proposition 3.2.8, where we use Proposition 3.1.4 and Proposition 3.1.5 above to characterize the adjointable and the compact-module maps for a representation of a right Hilbert module. Such representations are guaranteed to exist by Corollary 3.2.6 below. Roughly, this corollary states that for any right Hilbert A-module X, there are Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ and an isometric linear map $\pi_{\mathsf{X}}\colon \mathsf{X} \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ such that $\pi_{\mathsf{X}}(\mathsf{X})$ has the right Hilbert module structure from Example 3.1.1 above. We start by establishing what a representations of a Hilbert bimodule is. The following comes mostly from Definition 4.5 in [10].

Definition 3.2.1. Let X be a Hilbert A-B-bimodule. A *representation of* X *on a pair of Hilbert spaces* $(\mathcal{H}_0, \mathcal{H}_1)$ consists of a triple $(\pi_A, \pi_B, \pi_{\mathsf{X}})$, where π_A is a representation of A on $\mathcal{H}_1$, π_B is a representation of B on $\mathcal{H}_0$, and $\pi_{\mathsf{X}}\colon \mathsf{X} \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is a linear map, such that for all $a \in A, b \in B$, and $x, y \in \mathsf{X}$, the following compatibility conditions are satisfied.

1. $\pi_{\mathsf{X}}(ax) = \pi_A(a)\pi_{\mathsf{X}}(x)$,

2. $\pi_{\mathsf{X}}(xb) = \pi_{\mathsf{X}}(x)\pi_B(b)$,

3. $\pi_A({}_A\langle x, y\rangle) = \pi_{\mathsf{X}}(x)\pi_{\mathsf{X}}(y)^*$,

4. $\pi_B(\langle x, y\rangle_B) = \pi_{\mathsf{X}}(x)^*\pi_{\mathsf{X}}(y)$.

If π_{X} is an isometry, we say the representation $(\pi_A, \pi_B, \pi_{\mathsf{X}})$ is *isometric.*

Remark 3.2.2. The map π_{X} in Definition 3.2.1 is required to be bounded in Definition 4.5 in [10]. However, since both π_A and π_B are $*$-homomorphisms, boundedness of π_{X} follows either from compatibility condition (3) or (4). Indeed, for instance, compatibility condition (3) gives

$$\|\pi_X(x)\|^2 = \|\pi_{\mathsf{X}}(x)\pi_{\mathsf{X}}(x)^*\| = \|\pi_A({}_A\langle x, x\rangle)\| \leq \|{}_A\langle x, x\rangle\| = \|x\|^2.$$

Similarly, Proposition 4.6 in [10] shows that $(\pi_A, \pi_B, \pi_{\mathsf{X}})$ is an isometric representation of a Hilbert A-B-bimodule X whenever either π_A or π_B is faithful. Indeed, for example, if π_B is isometric, then by the compatibility condition (4) we have

$$\|\pi_{\mathsf{X}}(x)\|^2 = \|\pi_{\mathsf{X}}(x)^*\pi_{\mathsf{X}}(x)\| = \|\pi_B(\langle x, x\rangle_B)\| = \|\langle x, x\rangle_B\| = \|x\|^2.$$

The following theorem establishes the existence of representations for any Hilbert A-B-bimodule.

Theorem 3.2.3. *Let A and B be C^*-algebras, and let X be a Hilbert A-B-bimodule. Then for any nondegenerate representation π_B of B on a Hilbert space $\mathcal{H}_0$, there are a nondegenerate representation $\pi_A\colon A \to \mathcal{L}(\mathcal{H}_1)$ of A on a Hilbert space $\mathcal{H}_1$ and a bounded linear map $\pi_{\mathsf{X}}\colon \mathsf{X} \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$, such that $(\pi_A, \pi_B, \pi_{\mathsf{X}})$ is a representation of X on $(\mathcal{H}_0, \mathcal{H}_1)$.*

Proof. See Proposition 4.7 in [10]. ∎

Corollary 3.2.4. *Let A and B be C*-algebras, and let X be a Hilbert A-B-bimodule. Then there is an isometric representation $(\pi_A, \pi_B, \pi_{\mathsf{X}})$ of X on some pair of Hilbert spaces $(\mathcal{H}_0, \mathcal{H}_1)$.*

Proof. Let $\pi_B : B \to \mathcal{L}(\mathcal{H}_0)$ be the universal representation of B. Then π_B is faithful and nondegenerate. Hence, this follows at once from Theorem 3.2.3 and Remark 3.2.2. ■

We now present the definition for a representation of a right Hilbert module, which comes from looking at the conditions in Definition 3.2.1 that only deal with the right action and right inner product.

Definition 3.2.5. Let A be a C*-algebra and let X be a right Hilbert A-module. A *representation of* X *on a pair of Hilbert spaces* $(\mathcal{H}_0, \mathcal{H}_1)$ consists of a pair $(\pi_A, \pi_{\mathsf{X}})$ such that π_A is a representation of A on $\mathcal{H}_0$, and $\pi_{\mathsf{X}} : \mathsf{X} \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is a linear map, such that for all $a \in A$, and all $x, y \in \mathsf{X}$, the following compatibility conditions are satisfied.

1. $\pi_{\mathsf{X}}(xa) = \pi_{\mathsf{X}}(x)\pi_A(a)$,

2. $\pi_A(\langle x, y\rangle_A) = \pi_X(x)^*\pi_X(y)$.

If π_{X} is an isometry, we say the representation $(\pi_A, \pi_{\mathsf{X}})$ is *isometric.*

The map π_{X} in Definition 3.2.5 is always bounded and this follows exactly as in Remark 3.2.2. Similarly, faithfulness of π_A is sufficient for $(\pi_A, \pi_{\mathsf{X}})$ to be isometric. The following result establishes the existence of (isometric) representations for right Hilbert modules.

Corollary 3.2.6. *Let A be a C*-algebra and let X be a right Hilbert A-module. Then for any nondegenerate representation π_A of A on a Hilbert space $\mathcal{H}_0$, there* *are a Hilbert space $\mathcal{H}_1$ and a linear map $\pi_{\mathsf{X}}\colon \mathsf{X} \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ such that $(\pi_A, \pi_{\mathsf{X}})$ is a representation of X on $(\mathcal{H}_0, \mathcal{H}_1)$ as in Definition 3.2.5. Furthermore, if π_A is faithful, then (π_A, π_X) is isometric and in this case $\pi_{\mathsf{X}}(\mathsf{X})$ has the right Hilbert $\pi_A(A)$-module structure from Example 3.1.1.*

Proof. It is well known that a right Hilbert A-module X is also a Hilbert $\mathcal{K}_A(\mathsf{X})$-$A$-bimodule. Hence the desired result follows at once from Theorem 3.2.3. The isometric part of the statement follows from Remark 3.2.2. ■

Remark 3.2.7. There is a quite different approach to prove Corollary 3.2.6 that does not depend on Theorem 3.2.3. Indeed, one can take π_{X} to be the restriction to X of the map U from Theorem 2.6 in [29]. We are thankful to Julian Kranz for pointing out this reference to us.

We end this section by observing that our main results from Section III can be stated using the language of Definition 3.2.5.

Proposition 3.2.8. *Let A be a C^*-algebra, let X be any right Hilbert A-module, and let $(\pi_{\mathsf{X}}, \pi_A)$ be a representation of X on $(\mathcal{H}_0, \mathcal{H}_1)$, with π_A faithful. Suppose that $\pi_{\mathsf{X}}(\mathsf{X})\mathcal{H}_0$ is dense in $\mathcal{H}_1$. Then the C^*-algebras $\mathcal{K}_A(\mathsf{X})$ and $\mathcal{L}_A(\mathsf{X})$ can be represented on $\mathcal{H}_1$ via the maps described below.*

1. *There is a $*$-isomorphism from $\mathcal{K}_A(\mathsf{X})$ to*

$$\overline{\operatorname{span}\{\pi_{\mathsf{X}}(x)\pi_{\mathsf{X}}(y)^*\colon x, y \in \mathsf{X}\}} \subseteq \mathcal{L}(\mathcal{H}_1),$$

   which sends $\theta_{x,y}$ to $\pi_{\mathsf{X}}(x)\pi_{\mathsf{X}}(y)^*$ for $x, y \in \mathsf{X}$.

2. *We define*

$$B = \{b \in \mathcal{L}(\mathcal{H}_1)\colon b\pi_{\mathsf{X}}(x), b^*\pi_{\mathsf{X}}(x) \in \pi_{\mathsf{X}}(\mathsf{X}) \text{ for all } x \in \mathsf{X}\}.$$

For each $b \in B$ we get a map $\tau(b)\colon \pi_{\mathsf{X}}(\mathsf{X}) \to \pi_{\mathsf{X}}(\mathsf{X})$, given by $\tau(b)(\pi_{\mathsf{X}}(x)) = b\pi_{\mathsf{X}}(x)$. Then B is $*$-isomorphic to $\mathcal{L}_A(\mathsf{X})$, via the map that sends $b \in B$ to $\pi_{\mathsf{X}}^{-1} \circ \tau(b) \circ \pi_{\mathsf{X}}$, where π_{X}^{-1} is interpreted as the inverse of the linear bijection $\pi_{\mathsf{X}}\colon \mathsf{X} \to \pi_{\mathsf{X}}(\mathsf{X})$.

Proof. Since π_A is faithful, π_{X} is isometric. The result now follows immediately after replacing A with its isometric copy $\pi_A(A)$ and X with its isometric copy $\pi_{\mathsf{X}}(\mathsf{X})$ on Proposition 3.1.4 for part (1), and on Proposition 3.1.5 for part (2). ∎

Representations of C*-correspondences

In this section we define representations of C*-correspondences and present the main result of this paper, Theorem 3.3.2, which we will see is actually a generalization of Theorem 3.2.3. We then give two applications of this theorem. The first one, contained in Theorem 3.3.7, gives necessary and sufficient conditions for a general (A, B) C*-correspondence to admit a Hilbert A-B-bimodule structure. The second one, given in Theorem 3.3.12, shows that the interior tensor product of correspondences admits a representation as the product of suitable representations of the factors.

Recall that C*-correspondences are a generalization of Hilbert bimodules. Our goal is then to find a general version of Theorem 3.2.3 that works for the general C*-correspondence setting. For this, we need first to define what we mean by representations of C*-correspondences.

Definition 3.3.1. Let A and B be C*-algebras, and let (X, φ) be an (A, B) C*-correspondence. A *representation of* (X, φ) *on a pair of Hilbert spaces* $(\mathcal{H}_0, \mathcal{H}_1)$ consists of a triple $(\pi_A, \pi_B, \pi_{\mathsf{X}})$ where π_A is a representation of A on $\mathcal{H}_1$, π_B is a representation of B on $\mathcal{H}_0$, and $\pi_{\mathsf{X}}\colon \mathsf{X} \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is a linear map, such that for all $a \in A$, and all $x, y \in \mathsf{X}$, the following compatibility conditions are satisfied.

1. $\pi_{\mathsf{X}}(\varphi(a)x) = \pi_A(a)\pi_{\mathsf{X}}(x)$,
2. $\pi_{\mathsf{X}}(xb) = \pi_{\mathsf{X}}(x)\pi_B(b)$,
3. $\pi_B(\langle x, y\rangle_B) = \pi_{\mathsf{X}}(x)^*\pi_{\mathsf{X}}(y)$.

If π_{X} is an isometry, we say the representation $(\pi_A, \pi_B, \pi_{\mathsf{X}})$ is *isometric*.

As in Remark 3.2.2, the linear map π_{X} from Definition 3.3.1 is automatically bounded and faithfulness of π_B is sufficient for $(\pi_A, \pi_B, \pi_{\mathsf{X}})$ to be isometric.

We point out that Definition 3.3.1 agrees with the definitions of representations of C*-correspondences in the literature. Indeed, suppose that (X, φ) is an (A, A) C*-correspondence and that $(\pi_A, \pi_A, \pi_{\mathsf{X}})$ is a representation of (X, φ) as in Definition 3.3.1. Then $(\pi_A, \pi_{\mathsf{X}})$ is a representation of (X, φ) on $\mathcal{L}(\mathcal{H}_0)$ in the sense of Definition 2.1 in [14] and an isometric covariant representation of (X, φ) on $\mathcal{H}_0$ in the sense of Definition 2.11 in [18]. More generally, suppose that (X, φ) is an (A, B) C*-correspondence and that $(\pi_A, \pi_B, \pi_{\mathsf{X}})$ is a representation of (X, φ) on $(\mathcal{H}_0, \mathcal{H}_1)$ as in Definition 3.3.1. Then letting $C = \mathcal{L}(\mathcal{H}_0 \oplus \mathcal{H}_1)$, we get obvious maps $\widehat{\pi_A}\colon A \to C$, $\widehat{\pi_B}\colon B \to C$, and $\widehat{\pi_{\mathsf{X}}}\colon \mathsf{X} \to C$ induced by π_A, π_B, and π_{X}. It is clear that $(\widehat{\pi_A}, \widehat{\pi_B}, \widehat{\pi_{\mathsf{X}}})$ is in particular a rigged representation of (X, φ) on C in the sense of Definition 3.7 in [3].

Main results.

The following theorem establishes the existence of representations for any (A, B) C*-correspondence.

Theorem 3.3.2. *Let A and B be C*-algebras and let (X, φ) be an (A,B) C*-correspondence. Then for any nondegenerate representation π_B of B on a Hilbert space $\mathcal{H}_0$, there are a representation $\pi_A \colon A \to \mathcal{L}(\mathcal{H}_1)$ of A on a Hilbert space $\mathcal{H}_1$ and a bounded linear map $\pi_{\mathsf{X}} \colon \mathsf{X} \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$, such that $(\pi_A, \pi_B, \pi_{\mathsf{X}})$ is a representation of (X, φ) on $(\mathcal{H}_0, \mathcal{H}_1)$ as in Definition 3.3.1. If in addition A acts nondegenerately on X, then π_A is nondegenerate.*

Proof. Notice that $(\mathcal{H}_0, \pi_B)$ is a $(B, \mathbb{C})$ C*-correspondence. Let $\mathcal{H}_1 = \mathsf{X} \otimes_{\pi_B} \mathcal{H}_0$ be the interior tensor product of (X, φ) with $(\mathcal{H}_0, \pi_B)$, which is in particular a right Hilbert $\mathbb{C}$-module, that is, a Hilbert space. The representation of A on $\mathcal{H}_1$ comes from the left action of A on $\mathcal{H}_1$ gotten from equation (2.2.4) in the interior tensor product construction. Indeed, Proposition 2.66 of [25] gives $\pi_A \colon A \to \mathcal{L}(\mathcal{H}_1)$, a representation of A, such that for each $a \in A$, each $x \in \mathsf{X}$, and each $\xi \in \mathcal{H}_0$,

$$\pi_A(a)(x \otimes \xi) = \varphi(a)x \otimes \xi. \tag{3.3.1}$$

Furthermore, it is also shown in Proposition 2.66 of [25] that π_A is nondegenerate whenever A acts nondegenerately on X. We now establish the existence of π_{X}. This is motivated by the Fock space construction in [24]. Indeed, for each $x \in \mathsf{X}$, let $\pi_{\mathsf{X}}(x) \colon \mathcal{H}_0 \to \mathcal{H}_1$ be the creation operator

$$\pi_{\mathsf{X}}(x)\xi = x \otimes \xi. \tag{3.3.2}$$

Then it is clear that $x \mapsto \pi_{\mathsf{X}}(x)$ is a linear map from X to $\mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$. As in Remark 3.2.2, boundedness of π_{X} will follow once we check the compatibility conditions from Definition 3.3.1, which will in turn prove that $(\pi_A, \pi_B, \pi_{\mathsf{X}})$ is indeed a representation of (X, φ) on $(\mathcal{H}_0, \mathcal{H}_1)$. First we check condition (1). If $a \in A$,

$x \in \mathsf{X}$, and $\xi \in \mathcal{H}_0$, then

$$\pi_{\mathsf{X}}(\varphi(a)x)\xi = (\varphi(a)x) \otimes \xi = \pi_A(a)(x \otimes \xi) = \pi_A(a)\pi_{\mathsf{X}}(x)\xi.$$

That is, $\pi_{\mathsf{X}}(\varphi(a)x) = \pi_A(a)\pi_{\mathsf{X}}(x)$ as desired. Now for $b \in B$, $x \in \mathsf{X}$, and $\xi \in \mathcal{H}_0$, we use equation (2.2.3) at the second step and find

$$\pi_{\mathsf{X}}(xb)\xi = (xb) \otimes \xi = x \otimes \pi_B(b)\xi = \pi_{\mathsf{X}}(x)\pi_B(b)\xi.$$

This shows that $\pi_{\mathsf{X}}(xb) = \pi_{\mathsf{X}}(x)\pi_B(b)$, proving condition (2). Finally, notice that equation (2.2.2) shows that $\pi_{\mathsf{X}}(x)^*\colon \mathcal{H}_1 \to \mathcal{H}_0$ is the annihilation operator satisfying, for any $z \in \mathsf{X}$ and $\xi \in \mathcal{H}_0$,

$$\pi_{\mathsf{X}}(x)^*(z \otimes \xi) = \pi_B(\langle x, z\rangle_B)\xi. \tag{3.3.3}$$

Thus, for any $\xi \in \mathcal{H}_0$,

$$\pi_{\mathsf{X}}(x)^*\pi_{\mathsf{X}}(y)\xi = \pi_{\mathsf{X}}(x)^*(y \otimes \xi) = \pi_B(\langle x, y\rangle_B)\xi,$$

whence $\pi_{\mathsf{X}}(x)^*\pi_{\mathsf{X}}(y) = \pi_B(\langle x, y\rangle_B)$, which is compatibility condition (3), so we are done. ■

The method we used in the proof of Theorem 3.3.2 can be easily adapted to produce a different proof of Theorem 3.2.3. Thus, we present below a restatement of Theorem 3.2.3 followed by a proof along the lines of the proof of Theorem 3.3.2.

Theorem 3.3.3. *Let A and B be C^*-algebras, and let X be a Hilbert A-B-bimodule. Then for any nondegenerate representation π_B of B on a Hilbert space $\mathcal{H}_0$, there* *are a nondegenerate representation $\pi_A\colon A \to \mathcal{L}(\mathcal{H}_1)$ of A on a Hilbert space $\mathcal{H}_1$ and a bounded linear map $\pi_{\mathsf{X}}\colon \mathsf{X} \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$, such that $(\pi_A, \pi_B, \pi_{\mathsf{X}})$ is a representation of X on $(\mathcal{H}_0, \mathcal{H}_1)$ as in Definition 3.2.1.*

Proof. We get the Hilbert space $\mathcal{H}_1 = \mathsf{X} \otimes_{\pi_B} \mathcal{H}_0$ exactly as in the proof of Theorem 3.3.2. Since A acts on X via $\langle -, - \rangle_B$-adjointable operators (see Remark 2.2.3), we use Proposition 2.66 of [25] to get $\pi_A\colon A \to \mathcal{L}(\mathcal{H}_1)$, a representation of A, such that for each $a \in A$, each $x \in \mathsf{X}$, and each $\xi \in \mathcal{H}_0$,

$$\pi_A(a)(x \otimes \xi) = (ax) \otimes \xi.$$

Furthermore, since X is a left Hilbert A-module, it follows that A acts nondegenerately on X. Thus, Proposition 2.66 of [25] also guarantees that π_A is nondegenerate. Finally, compatibility condition (1) from Definition 3.2.1 is shown exactly as compatibility condition (1) from Definition 3.3.1 was shown in the proof of Theorem 3.3.2. Since compatibility conditions (2) and (4) from Definition 3.2.1 coincide with compatibility conditions (2) and (3) from Definition 3.3.1, we only need to make sure that compatibility condition (3) of Definition 3.2.1 is satisfied. Indeed, for any $x, y, z \in \mathsf{X}$, and any $\xi \in \mathcal{H}_0$, using equation (3.3.3) at the first step, equation (2.2.3) at the second step, and equation (2.2.1) at the third one, we get

$$\begin{aligned}
\pi_{\mathsf{X}}(x)\pi_{\mathsf{X}}(y)^*(z \otimes \xi) &= x \otimes \pi_B(\langle y, z \rangle_B)\xi \\
&= x\langle y, z \rangle_B \otimes \xi \\
&= {}_A\langle x, y \rangle z \otimes \xi \\
&= \pi_A({}_A\langle x, y \rangle)(z \otimes \xi).
\end{aligned}$$

Thus, $\pi_{\mathsf{X}}(x)\pi_{\mathsf{X}}(y)^* = \pi_A({}_A\langle x, y\rangle)$, as wanted. ∎

We give three remarks about the last two results. On those we explain how these results are useful to apply the main result of Section 3.2 and also how the proofs of these theorems compare to those known in the current literature.

Remark 3.3.4. The construction of the Hilbert space $\mathcal{H}_1$ given in the proofs of Theorems 3.3.2 and 3.3.3 above clearly implies that $\pi_X(\mathsf{X})\mathcal{H}_0$ is dense in $\mathcal{H}_1$. This is also true for the Hilbert space $\mathcal{H}_1$ obtained from Theorem 3.2.3. Indeed, according to the proof of Proposition 4.7 in [10], the space $\mathcal{H}_1$ is defined as follows. Let L be the linking algebra of X and $\iota\colon \mathsf{X} \to L$ the inclusion of X as the upper right corner of L. Then $\mathcal{H}_1$ is defined as the closure of $\pi(\iota(\mathsf{X}))\mathcal{H}_0$, where π is a suitable representation of L on a Hilbert space $\mathcal{H}$ that contains $\mathcal{H}_0$. For each $x \in \mathsf{X}$, the operator $\pi_{\mathsf{X}}(x) \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is then defined as the restriction of $\pi(\iota(x))$ to $\mathcal{H}_0$, so it follows that $\pi_{\mathsf{X}}(\mathsf{X})\mathcal{H}_0$ is dense in $\mathcal{H}_1$. This shows that the map π_{X}, no matter from which construction presented so far was obtained, satisfies the nondegeneracy condition on the hypothesis of Proposition 3.2.8.

Remark 3.3.5. The proof of Theorem 3.2.3 given in [10] does not appear to have an obvious modification to make it work for the C*-correspondence case. This is due to the fact that their proof relies on the linking algebra of the bimodule X, which does not exist in the general C*-correspondence setting due to the lack of an A-valued left inner product. We also believe that the arguments used in [29] to prove our Corollary 3.2.6 can't be modified to produce a proof of Theorem 3.3.2. Finally, we point out that the methods we employed to show Theorem 3.3.2 differ from those used in [29] and [10]. In particular, we have obtained in equation (3.3.2) a concise formula for π_{X} that might be useful to produce concrete representations of both Hilbert bimodules and modules. In particular, our formula for π_{X} has an immediate application presented in Theorem 3.3.7 below.

Remark 3.3.6. Suppose that the representation π_B in the hypotheses of Theorem 3.3.2 (or in Theorem 3.3.3) is faithful. Then $(\pi_A, \pi_B, \pi_{\mathsf{X}})$ is an isometric representation and a sufficient condition for the faithfulness of π_A is for A to act nondegenerately on X (this is always true in the Hilbert bimodule case). Indeed, for the sake of a contradiction, suppose that there is a nonzero $a \in A$ such that $\pi_A(a) = 0$. By nondegeneracy, we find a nonzero $x \in \mathsf{X}$ such that $\varphi(a)x \neq 0$ (for the Hilbert bimodule case we interpret $\varphi(a)x$ as ax). We can find nonzero elements $y \in \mathsf{X}$ and $b \in B$ such that $x = yb$ (see for example Proposition 2.31 in [25]). Then for any $\xi \in \mathcal{H}_0$, using (2.2.3) at the third step,

$$0 = \pi_A(a)(x \otimes \xi) = \varphi(a)yb \otimes \xi = \varphi(a)y \otimes \pi_B(b)\xi.$$

Since $\varphi(a)x \neq 0$, it follows that $\varphi(a)y \neq 0$ and therefore $\pi_B(b)\xi = 0$ for all $\xi \in \mathcal{H}_0$. Hence, faithfulness of π_B implies that $b = 0$, a contradiction.

We now present two applications of Theorem 3.3.2. The first application answers the problem of determining when a C*-correspondence can be given the structure of a Hilbert bimodule.

Theorem 3.3.7. *Let A and B be C*-algebras, and let (X, φ) be an (A, B) C*-correspondence such that A acts nondegenerately on X. Then there is an A-valued left inner product on X making it a Hilbert A-B bimodule if and only if $\mathcal{K}_B(\mathsf{X}) \subseteq \varphi(A)$.*

Proof. Suppose first that there is an A-valued left inner product on X making it a Hilbert A-B bimodule. That is, there is a map ${}_A\langle -, - \rangle \colon \mathsf{X} \times \mathsf{X} \to A$ such that for any $x, y, z \in \mathsf{X}$ we have $\varphi({}_A\langle x, y\rangle)z = x\langle y, z\rangle_B$. Since $x\langle y, z\rangle_B = \theta_{x,y}(z)$, this proves $\theta_{x,y} \in \varphi(A)$ for any $x, y \in \mathsf{X}$, which in turn implies $\mathcal{K}_B(\mathsf{X}) \subseteq \varphi(A)$.

Conversely, assume that $\mathcal{K}_B(\mathsf{X}) \subseteq \varphi(A)$. Let $\pi_B\colon B \to \mathcal{L}(\mathcal{H}_0)$ be the universal representation of B, which is faithful and nondegenerate. Then since A acts nondegenerately on X, Theorem 3.3.2 together with Remark 3.3.6 above gives the existence of a nondegenerate faithful representation $\pi_A\colon A \to \mathcal{L}(\mathcal{H}_1)$ of A on a Hilbert space $\mathcal{H}_1$ and a linear map $\pi_{\mathsf{X}}\colon \mathsf{X} \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$, such that $(\pi_A, \pi_B, \pi_{\mathsf{X}})$ is an isometric representation of (X, φ). Furthermore, the proof of Theorem 3.3.2 tells us that $\mathcal{H}_1 = \mathsf{X} \otimes_{\pi_B} \mathcal{H}_0$ and $\pi_{\mathsf{X}}(x)$ is the creation operator by $x \in \mathsf{X}$. Let $x, y, z \in \mathsf{X}$ and $\xi \in \mathcal{H}_0$. Then by our assumption, there is $a \in A$ such that $\theta_{x,y} = \varphi(a)$ and therefore using equation (3.3.3) at the first step, equation (2.2.3) at the second step, and equation (3.3.1) at the fifth step, we get

$$\begin{aligned}
\pi_{\mathsf{X}}(x)\pi_{\mathsf{X}}(y)^*(z \otimes \xi) &= x \otimes \pi_B(\langle y, z\rangle_B)\xi \\
&= x\langle y, z\rangle_B \otimes \xi \\
&= \theta_{x,y}(z) \otimes \xi \\
&= \varphi(a)z \otimes \xi \\
&= \pi_A(a)(z \otimes \xi).
\end{aligned}$$

This proves that $\pi_{\mathsf{X}}(x)\pi_{\mathsf{X}}(y)^* = \pi(a)$. Therefore, $\pi_{\mathsf{X}}(x)\pi_{\mathsf{X}}(y)^* \in \pi_A(A)$ for any $x, y \in \mathsf{X}$. Thus, for each $x, y \in \mathsf{X}$ we use the map $\pi_A^{-1}\colon \pi_A(A) \to A$ to define

$$ {}_A\langle x, y\rangle = \pi_A^{-1}\big(\pi_{\mathsf{X}}(x)\pi_{\mathsf{X}}(y)^*\big). \tag{3.3.4}$$

It is immediate to check that $(x, y) \mapsto {}_A\langle x, y\rangle$ is indeed an A-valued left inner product on X. Furthermore, if $x \in \mathsf{X}$, then

$$\|{}_A\langle x, x\rangle\| = \|\pi_{\mathsf{X}}(x)\pi_{\mathsf{X}}(x)^*\| = \|\pi_{\mathsf{X}}(x)\|^2 = \|x\|.$$

Hence, X is indeed a left Hilbert A-module. We now only need to check that this makes X into a Hilbert A-B-bimodule by verifying equation (2.2.1). That is, we need to prove that

$$\varphi({}_A\langle x, y\rangle)z = x\langle y, z\rangle_B.$$

for all $x, y, z \in \mathsf{X}$. To do so, we use the fact that $(\pi_A, \pi_B, \pi_{\mathsf{X}})$ is a representation of (X, φ) and notice that

$$\begin{aligned}
\pi_{\mathsf{X}}\big(\varphi({}_A\langle x, y\rangle)z\big) &= \pi_A({}_A\langle x, y\rangle)\pi_{\mathsf{X}}(z) \\
&= \pi_A\big(\pi_A^{-1}\big(\pi_{\mathsf{X}}(x)\pi_{\mathsf{X}}(y)^*\big)\big)\pi_{\mathsf{X}}(z) \\
&= \pi_{\mathsf{X}}(x)\pi_{\mathsf{X}}(y)^*\pi_{\mathsf{X}}(z) \\
&= \pi_{\mathsf{X}}(x)\pi_B(\langle y, z\rangle_B) \\
&= \pi_{\mathsf{X}}(x\langle y, z\rangle_B)
\end{aligned}$$

Since π_{X} is an isometry, this gives $\varphi({}_A\langle x, y\rangle)z = x\langle y, z\rangle_B$, as we wanted to check to finish the proof. ■

Remark 3.3.8. Suppose that we apply the proof of the "if" part of Theorem 3.3.7 to a Hilbert A-B-bimodule regarded as an (A, B) C*-correspondence. Then notice that the left A-valued inner product defined in (3.3.4) coincides with the left A-valued inner product already carried by X. Indeed, this follows at once from looking at the proof of Theorem 3.3.3, in which it is shown that $\pi_A({}_A\langle x, y\rangle) = \pi_{\mathsf{X}}(x)^*\pi_{\mathsf{X}}(y)$ if the representation $(\pi_A, \pi_B, \pi_{\mathsf{X}})$ was obtained using Theorem 3.3.2.

Remark 3.3.9. Recall that for a C*-correspondence (X, φ), the map φ is not required to be injective. However, if we require injectivity of φ in the hypotheses of Theorem 3.3.7, then the proof is straightforward and does not depend on the results of this paper. Indeed, it suffices to define the left A-valued inner product using the map $\varphi^{-1}\colon \varphi^{-1}(A) \to A$, by letting

$$ {}_A\langle x, y\rangle = \varphi^{-1}\big(\theta_{x,y}\big), $$

instead of using equation (3.3.4). We are not aware of any result in the literature like Theorem 3.3.7 either with or without the assumption of injectivity of φ.

As an application of Theorem 3.3.7, we give below an easy proof that a Hilbert space of dimension at least 2, thought of as a $(\mathbb{C}, \mathbb{C})$ C*-correspondence, can't be given the structure of a Hilbert $\mathbb{C}$-$\mathbb{C}$-bimodule.

Example 3.3.10. Let $\mathcal{H}$ be a Hilbert space with dimension at least 2. Clearly $\mathcal{H}$ is a right Hilbert $\mathbb{C}$-module, $\mathcal{L}_{\mathbb{C}}(\mathcal{H}) = \mathcal{L}(\mathcal{H})$, and $\mathcal{K}_{\mathbb{C}}(\mathcal{H}) = \mathcal{K}(\mathcal{H})$. For each $a \in \mathbb{C}$, we define $\varphi(a) = a \cdot \mathrm{id}_{\mathcal{H}}$. Then $\varphi\colon \mathbb{C} \to \mathcal{L}(\mathcal{H})$ makes $(\mathcal{H}, \varphi)$ into a $(\mathbb{C}, \mathbb{C})$ C*-correspondence. Furthermore, it's clear that $\varphi(\mathbb{C})\mathcal{H} = \mathcal{H}$, so the left action is nondegenerate. We claim that $\mathcal{K}(\mathcal{H}) \not\subseteq \varphi(A)$. Let $(\xi_j)_{j\in J}$ be an orthonormal basis for $\mathcal{H}$. By assumption $\mathrm{card}(J) \geq 2$ and therefore we can find $j, k \in J$ with $j \neq k$. Notice that $\theta_{\xi_j,\xi_k}(\xi_k) = \xi_j$, whence $\theta_{\xi_j,\xi_k} \neq \varphi(a)$ for all $a \in \mathbb{C}$, proving the claim. Therefore, Theorem 3.3.7 implies that (X, φ) can't be given the structure of a Hilbert $\mathbb{C}$-$\mathbb{C}$-bimodule.

Remark 3.3.11. A direct sum of Hilbert A-B-bimodules is not, in general, a Hilbert bimodule again. However, it is an (A, B) C*-correspondence. It is not hard to see that the C*-correspondence in Example 3.3.10 is a direct sum of Hilbert $\mathbb{C}$-$\mathbb{C}$-bimodules. We have not investigated which C*-correspondences can be decomposed as a direct sum of Hilbert bimodules, but we believe Theorem 3.3.7 might be a useful tool to tackle this problem.

We conclude this chapter with a second application of Theorem 3.3.2, which deals with how to get a representation for the interior tensor product of an (A, B) C*-correspondence $(\mathsf{X}, \varphi_{\mathsf{X}})$ with a (B, C) C*-correspondence $(\mathsf{Y}, \varphi_{\mathsf{Y}})$ using particular representations of the C*-correspondences $(\mathsf{X}, \varphi_{\mathsf{X}})$ and $(\mathsf{Y}, \varphi_{\mathsf{Y}})$. The main point is that we can always make the representation of the C*-algebra B from the representation of $(\mathsf{X}, \varphi_{\mathsf{X}})$ agree with the representation of B from the representation of $(\mathsf{Y}, \varphi_{\mathsf{Y}})$.

Theorem 3.3.12. *Let A, B, and C be C*-algebras, let $(\mathsf{X}, \varphi_{\mathsf{X}})$ be an (A,B) C*-correspondence, let $(\mathsf{Y}, \varphi_{\mathsf{Y}})$ be a (B,C) C*-correspondence such that B acts nondegenerately on Y, and let $\pi_C \colon C \to \mathcal{L}(\mathcal{H}_0)$ be any nondegenerate representation of C on a Hilbert space $\mathcal{H}_0$. Then:*

1. *There are Hilbert spaces $\mathcal{H}_1$,$\mathcal{H}_2$, maps $\pi_A \colon A \to \mathcal{L}(\mathcal{H}_2)$, $\pi_B \colon B \to \mathcal{L}(\mathcal{H}_1)$, $\pi_{\mathsf{X}} \colon \mathsf{X} \to \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, and $\pi_{\mathsf{Y}} \colon \mathsf{Y} \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$, such that $(\pi_A, \pi_B, \pi_{\mathsf{X}})$ is a representation of $(\mathsf{X}, \varphi_{\mathsf{X}})$ on $(\mathcal{H}_1, \mathcal{H}_2)$ and $(\pi_B, \pi_C, \pi_{\mathsf{Y}})$ is a representation of $(\mathsf{Y}, \varphi_{\mathsf{Y}})$ on $(\mathcal{H}_0, \mathcal{H}_1)$.*

2. *For every pair $((\pi_A, \pi_B, \pi_{\mathsf{X}}), (\pi_B, \pi_C, \pi_{\mathsf{Y}}))$ as in (1), there is a map $\pi \colon \mathsf{X} \otimes_{\varphi_{\mathsf{Y}}} \mathsf{Y} \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_2)$ satisfying $\pi(x \otimes y) = \pi_{\mathsf{X}}(x)\pi_{\mathsf{Y}}(y)$ and such that (π_A, π_C, π) is a representation of $(\mathsf{X} \otimes_{\varphi_{\mathsf{Y}}} \mathsf{Y}, \widetilde{\varphi_{\mathsf{X}}})$ on $(\mathcal{H}_0, \mathcal{H}_2)$.*

3. *The map π from (2) is an isomorphism from $\mathsf{X} \otimes_{\varphi_{\mathsf{Y}}} \mathsf{Y}$ to $\overline{\pi_{\mathsf{X}}(\mathsf{X})\pi_{\mathsf{Y}}(\mathsf{Y})}$.*

4. *If in addition π_C is also faithful and A acts nondegenerately on X, then (π_A, π_C, π) is isometric.*

Proof. Since $\pi_C \colon C \to \mathcal{L}(\mathcal{H}_0)$ is nondegenerate and B acts nondegenerately on Y, Theorem 3.3.2 gives a Hilbert space $\mathcal{H}_1$, a nondegenerate representation $\pi_B \colon B \to \mathcal{L}(\mathcal{H}_1)$, and a bounded linear map $\pi_{\mathsf{Y}} \colon \mathsf{Y} \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ such that $(\pi_B, \pi_C, \pi_{\mathsf{Y}})$ is a representation of $(\mathsf{Y}, \varphi_{\mathsf{Y}})$ on $(\mathcal{H}_0, \mathcal{H}_1)$. Hence, a second application of Theorem 3.3.2 gives a Hilbert space $\mathcal{H}_2$, a representation $\pi_A \colon A \to \mathcal{L}(\mathcal{H}_2)$, and a bounded linear map $\pi_{\mathsf{X}} \colon \mathsf{X} \to \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ such that $(\pi_A, \pi_B, \pi_{\mathsf{X}})$ is a representation of $(\mathsf{X}, \varphi_{\mathsf{X}})$ on $(\mathcal{H}_1, \mathcal{H}_2)$. This takes care of part (1). For part (2), we first prove the existence of the map π. First, fix $n \in \mathbb{Z}_{\geq 1}$, $x_1, \ldots, x_n \in \mathsf{X}$, and $y_1, \ldots, y_n \in \mathsf{Y}$. Then, using the fact that $(\pi_B, \pi_C, \pi_{\mathsf{Y}})$ is a representation at the second and third steps together with equation (2.2.2) at the sixth step, we find

$$\begin{aligned}
\Big\| \sum_{j=1}^{n} \pi_{\mathsf{X}}(x_j)\pi_{\mathsf{Y}}(y_j) \Big\|^2 &= \sup_{\|\xi\|=1} \sum_{j,k=1}^{n} \langle \pi_{\mathsf{X}}(x_k)\pi_{\mathsf{Y}}(y_k)\xi, \pi_{\mathsf{X}}(x_j)\pi_{\mathsf{Y}}(y_j)\xi \rangle \\
&= \sup_{\|\xi\|=1} \sum_{j,k=1}^{n} \langle \pi_{\mathsf{Y}}(y_j)^* \pi_{\mathsf{Y}}(\varphi_{\mathsf{Y}}(\langle x_j, x_k \rangle_B) y_k)\xi, \xi \rangle \\
&= \sup_{\|\xi\|=1} \sum_{j,k=1}^{n} \langle \pi_C(\langle y_j, \varphi_{\mathsf{Y}}(\langle x_j, x_k \rangle_B) y_k \rangle_C)\xi, \xi \rangle \\
&\leq \Big\| \sum_{j,k=1}^{n} \pi_C(\langle y_j, \varphi_{\mathsf{Y}}(\langle x_j, x_k \rangle_B) y_k \rangle_C) \Big\| \\
&\leq \Big\| \sum_{j,k=1}^{n} \langle y_j, \varphi_{\mathsf{Y}}(\langle x_j, x_k \rangle_B) y_k \rangle_C \Big\| \\
&= \Big\| \Big\langle \sum_{j=1}^{n} x_j \otimes y_j, \sum_{k=1}^{n} x_k \otimes y_k \Big\rangle_C \Big\| \\
&= \Big\| \sum_{j=1}^{n} x_j \otimes y_j \Big\|^2.
\end{aligned}$$

Moreover, a straightforward computation gives $\pi_{\mathsf{X}}(xb)\pi_{\mathsf{Y}}(y) = \pi_{\mathsf{X}}(x)\pi_{\mathsf{Y}}(\varphi_{\mathsf{Y}}(b)y)$ for any $x \in \mathsf{X}, y \in \mathsf{Y}$, and $b \in B$. Therefore, we can extend the map $x \otimes y \mapsto \pi_{\mathsf{X}}(x)\pi_{\mathsf{Y}}(y)$ to a well defined bounded linear map $\pi\colon \mathsf{X} \otimes_{\varphi_{\mathsf{Y}}} \mathsf{Y} \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_2)$. To finish part (2), It only remains to check that (π_A, π_C, π) is indeed a representation of the correspondence $(\mathsf{X} \otimes_{\varphi_{\mathsf{Y}}} \mathsf{Y}, \widetilde{\varphi_{\mathsf{X}}})$ on $(\mathcal{H}_0, \mathcal{H}_2)$. This follows from three immediate computations on elementary tensors using the fact that both $(\pi_A, \pi_B, \pi_{\mathsf{X}})$ and $(\pi_B, \pi_C, \pi_{\mathsf{Y}})$ are representations. Part (3) is now immediate from definition of π. Finally, to check part (4), since B acts nondegenerately on Y and π_C is faithful, Remark 3.3.6 then shows that π_B is faithful. Finally, since A acts nondegenerately on X, it follows again from Remark 3.3.6 that π_A is faithful, but as in Remark 3.2.2, this is enough for (π_A, π_C, π) to be isometric. ■

CHAPTER IV

PRELIMINARIES ON L^P OPERATOR ALGEBRAS

We start the chapter with a brief section on multiplier algebras for Banach Algebras, presenting basic definitions and results that will be useful later on. We then define L^p operator algebras, give examples of them, and state well known results that will be needed later. We also briefly discuss spatial partial isometries acting on L^p spaces in order to introduce the L^p-Cuntz algebras $\mathcal{O}_d^p$. This chapter includes, with no proof, some results from both [22] and [21] for future reference.

Multiplier Algebras

We briefly look at the multiplier algebra of a Banach algebra and some of its properties, which will come on handy in Chapters V and VI. The following definition comes from section 2.5 in [6].

Definition 4.1.1. Let A be a Banach algebra. A *double centralizer for* A is a pair (L, R) with $L, R \in \mathcal{L}(A)$ satisfying $L(ab) = L(a)b$, $R(ab) = aR(b)$, and $aL(b) = R(a)b$ for all $a, b \in A$. We define $M(A)$, the *multiplier algebra of* A, to be the subset of $\mathcal{L}(A) \times \mathcal{L}(A)^{\mathrm{op}}$ (equipped with the max norm) consisting of double centralizers.

It is clear that $M(A)$ is a unital Banach subalgebra of $\mathcal{L}(A) \times \mathcal{L}(A)^{\mathrm{op}}$. If A is a C*-algebra, an equivalent definition for $M(A)$ is as the set of two sided multipliers for A on any Hilbert space as long as A acts nondegenerately on it; see Definition 2.2.2 in [27] for instance. This will be also the case for Banach algebras that have contractive approximate identities and that can be nondegenerately represented on a Banach space. To prove this, we first state some precise definitions and prove a useful lemma.

Definition 4.1.2. Let A be a Banach algebra and E a Banach space. A *representation* of A on E is a continuous homomorphism $\pi\colon A \to \mathcal{L}(E)$.

1. We say that π is *contractive* if $\|\pi(a)\| \leq \|a\|$ for all $a \in A$.

2. We say that π is *isometric* if $\|\pi(a)\| = \|a\|$ for all $a \in A$.

3. We say that π is *separable* if E is separable, and that A is *separably representable* if it has a separable isometric representation.

4. We say that π is *nondegenerate* if

$$\pi(A)E = \operatorname{span}(\{\pi(a)\xi\colon a \in A \text{ and } \xi E\}),$$

is dense in E, and that A is *nondegenerately representable* if it has a nondegenerate isometric representation.

Definition 4.1.3. Let A be a Banach Algebra. We say that A has a *contractive approximate identity (c.a.i.)* if there is a net $(e_\lambda)_{\lambda\in\Lambda}$ such that $\|e_\lambda\| \leq 1$ for all $\lambda \in \Lambda$ and for all $a \in A$,

$$\lim_{\lambda\in\Lambda} \|ae_\lambda - a\| = \lim_{\lambda\in\Lambda} \|e_\lambda a - a\| = 0.$$

Lemma 4.1.4. *Let A be a Banach algebra with a c.a.i. Then for any $a \in A$,*

$$\|a\| = \sup_{\|b\|=1} \|ab\| = \sup_{\|b\|=1} \|ba\|.$$

Proof. Let $(e_\lambda)_{\lambda\in\Lambda}$ be a c.a.i. for A. Then for any $a \in A$,

$$\|a\| \geq \sup_{\|b\|=1} \|ba\| \geq \sup_{\lambda\in\Lambda} \frac{\|e_\lambda a\|}{\|e_\lambda\|} \geq \sup_{\lambda\in\Lambda} \|e_\lambda a\| = \|a\|.$$

The other equality is proved analogously. ■

Proposition 4.1.5. *Let A be a Banach algebra with a c.a.i., and let $(L, R) \in M(A)$. Then $\|L\| = \|R\|$. Furthermore, the map $\iota\colon A \mapsto M(A)$, given by $\iota(a) = (L_a, R_a)$, where $L_a(b) = ab$ and $R_a(b) = ba$, is isometric, and $\iota(A)$ is a closed ideal in $M(A)$.*

Proof. Lemma 4.1.4 gives

$$\|L(a)\| = \sup_{\|b\|=1} \|bL(a)\| = \sup_{\|b\|=1} \|R(b)a\| \leq \|R\|\|a\|.$$

Thus, $\|L\| \leq \|R\|$. Similarly,

$$\|R(a)\| = \sup_{\|b\|=1} \|R(a)b\| = \sup_{\|b\|=1} \|aL(b)\| \leq \|a\|\|L\|,$$

whence $\|R\| \leq \|L\|$ and therefore $\|L\| = \|R\|$. Once again Lemma 4.1.4 gives $\|L_a\| = \|R_a\| = \|a\|$, showing that ι is isometric and that $\iota(A)$ is closed in $M(A)$. Finally, direct computations show that for any $a \in A$ and any $(L, R) \in M(A)$, $\iota(a)(L, R) = \iota(R(a))$ and $(L, R)\iota(a) = \iota(L(a))$, whence $\iota(A)$ is indeed an ideal in $M(A)$, finishing the proof. ■

We are now ready to prove an important result, which implies that when A is a nondegenerately representable on a Banach space E Banach algebra with a c.a.i., then, just as in the C*-case, the algebra of two sided multipliers is independent of the Banach space E on which the algebra is nondegenerately represented.

Theorem 4.1.6. *Let A be a Banach algebra with a c.a.i. and that is nondegenerately represented on a Banach space E via $\pi\colon A \to \mathcal{L}(E)$. Then $M(A)$ is isometrically isomorphic to $\{t \in \mathcal{L}(E)\colon t\pi(A) \subseteq \pi(A), \pi(A)t \subseteq \pi(A)\}$, the algebra of two sided multipliers for A.*

Proof. For convenience, we identify A with its isometric copy $\pi(A)$ in $\mathcal{L}(E)$. Let $B = \{t \in \mathcal{L}(E)\colon tA \subseteq A, At \subseteq A\}$. For any $t \in B$, we define $D(t) = (L_t, R_t)$ where $L_t\colon A \to A$, $R_t\colon A \to A$ are given by $L_t(a) = ta$ and $R_t(a) = at$. For any $a, b \in A$, we easily check that $L_t(ab) = L_t(a)b$, $R_t(ab) = aR_t(b)$, and $aL_t(b) = R_t(a)b$. Hence, $D(B) \subseteq M(A)$. Furthermore, we check below that $t \in B$ implies that $\|L_t\| = \|R_t\| = \|t\|$. This will be done using the fact that that A has a c.a.i, say $(e_\lambda)_{\lambda\in\Lambda}$, and that A sits nondegenerately in $\mathcal{L}(E)$. Indeed, that $\|L_t\| \leq \|t\|$ and $\|R_t\| \leq \|t\|$ is obvious. For the reverse inequalities, notice first that for any $\eta \in AE$ with $\|\eta\| = 1$ we have

$$\|te_\lambda\| \geq \|te_\lambda\eta\| \quad\text{and}\quad \|e_\lambda t\| \geq \|e_\lambda t\eta\|.$$

However, since $\eta \in AE$ and $(e_\lambda)_{\lambda\in\Lambda}$ is an approximate identity for A, it follows that both $(te_\lambda\eta)_{\lambda\in\Lambda}$ and $(e_\lambda t\eta)_{\lambda\in\Lambda}$ converge to $t\eta$ in E. Thus,

$$\lim_{\lambda\in\Lambda}\|te_\lambda\| \geq \|t\eta\| \quad\text{and}\quad \lim_{\lambda\in\Lambda}\|e_\lambda t\| \geq \|t\eta\|,$$

and since AE is dense in E and $\|\eta\| = 1$, this gives at once

$$\lim_{\lambda\in\Lambda}\|te_\lambda\| \geq \|t\| \quad\text{and}\quad \lim_{\lambda\in\Lambda}\|e_\lambda t\| \geq \|t\|.$$

Now, for any $\lambda \in \Lambda$, we clearly have $\|L_t\| \geq \|te_\lambda\|$, and $\|R_t\| \geq \|e_\lambda t\|$, whence the reverse inequalities also hold. Therefore, $\|D(t)\| = \|t\|$, making $D\colon B \to M(A)$ an isometric map. It remains to show that D is surjective. Fix an arbitrary $(L, R) \in M(A)$. For any $\xi \in E$, we claim that the net $(L(e_\lambda)\xi)_{\lambda\in\Lambda}$ converges in E. To prove the claim, notice that by nondegeneracy, it suffices to show that $(L(e_\lambda)a\eta)_{\lambda\in\Lambda}$ is a Cauchy net in E for any $a \in A$ and any $\eta \in E$. Let $\varepsilon > 0$. Then there is $\lambda_0 \in \Lambda$ such that $\|ae_{\lambda_1} - ae_{\lambda_2}\| < \frac{\varepsilon}{\|L\|\|\eta\|+1}$ for all $\lambda_1, \lambda_2 \geq \lambda_0$. Hence,

$$\|L(e_{\lambda_1})a\eta - L(e_{\lambda_2})a\eta\| = \|L(e_{\lambda_1}a - e_{\lambda_2}a)\eta\| \leq \|L\|\|ae_{\lambda_1} - ae_{\lambda_2}\|\|\eta\| < \varepsilon,$$

proving the claim. Thus, we can define an element $t_0 \in \mathcal{L}(E)$ by letting

$$t_0\xi = \lim_{\lambda\in\Lambda} L(e_\lambda)\xi,$$

for any $\xi \in E$. Moreover, for any $a \in A$ and $\xi \in E$, we find

$$(t_0a)\xi = \lim_{\lambda\in\Lambda} L(e_\lambda)a\xi = \lim_{\lambda\in\Lambda} L(e_\lambda a)\xi = L(a)\xi,$$

whence $t_0a = L(a) \in A$. Similarly, using that $aL(e_\lambda) = R(a)e_\lambda$ we get $at_0 = R(a) \in A$. Therefore, it now follows at once that $t_0 \in B$ and that $D(t_0) = (L, R)$, proving surjectivity as we wanted to show. ∎

Corollary 4.1.7. *Let A be a Banach algebra with a c.a.i. and let $\pi\colon A \to \mathcal{L}(E)$ be a contractive nondegenerate representation of A on a Banach space E. Then π induces a nondegenerate contractive representation $\widehat{\pi}\colon M(A) \to \mathcal{L}(E)$ such that, if ι is as in Proposition 4.1.5, then $\pi = \widehat{\pi} \circ \iota$*

Proof. Let $(e_\lambda)_{\lambda\in\Lambda}$ be a c.a.i. for A. For each $(L,R) \in M(A)$ we define $\widehat{\pi}(L,R) \in \mathcal{L}(E)$ by letting, for any $\xi \in E$,

$$\widehat{\pi}(L,R)\xi = \lim_{\lambda\in\Lambda} \pi(L(e_\lambda))\xi.$$

That $\widehat{\pi}(L,R)$ is well defined is verified exactly as for the map t_0 in the proof of Theorem 4.1.6 when surjectivity of D was established. It follows now from standard verifications that $\widehat{\pi}\colon M(A) \to \mathcal{L}(E)$ is a nondegenerate contractive representation of $M(A)$ satisfying $\pi = \widehat{\pi} \circ \iota$. ■

We will need a final result that gives a map from the multiplier algebra of A to the multiplier algebras of quotients of A.

Proposition 4.1.8. *Let A be a Banach algebra with a c.a.i. and let $I \subseteq A$ a closed two sided ideal. Then the quotient map $q_0\colon A \to A/I$ induces a contractive homomorphism $\widetilde{q_0}\colon M(A) \to M(A/I)$.*

Proof. Let $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate identity for A and let $(L,R) \in M(A)$. Then for any $b \in I$ we have

$$L(b) = \lim_{\lambda} L(e_\lambda b) = \lim_{\lambda} L(e_\lambda)b \in I.$$

From here it follows that $L(I) \subseteq I$ and an analogous argument shows that $R(I) \subseteq I$. Thus, we get well defined linear maps $\widetilde{L}\colon A/I \to A/I$ and $\widetilde{R}\colon A/I \to A/I$ such that $\widetilde{L}\circ q_0 = q_0 \circ L$ and $\widetilde{R}\circ q_0 = q_0 \circ R$. Furthermore, for any $a \in A$ with $\|q_0(a)\| = 1$, we have

$$\|\widetilde{L}(q_0(a))\| = \|q_0(L(a))\| = \lim_{\lambda\in\Lambda} \|q_0(L(e_\lambda a))\| \leq \lim_{\lambda\in\Lambda} \|q_0(L(e_\lambda))\|\|q_0(a)\| \leq \|L\|,$$

whence $\|\widetilde{L}\| \leq \|L\|$ and similarly $\|\widetilde{R}\| \leq \|R\|$. Thus, $\widetilde{L}, \widetilde{R} \in \mathcal{L}(A/I)$ and, moreover, direct calculations show that $(\widetilde{L}, \widetilde{R}) \in M(A/I)$. Therefore we have shown that the map $\tilde{q}_0 \colon M(A) \to M(A/I)$ given by $\widetilde{q_0}(L, R) = (\widetilde{L}, \widetilde{R})$ is contractive. Another direct computation shows that $\widetilde{q_0}(L_1, R_1)\widetilde{q_0}(L_2, R_2) = \widetilde{q_0}(L_1L_2, R_2R_1)$ for any $(L_1, R_1), (L_2, R_2) \in M(A)$, whence $\widetilde{q_0}$ is indeed a homomorphism, as wanted. ■

L^p-operator algebras.

If $(\Omega, \mathfrak{M}, \mu)$ is a measure space, we define $L^0(\Omega, \mathfrak{M}, \mu)$ to be the space of complex valued measurable functions modulo functions that vanish a.e $[\mu]$. If $p \in [1, \infty) \cup \{0\}$, we sometimes write $L^p(\mu)$ for $L^p(\Omega, \mathfrak{M}, \mu)$. Also, if ν_I is counting measure on a set I, we write $\ell^p(I)$ instead of $L^p(I, 2^I, \nu_I)$. In particular, when $I = \{1, \ldots, d\}$ for some $d \in \mathbb{Z}_{\geq 1}$, we simply write ℓ^p_d to mean $\ell^p(\{1, \ldots, d\})$.

Definition 4.2.1. Let $p \in [1, \infty)$, let $(\Omega, \mathfrak{M}, \mu)$ be a measure space, let A be a Banach algebra, and let $\pi \colon A \to \mathcal{L}(L^p(\mu))$ be a representation of A on $L^p(\mu)$. We say π is *σ-finite* if $(\Omega, \mathfrak{M}, \mu)$ is σ-finite, and that A is *σ-finitely representable* if it has a σ-finite isometric representation.

Remark 4.2.2. If A is separably representable on an L^p-space, then it is σ-finitely representable. Indeed, let $(\Omega, \mathfrak{M}, \mu)$ be a measure space such that $L^p(\mu)$ is separable. Then we claim that there is a σ-finite measure space $(\Omega_0, \mathfrak{M}_0, \mu_0)$ such that $L^p(\mu)$ is isometrically isomorphic to $L^p(\mu_0)$. To verify the claim, let $\{\xi_n \colon n \in \mathbb{Z}_{\geq 1}\}$ be a countable dense subset in $L^p(\mu)$ and put

$$\Omega_0 = \bigcup_{n=1}^{\infty} \operatorname{supp}(\xi_n), \quad \mathfrak{M}_0 = \mathfrak{M}|_{\Omega_0}, \quad \text{and } \mu_0 = \mu|_{\mathfrak{M}_0}.$$

Since, for each $n \in \mathbb{Z}_{\geq 1}$, $\xi_n \in L^p(\mu)$, it follows that $\operatorname{supp}(\xi_n)$ is σ-finite with respect to μ and therefore Ω_0 is σ-finite with respect to μ_0.

Definition 4.2.3. Let $p \in [1, \infty)$. A Banach algebra A is an L^p *operator algebra* if there is a measure space $(\Omega, \mathfrak{M}, \mu)$ and an isometric representation of A on $L^p(\mu)$.

Example 4.2.4. Let $p \in [1, \infty)$.

1. For any measure space $(\Omega, \mathfrak{M}, \mu)$, the algebra $\mathcal{L}\big(L^p(\mu)\big)$ is trivially an L^p-operator algebra.

2. For any measure space $(\Omega, \mathfrak{M}, \mu)$, the algebra $\mathcal{K}\big(L^p(\mu)\big)$ of compact operators on $L^p(\mu)$ is an L^p-operator algebra.

3. Any C^*-algebra is an L^2 operator algebra. However, a general L^2 operator algebra is a not necessarily a self-adjoint algebra.

4. Equip M_n, the set of $n \times n$ complex matrices, with the operator norm from acting on $(\mathbb{C}^n, \| - \|_p)$. Then M_n is equal to $\mathcal{L}(\ell^p(\{1, \ldots, n\}))$. To emphasize the dependence on the p-norm, this space is denoted by M_n^p and it is clearly an L^p-operator algebra.

5. For $j, k \in \{1, \ldots, n\}$, let $e_{j,k} \in M_n^p$ be the matrix whose only non-zero entry is the entry (j, k) which is equal to 1. Then the set $T_n^p = \operatorname{span}\{e_{j,k} : 1 \leq j \leq k \leq n\}$ of upper triangular matrices is a subalgebra of M_n^p, which is also an L^p-operator algebra.

6. Let Ω be a locally compact topological space. Then $C_0(\Omega)$, with the usual supremum norm, is an L^p-operator algebra.

The following result is part of Proposition 1.25 of [22].

Proposition 4.2.5. *Let $p \in [1, \infty)$, and let A be a separable L^p operator algebra. Then A is separably representable.*

Spatial Tensor Product

For $p \in [1, \infty)$, there is a tensor product, called the *spatial tensor product* and denoted by $\otimes_p$. We refer the reader to Section 7 of [7] for complete details on this tensor product. We only describe below the properties we will need. If $(\Omega_0, \mathfrak{M}_0, \mu_0)$ is a measure space and E is a Banach space, then there is an isometric isomorphism

$$L^p(\mu_0) \otimes_p E \cong L^p(\mu_0, E) = \{g \colon \Omega_0 \to E \text{ measurable} \colon \textstyle\int_{\Omega_0} \|g(\omega)\|^p \, d\mu_0(\omega) < \infty\}$$

such that for any $\xi \in L^p(\mu_0)$ and $\eta \in E$, the elementary tensor $\xi \otimes \eta$ is sent to the function $\omega \mapsto \xi(\omega)\eta$. Furthermore, if $(\Omega_1, \mathfrak{M}_1, \mu_1)$ is another measure space and $E = L^p(\mu_1)$, then there is an isometric isomorphism

$$L^p(\mu_0) \otimes_p L^p(\mu_1) \cong L^p(\Omega_0 \times \Omega_1, \mu_0 \times \mu_1),$$

sending $\xi \otimes \eta$ to the function $(\omega_0, \omega_1) \mapsto \xi(\omega_0)\eta(\omega_1)$ for every $\xi \in L^p(\mu_0)$ and $\eta \in L^p(\mu_1)$. We describe its main properties below. The following is Theorem 2.16 in [21], except that we have removed the the σ-finiteness assumption as in the proof in Theorem 1.1 in [11].

1. Under the identification above, $\operatorname{span}\{\xi \otimes \eta \colon \xi \in L^p(\mu_0), \eta \in L^p(\mu_1)\}$ is a dense subset of $L^p(\Omega_0 \times \Omega_1, \mu_0 \times \mu_1)$.

2. $\|\xi \otimes \eta\|_p = \|\xi\|_p \|\eta\|_p$ for every $\xi \in L^p(\mu_0)$ and $\eta \in L^p(\mu_1)$.

3. Suppose that for $j \in \{0,1\}$ we have measure spaces $(\Omega_j, \mathfrak{M}_j, \mu_j)$, $(\Lambda_j, \mathfrak{N}_j, \nu_j)$, $a \in \mathcal{L}(L^p(\mu_0), L^p(\nu_0))$ and $b \in \mathcal{L}(L^p(\mu_1), L^p(\nu_1))$. Then there is a unique map $a \otimes b \in \mathcal{L}(L^p(\mu_0 \times \mu_1), L^p(\nu_0 \times \nu_1))$ such that

$$(a \otimes b)(\xi \otimes \eta) = a\xi \otimes b\eta$$

for every $\xi \in L^p(\mu_0)$ and $\eta \in L^p(\mu_1)$. Further, $\|a \otimes b\| = \|a\|\|b\|$.

4. The tensor product of operators defined in (3) is associative, bilinear, and satisfies (when the domains are appropriate) $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$.

Definition 4.3.1. Let $p \in [1, \infty)$ and let $A \subseteq \mathcal{L}(L^p(\mu))$ and $B \subseteq \mathcal{L}(L^p(\nu))$ be L^p-operator algebras. We define $A \otimes_p B$ to be the closed linear span, in $\mathcal{L}\big(L^p(\mu \times \nu)\big)$, of all $a \otimes b$ for $a \in A$ and $b \in B$.

Spatial partial isometries.

We start by introducing some language and notation. This comes mostly from Sections 5 and 6 in [21].

Definition 4.4.1. Let $(\Omega_0, \mathfrak{M}, \mu)$ and $(\Omega_1, \mathfrak{N}, \nu)$ be measure spaces. A *measurable set transformation* is a map $S \colon \mathfrak{M} \to \mathfrak{N}$, defined modulo null sets, such that

(i) For any $E \in \mathfrak{M}$, $\nu(S(E)) = 0$ if and only if $\mu(E) = 0$.

(ii) For any $E \in \mathfrak{M}$, $S(\Omega_0 \setminus E) = S(\Omega_0) \setminus S(E)$.

(iii) For any pairwise disjoint $E_1, E_2, \ldots \in \mathfrak{M}$, $S(\bigcup_{j=1}^{\infty} E_j) = \bigcup_{j=1}^{\infty} S(E_j)$.

Remark 4.4.2. Since a measurable set transformation is defined modulo null sets, we are in fact abusing notation in the definition above. Indeed, when we say $S(E) = F$ we really mean that the class $[E] = \{E' \in \mathfrak{M}\colon \mu(E'\triangle E) = 0\}$ gets mapped by S to the class $[F] = \{F' \in \mathfrak{N}\colon \nu(F'\triangle F) = 0\}$. Thus, we must specify the measure when dealing with a measurable set transformation. For this reason, from now on, we will write $S\colon (\Omega_0, \mathfrak{M}, \mu) \to (\Omega_1, \mathfrak{N}, \nu)$ instead of simply $S\colon \mathfrak{M} \to \mathfrak{N}$.

Remark 4.4.3. Let $S\colon (\Omega_0, \mathfrak{M}, \mu) \to (\Omega_1, \mathfrak{N}, \nu)$ be a measurable set transformation. Then $\operatorname{ran}(S) = \{S(E) \in \mathfrak{N}\colon E \in \mathfrak{M}\}$ is a sub σ-algebra of $\mathfrak{N}$ and S is surjective if and only if $\operatorname{ran}(S) = \mathfrak{N}$.

The following is part of Proposition 5.6. in [21].

Proposition 4.4.4. *Let $S\colon (\Omega_0, \mathfrak{M}, \mu) \to (\Omega_1, \mathfrak{N}, \nu)$ be a measurable set transformation. Then there is a unique linear map $S_*\colon L^0(\mu) \to L^0(\nu)$ such that*

1. *$S_*(\chi_E) = \chi_{S(E)}$ for all $E \in \mathfrak{M}$.*

2. *If $(\xi_n)_{n=1}^{\infty}$ is a sequence of measurable functions on Ω_0 converging pointwise a.e $[\mu]$ to ξ, then $S_*(\xi_n) \to S_*(\xi)$ pointwise a.e $[\nu]$.*

Moreover,

3. *$S_*(\xi \cdot \eta) = S_*(\xi) \cdot S_*(\eta)$ and $S_*(\overline{\xi}) = \overline{S_*(\xi)}$.*

4. *$S_*(L^0(\mu)) = L^0(\nu|_{\operatorname{ran}(S)})$*

5. *S_* is injective if and only if S is injective.*

6. *$S(\xi^{-1}(B)) = S_*(\xi)^{-1}(B)$ for any Borel set B in $\mathbb{C}$.*

7. *If $T\colon (\Omega_1, \mathfrak{N}, \nu) \to (\Omega_2, \mathfrak{P}, \lambda)$ is another measurable set transformation, then $(T \circ S)_* = T_* \circ S_*$.*

Definition 4.4.5. For a measure space $(\Omega, \mathfrak{M}, \mu)$ we define $\mathrm{ACM}(\Omega, \mathfrak{M}, \mu)$ to be the set of all measures on $(\Omega, \mathfrak{M})$ that are absolutely continuous with respect to μ.

The following is part of Lemma 5.9. in [21].

Proposition 4.4.6. *Let $S\colon (\Omega_0, \mathfrak{M}, \mu) \to (\Omega_1, \mathfrak{N}, \nu)$ be a measurable set transformation. Then there is a unique well defined map $S^*\colon \mathrm{ACM}(\Omega_1, \mathfrak{N}, \nu) \to \mathrm{ACM}(\Omega_0, \mathfrak{M}, \mu)$ such that if $E \in \mathfrak{M}$, $F \in \mathfrak{N}$ satisfy $S(E) = F$, then*

$$S^*(\lambda)(E) = \lambda(F),$$

whenever $\lambda \in \mathrm{ACM}(\Omega_1, \mathfrak{N}, \nu)$.

We really need to push measures forward rather than pull them back. For this, we require $S\colon (\Omega_0, \mathfrak{M}, \mu) \to (\Omega_1, \mathfrak{N}, \nu)$ to be an injective measurable transformation. In this case, we get a measurable transformation $S^{-1}\colon (\Omega_1, \mathrm{ran}(S), \nu|_{\mathrm{ran}(S)}) \to (\Omega_0, \mathfrak{M}, \mu)$. This allows us to define the *pushforward on* $\mathrm{ACM}(\Omega_0, \mathfrak{M}, \mu)$ *induced by* S as follows: $S_*\colon \mathrm{ACM}(\Omega_0, \mathfrak{M}, \mu) \to \mathrm{ACM}(\Omega_1, \mathrm{ran}(S), \nu|_{\mathrm{ran}(S)})$ is given by

$$S_* = (S^{-1})^*.$$

Corollaries 5.13 and 5.14 in [21] give important properties of this pushforward of measures for biyective measurable set transformations, which we present in the following proposition.

Proposition 4.4.7. *Let $(\Omega_0, \mathfrak{M}, \mu)$ and $(\Omega_1, \mathfrak{N}, \nu)$ be σ-finite measure spaces and let $S\colon (\Omega_0, \mathfrak{M}, \mu) \to (\Omega_1, \mathfrak{N}, \nu)$ be a bijective measurable transformation.*

1. *If* $h = \left[\frac{dS_*(\mu)}{d\nu}\right]$ *and* $\xi \in L^0(\mu)$ *is nonnegative or if one of the integrals in the following exists (in which case they all do), then*

$$\int_{\Omega_0} \xi \; d\mu = \int_{\Omega_1} S_*(\xi) \; dS_*(\mu) = \int_{\Omega_1} S_*(\xi) h \; d\nu$$

2. *If* $\lambda \in \mathrm{ACM}(\Omega_0, \mathfrak{M}, \mu)$ *is* σ*-finite, then so is* $S_*(\lambda)$*.*

3. *If* $\lambda_1, \lambda_2 \in \mathrm{ACM}(\Omega_0, \mathfrak{M}, \mu)$ *are mutually absolutely continuous, then*

$$\left[\frac{dS_*(\lambda_1)}{dS_*(\lambda_2)}\right] = S_*\left(\left[\frac{d\lambda_1}{d\lambda_2}\right] \right) \quad a.e \; [S_*(\lambda_2)]$$

Furthermore, if $T\colon (\Omega_1, \mathfrak{N}, \nu) \to (\Omega_2, \mathfrak{P}, \lambda)$ *is an injective measurable set transformation, then* $(T \circ S)_* = T_* \circ S_*$*.*

The importance of injective measurable set transformations is that they give rise to isometric operators between L^p-spaces, as shown in the next lemma.

Lemma 4.4.8. *Let* $p \in [1, \infty)$*, let* $S\colon (\Omega_0, \mathfrak{M}, \mu) \to (\Omega_1, \mathfrak{N}, \nu)$ *be an injective measurable set transformation such that* $\nu|_{\mathrm{ran}(S)}$ *is* σ*-finite, and let* g *be a measurable function on* Ω_1 *such that* $|g| = 1$ *a.e.* $[\nu]$*. Let* $s\colon L^p(\mu) \to L^p(\nu)$ *be given by*

$$s(\xi) = \left[\frac{dS_*(\mu)}{d\nu|_{\mathrm{ran}(S)}}\right]^{1/p} S_*(\xi) g.$$

Then $s \in \mathcal{L}(L^p(\mu), L^p(\nu))$ *and* $\|s(\xi)\|_p = \|\xi\|_p$*, that is,* s *is an isometry.*

Proof. It is clear that s is a linear map. One then easily checks that $\left[\frac{dS_*(\mu)}{d\nu|_{\mathrm{ran}(S)}}\right]$ is non-negative. The rest is a direct computation using the properties of the pushforwards on $L^0(\mu)$ and on $\mathrm{ACM}(\Omega_0, \mathfrak{M}, \mu)$ stated on Propositions 4.4.4 and

4.4.7. Indeed,

$$\|s(\xi)\|_p^p = \int_{\Omega_1} \left(\frac{dS_*(\mu)}{d\nu|_{\mathrm{ran}(S)}}\right)|S_*(\xi)|^p \, d\nu|_{\mathrm{ran}(S)} = \int_X S_*(|\xi|^p) \, dS_*(\mu) = \|\xi\|_p^p,$$

as wanted. ■

Lemma 4.4.8 gives rise to a more general type of operator known as semispatial partial isometries, which we define below.

Definition 4.4.9. Let $(\Omega_0, \mathfrak{M}, \mu)$ and $(\Omega_1, \mathfrak{N}, \nu)$ be σ-finite measure spaces.

(1) A *semispatial system* for $(\Omega_0, \mathfrak{M}, \mu)$ and $(\Omega_1, \mathfrak{N}, \nu)$ is a quadruple (E, F, S, g) in which $E \in \mathfrak{M}$, $F \in \mathfrak{N}$, $S\colon (E, \mathfrak{M}|_E, \mu|_E) \to (F, \mathfrak{N}|_F, \nu|_F)$ is an injective measurable set transformation with $\nu|_{\mathrm{ran}(S)}$ σ-finite, and $g\colon F \to \mathbb{C}$ a $\mathfrak{N}|_F$-measurable function such that $|g| = 1$ a.e. $[\nu|_F]$. If in addition S is bijective, we call (E, F, S, g) a *spatial system.*

(2) If $p \in [1, \infty)$, a linear map $s\colon L^p(\mu) \to L^p(\nu)$ is said to be a *(semi)spatial partial isometry* if there is a (semi)spatial system (E, F, S, g) such that

$$s(\xi) = \begin{cases} \left(\frac{dS_*(\mu|_E)}{d\nu|_{\mathrm{ran}(S)}}\right)^{1/p} S_*(\xi|_E)g & \text{on } F \\ 0 & \text{on } \Omega_1 \setminus F \end{cases}.$$

If in addition $\mu(\Omega_0 \setminus E) = 0$, we say s is simply a *(semi)spatial isometry.*

As in Lemma 4.4.8, we see that if s is a (semi)spatial partial isometry, then $\|s(\xi)\|_p = \|\xi|_E\|_p$. Thus, any (semi)spatial isometry is actually an isometry. The following theorem (due to Lamperti [16]) states that for $p \in [1, \infty) \setminus \{2\}$, any isometry $s \in \mathcal{L}\big(L^p(\mu), L^p(\nu)\big)$ is a semispatial isometry. However, Lamperti's original statement is slightly in error. For a complete proof we refer the reader to Theorem 3.2.5 in [12].

Theorem 4.4.10. *(Lamperti) Let $p \in [1,\infty)\setminus\{2\}$, and let $(\Omega_0, \mathfrak{M}, \mu)$ and $(\Omega_1, \mathfrak{N}, \nu)$ be σ-finite measure spaces. If $s \in \mathcal{L}(L^p(\mu), L^p(\nu))$ is an isometry, then s is a semispatial isometry. If in addition s is an isometric isomorphism, then s is a spatial isometry.*

We will mostly need spatial partial isometries, that is those that come from bijective measurable set transformations. The following is part of Lemma 6.12 in [21].

Lemma 4.4.11. *Let $(\Omega_0, \mathfrak{M}, \mu)$ and $(\Omega_1, \mathfrak{N}, \nu)$ be σ-finite measure spaces. Let $p \in [1,\infty)$, and let (E, F, S, g) be a spatial system for $(\Omega_0, \mathfrak{M}, \mu)$ and $(\Omega_1, \mathfrak{N}, \nu)$.*

1. *There is a unique spatial partial isometry $s \in \mathcal{L}\big(L^p(\mu), L^p(\nu)\big)$ whose spatial system is (E, F, S, g).*

2. *Furthermore, there is a unique spatial partial isometry $t \in \mathcal{L}\big(L^p(\nu), L^p(\mu)\big)$ whose spatial system is $(F, E, S^{-1}, (S^{-1})_*(g)^{-1})$.*

Definition 4.4.12. The element t from part (2) in Lemma 4.4.11 is called the *reverse of s*.

The following result is part of Lemma 6.18 of [21].

Lemma 4.4.13. *Let $(\Omega, \mathfrak{M}, \mu)$ be a σ-finite measure space. Then an idempotent $e \in \mathcal{L}(L^p(\mu))$ is a spatial partial isometry if and only if there is $E \in \mathfrak{M}$ such that $e = m(\chi_E)$.*

The next proposition provides a mostly algebraic criterion to check that, for $p \in [1,\infty) \setminus \{2\}$, an element in $\mathcal{L}(L^p(X,\mu))$ is a spatial partial isometry. We give a complete proof here as we believe there is not one available in the current literature.

Proposition 4.4.14. *Let $(\Omega, \mathfrak{M}, \mu)$ be a σ-finite measure space and let $p \in [1, \infty) \setminus \{2\}$. Then $s \in \mathcal{L}(L^p(\mu))$ is a spatial partial isometry if and only if there is $t \in \mathcal{L}(L^p(\mu))$ and there are idempotents $e, f \in \mathcal{L}(L^p(\mu))$ such that:*

1. *e and f are spatial partial isometries.*

2. *$st = f$ and $ts = e$.*

3. *$fse = s$ and $etf = t$.*

4. *$\|s\| \leq 1$ and $\|t\| \leq 1$.*

Proof. Suppose first that s is a spatial partial isometry. For this direction we do not use $p \neq 2$. Let (E, F, S, g) be the spatial system of s, and let $t \in \mathcal{L}(L^p(\mu))$ be its reverse, both guaranteed to exist by Lemma 4.4.11. That $\|s\| \leq 1$ and $\|t\| \leq 1$ is a consequence of Lemma 4.4.8, so part (4) follows. Define $f = st$ and $e = ts$. We claim that $f = m(\chi_F)$ and $e = m(\chi_E)$. Indeed, using the properties of the pushforwards on $L^0(\mu)$ and on $\mathrm{ACM}(\Omega, \mathfrak{M}, \mu)$ stated in Propositions 4.4.4 and 4.4.7 respectively, we find that for any $\xi \in L^p(\mu)$ and any $\omega \in F$,

$$\begin{aligned}
\big(s(t\xi)\big)(\omega) &= \left[\frac{dS_*(\mu|_E)}{d\mu|_F}(\omega)\right]^{1/p} S_*(t(\xi)|_E)(\omega)g(\omega) \\
&= \left[\frac{dS_*(\mu|_E)}{d\mu|_F}(x)\right]^{1/p} S_*\left(\left[\frac{d(S^{-1})_*(\mu|_F)}{d\mu|_F}(\omega)\right]^{1/p} (S^{-1})_*(\xi|_F g^{-1})\right)(\omega)g(\omega) \\
&= \left[\left(\frac{dS_*(\mu|_E)}{d\mu|_F}\right)(\omega)\left(\frac{d\mu|_F}{dS_*(\mu|_E)}\right)(\omega)\right]^{1/p} \xi|_F(\omega) \\
&= (\xi|_F)(\omega),
\end{aligned}$$

whereas if $\omega \notin F$ it is clear that $\big(s(t\xi)\big)(\omega) = 0$. Thus, $f = m(\chi_F)$. An analogous argument shows that $e = m(\chi_E)$, so the claim is proved. It follows from Lemma 4.4.13 that e and f are indeed idempotent spatial partial isometries. Since e is multiplication by χ_E and clearly $s(\xi) = s(\xi|_E)$, it follows that $se = s$. Similarly, $tf = t$. Hence, $fse = se^2 = se = s$ and $etf = tf^2 = tf = t$, giving part (3). This finishes the forward direction.

Conversely, assume that there is t and that there are idempotents e, f satisfying (1)-(4) in the statement. By Lemma 4.4.13, there are $E, F \in \mathfrak{M}$ such that $e = m(\chi_E)$ and $f = m(\chi_F)$. Notice that (2), (3) and (4) say that s is an isometric isomorphism from $\mathrm{Range}(e) = L^p(E, \mu|_E)$ to $\mathrm{Range}(f) = L^p(F, \mu|_F)$ with inverse t. Since $p \neq 2$, it follows from Theorem 4.4.10 (Lamperti's Theorem) that $s|_{L^p(E,\mu_E)} \colon L^p(E, \mu|_E) \to L^p(F, \mu|_F)$ is a spatial isometry, and therefore s is a spatial partial isometry. ■

Corollary 4.4.15. *Let $(\Omega, \mathfrak{M}, \mu)$ be a σ-finite measure space and let $p \in [1, \infty) \setminus \{2\}$. If $s, t \in \mathcal{L}(L^p(\mu))$ satisfy (1)-(4) of Proposition 4.4.14, then both s and t are spatial partial isometries.*

Proof. This is immediate from the fact that the statement of Proposition 4.4.14 is symmetric in s and t. ■

Remark 4.4.16. It was shown by Phillips and Viola, see Lemma 5.8 in [23], that for $(\Omega, \mathfrak{M}, \mu)$ a σ-finite measure space and $p \in [1, \infty) \setminus \{2\}$, an idempotent $e \in \mathcal{L}(L^p(\mu))$ is spatial partial isometry if and only if e is a hermitian idempotent. Since it is possible to define hermitian idempotents for any unital Banach algebra, Proposition 4.4.14 above could be taken as the definition of a spatial partial isometry in a unital Banach algebra.

Definition 4.4.17. Let $d \in \mathbb{Z}_{\geq 2}$. We define the *Leavitt algebra* L_d to be the universal complex unital algebra generated by elements $s_1, \ldots, s_d, t_1, \ldots, t_d$ subject to the relations:

1. For any $j \in \{1, \ldots, d\}$,
$$t_j s_j = 1.$$

2. For $j \neq k$ in $\{1, \ldots, d\}$,
$$t_j s_k = 0.$$

3. $$\sum_{j=1}^{d} s_j t_j = 1.$$

Definition 4.4.18. Let $d \in \mathbb{Z}_{\geq 2}$ and let B be a non-zero Banach space. A *representation* of L_d on B is a unital algebra homomorphism $\rho\colon L_d \to \mathcal{L}(B)$. If $p \in [1, \infty)$ and $B = L^p(\mu)$ for a σ-finite measure space $(\Omega, \mathfrak{M}, \mu)$, we say that ρ is a *spatial representation* if for each $j \in \{1, \ldots, d\}$ the element $\rho(s_j)$ is a spatial partial isometry with reverse given by $\rho(t_j)$.

The following are Theorem 8.7 and Definition 8.8 in [21] respectively

Theorem 4.4.19. *Let $d \in \mathbb{Z}_{\geq 2}$, let $p \in [1, \infty)$, let $(\Omega_0, \mathfrak{M}, \mu)$ and $(\Omega_1, \mathfrak{N}, \nu)$ be σ-finite measure spaces, and let $\rho\colon L_d \to \mathcal{L}(L^p(\mu))$ and $\varphi\colon L_d \to \mathcal{L}(L^p(\nu))$ be spatial representations. Then the map $\rho(s_j) \mapsto \varphi(s_j)$ and $\rho(t_j) \mapsto \varphi(t_j)$, for $j \in \{1, \ldots, d\}$, extends to an isometric isomorphism $\overline{\rho(L_d)} \to \overline{\varphi(L_d)}$.*

The previous theorem justifies the following definition.

Definition 4.4.20. Let $d \in \mathbb{Z}_{\geq 2}$ and let $p \in [1, \infty)$. We define the L^p *Cuntz algebra* $\mathcal{O}_d^p$ to be the completion of L_d in the norm $a \mapsto \|\rho(a)\|$ for any spatial representation ρ of L_d on $L^p(\mu)$ for a σ-finite measure space $(\Omega, \mathfrak{M}, \mu)$.

L^p-Crossed Products

The following material is taken mostly from Section 3 of [9] and Section 3 of [22].

Let A be an L^p-operator algebra, let G be a second countable locally compact group with left Haar measure ν, and let $\alpha : G \to \mathrm{Aut}(A)$ be an isometric action (i.e., for each $g \in G$ the map $\alpha(g) = \alpha_g \colon A \to A$ is isometric and for each $a \in A$, $g \mapsto \alpha_g(a)$ is a continuous map $G \to A$). The triple (G, A, α) is called an *isometric G-L^p-operator algebra* or an *L^p-dynamical system*. We denote by $C_c(G, A, \alpha)$ the space $L^1(G, A) = L^1(G, \nu) \otimes_1 A$ equipped with twisted convolution:

$$(x * y)(g) = \int_G x(h)\alpha_h(y(h^{-1}g))d\nu(h).$$

It is known that $L^1(G, A, \alpha)$ is a normed algebra that contains $C_c(G, A, \varphi)$ as a dense subalgebra. Let (G, A, α) be an isometric G-L^p-operator algebra and let $(\Omega, \mathfrak{M}, \mu)$ be a measure space. A *covariant representation* of (G, A, α) on $L^p(\mu)$ consists of a pair (π, u) where

1. π is a representation of A on $L^p(\mu)$,

2. $u : G \to \mathrm{Inv}(L^p(\mu))$ is a group homomorphism with $g \mapsto u(g)\xi = u_g\xi$ a continuous map for all $\xi \in L^p(\mu)$,

3. $\pi(\alpha_g(a)) = u_g\pi(a)u_g^{-1}$ for all $g \in G$, $a \in A$.

A covariant representations (π, u) induces a representation of $L^1(G, A, \alpha)$ on $L^p(\mu)$ via

$$\big((\pi \rtimes u)x\big)\xi = \int_G \pi(x(g))u_g\xi d\nu(g).$$

for any $\xi \in L^p(\mu)$. A contractive representation π_0 of A on $L^p(\mu)$ induces a contractive covariant representation (π, v) of (A, G, α) on $L^p(\nu \times \mu)$ given by

$$(\pi(a)\eta)(g) = \pi_0(\alpha_g^{-1}(a))(\eta(g))$$

for $\eta \in C_c(G, L^p(\mu)) \subseteq L^p(\nu \times \mu)$, and

$$(v_g\xi)(h, \omega) = \xi(g^{-1}h, \omega)$$

for any $\xi \in L^p(\nu \times \mu)$.

Definition 4.5.1. Given an isometric G-L^p-operator algebra, we define a seminorm on $C_c(G, A, \alpha)$ by

$$\sigma_{\rtimes}(x) = \sup_{\substack{(\pi,u) \text{ is a } \sigma\text{-finite, nondegenerate,} \\ \text{contractive, covariant representation of } (G,A,\alpha)}} \|(\pi \rtimes u)x\|$$

The full crossed product $F^p(G, A, \alpha)$ is the completion of $C_c(G, A, \alpha)/\ker(\sigma_{\rtimes})$ with respect to $\| - \|_{\rtimes}$, the norm induced by $\sigma_{\rtimes}$. Similarly, we define a seminorm by

$$\sigma_{\mathrm{r}}(x) = \sup_{\substack{\pi_0 \text{ is a } \sigma\text{-finite, nondegenerate,} \\ \text{contractive representation of } A}} \|(\pi \rtimes v)x\|$$

Then we define $F_{\mathrm{r}}(G, A, \alpha)$, the reduced crossed product, to be the completion of $C_c(G, A, \alpha)/\ker(\sigma_{\mathrm{r}})$ with respect to $\| - \|_{\mathrm{r}}$, the norm induced by σ_{r}

It is now known that $F^p(G,A,\alpha)$ is isometrically isomorphic to $F_{\mathrm{r}}(G,A,\alpha)$ when G is amenable. A proof of this is analogous to the C*-case and will appear in an updated version of [22].

The following result is an analogue of a well known fact about C*-crossed product and follows essentially by the same arguments.

Proposition 4.5.2. *Let A be an L^p-operator algebra with a c.a.i. and let (G,A,α) be an isometric G-L^p-operator algebra. If $\pi\colon L^1(G,A,\alpha) \to \mathcal{L}(L^p(\mu))$ is a nondegenerate contractive representation, then $\|\pi(x)\| \leq \|x\|_{\rtimes}$ for all $x \in L^1(G,A,\alpha)$.*

Proof. Since both A and $L^1(G)$ have c.a.i.'s, essentially the same arguments presented in the proof for the converse of Proposition 7.6.4 in [20] (ignoring $*$-preserving and replacing unitary by invertible), show that $\pi = \pi_0 \rtimes u$ for a covariant representation (π_0, u) of (G,A,α). Then $\|\pi(x)\| = \|(\pi_0 \rtimes u)(x)\| \leq \|x\|_{\rtimes}$, as wanted. ■

CHAPTER V

L^P-MODULES OVER L^P OPERATOR ALGEBRAS

In this chapter we initiate the study of a type of module over L^p-operator algebras that generalizes Hilbert modules over C*-algebras. The definitions here are motivated by our results from Chapter III.

L^p-modules and C*-like L^p-modules

For our main definition, it is worth revisiting Example 3.1.1 from Chapter III. Recall that if $A \subseteq \mathcal{L}(\mathcal{H}_0)$ is a concrete C*-algebra, then any closed subspace $\mathsf{X} \subseteq \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ satisfying

1. $xa \in \mathsf{X}$ for all $x \in \mathsf{X}$, $a \in A$,

2. $x^*y \in A$ for all $x, y \in \mathsf{X}$,

is a right Hilbert A-module. Furthermore, observe that the space $\mathsf{X}^* = \{x^* \colon x \in \mathsf{X}\}$ is a closed subspace of $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_0)$ satisfying

3. $ay \in \mathsf{X}^*$ for all $a \in A$, $y \in \mathsf{X}^*$.

Finally, by standard Hilbert module arguments we also know that the norm of an element x in any right Hilbert A-module X can be computed as an operator norm of the map $y \mapsto \langle x, y\rangle_A$ which is in $\mathcal{L}_A(\mathsf{X}, A)$ with adjoint given by $a \mapsto xa$. In Example 3.1.1, this is equivalent to

4. $\|x\| = \sup_{y\in \mathsf{X}^*, \|y\|=1} \|yx\|$, and $\|x^*\| = \sup_{y \in \mathsf{X}, \|y\|=1} \|x^*y\|$.

The next definition is motivated by the behavior we just described for the pair $(\mathsf{X}, \mathsf{X}^*)$ coming from Example 3.1.1.

Definition 5.1.1. Let $(\Omega_0, \mathfrak{M}_0, \mu_0)$ and $(\Omega_1, \mathfrak{M}_1, \mu_1)$ be measure spaces, let $p \in [1, \infty)$, and let $A \subseteq \mathcal{L}(L^p(\mu_0))$ be an L^p operator algebra. An *L^p-module* over A is a pair (X, Y), where $\mathsf{X} \subseteq \mathcal{L}(L^p(\mu_0), L^p(\mu_1))$ and $\mathsf{Y} \subseteq \mathcal{L}(L^p(\mu_1), L^p(\mu_0))$ are closed subspaces satisfying

1. $xa \in \mathsf{X}$ for all $x \in \mathsf{X}$, $a \in A$,

2. $yx \in A$ for all $x \in \mathsf{X}$, $y \in \mathsf{Y}$,

3. $ay \in \mathsf{Y}$ for all $y \in \mathsf{Y}$, $a \in A$.

If in addition for every $x \in \mathsf{X}$ and $y \in \mathsf{Y}$ we have

4. $\|x\| = \sup_{y \in \mathsf{Y}, \|y\|=1} \|yx\|$ and $\|y\| = \sup_{x \in \mathsf{X}, \|x\|=1} \|yx\|$,

then we say that (X, Y) is a *C^*-like L^p-module.*

Notation 5.1.2. If (X, Y) is an L^p-module over A, it comes naturally equipped with a pairing $\mathsf{Y} \times \mathsf{X} \to A$ via $(y, x) \mapsto yx$. It will be convenient to sometimes denote the operator $yx : L^p(\mu_0) \to L^p(\mu_0)$ by $(y \mid x)_A$.

We now present examples of L^p-modules.

Example 5.1.3. Let A be a C*-algebra and let X be any right Hilbert A module. If $(\pi_A, \pi_{\mathsf{X}})$ is an isometric representation of X on a pair of Hilbert spaces $(\mathcal{H}_0, \mathcal{H}_1)$ as in Definition 3.2.5, then $(\pi_{\mathsf{X}}(\mathsf{X}), \pi_{\mathsf{X}}(X)^*)$ is clearly a C*-like L^2-module over $\pi_A(A)$.

Example 5.1.4. Let $p \in [1, \infty)$, let $(\Omega, \mathfrak{M}, \mu)$ be a measure space, and let $A \subseteq \mathcal{L}(L^p(\mu))$ be an L^p operator algebra. Then (A, A) is trivially an L^p module over A.

However, (A, A) is not always C*-like, as condition (4) does not hold in general for non-unital A. Indeed, if

$$A = \left\{ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} : z \in \mathbb{C} \right\} \subset M_2^p(\mathbb{C}) = \mathcal{L}(\ell^p(\{1, 2\})),$$

then

$$\left\| \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\| = 1 > 0 = \sup_{|z|=1} \left\| \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\|.$$

Nevertheless, if A has a contractive approximate identity $(e_\lambda)_{\lambda\in\Lambda}$, then (A, A) is C*-like. Indeed, in this case condition (4) holds thanks to Lemma 4.1.4.

Example 5.1.5. Let $(\Omega, \mathfrak{M}, \mu)$ be a measure space, let $p \in (1, \infty)$, and consider the L^p-operator algebra $A = \mathcal{L}(\ell_1^p)$. Observe that A can be identified with $\mathbb{C}$ via $a \mapsto a(1)$ and that $\|a\| = |a(1)|$ for any $a \in A$, whence the identification is isometric. Now let $\mathsf{X} = L^p(\mu)$, which we isometrically identify with $\mathcal{L}(\ell_1^p, L^p(\mu))$ via $\xi \mapsto (z \mapsto z\xi)$ for any $\xi \in L^p(\mu)$ and $z \in \ell_1^p$. Similarly, if q is the Hölder conjugate of p, then $\mathsf{Y} = L^q(\mu)$ is isometrically identified with $\mathcal{L}(L^p(\mu), \ell_1^p)$ via the usual dual pairing $\eta \mapsto (\xi \mapsto \langle \eta, \xi \rangle = \int_\Omega \eta\xi d\mu)$ for $\eta \in L^q(\mu)$ and $\xi \in L^p(\mu)$. Under these identifications, we claim that (X, Y) is a C*-like L^p-module over A. Clearly X and Y are closed subsets of $\mathcal{L}(\ell_1^p, L^p(\mu))$ and $\mathcal{L}(L^p(\mu), \ell_1^p)$. We check that conditions (1)-(4) from Definition 5.1.1 hold. Let $\xi \in \mathsf{X}$ and $a \in A$. Then the composition $\xi a : \ell_1^p \to L^p(\mu)$ is clearly a bounded linear map, proving condition (1). If $\eta \in \mathsf{Y}$ and $\xi \in \mathsf{X}$, the composition $(\eta \mid \xi)_A \colon \ell_1^p \to \ell_1^p$ agrees with $\langle \eta, \xi \rangle$ as an element of A, so condition (2) follows. Similarly, for $a \in A$ and $\eta \in Y$, we note that the composition $a\eta \colon L^p(\mu) \to \ell_1^p$ is a bounded linear map and therefore condition (3) is done. Finally, it's well known that for any $\xi \in \mathsf{X}$ and $\eta \in Y$ $\|\xi\|_p = \sup_{\|\eta\|_q=1} |\langle \eta, \xi \rangle|$, so condition (4) also follows.

Example 5.1.6. Let $d \in \mathbb{Z}_{\geq 1}$, let $p \in [1, \infty)$, and let q be the Hölder conjugate of p. As particular instance of Example 5.1.5, we see that (ℓ_d^p, ℓ_d^q) is a C*-like L^p-module over $\mathbb{C}$. Notice that we are now able to include $p = 1$ because the dual of ℓ_d^1 is ℓ_d^∞.

Example 5.1.7. Let $(\Omega, \mathfrak{M}, \mu)$ be a measure space, let $p \in (1, \infty)$, and consider the L^p-operator algebra $A = \mathcal{K}(L^p(\mu))$. We can switch the modules in Example 5.1.5 and still get an L^p-module but over $\mathcal{K}(L^p(\mu))$ instead of $\mathbb{C}$. Indeed, let $\mathsf{X} = L^q(\mu)$, identified as before with $\mathcal{L}(L^p(\mu), \ell_1^p)$, and let $\mathsf{Y} = L^p(\mu)$ which is identified again with $\mathcal{L}(\ell_1^p, L^p(\mu))$. For any $a \in \mathcal{L}(L^p(\mu))$ there is $a' \in \mathcal{L}(L^q(\mu))$ given by $\langle a'\eta, \xi \rangle = \langle \eta, a\xi \rangle$ and it is clear that $xa = a'x \in \mathsf{X}$ for any $x \in L^q(\mu)$, whence condition (1) in Definition 5.1.1 follows. Condition (2) also holds, for a direct calculation shows that $yx = \theta_{y,x} \in \mathcal{K}(L^p(\mu)) = A$. Condition (3) follows at once from the fact that A naturally acts on $L^p(\mu)$ on the left as bounded operators. Finally, since $\|\theta_{y,x}\| = \|y\|_p \|x\|_q$, it is also clear that $(L^q(\mu), L^p(\mu))$ is a C*-like module over $\mathcal{K}(L^p(\mu))$.

Example 5.1.8. Let $d \in \mathbb{Z}_{\geq 1}$, let $p \in [1, \infty)$ and let q be the Hölder conjugate of p. As particular instance of Example 5.1.7 we get that (ℓ_d^q, ℓ_d^p) is a C*-like L^p-module over $\mathcal{L}(\ell_d^p) = M_d^p$. We are again able to include $p = 1$ because the dual of ℓ_d^1 is ℓ_d^∞.

Example 5.1.9. In this example we combine, via the spatial tensor product, Example 5.1.6 with Example 5.1.4. This is a particular case of the external tensor product construction discussed in Section 5.4 below. Let $d \in \mathbb{Z}_{\geq 2}$, let $p \in (1, \infty)$, and let $(\Omega, \mathfrak{M}, \mu)$ be a measure space. If ν_d is counting measure on $\{1, \ldots, d\}$, then we have the following isometric isomorphisms

$$\ell_d^p \otimes_p L^p(\mu) \cong L^p(\nu_d \times \mu) \cong L^p(\mu)^d.$$

The last one comes from $\xi \mapsto (\xi_1, \dots, \xi_d)$ where, for each $j \in \{1, \dots, d\}$, $\xi_j \in L^p(\mu)$ is given by $\xi_j(\omega) = \xi(j, \omega)$, and the norm on $L^p(\mu)^d$ is given by $\|(\xi_1, \dots, \xi_d)\|^p = \sum_{j=1}^d \|\xi_j\|^p$. Now let $A \subseteq \mathcal{L}(L^p(\mu))$ be an L^p operator algebra. We define $\mathsf{X} \subseteq \mathcal{L}(L^p(\mu), L^p(\mu)^d)$ and $\mathsf{Y} \subseteq \mathcal{L}(L^p(\mu)^d, L^p(\mu))$ by

$$\mathsf{X} = \ell_d^p \otimes_p A = \mathcal{L}(\ell_1^p, \ell_d^p) \otimes_p A \quad \text{and} \quad \mathsf{Y} = \ell_d^q \otimes_p A = \mathcal{L}(\ell_d^p, \ell_1^p) \otimes_p A.$$

Observe that X is identified with A^d, with norm given by

$$\|(a_1, \dots, a_d)\| = \sup_{\|\xi\|=1} \Big(\sum_{j=1} \|a_j \xi\|^p \Big)^{1/p},$$

where the supremum is taken over $\xi \in L^p(\mu)$. Similarly, Y is also identified with A^d, but equiped with the norm

$$\|(b_1, \dots, b_d)\| = \sup_{\|(\xi_1, \dots, \xi_d)\|=1} \Big\| \sum_{j=1}^d b_j \xi_j \Big\|,$$

where the supremum is taken over $(\xi_1, \dots, \xi_d) \in L^p(\mu)^d$. Since X and Y are closed by construction, we automatically have the closure conditions of Definition 5.1.1. For condition (1), take $z \in \ell_d^p$ and $a_1, a_2 \in A$. We have

$$(z \otimes a_1)a_2 = z \otimes a_1 a_2 \in \mathsf{X}.$$

Therefore the composition xa is in X for all $x \in \mathsf{X}$ and all $a \in A$. Similarly, to verify condition (2), notice that for $w \in \ell_d^q$, $z \in \ell_d^p$, and $a_1, a_2 \in A$, we have

$$(w \otimes a_1)(z \otimes a_2) = \Big(\sum_{j=1}^{d} w(j)z(j) \Big) \, a_1 a_2 \in A.$$

Hence, it follows that $(y \mid x)_A \in A$ for all $y \in \mathsf{Y}$ and all $x \in \mathsf{X}$. Condition (3) follows similarly. Indeed, if $a_1, a_2 \in A$ and $w \in \ell_d^q$ we get

$$a_1(w \otimes a_2) = w \otimes a_1 a_2 \in \mathsf{Y},$$

whence $ay \in \mathsf{Y}$ for all $a \in A$ and $y \in \mathsf{Y}$. Thus, (X, Y) is an L^p module over A. We suspect that (X, Y) is C*-like whenerver (A, A) is, but we haven't been able to prove this or find a counter example. We will say more about this example and its potential C*-likeness in Chapter VII.

Finite Direct Sum of L^p-modules

Let $p \in (1, \infty)$. Example 5.1.9 can be realized as the direct sum of d copies of the L^p module from Example 5.1.4. We now describe such direct sum in its full generality. Let $p \in [1, \infty)$, let $d \in \mathbb{Z}_{\geq 2}$, and for each $j \in \{1, \ldots, d\}$ let $(\mathsf{X}_j, \mathsf{Y}_j)$ be an L^p module over an L^p operator algebra $A \subseteq \mathcal{L}(L^p(\mu_0))$. For $j \in \{1, \ldots, d\}$, we have measure spaces $(\Omega_j, \mathfrak{M}_j, \mu_j)$ such that X_j is a closed subspace of $\mathcal{L}(L^p(\mu_0), L^p(\mu_j))$ and Y_j is a closed subspace of $\mathcal{L}(L^p(\mu_j), L^p(\mu_0))$. Consider the algebraic direct sums $\mathsf{X} = \bigoplus_{j=1}^{d} \mathsf{X}_j$ and $\mathsf{Y} = \bigoplus_{j=1}^{d} \mathsf{Y}_d$. The pair (X, Y) has a natural structure of

L^p module over A. Indeed,

$$\mathsf{X} \subseteq \mathcal{L}\Big(L^p(\mu_0), \bigoplus_{j=1}^{d} L^p(\mu_j)\Big),$$

where each $(x_1, \ldots, x_d) \in \mathsf{X}$ acts on $\xi \in L^p(\mu_0)$ by

$$(x_1, \ldots, x_d)\xi = (x_1\xi, \ldots, x_d\xi).$$

This endows X with the operator norm satisfying

$$\max_{j=1,\ldots,d} \|x_j\| \leq \|(x_1, \ldots, x_d)\| \leq \Big(\sum_{j=1}^{d} \|x_j\|^p\Big)^{1/p}.$$

Even though in general neither equality is true, this shows that X is a closed subspace of $\mathcal{L}\big(L^p(\mu_0), \bigoplus_{j=1}^{d} L^p(\mu_j)\big)$. Similarly,

$$\mathsf{Y} \subseteq \mathcal{L}\Big(\bigoplus_{j=1}^{d} L^p(\Omega_j, \mu_j), L^p(\Omega_0, \mu_0)\Big)$$

where each $(y_1, \ldots, y_d) \in \mathsf{Y}$ acts on $(\eta_1, \ldots, \eta_d) \in \bigoplus_{j=1}^{d} L^p(\mu_j)$ by

$$(y_1, \ldots, y_d)(\eta_1, \ldots, \eta_d) = \sum_{j=1}^{d} y_j\eta_j.$$

Thus, the operator norm inherited by Y satisfies

$$\max_{j=1,\ldots,d} \|y_j\| \leq \|(y_1, \ldots, y_d)\| \leq \Big(\sum_{j=1}^{d} \|y_j\|^q\Big)^{1/q}.$$

where q is the Hölder conjugate for p. Once again, equality in both ends of the last inequality does not always hold, but it follows that Y is a closed subspace of

$\mathcal{L}\big(\bigoplus_{j=1}^{d} L^p(\mu_j), L^p(\mu_0)\big)$. For each $(x_1, \ldots, x_d) \in \mathsf{X}$ and $a \in A$, it is clear that condition (1) in Definition 5.1.1 holds:

$$(x_1, \ldots, x_d)a = (x_1 a, \ldots, x_d a) \in \mathsf{X}$$

For condition (2), if $(y_1, \ldots, y_d) \in \mathsf{Y}$, we get

$$(y_1, \ldots, y_d)(x_1, \ldots, x_d) = \sum_{j=1}^{d} (y_j \mid x_j)_A \in A.$$

We now check condition (3). Indeed, it is clear that if $(y_1, \ldots, y_d) \in \mathsf{Y}$, $a \in A$, then $ay_j \in \mathsf{Y}_j$ for each $j \in \{1, \ldots, d\}$, and therefore we have

$$a(y_1, \ldots, y_d) = (ay_1, \ldots, ay_d) \in \mathsf{Y},$$

Hence, (X, Y) is an L^p-module over A. As before, we conjecture that (X, Y) will be C*-like whenever A has a c.a.i. and $(\mathsf{X}_j, \mathsf{Y}_j)$ is C*-like for each j.

Countable Direct Sums of L^p-modules

We start by discussing a naive attempt of a definition of countable direct sums of L^p-modules that generalizes the finite dimensional case. We then give an example to show why this fails in general. We finish the section with the correct definition and a result that shows that this definition generalizes direct sums of Hilbert modules.

Let $p \in [1, \infty)$. Suppose now that we have measure spaces $(\Omega_j, \mathfrak{M}_j, \mu_j)$ for each $j \in \mathbb{Z}_{\geq 0}$ and that we have countably infinitely many L^p modules $((\mathsf{X}_j, \mathsf{Y}_j))_{j=1}^{\infty}$ over $A \subseteq \mathcal{L}(L^p(\mu_0))$ where, for each each $j \in \mathbb{Z}_{\geq 1}$, the module X_j is a closed subspace of $\mathcal{L}(L^p(\mu_0), L^p(\mu_j))$. An immediate generalization from the finite case will be to consider the pair $(\mathsf{X}_{\mathrm{w}}, \mathsf{Y}_{\mathrm{w}})$ where

$$\mathsf{X}_{\mathrm{w}} = \left\{ (x_j)_{j=1}^{\infty} \colon x_j \in \mathsf{X}_j,\ \sup_{\|\xi\|_p=1} \sum_{j=1}^{\infty} \|x_j\xi\|^p < \infty \right\},$$

$$\mathsf{Y}_{\mathrm{w}} = \left\{ (y_j)_{j=1}^{\infty} \colon y_j \in \mathsf{Y}_j,\ \sup_{\sum_{j=1}^{\infty} \|\eta_j\|_p^p=1} \Big\| \sum_{j=1}^{\infty} y_j\eta_j \Big\|_p < \infty \right\},$$

where the supremum for elements in X_{w} is taken over elements $\xi \in L^p(\mu_0)$ and the one for elements in Y_{w} is taken considering elements $\eta_j \in L^p(\mu_j)$ for each $j \in \mathbb{Z}_{\geq 0}$. If we equip $\bigoplus_{j=1}^{\infty} L^p(\mu_j)$ with the usual p-norm, then X_{w} is a closed subspace of $\mathcal{L}\big(L^p(\mu_0), \bigoplus_{j=1}^{\infty} L^p(\mu_j)\big)$ and Y_{w} is a closed subspace of $\mathcal{L}\big(\bigoplus_{j=1}^{\infty} L^p(\mu_j), L^p(\mu_0)\big)$ (this will follow from Theorem 5.3.3). Furthermore, we can check that $(\mathsf{X}_{\mathrm{w}}, \mathsf{Y}_{\mathrm{w}})$ satisfies conditions (1) and (3) in Definition 5.1.1. However, condition (2) might fail. Indeed, in the following example we will see that, in general, it is not true that requiring $(x_j)_{j=1}^{\infty} \in \mathsf{X}_{\mathrm{w}}$ and $(y_j)_{n=1}^{\infty} \in \mathsf{Y}_{\mathrm{w}}$ implies that

$$(y_j)_{n=1}^{\infty}(x_j)_{j=1}^{\infty} = \sum_{j=1}^{\infty} (y_j \mid x_j)_A$$

converges to an element of A.

Example 5.3.1. Let $p \in [1, \infty)$ and consider $(\ell^q(\mathbb{Z}_{\geq 1}), \ell^p(\mathbb{Z}_{\geq 1}))$, which is a C*-like L^p-module over $\mathcal{K}(\ell^p(\mathbb{Z}_{\geq 1}))$, as shown in Example 5.1.7 (we are able to include $p = 1$ because the dual of $\ell^1(\mathbb{Z}_{\geq 1})$ is $\ell^\infty(\mathbb{Z}_{\geq 1})$). For each $j \in \mathbb{Z}_{\geq 1}$ we let $(\mathsf{X}_j, \mathsf{Y}_j) = (\ell^q(\mathbb{Z}_{\geq 1}), \ell^p(\mathbb{Z}_{\geq 1}))$ and consider X_{w} and Y_{w} as above. For each $j \in \mathbb{Z}_{\geq 1}$ define $x_j \colon \ell^p(\mathbb{Z}_{\geq 1}) \to \ell^p_1$ by $x_j\xi = \xi(j)$ and $y_j \colon \ell^p_1 \to \ell^p(\mathbb{Z}_{\geq 1})$ by $y_j\zeta = \zeta\delta_j$, where $\{\delta_j \colon j \in \mathbb{Z}_{\geq 1}\}$ is the canonical basis of $\ell^p(\mathbb{Z}_{\geq 1})$ (notice that for $p = 2$, y_j is actually

x_j^*). Then $x_j \in \mathsf{X}_j$ and $y_j \in \mathsf{Y}_j$ for each $j \geq 1$. Furthermore,

$$\sup_{\|\xi\|_p=1} \sum_{j=1}^{\infty} |x_j \xi|^p = \sup_{\|\xi\|_p=1} \|\xi\|_p^p = 1,$$

and

$$\sup_{\sum_{j=1}^{\infty} |\zeta_j|_p^p=1} \Big\| \sum_{j=1}^{\infty} y_j \zeta_j \Big\|_p^p = \sup_{\sum_{j=1}^{\infty} |\zeta_j|^p=1} \sum_{j=1}^{\infty} |\zeta_j|^p = 1.$$

Therefore $(x_j)_{j=1}^{\infty} \in \mathsf{X}_{\mathrm{w}}$ and $(y_j)_{j=1}^{\infty} \in \mathsf{Y}_{\mathrm{w}}$. Moreover, for each $j \in \mathbb{Z}_{\geq 1}$ we clearly have $y_j x_j \xi = \xi(j)\delta_j$ and therefore $y_j x_j = \theta_{\delta_j,\delta_j} \in \mathcal{K}(\ell^p(\mathbb{Z}_{\geq 1}))$. However, $\| \sum_{j=n}^{m} \theta_{\delta_j,\delta_j} \| = 1$ for any $m \geq n \geq 1$, and therefore $\sum_{j=1}^{\infty} y_j x_j = \sum_{j=1}^{\infty} \theta_{\delta_j,\delta_j}$ does not converge in $\mathcal{K}(\ell^p(\mathbb{Z}_{\geq 1}))$.

Thus, in general $(\mathsf{X}_{\mathrm{w}}, \mathsf{Y}_{\mathrm{w}})$ is not an L^p-module over A. We actually need to work with subspaces of X_{w} and Y_{w} to make things work. The motivation for the following definition for countable direct sums of L^p-modules will be clear once we introduce the external tensor product in Section 5.4 and prove Proposition 5.4.2.

Definition 5.3.2. Let $p \in [1, \infty)$, for each $j \in \mathbb{Z}_{\geq 0}$ let $(\Omega_j, \mathfrak{M}_j, \mu_j)$ be a measure space, and let $(\mathsf{X}_j, \mathsf{Y}_j)_{j=1}^{\infty}$ be a countable family of L^p-modules over $A \subseteq \mathcal{L}(L^p(\mu_0))$ such that for $j \in \mathbb{Z}_{\geq 1}$, the module X_j is a closed subspace of $\mathcal{L}(L^p(\mu_0), L^p(\mu_j))$. Then we define $\bigoplus_{j=1}^{\infty}(\mathsf{X}_j, \mathsf{Y}_j)$ to be the pair (X, Y) where

$$\mathsf{X} = \left\{ (x_j)_{j=1}^{\infty} \in \mathsf{X}_{\mathrm{w}} \colon \lim_{n,m \to \infty} \sup_{\|\xi\|_p=1} \sum_{j=n}^{m} \|x_j \xi\|_p^p = 0 \right\},$$

$$\mathsf{Y} = \left\{ (y_j)_{j=1}^{\infty} \in \mathsf{Y}_{\mathrm{w}} \colon \lim_{n,m \to \infty} \sup_{\sum_{j=1}^{\infty} \|\eta_j\|_p^p=1} \Big\| \sum_{j=n}^{m} y_j \eta_j \Big\|_p = 0 \right\}.$$

In Theorem 5.3.3 below we show that $\bigoplus_{j=1}^{\infty}(\mathsf{X}_j, \mathsf{Y}_j)$ is indeed an L^p-module over A that agrees with the usual definition of direct sums of Hilbert modules when A is a C*-algebra and X_j is a right Hilbert A module represented on the pair $(\mathcal{H}_0, \mathcal{H}_j)$ for each $j \geq 1$.

Theorem 5.3.3. *Let* $(\mathsf{X}, \mathsf{Y}) = \bigoplus_{j=1}^{\infty} (\mathsf{X}_j, \mathsf{Y}_j)$ *be as in Definition 5.3.2. Then:*

1. (X, Y) *is an* L^p*-module over* A.

2. *Let* $p = 2$*, let* A *be a C*-algebra, and for each* $j \geq 1$ *let* X_j *be a Hilbert* A*-module isometrically represented in* $(\mathcal{H}_0, \mathcal{H}_j)$ *via* $\pi_{\mathsf{X}_j} \colon \mathsf{X}_j \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$*, as in Definition 3.2.5, with* $\mathsf{X}_j \mathcal{H}_0$ *dense in* $\mathcal{H}_j$*. Then*

$$\big((\pi_{\mathsf{X}_j}(x_j))_{j=1}, (\pi_{\mathsf{X}_j}(x_j)^*)_{j=1} \big) \in \bigoplus_{j=1}^{\infty} (\pi_{\mathsf{X}_j}(\mathsf{X}_j), \pi_{\mathsf{X}_j}(\mathsf{X}_j)^*)$$

   when $\sum_{j=1}^{\infty} \langle x_j, x_j \rangle_A$ *converges in* A.

Proof. To prove the first statement, we first check that X is a closed subspace of $\mathcal{L}\big(L^p(\mu_0), \bigoplus_{j=1}^{\infty} L^p(\mu_j)\big)$ and that Y is a closed subspace of $\mathcal{L}\big(\bigoplus_{j=1}^{\infty} L^p(\mu_j), L^p(\mu_0)\big)$. To do so, let $(x^{(n)})_{n=1}^{\infty}$ be a Cauchy sequence in X. Then a direct check shows that for each $j \in \mathbb{Z}_{\geq 1}$, $\|x_j^{(n)} - x_j^{(m)}\| \leq \|x^{(n)} - x^{(m)}\|$ and therefore $(x_j^{(n)})_{n=1}^{\infty}$ is a Cauchy sequence in X_j. Thus, by completeness, we get for each $j \in \mathbb{Z}_{\geq 1}$ an element $x_j \in \mathsf{X}_j$ such that $x_j^{(n)} \to x_j$ as $n \to \infty$. Define $x = (x_j)_{j=1}^{\infty}$. We claim that $(x^{(n)})_{n=1}^{\infty}$ converges to x. Let $\varepsilon > 0$ and choose $N \in \mathbb{Z}_{\geq 1}$ such that $\|x^{(n)} - x^{(m)}\|^p < \varepsilon$ whenever $m \geq n \geq N$. Now take any $\xi \in L^p(\mu_0)$ with $\|\xi\| = 1$, and observe that

$$\sum_{j=1}^{\infty} \|(x_j^{(n)} - x_j^{(m)})\xi\|^p \leq \|x^{(n)} - x^{(m)}\|^p < \varepsilon.$$

Letting $m \to \infty$ on both ends of the previous inequality gives

$$\sum_{j=1}^{\infty} \|(x_j^{(n)} - x_j)\xi\|^p < \varepsilon,$$

and taking supremum over all $\|\xi\| = 1$ yields $\|x^{(n)} - x\| < \varepsilon$ whenever $n \geq N$. Thus, $x^{(n)}$ converges to x. Similarly, if we let $(y^{(n)})_{n=1}^{\infty}$ be a Cauchy sequence in Y, for each j we see that $(y_j^{(n)})_{n=1}^{\infty}$ is a Cauchy sequence in Y_j and therefore we get an element $y_j \in \mathsf{Y}_j$ such that $y_j^{(n)} \to y_j$. A similar argument shows that, if we define $y = (y_j)_{j=1}^{\infty}$, then $y^{(n)}$ converges to y. We still need to check that $x \in \mathsf{X}$ and $y \in \mathsf{Y}$. For any $\xi \in L^p(\mu_0)$ with $\|\xi\| = 1$ and for any $m > n \geq 1$ we repeatedly apply Minkowski's inequality (both for $L^p(\mu_j)$ and for $\mathbb{R}^{m-n}$) to get

$$\begin{aligned}\Big(\sum_{j=n}^{m} \|x_j\xi\|^p\Big)^{1/p} &\leq \Big(\sum_{j=n}^{m}(\|x_j\xi - x_j^{(k)}\xi\|^p\Big)^{1/p} + \Big(\sum_{j=n}^{m} \|x_j^{(k)}\xi\|^p\Big)^{1/p} \\ &\leq \Big(\sum_{j=n}^{m}(\|x_j\xi - x_j^{(k)}\xi\|^p\Big)^{1/p} + \Big(\sum_{j=n}^{m} \|x_j^{(k)}\xi\|^p\Big)^{1/p} \\ &\leq \|x - x^{(k)}\| + \Big(\sum_{j=n}^{m} \|x_j^{(k)}\xi\|^p\Big)^{1/p}.\end{aligned}$$

Then, since $x^{(k)} \in \mathsf{X}$, the previous inequality can be used to show that $x \in \mathsf{X}$, proving closure of X. Similarly, if $(\eta_j)_{j=1}^{\infty}$ is a norm one element of $\bigoplus_{j=1}^{\infty} L^p(\mu_j)$ and $m \geq n \geq 1$, a direct application of Minkowsky's inequality in $L^p(\mu_0)$ gives

$$\Big\|\sum_{j=n}^{m} y_j\eta_j\Big\| \leq \Big\|\sum_{j=n}^{m}(y_j - y_j^{(k)})\eta_j\Big\| + \Big\|\sum_{j=n}^{m} y_j^{(k)}\eta_j\Big\| \leq \|y - y^{(k)}\| + \Big\|\sum_{j=n}^{m} y_j^{(k)}\eta_j\Big\|.$$

Hence, using the previous inequality and the fact that $y^{(k)} \in \mathsf{Y}$, implies that $y \in \mathsf{Y}$, proving that Y is also closed.

It still remains for us to check that conditions (1)-(3) in Definition 5.1.1 are satisfied. Condition (2) is the only one that requires some work. Let $(x_j)_{j=1}^{\infty} \in \mathsf{X}$ and $(y_j)_{j=1}^{\infty} \in \mathsf{Y}$. We claim that $\sum_{j=1}^{\infty}(y_j \mid x_j)_A$ converges in A. If we prove the claim, the element to which this series converges is in fact

$((y_j)_{j=1}^\infty \mid (x_j)_{j=1}^\infty)_A \colon L^p(\mu_0) \to L^p(\mu_0)$, and condition (2) will follow. Let $K = \sup_{\sum_{j=1}^\infty \|\eta_j\|^p=1} \| \sum_{j=1}^\infty y_j\eta_j\|$ and for each $m \geq n \geq 1$ let $M_{n,m}(\xi) = \sum_{j=n}^m \|x_j\xi\|^p$. Then $K < \infty$ and $\lim_{m,n\to\infty} \sup_{\|\xi\|=1} M_{n,m}(\xi) = 0$. Now for any $\xi \in L^p(\mu_0)$ with $\|\xi\| = 1$, we find

$$\Big\| \sum_{j=n}^m y_j x_j \xi \Big\| \leq K M_{m,n}(\xi).$$

Hence,

$$\Big\| \sum_{j=n}^m y_j x_j \Big\| \leq K \sup_{\|\xi\|=1} M_{m,n}(\xi),$$

from which it follows that $(\sum_{j=1}^n y_j x_j)_{n=1}^\infty$ is a Cauchy sequence in A and therefore convergent, proving our claim.

For the second statement, we identify A with its isometric copy in $\mathcal{L}(\mathcal{H}_0)$ and similarly for each $j \in \mathbb{Z}_{\geq 1}$ we identify X_j with its isometric copy in $\mathcal{L}(\mathcal{H}_0, \mathcal{H}_j)$ so that $\mathsf{X}_j^* \subseteq \mathcal{L}(\mathcal{H}_j, \mathcal{H}_0)$. We have to show that convergence of $\sum_{j=1}^\infty x_j^* x_j$ in A implies the following two conditions

(a) $\displaystyle \sup_{\|\xi\|_2=1} \sum_{j=1}^\infty \|x_j\xi\|_2^2 < \infty$ and $\displaystyle \lim_{n,m\to\infty} \sup_{\|\xi\|_2=1} \sum_{j=n}^m \|x_j\xi\|_2^2 = 0$,

(b) $\displaystyle \sup_{\sum_{j=1}^\infty \|\eta_j\|_p^p=1} \Big\| \sum_{j=1}^\infty x_j^*\eta_j \Big\|_2 < \infty$ and $\displaystyle \lim_{n,m\to\infty} \sup_{\sum_{j=1}^\infty \|\eta_j\|_p^p=1} \Big\| \sum_{j=n}^m x_j^*\eta_j \Big\|_2 = 0.$

To check condition (a), let $\xi \in L^p(\mu_0)$ have norm 1 and let $m \geq n \geq 1$. Then,

$$\sum_{j=n}^m \|x_j\xi\|_2^2 = \sum_{j=n}^m \langle \xi, x_j^* x_j \xi\rangle = \Big\langle \xi, \sum_{j=n}^m x_j^* x_j \xi \Big\rangle \leq \Big\| \sum_{j=n}^m x_j^* x_j \Big\|,$$

and also

$$\sum_{j=1}^\infty \|x_j\xi\|_2^2 \leq \Big\| \sum_{j=1}^\infty x_j^* x_j \Big\|.$$

Hence, convergence of $\sum_{j=1}^\infty x_j^* x_j$ in A does imply condition (a). For condition (b), let $(\eta_j)_{j=1}^\infty$ be a norm 1 element of $\bigoplus_{j=1}^\infty \mathcal{H}_j$. In addition, for fixed $m \geq n \geq 1$, define $\boldsymbol{\eta} = (\eta_n, \ldots, \eta_m) \in \bigoplus_{j=n}^{m} \mathcal{H}_j$. Observe that $\|\boldsymbol{\eta}\| \leq \|(\eta_j)_{j=1}^{\infty}\| = 1$. Then

$$\begin{aligned} \Big\| \sum_{j=n}^{m} x_j^* \eta_j \Big\|^2 &= \Big\langle \sum_{j=n}^{m} x_j^* \eta_j, \sum_{k=n}^{m} x_k^* \eta_k \Big\rangle \\ &= \sum_{j=n}^{m} \sum_{k=n}^{m} \langle \eta_j, x_j x_k^* \eta_k \rangle \\ &= \langle \boldsymbol{\eta}, (x_j x_k^*)_{j,k=n}^{m} \boldsymbol{\eta} \rangle \\ &\leq \|(x_j x_k^*)_{j,k=n}^{m}\|. \end{aligned}$$

Both statements in condition (b) now follow at once from the convergence of $\sum_{j=1}^{\infty} x_j^* x_j$ and Lemma 3.1.6 which guarantees $\|(x_j x_k^*)_{j,k=n}^{m}\| = \|\sum_{j=n}^{m} x_j^* x_j\|$. ■

External Tensor Product of L^p-modules

We now present an analogue of the external tensor product for Hilbert modules. This generalizes the construction from Example 5.1.9. Moreover, Proposition 5.4.2 below was in fact the main motivation for the definition for countable direct sums presented above (see Definition 5.3.2). The external tensor product will also be used later in Definition 6.3.30, although it is then shown in Remark 6.3.31 that we could have simply used the correspondence version from Definition 6.2.1.

Definition 5.4.1. For $j = 0, 1$, let $(\Omega_j, \mathfrak{M}_j, \mu_j)$ and $(\Lambda_j, \mathfrak{N}_j, \nu_j)$ be measures spaces, let $p \in (1, \infty)$, let (X, Y) be an L^p-module over an L^p operator algebra $A \subseteq \mathcal{L}(L^p(\mu_0))$ with $\mathsf{X} \subseteq \mathcal{L}(L^p(\mu_0), L^p(\mu_1))$, and let (V, W) be an L^p-module over an L^p operator algebra $B \subseteq \mathcal{L}(L^p(\nu_0))$ with $\mathsf{V} \subseteq \mathcal{L}(L^p(\mu_0), L^p(\mu_1))$. Using the spatial tensor product for operators acting on L^p-spaces, we define the *external* *tensor product of* (X, Y) *with* (V, W) by letting

$$(\mathsf{X}, \mathsf{Y}) \otimes_p (\mathsf{V}, \mathsf{W}) = (\mathsf{X} \otimes_p \mathsf{V}, \mathsf{V} \otimes_p \mathsf{W}).$$

It is routine to check that all the conditions in Definition 5.1.1 needed to make $(\mathsf{X} \otimes_p \mathsf{V}, \mathsf{V} \otimes_p \mathsf{W})$ an L^p-module over $A \otimes_p B$ are met.

Proposition 5.4.2. *Let* $p \in (1, \infty)$ *and let* (X, Y) *be an* L^p*-module over* $A \subseteq \mathcal{L}(L^p(\mu_0))$ *with* $\mathsf{X} \subseteq \mathcal{L}(L^p(\mu_0), L^p(\mu_1))$*. Then*

$$(\ell^p(\mathbb{Z}_{\geq 1}), \ell^q(\mathbb{Z}_{\geq 1})) \otimes_p (\mathsf{X}, \mathsf{Y}) = \bigoplus_{j=1}^{\infty} (\mathsf{X}, \mathsf{Y}).$$

Proof. Recall that $\bigoplus_{j=1}^{\infty} (\mathsf{X}, \mathsf{Y}) = (\mathsf{Z}_{\mathsf{X}}, \mathsf{Z}_{\mathsf{Y}})$ where

$$\mathsf{Z}_{\mathsf{X}} = \left\{ (x_j)_{j=1}^{\infty} \colon x_j \in \mathsf{X}, \lim_{n,m \to \infty} \sup_{\|\xi\|_p = 1} \sum_{j=n}^{m} \|x_j \xi\|_p^p = 0 \right\} \subseteq \mathcal{L}\Big(L^p(\mu_0), \bigoplus_{j=1}^{\infty} L^p(\mu_1) \Big),$$

and

$$\mathsf{Z}_{\mathsf{Y}} = \left\{ (y_j)_{j=1}^{\infty} \colon y_j \in \mathsf{Y}, \lim_{n,m \to \infty} \sup_{\sum_{j=1}^{\infty} \|\eta_j\|_p^p = 1} \Big\| \sum_{j=n}^{m} y_j \eta_j \Big\| = 0 \right\} \subseteq \mathcal{L}\Big(\bigoplus_{j=1}^{\infty} L^p(\mu_1), L^p(\mu_0) \Big).$$

Let ι_{X} and ι_{Y} be the following natural inclusions:

$$\iota_{\mathsf{X}} \colon \ell^p(\mathbb{Z}_{\geq 1}) \otimes_p \mathsf{X} \to \mathcal{L}\Big(L^p(\mu_0), \bigoplus_{j=1}^{\infty} L^p(\mu_1) \Big),$$

and

$$\iota_{\mathsf{Y}} \colon \ell^q(\mathbb{Z}_{\geq 1}) \otimes_p \mathsf{Y} \to \mathcal{L}\Big(\bigoplus_{j=1}^{\infty} L^p(\mu_1), L^p(\mu_0) \Big).$$

It suffices to show that the image of ι_{X} is Z_{X} and that the image of ι_{Y} is Z_{Y}. For any $\zeta \in \ell^p(\mathbb{Z}_{\geq 1})$, any $x \in \mathsf{X}$, and any $\xi \in L^p(\mu_0)$ we have $\iota_{\mathsf{X}}(\zeta \otimes x)\xi = (\zeta(j)x\xi)_{j=1}^{\infty} \in \bigoplus_{j=1}^{\infty} L^p(\mu_1)$. Furthermore,

$$\lim_{m,n\to\infty} \sup_{\|\xi\|=1} \sum_{j=n}^{m} \|\zeta(j)x\xi\|^p = \|x\| \lim_{m,n\to\infty} \sum_{j=n}^{m} |\zeta(j)|^p = 0.$$

From this it is clear that $\iota_{\mathsf{X}}(\xi \otimes x) \in \mathsf{Z}_{\mathsf{X}}$. Since Z_{X} is closed in $\mathcal{L}\big(L^p(\mu_0), \bigoplus_{j=1}^{\infty} L^p(\mu_1)\big)$ (see Theorem 5.3.3), we conclude that $\iota_{\mathsf{X}}(\ell^p(\mathbb{Z}_{\geq 1}) \otimes_p \mathsf{X}) \subseteq \mathsf{Z}_{\mathsf{X}}$. For the reverse inclusion, suppose that $(x_j)_{j=1}^{\infty}$ is in Z_{X}. We claim that $\sum_{j=1}^{\infty} \delta_j \otimes x_j$ is an element of $\ell^p(\mathbb{Z}_{\geq 1}) \otimes \mathsf{X}$. Indeed, for any $m \geq n \geq 1$ we have

$$\Big\| \sum_{j=n}^{m} \delta_j \otimes x_j \Big\|^p = \sup_{\|\xi\|=1} \sum_{k=1}^{\infty} \int_{\Omega_1} \Big| \sum_{j=n}^{m} \delta_j(k)(x_j\xi)(\omega) \Big|^p d\mu_1(\omega) = \sup_{\|\xi\|=1} \sum_{j=n}^{m} \|x_j\xi\|^p.$$

After taking the limit as $m, n \to \infty$ and using the fact that $(x_j)_{j=1}^{\infty}$ is in Z_{X}, we see that $(\sum_{j=1}^{n} \delta_j \otimes x_j)_{n=1}^{\infty}$ is a Cauchy sequence in $\ell^p(\mathbb{Z}_{\geq 1}) \otimes \mathsf{X}$, so our claim follows. It is immediate to check that $\iota_{\mathsf{X}}(\sum_{j=1}^{\infty} \delta_j \otimes x_j) = (x_j)_{j=1}^{\infty}$, and therefore we have shown that $\iota_{\mathsf{X}}(\ell^p(\mathbb{Z}_{\geq 1}) \otimes_p \mathsf{X}) = \mathsf{Z}_{\mathsf{X}}$ as wanted. Similarly, notice that for any $\upsilon \in \ell^q(\mathbb{Z}_{\geq 1})$, $y \in \mathsf{Y}$, and $(\eta_j)_{j=1}^{\infty} \in \bigoplus_{j=1}^{\infty} L^p(\mu_1)$ we have

$$\iota_{\mathsf{Y}}(\upsilon \otimes y)(\eta_j)_{j=1}^{\infty} = \sum_{j=1}^{\infty} \upsilon(j) y \eta_j.$$

Hence, using the finite dimensional version of Hölder's inequality we see that for $m \geq n \geq 1$,

$$\sup_{\sum_{j=1}^{\infty} \|\eta_j\|_p^p = 1} \Big\| \sum_{j=n}^{m} \upsilon(j) y \eta_j \Big\| \leq \|y\| \Big(\sum_{j=n}^{m} |\upsilon(j)|^q \Big)^{1/q}.$$

Thus, taking limit when $m, n \to \infty$ shows that $\iota_Y(v \otimes y) \in \mathsf{Z}_{\mathsf{Y}}$. Since Z_{Y} is closed, this is enough to show that $\iota_Y(\ell^q(\mathbb{Z}_{\geq 1}) \otimes_p \mathsf{Y}) \subseteq \mathsf{Z}_{\mathsf{Y}}$. For the reverse inclusion, once again it suffices to show that $\sum_{j=1}^{\infty} \delta_j \otimes y_j$ defines an element in $\ell^q(\mathbb{Z}_{\geq 1}) \otimes_p \mathsf{Y}$ when $(y_j)_{j=1}^{\infty} \in \mathsf{Z}_{\mathsf{Y}}$. Let $m \geq n \geq 1$ and notice that

$$\Big\| \sum_{j=n}^{m} \delta_j \otimes y_j \Big\| = \sup_{\sum_{k=1}^{\infty} \|\eta_k\|_p^p = 1} \Big\| \sum_{k=1}^{\infty} \sum_{j=n}^{m} \delta_j(k) y_j \eta_k \Big\| = \sup_{\sum_{k=1}^{\infty} \|\eta_k\|_p^p = 1} \Big\| \sum_{k=n}^{m} y_k \eta_k \Big\|.$$

Thus, letting $m, n \to \infty$ shows that $(\sum_{j=1}^{n} \delta_j \otimes y_j)_{n=1}^{\infty}$ is Cauchy in $\ell^q(\mathbb{Z}_{\geq 1}) \otimes_p \mathsf{Y}$ and we are done. ■

Morphisms of L^p-modules

Let $(\Omega_0, \mathfrak{M}_0, \mu_0)$ and $(\Omega_1, \mathfrak{M}_1, \mu_1)$ be measures spaces, let $p \in [1, \infty)$, and let (X, Y) be an L^p-module over an L^p operator algebra $A \subseteq \mathcal{L}(L^p(\mu_0))$, as in Definition 5.1.1. Motivated by Proposition 3.1.5, we set

$$\mathcal{L}_A((\mathsf{X}, \mathsf{Y})) = \{t \in \mathcal{L}(L^p(\mu_1)) \colon tx \in \mathsf{X} \text{ and } yt \in \mathsf{Y} \text{ for all } x \in \mathsf{X}, y \in \mathsf{Y}\}. \tag{5.5.1}$$

It is clear that $\mathcal{L}_A((\mathsf{X}, \mathsf{Y}))$ is an L^p operator algebra. For each $x \in \mathsf{X}$ and $y \in \mathsf{Y}$ the composition $xy \in \mathcal{L}(L^p(\mu_1))$ satisfies $(xy)z = x(y \mid z)_A \in X$ for all $z \in \mathsf{X}$ and also $w(xy) = (w \mid x)_A y \in \mathsf{Y}$ for all $w \in \mathsf{Y}$. Therefore, $xy \in \mathcal{L}_A((\mathsf{X}, \mathsf{Y}))$ for any $x \in \mathsf{X}$ and $y \in \mathsf{Y}$. We will sometimes denote the operator xy by $\theta_{x,y} \in \mathcal{L}(L^p(\mu_1))$ and think of it as a "rank one" operator on X:

$$\theta_{x,y} z = (xy)z = x(y \mid z)_A \in \mathsf{X}$$

for all $z \in \mathsf{X}$. Moreover, these operators are also module maps, that is, $\theta_{x,y}(za) = \theta_{x,y}(z)a$ for $x, z \in \mathsf{X}$, $y \in \mathsf{Y}$, and $a \in A$. Since $\mathrm{span}\{\theta_{x,y} \colon x \in \mathsf{X} \text{ and } y \in \mathsf{Y}\}$ is closed under multiplication (because $\theta_{x_1,y_1}\theta_{x_2,y_2} = \theta_{x_1(y_1x_2),y_2}$), motivated by Proposition 3.1.4, we define the Banach algebra

$$\mathcal{K}_A((\mathsf{X},\mathsf{Y})) = \overline{\mathrm{span}\{\theta_{x,y} \colon x \in \mathsf{X} \text{ and } y \in \mathsf{Y}\}} \subseteq \mathcal{L}(L^p(\mu_1)). \tag{5.5.2}$$

Thus, $\mathcal{K}_A((\mathsf{X},\mathsf{Y}))$ is naturally an L^p-operator algebra and by definition $\mathcal{K}_A((\mathsf{X},\mathsf{Y})) \subseteq \mathcal{L}_A((\mathsf{X},\mathsf{Y}))$.

Proposition 5.5.1. $\mathcal{K}_A((\mathsf{X},\mathsf{Y}))$ *is a closed two sided ideal in* $\mathcal{L}_A((\mathsf{X},\mathsf{Y}))$.

Proof. By construction, $\mathcal{K}_A((\mathsf{X},\mathsf{Y}))$ is a closed subset of $\mathcal{L}_A((\mathsf{X},\mathsf{Y}))$. Let $x \in \mathsf{X}$, $y \in \mathsf{Y}$, and $t \in \mathcal{L}_A((\mathsf{X},\mathsf{Y}))$. Then it follows at once that $\theta_{x,y}t = \theta_{x,yt} \in \mathcal{K}_A((\mathsf{X},\mathsf{Y}))$ and $t\theta_{x,y} = \theta_{tx,y} \in \mathcal{K}_A((\mathsf{X},\mathsf{Y}))$. ■

Below we will compute $\mathcal{L}_A((\mathsf{X},\mathsf{Y}))$ and $\mathcal{K}_A((\mathsf{X},\mathsf{Y}))$ for some of our known examples.

Example 5.5.2. Let A be an L^p-operator algebra and let (A, A) be the L^p-module over A from Example 5.1.4. Furthermore, suppose that A has a c.a.i.. Then the Cohen-Hewitt factorization theorem (in fact, we only need Theorem 1 in [4]) implies at once that $\mathcal{K}_A((A,A))$ coincides isometrically with A via the map $\theta_{a,b} \mapsto ab$. If, in addition, we require that A sits nondegenerately in $\mathcal{L}(L^p(\mu))$ (i.e., $AL^p(\mu)$ is a dense subset of $L^p(\mu)$), then there is an isometric isomorphism between $\mathcal{L}_A((A,A))$ and $M(A)$ from Definition 4.1.1. Indeed, notice first that equation (5.5.1) becomes

$$\mathcal{L}_A((A,A)) = \{t \in \mathcal{L}(L^p(\mu)) \colon ta \in A, at \in A \text{ for all } a \in A\}.$$

Hence, Theorem 4.1.6 shows that $M(A)$ and $\mathcal{L}_A((A,A))$ are isometrically isomorphic.

The previous example also gives an answer to when the multiplier algebra of an L^p-operator algebra is also an L^p-operator algebra.

Corollary 5.5.3. *Let A be an L^p-operator algebra with a c.a.i. that is nondegenerately represented on $L^p(\mu)$. Then $M(A)$ is an L^p-operator algebra.*

Proof. Identify A with its isometric copy in $\mathcal{L}(L^p(\mu))$. The desired result follows at once from Example 5.1.9 in which we saw that $M(A)$ is isometrically isomorphic to $\mathcal{L}_A((A,A)) \subseteq \mathcal{L}(L^p(\mu))$, which is an L^p-operator algebra. ■

Example 5.5.4. Let $(\Omega, \mathfrak{M}, \mu)$ be a measure space, let $p \in (1,\infty)$, and let $(L^p(\mu), L^q(\mu))$ be the C*-like L^p-module over $A = \mathcal{L}(\ell_1^p)$ presented in Example 5.1.5. Then $\mathcal{L}_A((L^p(\mu), L^q(\mu))) = \mathcal{L}(L^p(\mu))$. Indeed, in this case equation (5.5.1) implies that $\mathcal{L}_{\mathbb{C}}((L^p(\mu), L^q(\mu)))$ is defined by

$$\{t \in \mathcal{L}(L^p(\mu)) \colon t\xi \in L^p(\mu) \text{ for all } \xi \in L^p(\mu) \ \text{ and } \ \eta t \in L^q(\mu) \text{ for all } \eta \in L^q(\mu)\}.$$

For any $t \in \mathcal{L}(L^p(\mu))$, there is $t' \in \mathcal{L}(L^q(\mu))$ given by $t'(\eta)(\xi) = \langle \eta, t(\xi)\rangle$. Thus, it is clear that if $\xi \in L^p(\mu)$ and $\eta \in L^q(\mu)$, then $t\xi = t(\xi) \in L^p(\mu)$ and $\eta t = t'(\eta) \in L^q(\mu)$. This proves that $\mathcal{L}_{\mathbb{C}}((L^p(\mu), L^q(\mu))) = \mathcal{L}(L^p(\mu))$. We now claim that $\mathcal{K}_{\mathbb{C}}((L^p(\mu), L^q(\mu))) = \mathcal{K}(L^p(\mu))$. Indeed, since $L^p(\mu)$ has the the approximation property (see Example 4.5 in [26]), then $\mathcal{K}(L^p(\mu))$ is the closure of the finite rank operators. Any rank one operator on $L^p(\mu)$ is given by a pair $(\xi, \eta) \in L^p(\mu) \times L^q(\mu)$ via $\xi_0 \mapsto \xi\langle \eta, \xi_0\rangle = \theta_{\xi,\eta}\xi_0$. Thus,

$$\mathcal{K}_{\mathbb{C}}((L^p(\mu), L^q(\mu))) = \overline{\operatorname{span}\{\theta_{\xi,\eta} \colon \xi \in L^p(\mu), \eta \in L^q(\mu)\}} = \mathcal{K}(L^p(\mu)),$$

as wanted.

The symmetry between Example 5.1.5 and Example 5.1.7 is actually a particular case of the following result.

Proposition 5.5.5. *Let $p \in [1, \infty)$, let $A \subseteq \mathcal{L}(L^p(\mu_0))$ be an L^p-operator algebra, and let (X, Y) be an L^p-module over A with $\mathsf{X} \subseteq \mathcal{L}(L^p(\mu_0), L^p(\mu_1))$. Then (Y, X) is an L^p-module over $\mathcal{K}_A((\mathsf{X}, \mathsf{Y})) \subseteq \mathcal{L}(L^p(\mu_1))$.*

Proof. We only need to verify conditions (1)-(3) in Definition 5.1.1. For any $t \in \mathcal{K}_A((\mathsf{X}, \mathsf{Y}))$ we have $t \in \mathcal{L}_A((\mathsf{X}, \mathsf{Y}))$ and therefore $yt \in \mathsf{Y}$ for any $y \in \mathsf{Y}$, and $tx \in \mathsf{X}$ for any $x \in \mathsf{X}$. This proves both condition (1) and (3). Finally, since $xy = \theta_{x,y} \in \mathcal{K}_A((\mathsf{X}, \mathsf{Y}))$, condition (2) holds and we are done. ∎

CHAPTER VI

L^P CORRESPONDENCES AND THEIR L^P OPERATOR ALGEBRAS

In this chapter we define the extra structure needed on L^p-modules to obtain L^p-correspondences. We then present an interior tensor product construction for these correspondences and the remainder of the chapter is devoted to the L^p-version of Fock representation and the algebras they generate. This should be compared with section 2.4 in Chapter II.

L^p-correspondences

Having defined the morphisms for an L^p module in the previous chapter, we are now ready to give a definition for correspondences over L^p motivated by the framework of representations of C*-correspondences on pairs of Hilbert spaces.

Definition 6.1.1. Let $(\Omega_0, \mathfrak{M}_0, \mu_0), (\Omega_1, \mathfrak{M}_1, \mu_1)$ be measure spaces, let $p \in [1, \infty)$, let A be an L^p-operator algebra, and let $B \subseteq \mathcal{L}(L^p(\mu_0))$ be a concrete L^p operator algebra. An (A, B) L^p*-correspondence* is a pair $((\mathsf{X}, \mathsf{Y}), \varphi)$ where (X, Y) is an L^p module over B with $\mathsf{X} \subseteq \mathcal{L}(L^p(\mu_0), L^p(\mu_1))$ and $\varphi\colon A \to \mathcal{L}_B((\mathsf{X}, \mathsf{Y}))$ is a contractive homomorphism. When $A = B$ we say that $((\mathsf{X}, \mathsf{Y}), \varphi)$ is an L^p correspondence over A.

We now look back at our examples of L^p-modules and make them into L^p correspondences.

Example 6.1.2. Let (A, A) be the L^p-module from Example 5.1.4. Let φ_A be a contractive automorphism of A. Notice that for any $a, b \in A$, $\varphi_A(a)b \in A$ and

$b\varphi_A(a) \in A$. Therefore, $\varphi_A(a) \in \mathcal{L}_A((A, A))$ for all $a \in A$. Thus, $((A, A), \varphi_A)$ can be regarded as an L^p-correspondence over A.

Example 6.1.3. Let $(\Omega, \mathfrak{M}, \mu)$ be a measure space, let $p \in (1, \infty)$ with Hölder conjugate q, and let $(L^p(\mu), L^q(\mu))$ be the C*-like L^p-module from Example 5.1.5. For each $z \in \mathbb{C}$, define $\varphi_{\mathbb{C}} \colon L^p(\mu) \to L^p(\mu)$ by $\varphi_{\mathbb{C}}(z) = z \cdot \mathrm{id}_{L^p(\mu)}$. Then it is clear that $\varphi_{\mathbb{C}}(z)\xi \in \mathcal{L}(\ell_1^p, L^p(\mu))$ and $\eta\varphi_{\mathbb{C}}(z) \in \mathcal{L}(L^p(\mu), \ell_1^p)$ for all $z \in \mathbb{C}$, $\xi \in \mathcal{L}(\ell_1^p, L^p(\mu))$, and $\eta \in \mathcal{L}(L^p(\mu), \ell_1^p)$. Hence, $\varphi_{\mathbb{C}}(z) \in \mathcal{L}_{\mathbb{C}}((L^p(\mu), L^q(\mu)))$. Finally, since $\|\varphi_{\mathbb{C}}(z)\| = |z|$, it follows that $((L^p(\mu), L^q(\mu)), \varphi_{\mathbb{C}})$ is an L^p-correspondence over $\mathbb{C}$.

Example 6.1.4. Let $p \in (1, \infty)$, let let $p \in (1, \infty)$ with Hölder conjugate q, and (ℓ_d^p, ℓ_d^q) be the C*-like L^p-module from Example 5.1.6. For each $z \in \mathbb{C}$ let $\varphi_d(z) : \ell_d^p \to \ell_d^p$ be given by

$$\varphi_d(z)(\zeta(1), \ldots, \zeta(d)) = (z\zeta(1), \ldots, z\zeta(d))$$

Then this is a particular example of Example 6.1.3, so it follows that $((\ell_d^p, \ell_d^q), \varphi_d)$ is an L^p-correspondence over $\mathbb{C}$.

Example 6.1.5. Let $(\ell_d^p \otimes_p A, \ell_d^q \otimes_p A)$ be the L^p-module from Example 5.1.9. For each $a \in A$ let $\varphi(a) : L^p(\Omega, \mu)^d \to L^p(\Omega, \mu)^d$ be given by

$$\varphi(a)(\xi_1, \ldots, \xi_d) = (a\xi_1, \ldots, a\xi_d)$$

Then it is clear that $\varphi(a)x \in \ell_d^p \otimes_p A$ and $y\varphi(a) \in \ell_d^q \otimes_p A$ for all $x \in \ell_d^p \otimes_p A$ and $y \in \ell_d^q \otimes_p A$. Since $\|\varphi(a)\| \leq \|a\|$, it follows that $((\ell_d^p \otimes_p A, \ell_d^q \otimes_p A), \varphi)$ is an L^p-correspondence over A.

Tensor Product of L^p-correspondences

We define an interior tensor product motivated by Proposition 3.3.12 for the C* case. Before giving our definition for the L^p-case, we briefly recall the setting for the C*-case guaranteed by Proposition 3.3.12. If $(\mathsf{X}, \varphi_{\mathsf{X}})$ is an (A, B) C*-correspondence represented by $(\pi_A, \pi_B, \pi_{\mathsf{X}})$ on $(\mathcal{H}_1, \mathcal{H}_2)$, and $(\mathsf{Y}, \varphi_{\mathsf{Y}})$ is a (B, C) C*-correspondence represented by $(\pi_B, \pi_C, \pi_{\mathsf{Y}})$ on $(\mathcal{H}_0, \mathcal{H}_1)$. Then given some nondegeneracy conditions, $(\mathsf{X} \otimes_{\varphi_{\mathsf{Y}}} \mathsf{Y}, \widetilde{\varphi_{\mathsf{X}}})$ can be represented on $(\mathcal{H}_0, \mathcal{H}_2)$ via $x \otimes y \mapsto \pi_X(x)\pi_{\mathsf{Y}}(y)$. Furthermore, in this scenario, if κ_C is the isomorphism from $\mathcal{L}_{\pi_C(Y)}(\pi_{\mathsf{Y}}(\mathsf{Y}))$ to $\mathcal{L}_C(\mathsf{Y})$ given by part 2 in Proposition 3.2.8, then it is not hard to check that

$$\varphi_{\mathsf{Y}}(\langle x_1, x_2\rangle_B) = \kappa_C(\pi_X(x_1)^*\pi_X(x_2)).$$

This essentially means that, at least for the concrete version, the left action φ_{Y} acts as the identity on $\langle \mathsf{X}, \mathsf{X}\rangle_B$. Translating all this to the L^p-case gives rise to the following definition.

Definition 6.2.1. Let $p \in [1, \infty)$ and for each $j = 0, 1, 2$ let $(\Omega_j, \mathfrak{M}_j, \mu_j)$ be a measure space. Set $E_j = L^p(\mu_j)$ for $j = 0, 1, 2$ and let A be an L^p-operator algebra, and let $B \subseteq \mathcal{L}(E_1)$ and $C \subseteq \mathcal{L}(E_0)$ be concrete L^p-operator algebras. Suppose $((\mathsf{X}, \mathsf{Y}), \varphi)$ is an (A, B) L^p-correspondence with $\mathsf{X} \subseteq \mathcal{L}(E_1, E_2)$ and $\mathsf{Y} \subseteq \mathcal{L}(E_2, E_1)$. Suppose also that $((\mathsf{V}, \mathsf{W}), \rho)$ is a (B, C) L^p-correspondence with $\mathsf{V} \subseteq \mathcal{L}(E_0, E_1)$, $\mathsf{W} \subseteq \mathcal{L}(E_1, E_0)$, and such that $\rho((y \mid x)_B) = yx$ for all $x \in \mathsf{X}$ and $y \in \mathsf{Y}$. Then we define an (A, C)-L^p correspondence

$$((\mathsf{X}, \mathsf{Y}), \varphi) \otimes_\rho ((\mathsf{V}, \mathsf{W}), \rho) = ((\mathsf{X} \otimes_B \mathsf{V}, \mathsf{Y} \otimes_B \mathsf{W}), \widetilde{\varphi})$$

by letting $\mathsf{X} \otimes_B \mathsf{V} = \overline{\mathsf{XV}} \subseteq \mathcal{L}(E_0, E_2)$, $\mathsf{Y} \otimes_B \mathsf{W} = \overline{\mathsf{WY}} \subseteq \mathcal{L}(E_2, E_0)$, and $\widetilde{\varphi} : A \to \mathcal{L}_C((\mathsf{X} \otimes_B \mathsf{V}, \mathsf{Y} \otimes_B \mathsf{W}))$ be determined by

$$\widetilde{\varphi}(a)\xi = \varphi(a)\xi,$$

for any $\xi \in E_2$.

We now check that the objects defined in Definition 6.2.1 form indeed an (A, C) L^p-correspondence. We first check that $(\mathsf{X} \otimes_B \mathsf{V}, \mathsf{Y} \otimes_B \mathsf{W})$ is indeed an L^p-module over C. By Definition $\overline{\mathsf{XV}}$ and $\overline{\mathsf{WY}}$ are closed subspaces of bounded operators of $\mathcal{L}(E_2, E_0)$ and $\mathcal{L}(E_0, E_2)$. We now check all the conditions in Definition 5.1.1. Let $x \in \mathsf{X}$, $v \in \mathsf{V}$ and $c \in C$. Then we know that $vc \in \mathsf{V}$ and therefore $x(vc) \in \mathsf{XV}$. This is enough to see that $\overline{\mathsf{XV}}C \subseteq \overline{\mathsf{XV}}$, giving condition 1. For condition 2, take $x \in \mathsf{X}$, $v \in \mathsf{V}$, $y \in \mathsf{Y}$ and $w \in \mathsf{W}$. Then since $yx \in B$ satisfies $\rho((y \mid x)_B) = yx$, it follows that

$$(wy \mid xv)_C = (wy)(xv) = w\rho((y \mid x)_B)v \in \mathsf{WV} \subseteq C,$$

because $\rho(b)v \in \mathsf{V}$ for any $b \in B$. Finally, if $c \in C$, $y \in \mathsf{Y}$ and $w \in \mathsf{W}$ we get $c(wy) = (cw)y \in \mathsf{WY}$, from where condition 3 follows. We still need to check that $\widetilde{\varphi}(a) \in \mathcal{L}_C((\mathsf{X} \otimes_B \mathsf{V}, \mathsf{Y} \otimes_B \mathsf{W}))$ for any $a \in A$. Indeed, it is clear that for any $x \in \mathsf{X}$ and $v \in \mathsf{V}$

$$\widetilde{\varphi}(a)xv = (\varphi(a)x)v \in \mathsf{XV},$$

and also that for each $y \in \mathsf{Y}$ and $w \in \mathsf{W}$

$$(wy)\widetilde{\varphi}(a) = w(y\varphi(a)) \in \mathsf{WY}.$$

Finally, since $\|\widetilde{\varphi}(a)\| = \|\varphi(a)\|$, it now follows that $\widetilde{\varphi}(a) \in \mathcal{L}_C((\mathsf{X} \otimes_B \mathsf{V}, \mathsf{Y} \otimes_B \mathsf{W}))$. Therefore, the ingredients in Definition 6.2.1 do give rise to an (A, C) L^p-correspondence.

L^p-Fock representations

The main purpose of this section is to investigate the analogue of Fock representations (see Definition 2.4.1) for L^p-correspondences. This in turn will give rise to potential L^p version of the Toeplitz and Cuntz-Pimsner algebras.

Definition 6.3.1. Let $p \in [1, \infty)$, let A be an L^p-operator algebra, and let $((\mathsf{X}, \mathsf{Y}), \varphi)$ an L^p-correspondence over A. An *L^p-Fock representation* of $((\mathsf{X}, \mathsf{Y}), \varphi)$ consist of $(B, \pi_A, \pi_{\mathsf{X}}, \pi_{\mathsf{Y}})$ where B is an L^p-operator algebra, $\pi_A \colon A \to B$ a contractive homomorphism, and both $\pi_{\mathsf{X}} \colon \mathsf{X} \to B$ and $\pi_{\mathsf{Y}} \colon \mathsf{Y} \to B$ are contractive linear maps satisfying the following conditions:

1. $\pi_{\mathsf{X}}(xa) = \pi_{\mathsf{X}}(x)\pi_A(a)$, and $\pi_{\mathsf{X}}(\varphi(a)x) = \pi_A(a)\pi_{\mathsf{X}}(x)$, for all $x \in \mathsf{X}$, $a \in A$,

2. $\pi_Y(ay) = \pi_A(a)\pi_{\mathsf{Y}}(y)$, and $\pi_Y(y\varphi(a)) = \pi_{\mathsf{Y}}(y)\pi_A(a)$, for all $y \in \mathsf{Y}$, $a \in A$,

3. $\pi_A((y \mid x)_A) = \pi_{\mathsf{Y}}(y)\pi_{\mathsf{X}}(x)$, for all $x \in \mathsf{X}$, $y \in \mathsf{Y}$.

We denote by $F^p(B, \pi_A, \pi_{\mathsf{X}}, \pi_{\mathsf{Y}})$ to the closed subalgebra in B generated by $\pi_A(A)$, $\pi_{\mathsf{X}}(\mathsf{X})$ and $\pi_{\mathsf{Y}}(\mathsf{Y})$.

Remark 6.3.2. Observe that by construction, $F^p(B, \pi_A, \pi_{\mathsf{X}}, \pi_{\mathsf{Y}})$ is an L^p operator algebra. Furthermore, if $(\mathsf{Y} \mid \mathsf{X})_A = \operatorname{span}\{(y \mid x)_A \colon x \in \mathsf{X}, y \in \mathsf{Y}\}$ is a dense subset of A, then condition 3 in Definition 6.3.1 implies that $F^p(B, \pi_A, \pi_{\mathsf{X}}, \pi_{\mathsf{Y}})$ is simply generated by $\pi_{\mathsf{X}}(\mathsf{X})$ and $\pi_{\mathsf{Y}}(\mathsf{Y})$.

Definition 6.3.3. An L^p-Fock representation $(C, \rho_A, \rho_{\mathsf{X}}, \rho_{\mathsf{Y}})$ for $((\mathsf{X}, \mathsf{Y}), \varphi)$ is the universal L^p-Fock representation if for any other L^p-Fock representation $(B, \pi_A, \pi_{\mathsf{X}}, \pi_{\mathsf{Y}})$ there is a contractive homomorphism $\sigma\colon F^p(C, \rho_A, \rho_{\mathsf{X}}, \rho_{\mathsf{Y}}) \to F^p(B, \pi_A, \pi_{\mathsf{X}}, \pi_{\mathsf{Y}})$ such that $\sigma \circ \rho_E = \pi_E$, for any $E \in \{A, \mathsf{X}, \mathsf{Y}\}$. Such universal representation exists, and we define the Toeplitz L^p-algebra of $((\mathsf{X}, \mathsf{Y}), \varphi)$ by $\mathcal{T}^p((\mathsf{X}, \mathsf{Y}), \varphi) = F^p(C, \rho_A, \rho_{\mathsf{X}}, \rho_{\mathsf{Y}})$.

We now define an L^p-version of Katsura's ideal (see Definition 2.4.9).

Definition 6.3.4. Let $((\mathsf{X}, \mathsf{Y}), \varphi)$ be an L^p-correspondence over an L^p-operator algebra A. We define

$$J_{(\mathsf{X},\mathsf{Y})} = \{a \in A\colon \varphi(a) \in \mathcal{K}_A((\mathsf{X}, \mathsf{Y})) \text{ and } ab = 0 \text{ for all } b \in \ker(\varphi)\}$$

Just as in the C*-case, the following three statements follow directly from Definition 6.3.4:

1. $J_{(\mathsf{X},\mathsf{Y})}$ is a closed double sided ideal of A,
2. $J_{(\mathsf{X},\mathsf{Y})}$ the largest ideal I of A with the property that $\varphi|_I\colon I \to \mathcal{K}_A((\mathsf{X}, \mathsf{Y}))$ is injective,
3. if φ is injective, then $J_{(\mathsf{X},\mathsf{Y})} = \varphi^{-1}(\mathcal{K}_A((\mathsf{X}, \mathsf{Y})))$, and if in addition $\varphi(A) \subseteq \mathcal{K}_A((\mathsf{X}, \mathsf{Y})$, then $J_{(\mathsf{X},\mathsf{Y})} = A$.

Definition 6.3.5. Let $p \in [1, \infty)$, let A be an L^p-operator algebra, and let $(B, \pi_A, \pi_{\mathsf{X}}, \pi_{\mathsf{Y}})$ be an L^p-Fock representation for a L^p-correspondence $((\mathsf{X}, \mathsf{Y}), \varphi)$ over A. We say $(B, \pi_A, \pi_{\mathsf{X}}, \pi_{\mathsf{Y}})$ is *covariant* if there is a contractive homomorphism

$\pi_{\mathcal{K}} \colon \mathcal{K}_A((\mathsf{X},\mathsf{Y})) \to B$ satisfying

$$\pi_{\mathcal{K}}(\theta_{x,y}) = \pi_{\mathsf{X}}(x)\pi_{\mathsf{Y}}(y) \text{ for all } x \in \mathsf{X}, y \in \mathsf{Y}, \text{ and } \pi_{\mathcal{K}}(\varphi(a)) = \pi_A(a) \text{ for all } a \in J_{(\mathsf{X},\mathsf{Y})}$$

Definition 6.3.6. A covariant L^p-Fock representation $(D, \tau_A, \tau_{\mathsf{X}}, \tau_{\mathsf{Y}})$ for $((\mathsf{X},\mathsf{Y}),\varphi)$ is the universal covariant L^p-Fock representation if for any other covariant L^p-Fock representation $(B, \pi_A, \pi_{\mathsf{X}}, \pi_{\mathsf{Y}})$ there is a contractive homomorphism $\sigma \colon F^p(D, \tau_A, \tau_{\mathsf{X}}, \tau_{\mathsf{Y}}) \to F^p(B, \pi_A, \pi_{\mathsf{X}}, \pi_{\mathsf{Y}})$ such that $\sigma \circ \tau_E = \pi_E$, for any $E \in \{A, \mathsf{X}, \mathsf{Y}\}$. Such universal representation exists, and we define the L^p-Cuntz-Pimsner algebra of $((\mathsf{X},\mathsf{Y}),\varphi)$ by $\mathcal{O}^p((\mathsf{X},\mathsf{Y}),\varphi) = F^p(D, \tau_A, \tau_{\mathsf{X}}, \tau_{\mathsf{Y}})$.

In the next two subsections, we will show that the L^p-correspondences from Examples 6.1.2 and 6.1.4 admit (covariant) L^p-Fock representations and we will compute the algebras these generate. In both examples, $((\mathsf{X},\mathsf{Y}),\varphi)$ is an L^p-correspondence over an L^p-operator algebra A for which φ is injective and $\varphi(A) \subseteq \mathcal{K}_A((\mathsf{X},\mathsf{Y}))$. Thus, $J_{(\mathsf{X},\mathsf{Y})} = A$ and the covariance condition from Definition 6.3.5 becomes $\pi_{\mathcal{K}}(\varphi(a)) = \pi_A(a)$ for all $a \in A$.

L^p-Cuntz algebras come from covariant L^p-Fock representations

Let $p \in (1,\infty)$, $d \in \mathbb{Z}_{\geq 2}$, and let $((\ell^p_d, \ell^q_d), \varphi)$ be the L^p-correspondence form Example 6.1.4. We will show that $((\ell^p_d, \ell^q_d), \varphi_d)$ admits a covariant L^p-Fock representation and that the L^p-operator algebra generated by such representation is actually isometrically isomorphic to $\mathcal{O}^p_d$ from Definition 4.4.20.

First of all, in order to safely use that $J_{(\ell^p_d,\ell^q_d)} = \mathbb{C}$, we make sure that $\varphi_d : \mathbb{C} \to \mathcal{L}_{\mathbb{C}}((\ell^p_d, \ell^q_d))$ is injective and that $\varphi_d(\mathbb{C}) \subseteq \mathcal{K}_{\mathbb{C}}((\ell^p_d, \ell^q_d))$. Injectivity of φ is clear. Now, for any $j \in \{1, \ldots, d\}$, let $\delta_j : \{1, \ldots, d\} \to \{0, 1\}$ be given by $\delta_j(k) = \delta_{j,k}$, so that δ_j is interpreted as both an element of $\ell_d^p \cong \mathcal{L}(\ell_1^p, \ell_d^p)$ and of $\ell_d^q \cong \mathcal{L}(\ell_d^p, \ell_1^p)$. Then for each $j, k \in \{1, \ldots, d\}$, we can interpret expressions of the form $\delta_j \delta_k$ by regarding $\delta_j \in \ell_d^q$ and $\delta_k \in \ell_d^p$, whence $\delta_j \delta_k = \theta_{\delta_j, \delta_k} \in \mathcal{K}_{\mathbb{C}}((\ell_d^p, \ell_d^q))$. Thus, if $z \in \mathbb{C}$, it is clear that

$$\varphi_d(z) = \sum_{j=1}^{d} \theta_{\delta_j, \delta_j} z \in \mathcal{K}_{\mathbb{C}}((\ell_d^p, \ell_d^q))$$

Hence, $\varphi_d(\mathbb{C}) \subseteq \mathcal{K}_C((\ell_d^p, \ell_d^q))$ as wanted.

We are now ready to construct a covariant L^p-Fock representation for $((\ell_d^p, \ell_d^q), \varphi_d)$ as in Definition 6.3.5. What follows is a modification of the usual Fock space construction for C*-correspondences from Definition 2.4.4. In order to define the corresponding Fock modules, we will use the following lemma that allows us to tensor the L^p correspondence $((\ell_d^p, \ell_d^q), \varphi_d)$ with it self n times via Definition 6.2.1.

Lemma 6.3.7. *Let $n \in \mathbb{Z}_{\geq 1}$. Then we can make sense of the expression $((\ell_d^p, \ell_d^q), \varphi_d)^{\otimes n}$ via Definition 6.2.1. Furthermore*

$$((\ell_d^p, \ell_d^q), \varphi_d)^{\otimes n} = ((\ell_{d^n}^p, \ell_{d^n}^q), \varphi_{d^n})$$

Proof. The case $n = 1$ is obvious. Next, we verify $n = 2$. To make sense of $((\ell_d^p, \ell_d^q), \varphi_d)^{\otimes 2}$ we first notice that the module (ℓ_d^p, ℓ_d^q) can be represented in different L^p-spaces, which will allow us to perform the tensor product computation from Definition 6.2.1. Indeed, let $\mathsf{X}_r = \ell_d^p = \mathcal{L}(\ell_1^p, \ell_d^p)$ and $\mathsf{Y}_r = \ell_d^q = \mathcal{L}(\ell_d^p, \ell_1^q)$ where the identifications are exactly as in Example 5.1.6. That is, $(\mathsf{X}_r, \mathsf{Y}_r)$ is an L^p module over the L^p-operator algebra $\mathbb{C} = \mathcal{L}(\ell_1^p)$. Now, recall that $\ell_d^p \otimes_p \ell_d^p = \ell_{d^2}^p$ and let $\mathsf{X}_l = \ell_d^p \subseteq \mathcal{L}(\ell_d^p, \ell_{d^2}^p)$ via the isometric map $x \mapsto (\xi \mapsto x \otimes \xi)$. Similarly, we let $\mathsf{Y}_l = \ell_d^q \subseteq \mathcal{L}(\ell_d^p, \ell_1^q)$ via the isometric map $y \mapsto (\xi \otimes \eta \mapsto \langle y, \xi \rangle \eta)$. Then $(\mathsf{X}_l, \mathsf{Y}_l)$ is an L^p-module over the L^p-operator algebra $\mathbb{C} \subseteq \mathcal{L}(\ell_d^p)$, where the image of $\mathbb{C}$ in $\mathcal{L}(\ell_d^p)$

is given by scalar multiplication. Now notice that for any $x \in \mathsf{X}_l$ and any $y \in Y_l$ we have $\varphi_d((y \mid x)_{\mathbb{C}}) = yx \in \mathbb{C} \subset \mathcal{L}(\ell^p_d)$. Thus, we can follow the construction from Definition 6.2.1 to get

$$((\mathsf{X}_l, \mathsf{Y}_l), \varphi_{d^2}) \otimes_{\varphi_d} ((\mathsf{X}_l, \mathsf{Y}_l), \varphi_{d^2}) = ((\ell^p_{d^2}, \ell^q_{d^2}), \varphi_{d^2})$$

where $\ell^p_{d^2} = \mathcal{L}(\ell^p_1, \ell^p_{d^2})$ and $\ell^q_{d^2} = \mathcal{L}(\ell^p_{d^2}, \ell^p_1)$. The general case $n \in \mathbb{Z}_{\geq 1}$ follows by repeatedly applying the argument above. ■

The previous lemma suggests the following convention

$$((\ell^p_d, \ell^q_d), \varphi_d)^{\otimes 0} = ((\ell^p_1, \ell^q_1), \varphi_1).$$

Furthermore, the previous lemma also allows us to define L^p-Fock modules.

Definition 6.3.8. We define the sets $\mathcal{F}^p$ and $\mathcal{F}^q$ as the p and q direct sums of the ingredients on the tensor product correspondence from Lemma 6.3.7. To be more precise

$$\mathcal{F}^p = \left\{ (\kappa_n)_{n\geq 0} = (\kappa_0, \kappa_1, \ldots) \colon \kappa_n \in \ell^p_{d^n}, \sum_{n=0}^{\infty} \|\kappa_n\|_p^p < \infty \right\},$$

and

$$\mathcal{F}^q = \left\{ (\tau_n)_{n\geq 0} = (\tau_0, \tau_1, \ldots) \colon \tau_n \in \ell^q_{d^n}, \sum_{n=0}^{\infty} \|\tau_n\|_q^q < \infty \right\}.$$

We equip $\mathcal{F}^p$ and $\mathcal{F}^q$ with the obvious p-norm and q-norm, respectively.

In what follows, we will use I to denote the index set

$$I = \bigsqcup_{n\geq 0} \{1, 2, 3, ..., d^n\}, \tag{6.3.1}$$

where $\sqcup$ denotes disjoint union of sets. The importance of the index set I is that it allows us to identify $\mathcal{F}^p$ with $\ell^p(I)$ and $\mathcal{F}^q$ with $\ell^q(I)$. This is useful for computations but also to check that $(\mathcal{F}^p, \mathcal{F}^q)$ is a C*-like L^p-module. We record all this in the following proposition.

Proposition 6.3.9. *Let $p \in (1, \infty)$, let $d \in \mathbb{Z}_{\geq 2}$, let $\mathcal{F}^p$ and $\mathcal{F}^q$ be as in Definition 6.3.8, and let I be the index set from equation (6.3.1). Then $\mathcal{F}^p$ is canonically identified with $\ell^p(I)$ and $\mathcal{F}^q$ is canonically identified with $\ell^q(I)$. Furthermore, $(\mathcal{F}^p, \mathcal{F}^q)$ is a C*-like L^p-module over $\mathbb{C} = \mathcal{L}(\ell_1^p)$ with*

$$(\tau \mid \kappa)_{\mathbb{C}} = \sum_{n=0}^{\infty} \sum_{k=1}^{d^n} \tau_n(k)\kappa(k) \in \mathbb{C} \tag{6.3.2}$$

for any $\tau \in \mathcal{F}^q$ and $\kappa \in \mathcal{F}^p$.

Proof. The natural identifications follow from observing that any element in either $\mathcal{F}^p$ or $\mathcal{F}^q$ is also a function $I \to \mathbb{C}$. Indeed, for instance if $\kappa = (\kappa_n)_{n \geq 0} \in \mathcal{F}^p$, where, for each $n \geq 0$, $\kappa_n = (\kappa_n(1), \ldots, \kappa_n(d^n)) \in \mathbb{C}^{d^n}$. Then

$$\begin{aligned} \kappa &= (\kappa_0 \quad , \qquad \kappa_1 \qquad , \qquad \kappa_2 \qquad , \ \ldots) \in \mathcal{F}^p \\ &= (\kappa_0(1), \kappa_1(1), \ldots, \kappa_1(d), \kappa_2(1), \ldots, \kappa_2(d^2), \ldots) \in \ell^p(I). \end{aligned}$$

It is clear that the norm of κ regarded as an element in $\mathcal{F}^p$ coincides with its norm when thought of as an element of $\ell^p(I)$. Thus, from now on, we will safely use $\mathcal{F}^p = \ell^p(I)$ and $\mathcal{F}^q = \ell^q(I)$. This automatically makes, exactly as in Example 5.1.5, $(\mathcal{F}^p, \mathcal{F}^q)$ a C*-like L^p-module over $\mathbb{C} = \mathcal{L}(\ell_1^p)$ and the usual pairing between $\ell^q(I)$ and $\ell^p(I)$ translates precisely into equation (6.3.2) via the above identification. ∎

Corollary 6.3.10. *Let $p \in (1, \infty)$, let $d \in \mathbb{Z}_{\geq 2}$, let $\mathcal{F}^p$ and $\mathcal{F}^q$ be as in Definition 6.3.8. Then*

1. $\mathcal{L}_{\mathbb{C}}((\mathcal{F}^p, \mathcal{F}^q)) = \mathcal{L}(\ell^p(I))$, *and* $\mathcal{K}_{\mathbb{C}}((\mathcal{F}^p, \mathcal{F}^q)) = \mathcal{K}(\ell^p(I))$.

2. $\mathcal{L}_{\mathbb{C}}((\mathcal{F}^p, \mathcal{F}^q))$ *is a σ-finitely representable L^p-operator algebra,*

3. $\mathcal{L}_{\mathbb{C}}((\mathcal{F}^p, \mathcal{F}^q))/\mathcal{K}_{\mathbb{C}}((\mathcal{F}^p, \mathcal{F}^q))$ *is an L^p-operator algebra.*

Proof. Let ν_I be counting measure on the index set I from equation (6.3.1). Then $(I, 2^I, \nu_I)$ is σ-finite and therefore the first assertion is in Example 5.5.4. Also, $\mathcal{L}_{\mathbb{C}}((\mathcal{F}^p, \mathcal{F}^q)) = \mathcal{L}(L^p(\nu_I))$, proving the second assertion. Finally, by Proposition 6.3.9 we have

$$\mathcal{L}_{\mathbb{C}}((\mathcal{F}^p, \mathcal{F}^q))/\mathcal{K}_{\mathbb{C}}((\mathcal{F}^p, \mathcal{F}^q)) = \mathcal{L}(\ell^p(I))/\mathcal{K}(\ell^p(I)) = \mathcal{Q}(\ell^p(I)).$$

The conclusion for the third assertion now follows from part (1) of Lemma 4.5(1) in [1]. ■

Remark 6.3.11. From now on, we regard the sets $\mathcal{F}^p$ and $\ell^p(I)$ as equal, as well as $\mathcal{F}^q = \ell^q(I)$. Similarly, for each $n \geq 0$, we think of elements $z \in \mathbb{C}^{d^n}$ also as a functions $z \colon \{1, 2, 3, ..., d^n\} \to \mathbb{C}$. For some arguments (see for example the proof of Proposition 6.3.24 below) we will need to work with subsets of I. For instance, if J is a subset of $\{1, 2, 3, ..., d^n\}$, it makes sense to consider the multiplication operator $m(\chi_J) \colon \mathbb{C}^{d^n} \to \mathbb{C}^{d^n}$ in the obvious way. That is, if $j \in \{1, 2, 3, ..., d^n\}$, then

$$(m(\chi_J)z)(j) = \chi_J(j)z(j) = \begin{cases} z(j) & \text{if } \ j \in J \\ 0 & \text{if } \ j \notin J. \end{cases}$$

Thus, if for each $n \in \mathbb{Z}_{\geq 0}$ we have a subset $J^{(n)} \subset \{1, 2, 3, ..., d^n\}$, and we define $J = \bigsqcup_{n \geq 0} J^{(n)}$, which is a subset of I, then the usual multiplication operator $m(\chi_J)\colon \ell^p(I) \to \ell^p(I)$ acts as

$$m(\chi_J)(\kappa_n)_{n\geq 0} = \big(m(\chi_{J^{(n)}})\kappa_n\big)_{n\geq 0}.$$

We will define 4 maps to the L^p-operator algebra $\mathcal{L}_{\mathbb{C}}((\mathcal{F}^p, \mathcal{F}^q))/\mathcal{K}_{\mathbb{C}}((\mathcal{F}^p, \mathcal{F}^q))$ that will yield a covariant L^p-Fock representation. To do so, notice first of all that, under the identifications of Lemmas 6.3.7 and 6.3.9, any element $\kappa = (\kappa_n)_{n\geq 0} \in \mathcal{F}^p$ is such that for each $n \in \mathbb{Z}_{\geq 1}$, κ_n can be written uniquely in the form

$$\kappa_n = \sum_{j=1}^{d} \delta_j \otimes z_j^{(n)} \in \ell_d^p \otimes_p \ell_{d^{n-1}}^p,$$

where, as before, $\delta_1, \ldots, \delta_d$ is the canonical basis for ℓ_d^p. This fact will be used repeatedly below to define linear maps on $\mathcal{F}^p$ and several other computations.

Definition 6.3.12. Let $p \in (1, \infty)$ and let $d \in \mathbb{Z}_{\geq 2}$. For each $x \in \ell_d^p$, we define $c(x)\colon \mathcal{F}^p \to \mathcal{F}^p$ by

$$c(x)(\kappa) = \big(0, (x \otimes \kappa_n)_{n\geq 0}\big) = (0, x \otimes \kappa_0, x \otimes \kappa_1, \ldots) \in \mathcal{F}^p \tag{6.3.3}$$

for any $\kappa = (\kappa_n)_{n\geq 0} \in \mathcal{F}^p$. We call this the *creation operator by x on $\mathcal{F}^p$*.

Lemma 6.3.13. *Let $p \in (1, \infty)$ and let $d \in \mathbb{Z}_{\geq 2}$. For each $x \in \mathsf{X}$, $c(x) \in \mathcal{L}(\mathcal{F}^p)$ and $\|c(x)\| = \|x\|_p$. In fact, $c(x) = \|x\|_p u$ where $u : \mathcal{F}^p \to \mathcal{F}^p$ is an isometry. Furthermore, $c : \ell_d^p \to \mathcal{L}(\mathcal{F}^p)$ is a bounded linear map.*

Proof. That $c(x)$ is linear is clear form definition. Let $m \in \mathbb{Z}_{\geq 1}$. For any $z \in \ell^p_m$, we have $\|x \otimes z\|_p^p = \|x\|_p^p \|z\|_p^p$. Therefore,

$$\|c(x)(\kappa)\|^p = \sum_{n=0}^{\infty} \|x \otimes \kappa_n\|_p^p = \|x\|_p^p \sum_{n=0}^{\infty} \|\kappa_n\|_p^p = \|x\|_p^p \|\kappa\|.$$

Thus, $c(x)(\kappa) \in \mathcal{F}^p$ and $\|c(x)\| = \|x\|_p$, as wanted. Finally, if $x = 0$, it is clear that $c(x) = \|x\|_p 1_{\mathcal{F}^p}$, where $1_{\mathcal{F}^p(\mathsf{X})}$ is the identity map. Otherwise, for $x \neq 0$, we have actually shown above that $u = \frac{1}{\|x\|_p} c(x)$ is an isometry. Finally, it's clear that for any $x_1, x_2 \in \ell^p_d$ and $\lambda \in \mathbb{C}$ we have $c(x_1 + \lambda x_2) = c(x_1) + \lambda c(x_2)$ and therefore $c \colon \ell^p_d \to \mathcal{L}(\mathcal{F}^p)$ is indeed a bounded linear map. This finishes the proof. ∎

Recall that for any $x \in \ell^p_d$ and $y \in \ell^q_d$, we have the pairing

$$(y \mid x)_{\mathbb{C}} = \sum_{j=1}^{d} y(j)x(j)$$

By Hölder, $|(y \mid x)_{\mathbb{C}}| \leq \|y\|_q \|x\|_p$. Moreover, it's well known that

$$\|y\|_q = \sup_{\|x\|_p = 1} |(y \mid x)_{\mathbb{C}}| \tag{6.3.4}$$

For $n \in \mathbb{Z}_{\geq 1}$, $x \in \ell^p_d$ and $w \in \ell_{d^{n-1}}$, we define

$$v_n(y)(x \otimes w) = (y \mid x)_{\mathbb{C}} w \in \mathsf{X}^{\otimes(n-1)}.$$

This extends linearly to a well defined map $v_n(y) \colon \ell^p_{d^n} \to \ell^p_{d^{n-1}}$.

Definition 6.3.14. Let $p \in (1, \infty)$ and let $d \in \mathbb{Z}_{\geq 2}$. For each $y \in \ell_d^q$ we define $v(y)\colon \mathcal{F}^p \to \mathcal{F}^p$ by

$$v(y)(\kappa) = \big(v_n(y)(\kappa_n)\big)_{n\geq 1} = (v_1(y)(\kappa_1), v_2(y)(\kappa_2), \ldots) \in \mathcal{F}^p \tag{6.3.5}$$

We call this the *annihilation operator by y on $\mathcal{F}^p$*.

Lemma 6.3.15. *Let $p \in (1, \infty)$ and let $d \in \mathbb{Z}_{\geq 2}$. For each $y \in \ell_d^q$, $v(y) \in \mathcal{L}(\mathcal{F}^p)$ and $\|v(y)\| = \|y\|_q$. Furthermore, $v : \ell_d^q \to \mathcal{L}(\mathcal{F}^p)$ is a bounded linear map.*

Proof. That $v(y)$ is linear is clear from definition. Then notice that when $n \in \mathbb{Z}_{\geq 1}$,

$$v_n(y) = y \otimes \mathrm{id}_{\ell^p_{d^{n-1}}} : \ell_d^p \otimes_p \ell^p_{d^{n-1}} \to \ell_1^p \otimes_p \ell^p_{d^{n-1}} = \ell^p_{d^{n-1}}$$

Hence, it follows that $\|v_n(y)\| = \|y\|_q$. Finally, for any $\kappa = (\kappa_n)_{n\geq 0} \in \mathcal{F}^p$ we have,

$$\|v(y)(\kappa)\|^p = \sum_{n=1}^{\infty} \|v_n(y)(\kappa_n)\|_p^p \leq \|y\|_q^p \sum_{n=1}^{\infty} \|\kappa_n\|_p^p \leq \|y\|_q^p \sum_{n=0}^{\infty} \|\kappa_n\|_p^p = \|y\|_q^p \|\kappa\|^p.$$

Thus, $v(y)(\kappa) \in \mathcal{F}^p$ and $\|v(y)\| \leq \|y\|_q$. To prove equality, let $\varepsilon > 0$, and use equation (6.3.4) to find $x \in \ell_d^p$ such that $\|x\|_p = 1$ and $|(y \mid x)_{\mathbb{C}}| > \|y\|_q - \varepsilon$. Now let $\kappa \in \mathcal{F}^p$ be given by $\kappa_1 = x$ and $\kappa_n = 0$ for any $n \in \mathbb{Z}_{\geq 0}$ with $n \neq 1$. It's clear that $\|\kappa\| = 1$ and that $\|v(y)(\kappa)\| = |v_1(y)x| = |(y \mid x)_{\mathbb{C}}| > \|y\|_q - \varepsilon$. This shows that $\|v(y)\| = \|y\|_q$, as desired. Finally, if $y_1, y_2 \in \ell_d^q$ and $\lambda \in \mathbb{C}$, we clearly have $v(y_1 + \lambda y_2) = v(y_1) + \lambda v(y_2)$, proving that $v : \ell_d^q \to \mathcal{L}(\mathcal{F}^p)$ is in fact a bounded linear map. ∎

Definition 6.3.16. Let $p \in (1, \infty)$ and let $d \in \mathbb{Z}_{\geq 2}$. For each $z \in \mathbb{C}$ we define $\varphi^{\infty}(z)\colon \mathcal{F}^p \to \mathcal{F}^p$ by

$$\varphi^{\infty}(z)(\kappa_n)_{n\geq 0} = (\varphi_{d^n}(z)\kappa_n)_{n\geq 0} \tag{6.3.6}$$

Lemma 6.3.17. *Let $p \in (1, \infty)$ and let $d \in \mathbb{Z}_{\geq 2}$. For each $z \in \mathbb{C}$, $\varphi^{\infty}(z) \in \mathcal{L}(\mathcal{F}^p)$ and $\|\varphi^{\infty}(z)\| = |z|$. Furthermore, $\varphi^{\infty} : \mathbb{C} \to \mathcal{L}(\mathcal{F}^p)$ is a bounded homomorphism.*

Proof. All the assertions follow at once by observing that $\varphi^{\infty}(z)$ is actually coordinatewise multiplication by z on $\ell^p(I)$. That is, $\varphi^{\infty}(z) = z \cdot \mathrm{id}_{\mathcal{F}^p}$. ∎

If $n \in \mathbb{Z}_{\geq 1}$, $t \in \mathcal{L}(\ell^p_d)$, $x \in \ell^p_d$, and $w \in \ell^p_{d^{n-1}}$, we get a map $t^{(n)} \colon \ell^p_{d^n} \to \ell^p_{d^n}$ by linearly extending the assignment

$$x \otimes w \mapsto t^{(n)}(x \otimes w) = (tx) \otimes w$$

to all of $\ell^p_{d^n} = \ell^p_d \otimes_p \ell^p_{d^{n-1}}$. Furthermore, $t^{(n)} \in \mathcal{L}(\ell^p_{d^n})$ with $\|t^{(n)}\| \leq \|t\|$.

Definition 6.3.18. Let $p \in (1, \infty)$ and let $d \in \mathbb{Z}_{\geq 2}$. For each $k \in \mathcal{K}_{\mathbb{C}}((\ell^p_d, \ell^q_d))$ we define $\Theta(k) \colon \mathcal{F}^p \to \mathcal{F}^p$ by

$$\Theta(k)(\kappa_n)_{n \geq 0} = \big(0, (k^{(n)}\kappa_n)_{n \geq 1}\big). \tag{6.3.7}$$

Lemma 6.3.19. *Let $p \in (1, \infty)$ and let $d \in \mathbb{Z}_{\geq 2}$. For each $k \in \mathcal{K}_{\mathbb{C}}((\ell^p_d, \ell^q_d))$, $\Theta(k) \in \mathcal{L}(\mathcal{F}^p)$ and $\|\Theta(k)\| \leq \|k\|$. Furthermore, $\Theta : \mathcal{K}_{\mathbb{C}}((\ell^p_d, \ell^q_d)) \to \mathcal{L}(\mathcal{F}^p)$ is a bounded homomorphism.*

Proof. Let $k \in \mathcal{K}_{\mathbb{C}}((\ell^p_d, \ell^q_d))$ and $\kappa = (\kappa_n)_{n \geq 0} \in \mathcal{F}^q$. That $\Theta(k)$ is linear is clear. Now,

$$\|\Theta(k)\kappa\|^p = \sum_{n=1}^{\infty} \|k^{(n)}\kappa_n\|_p^p \leq \|k\|^p \|\kappa\|^p,$$

whence $\|\Theta(k)\| \leq \|k\|$. The homomorphism part consists of a straightforward verification that $\Theta(k_1)\Theta(k_2) = \Theta(k_1 k_2)$ and $\Theta(k_1 + \lambda k_2) = \Theta(k_1) + \lambda\Theta(k_2)$ for any $k_1, k_2 \in \mathcal{K}_{\mathbb{C}}((\ell^p_d, \ell^q_d))$ and any $\lambda \in \mathbb{C}$. ∎

Proposition 6.3.20. *Let $p \in (1,\infty)$, let $d \in \mathbb{Z}_{\geq 2}$, let I be the index set from 6.3.1, and let $q_0 \colon \mathcal{L}(\ell^p(I)) \to \mathcal{Q}(\ell^p(I))$ be the quotient map. Consider the maps $c \colon \ell^p_d \to \mathcal{L}(\mathcal{F}^p)$ from (6.3.3), $v \colon \ell^q_d \to \mathcal{L}(\mathcal{F}^p)$ from (6.3.5), $\varphi^\infty \colon \mathbb{C} \to \mathcal{L}(\mathcal{F}^p)$ from (6.3.6), and $\Theta : \mathcal{K}_{\mathbb{C}}((\ell^p_d, \ell^q_d)) \to \mathcal{L}(\mathcal{F}^p)$ from (6.3.7). Then $(\mathcal{L}(\mathcal{F}^p), \varphi_\infty, c, v)$ is an L^p-Fock representation of $((\ell^p_d, \ell^q_d), \varphi_d)$, and $(\mathcal{Q}(\ell^p(I)), q_0 \circ \varphi_\infty, q_0 \circ c, q_0 \circ v)$ is a covariant L^p-Fock representation of $((\ell^p_d, \ell^q_d), \varphi_d)$ via the map $q_0 \circ \Theta$.*

Proof. We first check that check that $(\mathcal{L}(\mathcal{F}^p), \varphi_\infty, c, v)$ satisfies the conditions on Definition 6.3.1. Clearly $\mathcal{L}(F^p)$ is an L^p-operator algebra. For condition 1, let $x \in \mathsf{X}$, $z \in \mathbb{C}$ and $\kappa \in \mathcal{F}^p$, then

$$c(xz)\kappa = \big(0, (\varphi_d(z)x \otimes \kappa_n)_{n\geq 0}\big) = \big(0, (x \otimes \varphi_d^n(z)\kappa_n)_{n\geq 0}\big) = c(x)\varphi^\infty(z)\kappa,$$

and since in this case $xz = \varphi_d(z)x$ and $c(x)$ commutes with $\varphi^\infty(z)$, we have also shown that $c(\varphi_d(z)x) = \varphi^\infty(z)c(x)$. Similarly, for condition 2 we find for any $y \in \mathsf{Y}$ and $z \in \mathbb{C}$

$$v(zy) = \varphi^\infty(z)v(y) = v(y)\varphi^\infty(z) = v(y\varphi_d(z)),$$

as wanted. Finally, to check condition 3, let $x \in \mathsf{X}$ and $y \in \mathsf{Y}$ and observe that for any $\kappa \in \mathcal{F}^p$

$$v(y)c(x)\kappa = v(y)(0, (x \otimes \kappa_n)_{n\geq 0}) = ((y \mid x)_{\mathbb{C}}\kappa_n)_{n\geq 0} = \varphi^\infty((y \mid x)_{\mathbb{C}})\kappa,$$

whence $v(y)c(x) = \varphi^\infty((y \mid x)_{\mathbb{C}})$. Now we check that $q_0 \circ \Theta$ makes $(\mathcal{Q}(\ell^p(I)), q_0 \circ \varphi_\infty, q_0 \circ c, q_0 \circ v)$ a covariant L^p-Fock representation. Indeed, we already showed in Corollary 6.3.10 that $\mathcal{Q}(\ell^p(I))$ is an L^p-operator algebra. Furthermore, since q_0 is an algebra homomorphism, it follows that $(\mathcal{Q}(\ell^p(I)), q_0 \circ \varphi_\infty, q_0 \circ c, q_0 \circ v)$ is also an L^p-Fock representation. It only remains to show that is covariant according to Definition 6.3.5. First of all, we claim that for any $n \in \mathbb{Z}_{\geq 1}$ we have $(x \otimes v_n(y))\kappa_n = \theta^{(n)}_{x,y}\kappa_n$. To see this, we write

$$\kappa_n = \sum_{k=1}^{d} \delta_k \otimes z_k^{(n)} \in \ell^p_d \otimes_p \ell^p_{d^{n-1}},$$

so that

$$\begin{aligned}
(x \otimes v_n(y))\kappa_n =& x \otimes \Big(\sum_{k=1}^{d} v_n(y)\delta_k \otimes z_k^{(n)} \Big) \\
&= x \otimes \Big(\sum_{k=1}^{d} (y \mid \delta_k)_{\mathbb{C}} z_k^{(n)} \Big) \\
&= \sum_{k=1}^{d} x(y \mid \delta_k)_{\mathbb{C}} \otimes z_k^{(n)} \\
&= \sum_{k=1}^{d} (\theta_{x,y}\delta_k) \otimes z_k^{(n)} = \theta^{(n)}_{x,y}\kappa_n,
\end{aligned}$$

proving our claim. Therefore,

$$c(x)v(y)\kappa = c(x)(v_n(y)\kappa_n)_{n\geq 1} = (0, (x \otimes v_n(y))\kappa_n)_{n \geq 1} = (0, (\theta^{(n)}_{x,y}\kappa_n)_{n\geq 0}) = \Theta(\theta_{x,y})\kappa.$$

This proves that $(q_0 \circ \Theta)(\theta_{x,y}) = (q_0 \circ c)(x)(q_0 \circ v)(y)$. Finally, for any $z \in \mathbb{C}$ we have

$$\Theta(\varphi_d(z))\kappa = \big(0, ((\varphi_d)^{(n)}\kappa_n)_{n\geq 1}\big) = (0, (\varphi_{d^n}(z)\kappa_n)_{n\geq 1}) = \varphi_\infty(z)(0, (\kappa_n)_{n\geq 1}).$$

Hence, if $\iota = (1, 0, 0, \ldots)$ is regarded as both an element of $\ell^p(I)$ and $\ell^q(I)$, we immediately see

$$\Theta(\varphi_d(z)) - \varphi_\infty(z) = \theta_{\iota,\iota} \in \mathcal{K}(\ell^p(I)).$$

Therefore, $(q_0 \circ \Theta)(\varphi_d(z)) = (q_0 \circ \varphi_\infty)(z)$ proving that indeed $(\mathcal{Q}(\ell^p(I)), q_0 \circ \varphi_\infty, q_0 \circ c, q_0 \circ v)$ is covariant. ■

Since $((\ell_d^p, \ell_d^q), \varphi_d)$ admits a covariant L^p-Fock representation, we get an L^p operator algebra $F^p(\mathcal{Q}(\ell^p(I)), q_0 \circ \varphi_\infty, q_0 \circ c, q_0 \circ v)$ which we denote for short $\mathcal{O}^p(\varphi_\infty, c, v)$. Notice that the set $\{(y \mid x)_{\mathbb{C}} \colon x \in \ell_d^p, y \in \ell_d^q\}$ is in fact equal to $\mathbb{C}$. Hence, following Remark 6.3.2 and for further reference, we record below a precise definition for $\mathcal{O}^p(\varphi_\infty, c, v)$.

Definition 6.3.21. Let $p \in (1, \infty)$, let $d \in \mathbb{Z}_{\geq 2}$, let I be the index set from equation (6.3.1), and let $q \colon \mathcal{L}(\ell^p(I)) \to \mathcal{Q}(\ell^p(I))$ be the quotient map. We define $\mathcal{O}^p(\varphi_\infty, c, v)$ as the closed subalgebra in $\mathcal{Q}(\ell^p(I))$ generated by $q_0(c(\ell_d^p))$ and $q_0(v(\ell_d^q))$.

We are finally ready to state a main result:

Theorem 6.3.22. *Let $p \in (1, \infty)$, let $d \in \mathbb{Z}_{\geq 2}$. Then $\mathcal{O}^p(\varphi_\infty, c, v)$ is isometrically isomorphic to $\mathcal{O}_d^p$.*

The idea of the proof is to produce a spatial representation of the Leavitt algebra from Definition 4.4.17 on $\mathcal{O}^p(\varphi_\infty, c, v)$. We break this into several steps.

Proposition 6.3.23. *Let $p \in (1, \infty)$ and let $d \in \mathbb{Z}_{\geq 2}$. Let $\delta_1, \ldots, \delta_d$ be the usual basis for ℓ_d^p, which is also the usual basis for ℓ_d^q. Then*

1. *For any $j \in \{1, \ldots, d\}$,*
$$v(\delta_j)c(\delta_j) = \mathrm{id}_{\mathcal{F}^p},$$
2. *For j, k in $\{1, \ldots, d\}$ with $j \neq k$,*
$$v(\delta_k)c(\delta_j) = 0.$$

3. *Regard* $\iota = (1, 0, 0, \ldots)$ *as an element of both* $\mathcal{F}^p$ *and* $\mathcal{F}^q$*. Then*

$$\sum_{j=1}^{d} c(\delta_j)v(\delta_j) = \mathrm{id}_{\mathcal{F}^p} - \theta_{\iota,\iota}.$$

Proof. By Proposition 6.3.20, $(\mathcal{L}(\mathcal{F}^p), \varphi_\infty, c, v)$ is an L^p-Fock representation and therefore we have

$$v(\delta_k)c(\delta_j) = \varphi^\infty((\delta_k \mid \delta_j)_{\mathbb{C}}) = (\delta_k \mid \delta_j)_{\mathbb{C}} \cdot \mathrm{id}_{\mathcal{F}^p}.$$

This proves both 1 and 2. Finally, to show 3, let $\kappa = (\kappa_n)_{n\geq 0} \in \mathcal{F}^p(\mathsf{X})$. For each $n \in \mathbb{Z}_{\geq 1}$, recall from before Definition 6.3.12 that we can write

$$\kappa_n = \sum_{k=1}^{d} \delta_k \otimes z_k^{(n)} \in \ell_d^p \otimes_p \ell_{d^{n-1}}^p.$$

Since we know from Proposition 6.3.5 that $c(x)v(y) = \Theta(\theta_{x,y})$, we compute

$$\begin{aligned}
\Big(\sum_{j=1}^{d} c(\delta_j)v(\delta_j)\Big)\kappa &= \Theta\Big(\sum_{j=1}^{d} \theta_{\delta_j,\delta_j}\Big)\kappa \\
&= \bigg(0, \bigg(\sum_{j=1}^{d} \theta_{\delta_j,\delta_j}^{(n)} \kappa_n\bigg)_{n\geq 1}\bigg) \\
&= \bigg(0, \bigg(\sum_{j=1}^{d}\sum_{k=1}^{d} (\theta_{\delta_j,\delta_j}\delta_k) \otimes z_k^{(n)}\bigg)_{n\geq 1}\bigg) \\
&= \bigg(0, \bigg(\sum_{j=1}^{d}\sum_{k=1}^{d} (\delta_j(\delta_j \mid \delta_k)_{\mathbb{C}}) \otimes z_k^{(n)}\bigg)_{n\geq 1}\bigg) \\
&= \bigg(0, \bigg(\sum_{j=1}^{d} \delta_j \otimes z_j^{(n)}\bigg)_{n\geq 1}\bigg) = \big(0, (\kappa_n)_{n\geq 1}\big).
\end{aligned}$$

On the other hand, $\theta_{\iota,\iota}\kappa = \iota(\iota \mid \kappa)_{\mathbb{C}} = (\kappa_0, 0, 0, \ldots) \in \mathcal{F}^p$. Thus, $\big(\mathrm{id}_{\mathcal{F}^p} - \theta_{\iota,\iota}\big)\kappa = \big(0, (\kappa_n)_{n\geq 1}\big)$, yielding part 3. This finishes the proof. ■

Proposition 6.3.24. *Let $p \in (1,\infty) \setminus \{2\}$, let I be the index set from equation (6.3.1), let $d \in \mathbb{Z}_{\geq 2}$, and let $\delta_1, \ldots, \delta_d$ be the usual basis for ℓ^p_d, which is also the usual basis for ℓ^q_d. Then, for each $j \in \{1, \ldots, d\}$, the operators $c(\delta_j)$ and $v(\delta_j)$, regarded as elements of $\mathcal{L}(\ell^p(I))$, are spatial partial isometries as in Definition 4.4.9.*

Proof. Fix $j \in \{1, \ldots, d\}$. We will show that $c(\delta_j)$ satisfies the properties for s in Proposition 4.4.14, with $v(\delta_j)$ playing the role of t. This will automatically show that both $c(\delta_j)$ and $v(\delta_j)$ are spatial partial isometries. That $\|c(\delta_j)\| \leq 1$ and $\|v(\delta_j)\| \leq 1$ follows immediately from Lemmas 6.3.13 and 6.3.15. Set $e = v(\delta_j)c(\delta_j)$ and $f = c(\delta_j)v(\delta_j)$. By part 1 in Proposition 6.3.23, $e = \mathrm{id}_{\ell^p(I)}$, which is clearly an idempotent and a spatial isometry. We next show that f is a spatial partial isometry using Lemma 4.4.13. That is, we will find a set $I_j \subseteq I$ such that $f = m(\chi_{I_j})$ (notice that this will automatically prove that f is also an idempotent). Since $d \geq 2$, for each $n \geq 1$ we write the set $\{1, 2, 3, ..., d^n\}$ as the disjoint union of d subsets each of size d^{n-1} in the following way:

$$\{1, 2, 3, ..., d^n\} = \bigsqcup_{k=1}^{d} \big\{(k-1)d^{n-1} + 1, \ldots, kd^{n-1}\big\}.$$

For each $k \in \{1, \ldots, d\}$, we define $I_k^{(n)} = \big\{(k-1)d^{n-1} + 1, \ldots, kd^{n-1}\big\}$. Now we define $I_j \subset I$ as

$$I_j = \bigsqcup_{n\geq 1} I_j^{(n)} = \bigsqcup_{n\geq 1} \big\{(j-1)d^{n-1} + 1, \ldots, jd^{n-1}\big\}.$$

We claim that $f = m(\chi_{I_j})$. To prove this claim we will need the notation introduced in Remark 6.3.11. Let $\kappa = (\kappa_n)_{n\geq 0}$. We have

$$m(\chi_{I_j})\kappa = \big(0, (m(\chi_{I_j^{(n)}})\kappa_n)_{n\geq 1}\big).$$

As before, we can uniquely write for each $n \geq 1$

$$\kappa_n = \sum_{k=1}^{d} \delta_k \otimes z_k^{(n)} \in \ell_d^p \otimes_p \ell_{d^{n-1}}^p.$$

By definition of $I_j^{(n)}$ we find

$$m(\chi_{I_j^{(n)}})\kappa_n = \delta_j \otimes z_j^{(n)}.$$

On the other hand, as in the proof of part 3 in Proposition 6.3.23, we get

$$f\kappa = c(\delta_j)v(\delta_j)(\kappa) = \big(0, (\delta_j \otimes z_j^{(n)})_{n\geq 1}\big) = \big(0, (m(\chi_{I_j^{(n)}})\kappa_n)_{n\geq 1}\big).$$

The claim follows. Finally, since $e = v(\delta_j)c(\delta_j) = \mathrm{id}_{\ell^p(I)}$, we have

$$fc(\delta_j)e = fc(\delta_j) = (c(\delta_j)v(\delta_j))c(\delta_j) = c(\delta_j)e = c(\delta_j),$$

and

$$ev(\delta_j)f = v(\delta_j)f = v(\delta_j)(c(\delta_j)v(\delta_j)) = ev(\delta_j) = v(\delta_j).$$

Thus, all the conditions from Proposition 4.4.14 are satisfied, finishing the proof.

■

Remark 6.3.25. Notice that $\mathcal{O}^p(\varphi_\infty, c, v)$ is a unital algebra as it contains the element $q(v(\delta_1))q(c(\delta_1)) = q(\mathrm{id}_{\ell^p(I)})$, which is the identity element of $\mathcal{Q}(\ell^p(I))$. Henceforth, we denote the unit $\mathcal{O}^p(\varphi_\infty, c, v)$ simply by 1.

Proposition 6.3.26. *Let $p \in (1, \infty)$ and let $d \in \mathbb{Z}_{\geq 2}$. Let $\delta_1, \ldots, \delta_d$ be the usual basis for ℓ^p_d, which is also the usual basis for ℓ^q_d, and let $q_0 \colon \mathcal{L}(\ell^p(I)) \to \mathcal{Q}(\ell^p(I))$ be the quotient map. Then*

1. *For any $j \in \{1, \ldots, d\}$,*
$$q_o(v(\delta_j))q_0(c(\delta_j)) = 1.$$
2. *For j, k in $\{1, \ldots, d\}$, with $j \neq k$*
$$q_0(v(\delta_j))q_0(c(\delta_k)) = 0.$$
3. $$\sum_{j=1}^{d} q_0(c(\delta_j))q_0(v(\delta_j)) = 1.$$

Proof. This is a direct consequence of Proposition 6.3.23. Indeed, parts 1 and 2 are immediate. For part 3 we have

$$\sum_{j=1}^{d} q_0(c(\delta_j))q_0(v(\delta_j)) = q_0\left(\sum_{j=1}^{d} c(\delta_j)v(\delta_j)\right) = q_0(\mathrm{id}_{\ell^p(I)} - \theta_{\iota,\iota}) = q_0(\mathrm{id}_{\ell^p(I)}) = 1,$$

where the second to last equality follows because $\theta_{\iota,\iota} \in \mathcal{K}(\ell^p(I))$. ∎

Proposition 6.3.27. *Let $p \in (1, \infty)$ and let $d \in \mathbb{Z}_{\geq 2}$. Then $\mathcal{O}^p(\varphi_\infty, c, v)$ is a σ-finitely representable L^p operator algebra.*

Proof. That $\mathcal{O}^p(\varphi_\infty, c, v)$ is an L^p operator algebra is clear by definition. Furthermore, $\mathcal{O}^p(\varphi_\infty, c, v)$ is finitely generated as a Banach algebra and therefore is separable. The desired result follows from Proposition 4.2.5 and the fact pointed out in Remark 4.2.2 that a separably representable L^p operator algebra is is σ-finitely representable. ■

From now on we can and will identify $\mathcal{O}^p(\varphi_\infty, c, v)$ as a closed subalgebra of $\mathcal{L}(L^p(\mu))$ for a fixed σ-finite measure space $(\Omega, \mathfrak{M}, \mu)$.

Proposition 6.3.28. *Let $p \in (1,\infty) \setminus \{2\}$, let $d \in \mathbb{Z}_{\geq 2}$, let $\delta_1, \ldots, \delta_d$ be the usual basis for ℓ_d^p, which is also the usual basis for ℓ_d^q, and let L_d be the Leavitt algebra from Definition 4.4.17 . If $\rho\colon L_d \to \mathcal{L}(L^p(\mu))$ is the algebra homomorphism determined by*

$$\rho(s_j) = q(c(\delta_j)) \quad \textit{and} \quad \rho(t_j) = q(v(\delta_j)),$$

for each $j \in \{1, \ldots, d\}$, then ρ is a spatial representation.

Proof. That ρ is indeed a representation of L_d (in the sense of Definition 4.4.18) follows from Proposition 6.3.26. Fix $j \in \{1, \ldots, d\}$. By Proposition 6.3.24, both $c(\delta_j)$ and $v(\delta_j)$ are spatial partial isometries. Since $q\colon \mathcal{L}(\mathcal{F}^p) \to \mathcal{Q}(\mathcal{F}^p)$ is a contractive homomorphism, the elements $q(c(\delta_j))$ and $q(v(\delta_j))$ satisfy the conditions in Proposition 4.4.14 (because $c(\delta_j)$ and $v(\delta_j)$ do), whence ρ is indeed spatial. ■

We now have all the tools to prove Theorem 6.3.22:

Proof of Theorem 6.3.22. If $p = 2$ and $j \in \{1, \ldots, d\}$, we see that $q(c(\delta_j))^* = q(v(\delta_j))$. Then it follows from Proposition 6.3.26 that the elements $q(c(\delta_1)), \ldots, q(c(\delta_d)) \in \mathcal{O}^p(\varphi_\infty, c, v)$ satisfy the universal property of $\mathcal{O}_2$. Thus, by simplicity of $\mathcal{O}_2$, we must have $\mathcal{O}^p(\varphi_\infty, c, v) \cong \mathcal{O}_2$. Phillips showed below Definition 8.8 of [21] that $\mathcal{O}_d^2 \cong \mathcal{O}_2$. So the case $p = 2$ is done.

Suppose now that $p \neq 2$. Let ρ be the spatial representation of Proposition 6.3.28 above. It follows from Theorem 4.4.19 and Definition 4.4.20 that $\overline{\rho(L_d)}$

is isometrically isomorphic to $\mathcal{O}_d^p$. On the other hand, by Definition 6.3.21, the algebra $\overline{\rho(L_d)}$ is equal to $\mathcal{O}^p(\varphi_\infty, c, v)$, and we are done. ∎

Given that when $p = 2$, the C*-algebra $\mathcal{O}^2(\varphi_\infty, c, v)$ is both the universal Cuntz-Pimsner C*-algebra for (ℓ_d^2, φ_d) and also $\mathcal{O}_d$ (see Theorem 2.4.13 and Example 2.4.16), the following question arises

Question 6.3.29. Let $p \in (1,\infty)$ and let $d \in \mathbb{Z}_{\geq 2}$. Is $O_d^p = \mathcal{O}^p(\varphi_\infty, c, v)$ isometrically isomorphic to $\mathcal{O}^p((\ell_d^p, \ell_d^q), \varphi_d)$? In other words, is $(\mathcal{Q}(\ell^p(I)), q_0 \circ \varphi_\infty, q_0 \circ c, q_0 \circ v)$ the universal covariant L^p-Fock representation for $((\ell_d^p, \ell_d^q), \varphi_d)$?

Do L^p-crossed products by $\mathbb{Z}$ come from L^p-Fock representations ?

Let $p \in (1, \infty)$, let $(\Omega, \mathfrak{M}, \mu)$ be a measure space, and let $A \subseteq \mathcal{L}(L^p(\mu))$ be an L^p-operator algebra with a contractive approximate identity $(e_\lambda)_{\lambda\in\Lambda}$. At some point, will also need assume that $(e_\lambda)_{\lambda\in\Lambda}$ is bicontractive, that is $\|\mathrm{id}_{L^p(\mu)} - e_\lambda\| \leq 1$ for all $\lambda \in \Lambda$, which makes A a biapproximately unital algebra in the sense of Definition 4.2 in [1].

We start by showing that $((A, A), \varphi_A)$, the L^p-correspondence over A from Example 6.1.2 has a L^p-Fock representations and a covariant one. To safely use that $J_{(A,A)} = A$, we only need to make sure that $\varphi_A(A) \subseteq \mathcal{K}_A((A, A))$ (φ_A is already injective being an automorphism). Let $a \in A$ and notice that, since A has an approximate identity, the Cohen-Hewitt factorization theorem implies that $\varphi(a) = a_0a_1$ for some $a_0, a_1 \in A$ (in fact, Theorem 1 in [4] suffices here). Then for any $b \in A$, $\varphi(a)b = a_0a_1b = a_0(a_1 \mid b)_A = \theta_{a_0,a_1}b$. Hence, $\varphi(a) \in \mathcal{K}_A((A, A))$ for any $a \in A$, and therefore $\varphi_A(A) \subseteq \mathcal{K}_A((A, A))$, as wanted.

Our first goal is to get an L^p-Fock representation for $((A, A), \varphi_A)$. This time, we will not follow the usual Fock space construction. Instead, we will take advantage of the results from Proposition 2.4.24 in the C*-case.

Definition 6.3.30. Let $A \subseteq \mathcal{L}(L^p(\mu))$ be an L^p-operator algebra with a contractive approximate identity and let ν denote counting measure on $\mathbb{Z}_{\geq 0}$. We use the external tensor product of L^p-modules (see Definition 5.4.1) to to define $(\mathcal{F}^p(A), \mathcal{F}^q(A)) = (\ell^p(\mathbb{Z}_{\geq 0}), \ell^q(\mathbb{Z}_{\geq 0})) \otimes_p (A, A)$ regarded as an L^p-module over $\mathbb{C} \otimes_p A = A$. That is, $\mathcal{F}^p(A) = \ell^p(\mathbb{Z}_{\geq 0}) \otimes_p A \subseteq \mathcal{L}(L^p(\mu), L^p(\nu \times \mu))$ and $\mathcal{F}^q(A) = \ell^q(\mathbb{Z}_{\geq 0}) \otimes_p A \subseteq \mathcal{L}(L^p(\nu \times \mu), L^p(\mu))$ after making the usual identifications $\ell^p(\mathbb{Z}_{\geq 0}) = \mathcal{L}(\ell_1^p, \ell^p(\mathbb{Z}_{\geq 0}))$ and $\ell^q(\mathbb{Z}_{\geq 0}) = \mathcal{L}(\ell^p(\mathbb{Z}_{\geq 0}), \ell_1^p)$.

Remark 6.3.31. Is worth mentioning that the pair $(\mathcal{F}^p(A), \mathcal{F}^q(A))$ from Definition 6.3.30 can be equivalently defined using the tensor product of L^p-correspondences introduced in Definition 6.2.1. Indeed, identify $\ell^p(\mathbb{Z}_{\geq 0})$ with a subspace of $\mathcal{L}(L^p(\mu), \ell^p(\mathbb{Z}_{\geq 0}) \otimes_p L^p(\mu))$ via $\xi \mapsto x \otimes \xi$ for any $x \in \ell^p(\mathbb{Z}_{\geq 0})$ and $\xi \in L^p(\mu)$. Similarly, identify $\ell^q(\mathbb{Z}_{\geq 0})$ with a subspace of $\mathcal{L}(\ell^p(\mathbb{Z}_{\geq 0}) \otimes_p L^p(\mu), L^p(\mu))$ via $x \otimes \xi \mapsto \langle y, x\rangle \xi$ for any $y \in \ell^q(\mathbb{Z}_{\geq 0})$ and $x \otimes \xi \in \ell^p(\mathbb{Z}_{\geq 0}) \otimes_p L^p(\mu)$. Then, under these identifications, $(\ell^p(\mathbb{Z}_{\geq 0}), \ell^q(\mathbb{Z}_{\geq 0}))$ is a C*-like L^p module over $\mathbb{C} \subseteq \mathcal{L}(L^p(\mu))$, where $\mathbb{C}$ is identified with the operator given by scalar multiplication. Further, if we let $\psi_C(z) = z \cdot \mathrm{id}_{L^p(\nu \times \mu)}$ for any $z \in \mathbb{C}$, then $((\ell^p(\mathbb{Z}_{\geq 0}), \ell^q(\mathbb{Z}_{\geq 0})), \psi_{\mathbb{C}})$ becomes an L^p-correspondence over $\mathbb{C}$. Now let (A, A) be the L^p-module from Definition 5.1.4 and make it a $(\mathbb{C}, A)$ L^p-correspondence via $\rho\colon \mathbb{C} \to \mathcal{L}_A((A, A))$ defined as $\rho(z) = z \cdot \mathrm{id}_{L^p(\mu)}$. Then, using Definition 6.2.1, we get

$$((\ell^p(\mathbb{Z}_{\geq 0}), \ell^q(\mathbb{Z}_{\geq 0})), \psi_{\mathbb{C}}) \otimes_\rho ((A, A), \rho),$$

a $(\mathbb{C}, A)$ L^p-correspondence. It's not hard to check that the underlying L^p module over A for this correspondence is actually the pair $(\mathcal{F}^p(A), \mathcal{F}^q(A))$ defined in Definition 6.3.30.

To define maps from specific sets to $\mathcal{L}_A((\mathcal{F}^p(A), \mathcal{F}^q(A)))$, we will adopt a general strategy. While we provide a detailed account of this strategy for the map φ_A^∞ defined below, we omit it for the maps c_A, v_A, Θ_A, and several others as the arguments for these maps are almost identical.

As before, $\{\delta_n \colon n \in \mathbb{Z}_{\geq 0}\}$ is the canonical basis for $\ell^p(\mathbb{Z}_{\geq 0})$. For each $a \in A$, each $x \in \ell^p(\mathbb{Z}_{\geq}0)$, and each $\xi \in L^p(\mu)$ we define

$$\varphi_A^\infty(a)(x \otimes \xi) = \sum_{n=0}^{\infty} x(n)\delta_n \otimes \varphi_A^n(a)\xi. \tag{6.3.8}$$

This formula is motivated from the one in equation (2.4.6) for the C*-case. We claim that $\varphi^\infty(a)(x \otimes \xi) \in L^p(\nu \times \mu)$ for any $a \in A$, and any $x \otimes \xi \in L^p(\nu \times \mu)$. To see this, take any $m, k \in \mathbb{Z}_{\geq 0}$ with $k < m$,

$$\begin{aligned}
\Big\| \sum_{n=k}^{m} x(n)\delta_n \otimes \varphi_A^n(a)\xi \Big\|_p^p &= \sum_{j=0}^{\infty} \int_\Omega \Big| \sum_{n=k}^{m} x(n)\delta_j(n)(\varphi_A^n(a)\xi)(\omega) \Big|^p d\mu(\omega) \\
&= \sum_{j=k}^{m} \int_\Omega |x(j)(\varphi_A^j(a)\xi)(\omega)|^p d\mu(\omega) \\
&= \sum_{j=k}^{m} |x(j)|^p \|\varphi_A^j(a)\xi\|_p^p \\
&\leq \|a\|^p \|\xi\|_p^p \sum_{j=k}^{m} |x(j)|^p
\end{aligned}$$

Since $x \in \ell^p(\mathbb{Z}_{\geq 0})$, this proves that the series in the RHS of equation (6.3.8) converges to an element of $L^p(\nu \times \mu)$.

Lemma 6.3.32. *The assignment $x \otimes \xi \mapsto \varphi_A^\infty(a)(x \otimes \xi)$ given by equation (6.3.8) extents to a bounded linear map in $\mathcal{L}(L^p(\nu \times \mu))$, also denoted by $\varphi_A^\infty(a)$, satisfying $\|\varphi_A^\infty(a)\| = \|a\|$.*

Proof. Take $k \in \mathbb{Z}_{\geq 1}$, let $x_1, \ldots, x_k \in \ell^p(\mathbb{Z}_{\geq 0})$, and let $\xi_1, \ldots, \xi_k \in L^p(\mu)$. Then

$$\begin{aligned}
\Big\| \sum_{j=1}^{k} \sum_{n=0}^{\infty} x_j(n)\delta_n \otimes \varphi_A^n(a)\xi_j \Big\|_p^p &= \sum_{m=0}^{\infty} \int_\Omega \Big| \sum_{j=1}^{k} \sum_{n=0}^{\infty} x_j(n)\delta_n(m)(\varphi_A^n(a)\xi_j)(\omega) \Big|^p d\mu(\omega) \\
&= \sum_{m=0}^{\infty} \int_\Omega \Big| \sum_{j=1}^{k} x_j(m)(\varphi_A^m(a)\xi_j)(\omega) \Big|^p d\mu(\omega) \\
&= \sum_{m=0}^{\infty} \int_\Omega \Big| \varphi_A^m(a)\big(\sum_{j=1}^{k} x_j(m)\xi_j \big)(\omega) \Big|^p d\mu(\omega) \\
&= \sum_{m=0}^{\infty} \Big\| \varphi_A^m(a)\big(\sum_{j=1}^{k} x_j(m)\xi_j \big) \Big\|_p^p \\
&\leq \|a\|^p \sum_{m=0}^{\infty} \Big\| \sum_{j=1}^{k} x_j(m)\xi_j \Big\|_p^p \\
&= \|a\|^p \Big\| \sum_{j=1}^{k} x_j \otimes \xi_j \Big\|_p^p.
\end{aligned}$$

Therefore, we can first extend the assignment in equation (6.3.8) by linearity to sums of elementary tensors and then to all of $L^p(\nu \times \mu)$ to get an element $\varphi_A^\infty(a)$ in $\mathcal{L}(L^p(\nu \times \mu))$ satisfying $\|\varphi_A^\infty(a)(\eta)\| \leq \|a\|\|\eta\|$ for all $\eta \in L^p(\nu \times \mu)$. Hence, $\|\varphi_A^\infty(a)\| \leq \|a\|$. For the reverse inequality, take $x_0 = (1, 0, 0, \ldots) \in \ell^p(\mathbb{Z}_{\geq 0})$ and $\xi \in L^p(\mu)$ with $\|\xi\|_p = 1$. Then $\|x_0 \otimes \xi\|_p = 1$ and therefore

$$\|\varphi_A^\infty(a)\| \geq \|\varphi_A^\infty(a)(x_0 \otimes \xi)\| = \|a\xi\|,$$

proving that $\|\varphi_A^\infty(a)\| \geq \|a\|$. ∎

Proposition 6.3.33. *Let $(\mathcal{F}^p(A), \mathcal{F}^p(A))$ be the L^p module from Definition 6.3.30. Then, for each $a \in A$, $\varphi^\infty(a) \in \mathcal{L}_A((\mathcal{F}^p(A), \mathcal{F}^q(A)))$. Furthermore, $\varphi_A^\infty \colon A \to \mathcal{L}_A((\mathcal{F}^p(A), \mathcal{F}^p(A)))$ is an isometric algebra homomorphism*

Proof. First we show that, for any $a, b \in A$, $x \in \ell^p(\mathbb{Z}_{\geq 0})$, and $y \in \ell^q(\mathbb{Z}_{\geq 0})$, the compositions

$$\varphi_A^\infty(a)(x \otimes b) \colon L^p(\mu) \to L^p(\nu \times \mu)$$

and

$$(y \otimes b)\varphi_A^\infty(a) \colon L^p(\nu \times \mu) \to L^p(\mu)$$

are in $\mathcal{F}^p(A)$ and $\mathcal{F}^q(A)$ respectively. To do so, notice that a similar computation to the one shown above Lemma 6.3.32 shows that

$$\sum_{n=0}^{\infty} x(n)\delta_n \otimes \varphi^n(a)b \in \mathcal{F}^p(A),$$

and it is clear that this operator coincides with $\varphi_A^\infty(a)(x \otimes b)$. Similarly, one checks that

$$\sum_{n=0}^{\infty} y(n)\delta_n \otimes b\varphi^n(a)$$

is an element of $\mathcal{F}^q(A)$ that coincides with $(y \otimes b)\varphi_A^\infty(a)$. That $\|\varphi_A^\infty(a)\| = \|a\|$ was shown in Lemma 6.3.32 and that φ_A^∞ is an algebra homomorphism is a direct computation done on elements of the form $x \otimes \xi \in L^p(\nu \times \mu)$. ■

We now have established the following.

Corollary 6.3.34. *Let $(\mathcal{F}^p(A), \mathcal{F}^p(A))$ be the L^p module from Definition 6.3.30. Then $((\mathcal{F}^p(A), \mathcal{F}^q(A)), \varphi_A^\infty)$ is an L^p-correspondence.*

By analogous arguments to those presented for Lemma 6.3.32 and in its preceding discussion, now looking at equations (2.4.7) and (2.4.8), for each $a \in A$ we define operators $c_A(a), v_A(a) \in \mathcal{L}(L^p(\nu \times \mu))$, with norm equal to $\|a\|$, such that when $x \in \ell^p(\mathbb{Z}_{\geq 0})$, and $\xi \in L^p(\mu)$,

$$c_A(a)(x \otimes \xi) = \sum_{n=1}^{\infty} (sx)(n)\delta_n \otimes \varphi_A^{n-1}(a)\xi, \tag{6.3.9}$$

where $s \in \mathcal{L}(\ell^p(\mathbb{Z}_{\geq 0}))$ is right translation; and

$$v_A(a)(x \otimes \xi) = \sum_{n=0}^{\infty} (tx)(n)\delta_n \otimes \varphi_A^{n}(a)\xi, \tag{6.3.10}$$

with $t \in \mathcal{L}(\ell^p(\mathbb{Z}_{\geq 0}))$ given by left translation.

Proposition 6.3.35. *Let $(\mathcal{F}^p(A), \mathcal{F}^p(A))$ be the L^p module from Definition 6.3.30. Then, for each $a \in A$, $c_A(a), v_A(a) \in \mathcal{L}_A((\mathcal{F}^p(A), \mathcal{F}^q(A)))$. Furthermore, the following conditions hold for any $a, b \in A$.*

1. *$c_A(ab) = c_A(a)\varphi_A^{\infty}(b)$, and $c_A(\varphi_A(a)b) = \varphi_A^{\infty}(a)c_A(b)$,*
2. *$v_A(ab) = \varphi_A^{\infty}(a)v_A(b)$, and $v_A(a\varphi_A(b)) = v_A(a)\varphi_A^{\infty}(b)$,*
3. *$v_A(a)c_A(b) = \varphi_A^{\infty}(ab)$.*

Proof. That $c_A(a), v_A(a) \in \mathcal{L}_A((\mathcal{F}^p(A), \mathcal{F}^q(A)))$ with norm equal to $\|a\|$ follows from similar arguments as those used in the proof of Proposition 6.3.33. We now only need to check that for any $a, b \in A$, parts 1-3 in the statement hold. Since all operators are elements in $\mathcal{L}(L^p(\nu \times \mu))$, suffices to check each equality only by acting on elements of the form $x \otimes \xi \in L^p(\nu \times \mu)$. Indeed, for part 1 we get,

$$c_A(a)\varphi_A^{\infty}(b)(x \otimes \xi) = c_A(a)\Big(\sum_{n=0}^{\infty} x(n)\delta_n \otimes \varphi^n(b)\xi \Big)$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} x(n)(s\delta_n)(k)\delta_k \otimes \varphi_A^{k-1}(a)\varphi_A^{n}(b)\xi \\
&= \sum_{n=1}^{\infty} x(n-1)\delta_n \otimes \varphi_A^{n-1}(a)\varphi_A^{n-1}(b)\xi \\
&= \sum_{n=1}^{\infty} (sx)(n)\delta_n \otimes \varphi_A^{n-1}(ab)\xi \\
&= c_A(ab)(x \otimes \xi),
\end{aligned}$$

and

$$\begin{aligned}
\varphi_A^{\infty}(a)c_A(b)(x\otimes\xi) &= \varphi_A^{\infty}(a)\Big(\sum_{n=1}^{\infty}(sx)(n)\delta_n \otimes \varphi^{n-1}(b)\xi\Big) \\
&= \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}(sx)(n)\delta_n(k)\delta_k \otimes \varphi_A^{k}(a)\varphi_A^{n-1}(b)\xi \\
&= \sum_{n=1}^{\infty}(sx)(n)\delta_n \otimes \varphi_A^{n}(a)\varphi_A^{n-1}(b)\xi \\
&= \sum_{n=1}^{\infty}(sx)(n)\delta_n \otimes \varphi_A^{n-1}(\varphi_A(a)b)\xi \\
&= c_A(\varphi(a)b)(x\otimes\xi).
\end{aligned}$$

For part 2, we find

$$\begin{aligned}
\varphi_A^{\infty}(a)v_A(b)(x\otimes\xi) &= \varphi_A^{\infty}(a)\Big(\sum_{n=0}^{\infty}(tx)(n)\delta_n \otimes \varphi^{n}(b)\xi\Big) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(tx)(n)\delta_n(k)\delta_k \otimes \varphi_A^{k}(a)\varphi_A^{n}(b)\xi \\
&= \sum_{n=0}^{\infty}(tx)(n)\delta_n \otimes \varphi_A^{n}(ab)\xi \\
&= v_A(ab)(x\otimes\xi),
\end{aligned}$$

and

$$\begin{aligned}
v_A(a)\varphi_A^\infty(b)(x \otimes \xi) &= v_A(a)\Big(\sum_{n=0}^{\infty} x(n)\delta_n \otimes \varphi^n(b)\xi\Big) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} x(n)(t\delta_n)(k)\delta_k \otimes \varphi_A^k(a)\varphi_A^n(b)\xi \\
&= \sum_{n=0}^{\infty} x(n+1)\delta_n \otimes \varphi_A^n(a)\varphi_A^{n+1}(b)\xi \\
&= \sum_{n=0}^{\infty} (tx)(n)\delta_n \otimes \varphi_A^n(a\varphi_A(b))\xi \\
&= v_A(a\varphi_A(b))(x \otimes \xi).
\end{aligned}$$

Finally, to show part 3 we compute

$$\begin{aligned}
v_A(a)c_A(b)(x \otimes \xi) &= v_A(a)\Big(\sum_{n=1}^{\infty} (sx)(n)\delta_n \otimes \varphi_A^{n-1}(b)\xi\Big) \\
&= \sum_{n=1}^{\infty} v_A(a)((sx)(n)\delta_n \otimes \varphi_A^{n-1}(b)\xi) \\
&= \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} (sx)(n)(t\delta_n)(k)\delta_k \otimes \varphi_A^k(a)\varphi_A^{n-1}(b)\xi \\
&= \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} x(n-1)\delta_n(k+1)\delta_k \otimes \varphi_A^k(a)\varphi_A^{n-1}(b)\xi \\
&= \sum_{k=0}^{\infty} x(k)\delta_k \otimes \varphi_A^k(ab)\xi \\
&= \varphi_A^\infty(ab)(x \otimes \xi).
\end{aligned}$$

This finishes the proof. ■

We have now proven the following statement:

Corollary 6.3.36. *Let $(\mathcal{F}^p(A), \mathcal{F}^p(A))$ be the L^p module from Definition 6.3.30. Then $(\mathcal{L}_A((\mathcal{F}^p(A), \mathcal{F}^p(A))), \varphi_A^\infty, c_A, v_A)$ is an L^p-Fock correspondence for $((A, A), \varphi_A)$.*

For convenience and future use, we record below a precise definition for $F^p(\mathcal{L}_A((\mathcal{F}^p(A), \mathcal{F}^q(A))), \varphi_A^\infty, c_A, v_A)$, the L^p-operator algebra generated by the L^p-Fock representation from Corollary 6.3.36. Since $(A \mid A)_A = A$, recall from 6.3.2 that in this case we only need $c_A(A)$ and $v_A(A)$.

Definition 6.3.37. Let $p \in (1, \infty)$, let $(\Omega, \mathfrak{M}, \mu)$ be a measure space, and let $A \subseteq \mathcal{L}(L^p(\mu))$ be an L^p-operator algebra with a contractive approximate identity. We define $\mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$ as the closed algebra in $\mathcal{L}_A((\mathcal{F}^p(A), \mathcal{F}^q(A)))$ generated by $c_A(A)$ and $v_A(A)$.

Remark 6.3.38. Let $\mathcal{T}_0^p(\varphi_A^\infty, c_A, v_A)$ denote the complex algebra generated by $\varphi_A^\infty(A)$, $c_A(A)$, $v_A(A)$ in $\mathcal{L}(L^p(\nu \times \mu))$. Then, using the relations from Proposition 6.3.35, we see that any element in $\mathcal{T}_0^p(\varphi_A^\infty, c_A, v_A)$ is a linear combination of elements of the form $\varphi_A^\infty(ab) = v_A(a)c_A(b)$ with $a, b \in A$ and of the form

$$c_A(a_1) \cdots c_A(a_n) v_A(b_1) \cdots v_A(b_k)$$

where $n, k \in \mathbb{Z}_{\geq 1}$ and $a_1, \ldots, a_n, b_1, \ldots, b_k \in A$. Then $\mathcal{T}_0^p(\varphi_A^\infty, c_A, v_A) \subseteq \mathcal{L}_A(\mathcal{F}^p(A), \mathcal{F}^q(A))$ and $\mathcal{T}_0^p(\varphi_A^\infty, c_A, v_A)$ is dense in $\mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$,

Next, we establish that the L^p-operator algebra $\mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$ just defined has a c.a.i. and is nondegenerately represented as long as A has both characteristics.

Proposition 6.3.39. *Let $p \in (1, \infty)$, let $A \subseteq \mathcal{L}(L^p(\mu))$ be an L^p-operator algebra with a c a.i., and assume that A sits nondegenerately in $\mathcal{L}(L^p(\mu))$. Then* *the L^p-operator algebra $\mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$ from Definition 6.3.37 has a c.a.i and sits nondegenerately in $\mathcal{L}(L^p(\nu \times \mu))$.*

Proof. Let $(e_\lambda)_{\lambda\in\Lambda}$ be a c.a.i. for A. We claim that $(\varphi_A^\infty(e_\lambda))$ is a c.a.i. for $\mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$. By Remark 6.3.38, suffices to prove that $(\varphi_A^\infty(e_\lambda))$ is an approximate identity for the algebra $\mathcal{T}_0^p(\varphi_A^\infty, c_A, v_A)$ and this follows at once from conditions 1 and 2 from Proposition 6.3.35, so the claim is proved. Next, we show that $\mathcal{T}^p(\varphi_A^\infty, c_A, v_A)L^p(\nu \times \mu)$ is dense in $L^p(\nu \times \mu)$. Since linear combinations of elementary tensors $x \otimes \xi \in L^p(\nu \times \mu)$ are dense in $L^p(\nu \times \mu)$, and since A sits nondegenerately in $L^p(\mu)$ it suffices to show that $x \otimes a\xi$ is in $\overline{\mathcal{T}^p(\varphi_A^\infty, c_A, v_A)L^p(\nu \times \mu)}$ for any $x \in \ell^p(\mathbb{Z})$, $a \in A$ and any $\xi \in L^p(\mu)$. Thus, fix $x \in \ell^p(\mathbb{Z})$, $a \in A$, $\xi \in L^p(\mu)$, and for each $n \in \mathbb{Z}_{\geq 0}$, let $x_n \in \ell^p(\mathbb{Z}_{\geq 0})$ be given by $x_n = x(n)\delta_n$ and let $t_n \in \mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$ be given by $t_n = \varphi_A^\infty(\varphi^{-n}(a))$. Then, for each $k \in \mathbb{Z}_{\geq 0}$, we get

$$\sum_{n=0}^{k} x(n)\delta_n \otimes a\xi = \sum_{k=0}^{n} t_n(x_n \otimes \xi) \in \mathcal{T}^p(\varphi_A^\infty, c_A, v_A)L^p(\nu \times \mu).$$

Since the the sum in the left hand side of the previous equation converges to $x \otimes a\xi$, we have shown that $x \otimes a\xi \in \overline{\mathcal{T}^p(\varphi_A^\infty, c_A, v_A)L^p(\nu \times \mu)}$, as we needed to do. ■

The following Lemma shows that the norm of a product of c_A's (and also v_A's) still follows the same pattern as the C*-case (see Lemma 2.4.21). Furthermore, it provides a tool for us to easily perform calculations in $\mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$.

Lemma 6.3.40. *Let $c_A, v_A \in \mathcal{L}(L^p(\nu \times \mu))$ as defined in (6.3.9) and (6.3.10). If $n, m \in \mathbb{Z}_{\geq 1}$ and $a_1, \ldots, a_n, b_1, \ldots, b_m \in A$, then for any $x \otimes \xi \in L^p(\nu \times \mu)$*

$$c_A(a_1) \cdots c_A(a_n)(x \otimes \xi) = \sum_{k=1}^{\infty} (sx)(k) \delta_{k+n-1} \otimes \varphi^{k-1}(\varphi^{n-1}(a_1) \cdots \varphi(a_{n-1}) a_n) \xi, \tag{6.3.11}$$

$$v_A(b_1) \cdots v_A(b_m)(x \otimes \xi) = \sum_{k=m-1}^{\infty} (tx)(k) \delta_{k-(m-1)} \otimes \varphi^{k-(m-1)}(b_1 \varphi(b_2) \cdots \varphi^{m-1}(b_m)) \xi, \tag{6.3.12}$$

$$c_A(a_1) \cdots c_A(a_n) v_A(b_1) \cdots v_A(b_m)(x \otimes \xi) = \sum_{k=m-1}^{\infty} (tx)(k) \delta_{k-m+n+1} \otimes \varphi^{k-(m-1)}(c) \xi, \tag{6.3.13}$$

where $c = a_1 \varphi(a_2) \cdots \varphi^{n-1}(a_n) b_1 \varphi(b_2) \cdots \varphi^{m-1}(b_m)$. In particular,

$$\|c_A(a_1) \cdots c_A(a_n)\| = \|\varphi^{n-1}(a_1) \cdots \varphi(a_{n-1}) a_n\|,$$

$$\|v_A(b_1) \cdots v_A(b_m)\| = \|b_1 \varphi(b_2) \cdots \varphi^{m-1}(b_m)\|,$$

$$\|c_A(a_1) \cdots c_A(a_n) v_A(b_1) \cdots v_A(b_m)\| = \|c\|.$$

Proof. Recursive applications of the formulas from (6.3.9) and (6.3.10) give the first three equations. The desired norm equalities follow at once by the same methods used in Lemma 6.3.32 to show that $\|\varphi^{\infty}(a)\| = \|a\|$. ■

Next, we define the necessary maps to obtain a covariant L^p-Fock representation. Yet again by the same techniques used in Lemma 6.3.32 and in its preceding discussion, for each $a \in A$ we get a map $\Theta_A(a) \in \mathcal{L}(L^p(\nu \times \mu))$, with norm bounded by a, such that whenever $x \in \ell^p(\mathbb{Z}_{\geq 0})$ and $\xi \in L^p(\mu)$,

$$\Theta_A(a)(x \otimes \xi) = \sum_{n=1}^{\infty} x(n)\delta_n \otimes \varphi^{n-1}(a)\xi. \tag{6.3.14}$$

Recall from Example 5.5.2 that $\mathcal{K}_A((A, A))$ is identified with A.

Proposition 6.3.41. *Let $(\mathcal{F}^p(A), \mathcal{F}^p(A))$ be the L^p module from Definition 6.3.30. Then the map $a \mapsto \Theta_A(a)$ defines a contractive homomorphism $\mathcal{K}_A((A, A)) \to \mathcal{L}_A((\mathcal{F}^p(A), \mathcal{F}^q(A)))$. Furthermore, $c_A(a)v_A(b) = \Theta_A(ab)$ for any $a, b \in A$.*

Proof. Let $a \in \mathcal{K}_A((A, A)) = A$. That $\Theta_A(a) \in \mathcal{L}_A((\mathcal{F}^p(A), \mathcal{F}^q(A)))$ and that Θ_A is a homomorphism follows by the same reasoning used for φ_A^{∞}. We now only need to check that for any $a, b \in A$, $c_A(a)v_A(b) = \Theta_A(ab)$ as elements in $\mathcal{L}(L^p(\nu \times \mu))$. As before, it is enough to check equality by acting on elements of the form $x \otimes \xi \in L^p(\nu \times \mu)$:

$$\begin{aligned}
c_A(a)v_A(b)(x \otimes \xi) &= c_A(a)\Big(\sum_{n=0}^{\infty}(tx)(n)\delta_n \otimes \varphi_A^n(b)\xi\Big) \\
&= \sum_{n=0}^{\infty} c_A(a)((tx)(n)\delta_n \otimes \varphi_A^n(b)\xi) \\
&= \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(tx)(n)(s\delta_n)(k)\delta_k \otimes \varphi_A^{k-1}(a)\varphi_A^n(b)\xi \\
&= \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} x(n+1)\delta_n(k-1)\delta_k \otimes \varphi_A^{k-1}(a)\varphi_A^n(b)\xi \\
&= \sum_{k=1}^{\infty} x(k)\delta_k \otimes \varphi_A^{k-1}(ab)\xi \\
&= \Theta_A(ab)(x \otimes \xi).
\end{aligned}$$

This finishes the proof. ∎

Proposition 6.3.42. *Let $(\mathcal{F}^p(A), \mathcal{F}^p(A))$ be the L^p module from Definition 6.3.30. Then $\mathcal{L}_A((\mathcal{F}^p(A), \mathcal{F}^q(A)))/\mathcal{K}_A((\mathcal{F}^p(A), \mathcal{F}^q(A)))$ is an L^p-operator algebra.*

Proof. We will show that $\mathcal{K}_A((\mathcal{F}^p(A), \mathcal{F}^q(A)))$ has a bicontractive approximate identity. The desired result will then follow from part (1) of Lemma 4.5 in [1]. For each $n \in \mathbb{Z}_{\geq 0}$, let $p_n \in \mathcal{L}(\ell^p(\mathbb{Z}_{\geq 0}))$ be the projection onto the subspace spanned by $\{\delta_0, \ldots, \delta_n\}$. That is, for any $x \in \ell^p(\mathbb{Z}_{\geq 0})$,

$$p_n x = \sum_{j=0}^{n} x(j)\delta_j.$$

It's well known that $(p_n)_{n\geq 0}$ is a bicontractive identity for $\mathcal{K}(\ell^p(\mathbb{Z}_{\geq 0}))$. Recall that A has a bicontractive approximate identity $(e_\lambda)_{\lambda\in\Lambda}$. First of all, we check that $p_n \otimes e_\lambda$, which lies in $\mathcal{L}(L^p(\nu \times \mu))$, is an element of $\mathcal{K}_A((\mathcal{F}^p(A), \mathcal{F}^q(A)))$. Indeed, for each $\lambda \in \Lambda$, we use the Cohen-Hewitt factorization theorem (again Theorem 1 in [4] is enough), to find $u_\lambda, v_\lambda \in A$ such that $u_\lambda v_\lambda = e_\lambda$. Then, for each $n \in \mathbb{Z}_{\geq 0}$ and $\lambda \in \Lambda$, we have

$$p_n \otimes e_\lambda = \sum_{j=0}^{n} \theta_{\delta_j \otimes u_\lambda, \delta_j \otimes v_\lambda} \in \mathcal{K}_A((\mathcal{F}^p(A), \mathcal{F}^q(A))).$$

Furthermore, $\|p_n \otimes e_\lambda\| = \|p_n\|\|e_\lambda\| \leq 1$, and $\|\mathrm{id}_{L^p(\nu\times\mu)} - (p_n \otimes e_\lambda)\| = \|\mathrm{id}_{\ell^p(\mathbb{Z}_{\geq 0})} - p_n\|\|\mathrm{id}_{L^p(\mu)} - e_\lambda\| \leq 1$. Thus, it only remains for us to show that $(p_n \otimes e_\lambda)_{(\lambda,n)\in\Lambda\times\mathbb{Z}_{n\geq 0}}$ is a an approximate identity for $\mathcal{K}_A((\mathcal{F}^p(A), \mathcal{F}^q(A)))$. To do so, we show first that for any $\kappa \in \mathcal{F}^p(A)$, the net $(\|(p_n \otimes e_\lambda)\kappa - \kappa\|)_{(\lambda,n)\in\Lambda\times\mathbb{Z}_{n\geq 0}}$ converges to 0. By density, suffices to show it when $\kappa_0 = \sum_{j=1}^{k} x_j \otimes a_j$, for $x_1, \ldots, x_k \in \ell^p(\mathbb{Z}_{\geq 0})$ and $a_1, \ldots, a_k \in A$. Let $\varepsilon > 0$ and for each $j \in \{1, \ldots, k\}$ choose $\lambda_j \in \Lambda$ and $n_j \in \mathbb{Z}_{\geq 0}$

such that, whenever $(\lambda, n) \geq (\lambda_j, n_j)$,

$$\|a_j e_\lambda - a\| < \sqrt{\frac{\varepsilon}{k}},\ \|p_n x_j - x_j\| < \sqrt{\frac{\varepsilon}{k}}.$$

Then for $(\lambda, n) \geq \max_{1\leq j\leq k}(\lambda_j, n_j)$,

$$\|(p_n \otimes e_\lambda)\kappa_0 - \kappa_0\| \leq \sum_{j=1}^{k} \|p_n x_j - x_j\|\|a_j e_\lambda - a\| < \varepsilon$$

as wanted. Finally, for $\kappa_1, \ldots, \kappa_m \in \mathcal{F}^p(A)$ and $\tau_1, \ldots, \tau_m \in \mathcal{F}^q(A)$ let $t_0 = \sum_{j=1}^{m} \theta_{\kappa_j,\tau_j}$. Then we have

$$\|(p_n \otimes e_\lambda)t_0 - t_0\| = \left\| \sum_{j=1}^{m} \theta_{(p_n\otimes e_\lambda)\kappa_j - \kappa_j, \tau_j} \right\| \leq \sum_{j=1}^{m} \|(p_n \otimes e_\lambda)\kappa_j - \kappa_j\|\|\tau_j\|,$$

whence the net $(\|(p_n \otimes e_\lambda)t_0 - t_0\|)_{(\lambda,n)\in\Lambda\times\mathbb{Z}_{n\geq 0}}$ converges to 0. This shows that, for any $t \in \mathcal{K}_A((\mathcal{F}^p(A), \mathcal{F}^q(A)))$, the net $(\|(p_n \otimes e_\lambda)t - t\|)_{(\lambda,n)\in\Lambda\times\mathbb{Z}_{n\geq 0}}$ converges to 0, finishing the proof. ■

Remark 6.3.43. The proof of Proposition 6.3.42 relies on the fact that $\mathcal{K}_A((\mathcal{F}^p(A), \mathcal{F}^q(A)))$ has a bicontractive approximate identity which was produced by bicontractive approximate identities of $\mathcal{K}(\ell^p(\mathbb{Z}_{\geq 0}))$ and A. This works because $\mathcal{K}_A((\mathcal{F}^p(A), \mathcal{F}^q(A)))$, as a subspace of $\mathcal{L}(L^p(\nu \times \mu)$, is equal to $\mathcal{K}(\ell^p(\mathbb{Z}_{\geq 0})) \otimes_p A$. Indeed, this follows at once from the Cohen-Hewitt factorization Theorem and the fact that $\theta_{x\otimes a, y\otimes b} = \theta_{x,y} \otimes ab$ for any $x \in \ell^p(\mathbb{Z}_{\geq 0})$, $y \in \ell^q(\mathbb{Z}_{\geq 0})$, and $a, b \in A$. Therefore, this gives another analogue of a well known result for Hilbert modules, in which the compact module maps of the standard Hilbert A-module are given by $\mathcal{K} \otimes A$.

Let's write $\mathcal{Q}_A((\mathcal{F}^p(A),\mathcal{F}^q(A)))$ for $\mathcal{L}_A((\mathcal{F}^p(A),\mathcal{F}^q(A)))/\mathcal{K}_A((\mathcal{F}^p(A),\mathcal{F}^q(A)))$ and let $q_0\colon \mathcal{L}_A((\mathcal{F}^p(A),\mathcal{F}^q(A))) \to \mathcal{Q}_A((\mathcal{F}^p(A),\mathcal{F}^q(A)))$ be the quotient map. We are now ready to provide a covariant L^p-Fock representation.

Proposition 6.3.44. *Let $p \in (1,\infty)$, let $(\Omega,\mathfrak{M},\mu)$ be a measure space, let $A \subseteq \mathcal{L}(L^p(\mu))$ be an L^p-operator algebra with a bicontractive approximate identity, and let $(\mathcal{F}^p(A),\mathcal{F}^q(A))$ be the L^p-module from Definition 6.3.30. Then, via the map Θ_A from (6.3.14), $(\mathcal{Q}_A((\mathcal{F}^p(A),\mathcal{F}^q(A))), q_0\circ\varphi_A^\infty, q_0\circ c_A, q_0\circ v_A)$ forms a covariant L^p-Fock representation for $((A,A),\varphi_A)$, as defined in Definition 6.3.5.*

Proof. Using Corollary 6.3.36, Proposition 6.3.41, and Proposition 6.3.42, we see that it only remains to prove the covariance condition for $q_0\circ\Theta_A$. That is, we only need to show that $q_0(\Theta_A(\varphi_A(a))) = q_0(\varphi_A^\infty(a))$ for all $a\in A$. In other words, we are to establish that $\varphi_A^\infty(a) - \Theta_A(\varphi_A(a)) \in \mathcal{K}_A((\mathcal{F}^p(A),\mathcal{F}^q(A)))$. Fix $a \in A$ and use the Cohen-Hewitt factorization theorem (Theorem 1 in [4] is enough) to find $a_0, a_1 \in A$ such that $a = a_0a_1$. We claim that $\varphi_A^\infty(a) - \Theta_A(\varphi_A(a)) = \theta_{\delta_0\otimes a_0,\delta_0\otimes a_1} \in \mathcal{K}_A((\mathcal{F}^p(A),\mathcal{F}^q(A)))$. As usual, it suffices to prove the equality for elementary tensors $x\otimes\xi \in L^p(\nu\times\mu)$, so we compute

$$(\varphi_A^\infty(a)-\Theta_A(\varphi_A(a))(x\otimes\xi) = \sum_{n=0}^\infty x(n)\delta_n\otimes\varphi^n(a)\xi - \sum_{n=1}^\infty x(n)\delta_n\otimes\varphi^n(a)\xi = x(0)\delta_0\otimes a\xi.$$

On the other hand,

$$\theta_{\delta_0\otimes a_0,\delta_0\otimes a_1}(x\otimes\xi) = (\delta_0\otimes a_0)(\delta_0\otimes a_1)(x\otimes\xi) = (\delta_0\otimes a_0)(x(0)\otimes a_1\xi) = x(0)\delta_0\otimes a_0a_1\xi,$$

which proves the claim and therefore finishes the proof. ∎

Given that $((A,A),\varphi_A)$ admits a covariant L^p-Fock representation, we get an L^p-operator algebra $F^p(\mathcal{Q}_A((\mathcal{F}^p(A),\mathcal{F}^q(A))), q_0\circ\varphi_A^\infty, q_0\circ c_A, q_0\circ v_A)$, which we denote for short $\mathcal{O}^p(\varphi_A^\infty, c_A, v_A)$. Furthermore, since $(A\mid A)_A$ is clearly equal to A, we follow Remark 6.3.2 and record below a precise definition for $\mathcal{O}^p(\varphi_A^\infty, c_A, v_A)$.

Definition 6.3.45. Let $p\in(1,\infty)$, let $(\Omega,\mathfrak{M},\mu)$ be a measure space, let $A\subseteq\mathcal{L}(L^p(\mu))$ be an L^p-operator algebra with a bicontractive approximate identity, and let $q_0:\mathcal{L}_A((\mathcal{F}^p(A),\mathcal{F}^q(A)))\to\mathcal{Q}_A((\mathcal{F}^p(A),\mathcal{F}^q(A)))$ be the quotient map. We define $\mathcal{O}^p(\varphi_A^\infty, c_A, v_A)$ as the closed algebra in $\mathcal{Q}_A((\mathcal{F}^p(A),\mathcal{F}^q(A)))$ generated by $q_0(c_A(A))$ and $q_0(v_A(A))$.

Remark 6.3.46. Observe that $\mathcal{O}^p(\varphi_A^\infty, c_A, v_A)$ is the quotient of $\mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$ from Definition 6.3.37 and the ideal $\mathcal{T}^p(\varphi_A^\infty, c_A, v_A)\cap\mathcal{K}_A((\mathcal{F}^p(A),\mathcal{F}^q(A)))$. We still denote the quotient map by $q_0\colon\mathcal{T}^p(\varphi_A^\infty, c_A, v_A)\to\mathcal{O}^p(\varphi_A^\infty, c_A, v_A)$.

An analogous question to Question 6.3.29 arises in this situation:

Question 6.3.47. Let $p\in(1,\infty)$, let $(\Omega,\mathfrak{M},\mu)$ be a measure space, and let $A\subseteq\mathcal{L}(L^p(\mu))$ be an L^p-operator algebra with a bicontractive approximate identity. Is $mathcalO^p(\varphi_A^\infty, c_A, v_A)$ isometrically isomorphic to $\mathcal{O}^p((A,A),\varphi_A)$? In other words, is $(\mathcal{Q}_A((\mathcal{F}^p(A),\mathcal{F}^q(A))), q_0\circ\varphi_A^\infty, q_0\circ c_A, q_0\circ v_A)$ the universal covariant L^p-Fock representation for $(A,A),\varphi_A)$?

Without having universality of $(\mathcal{Q}_A((\mathcal{F}^p(A),\mathcal{F}^q(A))), q_0\circ\varphi_A^\infty, q_0\circ c_A, q_0\circ v_A)$, the question on whether $\mathcal{O}^p(\varphi_A^\infty, c_A, v_A)$ is isometrically isomorphic to $F^p(A,\mathbb{Z},\varphi_A)$ also remains unanswered. The final goal for this chapter is to present the initial steps needed to potentially show the existence of such an isometric isomorphism.

At this moment we should also impose the extra assumption that φ_A^{-1} is also contractive. This is needed to make sense of the crossed product $F^p(A,\mathbb{Z},\varphi_A)$.

If we would like to work instead with a more general crossed product of Banach algebras, we could follow Definition 3.1 in [9], and require the existence of a function $M\colon \mathbb{Z} \to [0,\infty)$ such that M is bounded on finite subsets of $\mathbb{Z}$ and $\|\varphi^n\| \leq M(n)$ for all $n \in \mathbb{Z}$. Nevertheless in what follows we will stick to $M \equiv 1$ (that is, following Definition 4.5.1), which implies that φ_A^n is contractive for all $n \in \mathbb{Z}$ and therefore that φ is an isometric automorphism of A.

Following the C*-case, we need an analogue of the map defined by equation (2.4.5). Without an involution, we also need to determine where an element of the form au_1 is being mapped. Since $v_A(a)$ takes the role of $c_A(a^*)^*$, the desired output is forced on us. Indeed, with notation as in (2.4.3), for each $a \in A$ we define

$$\gamma_0(au_{-1}) = c_A(\varphi_A(a)),$$
$$\gamma_0(au_1) = v_A(a).$$

Exploiting the Cohen-Hewitt Factorization theorem and the multiplication rule (2.4.3), this automatically defines a rule for $\gamma_0(au_n)$ for any $n \in \mathbb{Z}$. Indeed, for any $a \in A$ we have to set

$$\gamma_0(au_0) = \varphi_A^{\infty}(a).$$

For $n \in \mathbb{Z}_{>1}$ and $a \in A$, by the Cohen-Hewitt Factorization theorem we can choose factorizations of a of the form

$$\begin{aligned} a &= a_1\varphi^{-1}(a_2)\dots\varphi^{-(n-1)}(a_n) \\ &= b_1\varphi(b_2)\dots\varphi^{n-1}(b_n) \end{aligned}$$

Thus, to make γ a homomorphism, we are forced to set

$$\gamma_0(au_{-n}) = c_A(\varphi_A(a_1))c_A(\varphi_A(a_2)) \cdots c_A(\varphi_A(a_n)),$$
$$\gamma_0(au_n) = v_A(b_1)v_A(b_2) \cdots v_A(b_n).$$

Furthermore, Lemma 6.3.40 shows that the definitions for $\gamma_0(au_{-n})$ and $\gamma_0(au_n)$ are independent of the factorizations of a chosen.

Definition 6.3.48. We have defined an algebra homomorphism $\gamma_0 \colon C_c(A, \mathbb{Z}, \varphi_A) \to \mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$. such that, when $J \subset \mathbb{Z}$ is a finite subset of $\mathbb{Z}$,

$$\sum_{j\in J} a_j u_j \mapsto \sum_{j \in J} \gamma_0(a_j u_j)$$

An immediate consequence of the definition of γ_0 and together with Lemma 6.3.40 implies that $\|\gamma_0(au_n)\| \leq \|a\|$ for any $n \in \mathbb{Z}$ and any $a \in A$. Hence,

$$\Big\| \sum_{j\in J} \gamma_0(a_j u_j) \Big\| \leq \sum_{j\in J} \|a_j\|,$$

and therefore, since $\mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$ is sits nondegenerately in $L^p(\nu \times \mu)$ (see Proposition 6.3.39), Proposition 4.5.2 implies that γ_0 extends to a contractive map $\gamma_0 \colon F^p(A, \mathbb{Z}, \varphi_A) \to \mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$. We then define a contractive algebra homomorphism $\gamma \colon F^p(A, \mathbb{Z}, \varphi_A) \to \mathcal{O}^p(\varphi_A^\infty, c_A, v_A)$ by letting $\gamma = q_0 \circ \gamma_0$.

The main complication in this situation arises when trying to construct a contractive map in the other direction that is an inverse for γ. Since we have not established that $(\mathcal{Q}_A((\mathcal{F}^p(A), \mathcal{F}^q(A))), q_0 \circ \varphi_A^\infty, q_0 \circ c_A, q_0 \circ v_A)$ is the universal covariant L^p-Fock representation for $((A, A), \varphi_A)$, we cannot follow an analogue path as the one used in the C*-case (see Example 2.4.17). We can, however, get a covariant representation of $(\mathbb{Z}, A, \varphi_A)$ on $M(\mathcal{O}^p(\varphi_A^\infty, c_A, v_A))$ which will be a good candidate to induce any other nondegenerate covariant representation of $(\mathbb{Z}, A, \varphi_A)$ (see Remark 6.3.52 below for a precise discussion). To do so, we need to add an extra assumption: we now require that the L^p-operator algebra A sits nondegenerately in $\mathcal{L}(L^p(\mu))$. The main reason for this extra assumption is that, by Proposition 6.3.39, the algebra $\mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$ is nondegenerately represented on $L^p(\nu \times \mu)$ and therefore, by Theorem 4.1.6, we can see $M(\mathcal{T}^p(\varphi_A^\infty, c_A, v_A))$ as the set of two sided multipliers for $\mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$ on $L^p(\nu \times \mu)$.

There are several steps needed to get the desired covariant representation. First one is to get a map $u_{\mathbb{Z}} \colon \mathbb{Z} \to M(\mathcal{T}^p(\varphi_A^\infty, c_A, v_A))$. For each $n \in \mathbb{Z}$ and $x \otimes \xi \in L^p(\nu \times \mu)$, we put

$$u_{\mathbb{Z}}(n)(x \otimes \xi) = \begin{cases} t^n x \otimes \xi & \text{if } n \geq 0 \\ s^{-n} x \otimes \xi & \text{if } n \leq 0 \end{cases}, \tag{6.3.15}$$

where as before, $t \in \mathcal{L}(\ell^p(\mathbb{Z}_{\geq 0}))$ is left translation and $s \in \mathcal{L}(\ell^p(\mathbb{Z}_{\geq 0}))$ is right translation.

Lemma 6.3.49. *For each $n \in \mathbb{Z}$, the assignment from (6.3.15) extends to a norm 1 linear map $u_{\mathbb{Z}}(n) \colon L^p(\nu \times \mu) \to L^p(\nu \times \mu)$, and satisfies the following conditions*

1. $u_{\mathbb{Z}}(n)\varphi_A^\infty(a)u_{\mathbb{Z}}(-n) = \varphi_A^\infty(\varphi_A^n(a))$ *for all* $n \geq 0$,
2. $u_{\mathbb{Z}}(-n)\varphi_A^\infty(a)u_{\mathbb{Z}}(n) = \varphi_A^\infty(\varphi_A^{-n}(a)) - \sum_{k=0}^{n-1} \theta_{\delta_k,\delta_k} \otimes \varphi_A^{k-n}(a)$ *for all* $n \geq 1$,
3. $u_{\mathbb{Z}}(n)u_{\mathbb{Z}}(m) = u_{\mathbb{Z}}(n+m)$ *if* $n \geq 0$ *and* $m \in \mathbb{Z}$ *or if* $n, m \leq 0$,
4. $u_{\mathbb{Z}}(n)u_{\mathbb{Z}}(m) = u_{\mathbb{Z}}(n+m) - \left(\sum_{j=0}^{-n} \theta_{\delta_j,\delta_{j+n+m}}\right) \otimes \mathrm{id}_{L^p(\mu)}$ *if* $n < 0$ *and* $m \geq 0$ *with* $n + m \geq 0$,

5. $u_{\mathbb{Z}}(n)u_{\mathbb{Z}}(m) = u_{\mathbb{Z}}(n+m) - \left(\sum_{j=0}^{m-1} \theta_{\delta_{j+1-n-m},\delta_j}\right) \otimes \mathrm{id}_{L^p(\mu)}$ *if* $n < 0$ *and* $m \geq 0$ *with* $n + m < 0$.

Proof. Let $k \in \mathbb{Z}_{\geq 1}$, let $x_1, \ldots, x_k \in \ell^p(\mathbb{Z}_{\geq 0})$, and let $\xi_1, \ldots, \xi_k \in L^p(\mu)$. Direct computations show that for any $n \in \mathbb{Z}$,

$$\Big\| \sum_{j=1}^{k} u_{\mathbb{Z}}(n)(x_j \otimes \xi_j) \Big\| \leq \Big\| \sum_{j=1}^{k} x_j \otimes \xi_j \Big\|,$$

and we get equality for $n < 0$. Thus, $u_{\mathbb{Z}}(n)$ extends to all of $L^p(\nu \times \mu)$ and satisfies $\|u_{\mathbb{Z}}(n)\| \leq 1$. In fact, since $\|u_{\mathbb{Z}}(n)(\delta_n \otimes \xi)\| = \|\delta_0 \otimes \xi\| = \|\xi\|$ for any $n \geq 0$, it follows that $\|u_{\mathbb{Z}}(n)\| = 1$ for all $n \in \mathbb{Z}$. We now check conditions 1-5. It suffices to show that they hold when the operators act on elementary tensors $x \otimes \xi \in L^p(\nu \times \mu)$. If $n \geq 0$,

$$\begin{aligned}
u_{\mathbb{Z}}(n)\varphi_A^{\infty}(a)u_{\mathbb{Z}}(-n)(x \otimes \xi) &= \sum_{k=0}^{\infty} (s^n x)(k) t^n \delta_k \otimes \varphi_A^k(a)\xi \\
&= \sum_{k=n}^{\infty} x(k-n)\delta_{k-n} \otimes \varphi_A^k(a)\xi \\
&= \sum_{k=0}^{\infty} x(k)\delta_k \otimes \varphi_A^{k+n}(a)\xi \\
&= \varphi_A^{\infty}(\varphi_A^n(a))(x \otimes \xi),
\end{aligned}$$

proving 1. For $n \geq 1$

$$\begin{aligned}
u_{\mathbb{Z}}(-n)\varphi_A^{\infty}(a)u_{\mathbb{Z}}(n)(x \otimes \xi) &= \sum_{k=0}^{\infty} (t^n x)(k) s^n \delta_k \otimes \varphi_A^k(a)\xi \\
&= \sum_{k=0}^{\infty} x(k+n)\delta_{k+n} \otimes \varphi_A^k(a)\xi \\
&= \sum_{k=n}^{\infty} x(k)\delta_k \otimes \varphi_A^{k-n}(a)\xi
\end{aligned}$$

$$= \Big(\varphi_A^\infty(\varphi_A^{-n}(a)) - \sum_{k=0}^{n-1} \theta_{\delta_k,\delta_k} \otimes \varphi_A^{k-n}(a)\Big)(x \otimes \xi),$$

so condition 2 is proved. Conditions 3-5 are direct routine calculations involving left and right translation. For instance, if $n \geq 0$ and $m = -n$

$$u_{\mathbb{Z}}(n)u_{\mathbb{Z}}(-n)(x \otimes \xi) = (t^n s^n x) \otimes \xi = x \otimes \xi.$$

Similarly, if $n \geq 1$

$$u_{\mathbb{Z}}(-n)u_{\mathbb{Z}}(n)(x \otimes \xi) = (s^n t^n x) \otimes \xi = \sum_{j=n}^{\infty} x(j)\delta_j \otimes \xi,$$

but on the other hand,

$$\Big(\mathrm{id}_{\ell^p(\mathbb{Z}_{\geq 0})} - \sum_{j=0}^{n-1} \theta_{\delta_j,\delta_j}\Big)x = x - \sum_{j=0}^{n} x(j)\delta_j = \sum_{j=n}^{\infty} x(j)\delta_j,$$

which yields a particular instance of condition 4. ■

Lemma 6.3.50. *Let $p \in (1, \infty)$ and let A be an L^p-operator algebra that sits nondegenerately in $\mathcal{L}(L^p(\mu))$ and has a c.a.i. Then $u_{\mathbb{Z}}(n) \in M(\mathcal{T}^p(\varphi_A^\infty, c_A, v_A))$ for every $n \in \mathbb{Z}$.*

Proof. Proposition 6.3.39 implies that $\mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$ has a c.a.i and sits nondegenerately in $L^p(\nu \times \mu)$. Hence, using Theorem 4.1.6 it suffices to show that for each $n \in \mathbb{Z}$, $u_{\mathbb{Z}}(n) \in \mathcal{L}(L^p(\nu \times \mu))$ is a two sided multiplier for $\mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$. To do so, in view of Remark 6.3.38 and Proposition 6.3.35, we only need to verify that $u_{\mathbb{Z}}(n)c_A(a), c_A(a)u_{\mathbb{Z}}(n), u_{\mathbb{Z}}(n)v_A(a), v_A(a)u_{\mathbb{Z}}(n) \in \mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$ for all $n \in \mathbb{Z}$ and $a \in A$.

We start with the computations for c_A. Let $a \in A$ and let $n \in \mathbb{Z}$. It is clear that $u_{\mathbb{Z}}(0)c_A(a) = c_A(a)u_{\mathbb{Z}}(0) = c_A(a) \in \mathcal{T}^p(\varphi_A^{\infty}, c_A, v_A)$, so the case $n = 0$ is done. We first prove that $u_{\mathbb{Z}}(\mathbb{Z}_{<0})c_A(a) \subseteq \mathcal{T}^p(\varphi_A^{\infty}, c_A, v_A)$. Let $n \geq 1$ and let $x \otimes \xi \in L^p(\nu \times \mu)$. Then

$$u_{\mathbb{Z}}(-n)c_A(a)(x \otimes \xi) = \sum_{k=1}^{\infty}(sx)(k)\delta_{k+n} \otimes \varphi_A^{k-1}(a)\xi$$

On the other hand, using the Cohen-Hewitt factorization theorem, we find a factorization of A of the form

$$a = \varphi^n(a_1)\cdots\varphi(a_n)a_{n+1},$$

and using equation (6.3.11) from Lemma 6.3.40 we find

$$\begin{aligned} c(a_1)\dots c(a_{n+1})(x \otimes \xi) &= \sum_{k=1}^{\infty}(sx)(k)\delta_{k+n} \otimes \varphi^{k-1}(\varphi^n(a_1)\cdots\varphi(a_n)a_{n+1})\xi \\ &= \sum_{k=1}^{\infty}(sx)(k)\delta_{k+n} \otimes \varphi^{k-1}(a)\xi. \end{aligned}$$

This shows that $u_{\mathbb{Z}}(-n)c_A(a) = c(a_1)\dots c(a_{n+1}) \in \mathcal{T}^p(\varphi_A^{\infty}, c_A, v_A)$ when $n \geq 1$. We now check that $u_{\mathbb{Z}}(\mathbb{Z}_{>0})c_A(a) \subseteq \mathcal{T}^p(\varphi_A^{\infty}, c_A, v_A)$. First,

$$u(1)c_A(a)(x \otimes \xi) = \sum_{k=0}^{\infty} x(k)\delta_k \otimes \varphi^n(\varphi^{n-1}(a))\xi = \varphi_A^{\infty}(\varphi_A^{-1}(a)),$$

whence $u(1)c_A(a) = \varphi_A^{\infty}(\varphi_A^{-1}(a)) \in \mathcal{T}^p(\varphi_A^{\infty}, c_A, v_A)$. If $n \geq 2$, then,

$$u_{\mathbb{Z}}(n)c_A(a)(x \otimes \xi) = \sum_{k=n}^{\infty}(sx)(k)\delta_{k-n} \otimes \varphi^{k-1}(a)$$

$$= \sum_{k=n}^{\infty} x(k-1)\delta_{k-n} \otimes \varphi^{k-n}(\varphi^{n-1}(a))$$
$$= \sum_{k=n-2}^{\infty} (tx)(k)\delta_{k-(n-2)} \otimes \varphi^{k-(n-2)}(\varphi^{n-1}(a)).$$

Now choose a factorization

$$\varphi^{n-1}(a) = a_1\varphi(a_2)\ldots\varphi^{n-2}(a_{n-1}),$$

so that equation (6.3.12) from Lemma 6.3.40 now yields

$$v_A(a_1)\cdots v_A(a_{n-1})(x \otimes \xi) = \sum_{k=n-2}^{\infty} (tx)(k)\delta_{k-(n-2)} \otimes \varphi^{k-(n-2)}(a_1\varphi(a_2)\cdots\varphi^{n-2}(a_{n-1}))\xi$$
$$= \sum_{k=n-2}^{\infty} (tx)(k)\delta_{k-(n-2)} \otimes \varphi^{k-(n-2)}(\varphi^{n-1}(a))\xi,$$

proving that $u_{\mathbb{Z}}(n)c_A(a) = v_A(a_1)\cdots v_A(a_{n-1}) \in \mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$ when $n \geq 2$. Next, we verify that $c_A(a)u_{\mathbb{Z}}(\mathbb{Z}_{<0}) \subseteq \mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$. On the one hand, for $n \geq 1$, we get

$$c_A(a)u_{\mathbb{Z}}(-n)(x \otimes \xi) = \sum_{k=1}^{\infty}(sx)(k)\delta_{k+n} \otimes \varphi^{k-1}(\varphi^n(a)).$$

Thus, if we choose a factorization for $\varphi^n(a)$ of the form

$$\varphi^n(a) = \varphi^n(a_1)\cdots\varphi(a_n)a_{n+1},$$

it follows from equation (6.3.11) in Lemma 6.3.40 that $c_A(a)u_{\mathbb{Z}}(-n) = c_A(a_1)\cdots c_A(a_{n+1}) \in \mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$ when $n \geq 1$. To finish with the calculations for c_A, it remains to show that $c_A(a)u_{\mathbb{Z}}(\mathbb{Z}_{>0}) \subseteq \mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$. For any $n \geq 1$ we first find

$$c_A(a)u_{\mathbb{Z}}(n)(x \otimes \xi) = \sum_{k=n-1} (tx)(k)\delta_{j-n+2} \otimes \varphi^{j-n+1}(a)\xi.$$

Thus, choosing a factorization for a of the form

$$a = a_1 b_1 \varphi(b_2) \cdots \varphi^{n-1}(b_n),$$

it follows now from equation (6.3.13) in Lemma 6.3.40 that $c_A(a)u_{\mathbb{Z}}(n) = c_A(a_1)v_A(b_1)\cdots v_A(b_n) \in \mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$ when $n \geq 1$.

We now check it for v_A. It's clear that $u_{\mathbb{Z}}(0)v_A(a) = v_A(a)u_{\mathbb{Z}}(0) = v_A(a)$, so the case $n = 0$ is done. If $a = a_1 a_2$, then the computation from the proof of Proposition 6.3.41 gives

$$u_{\mathbb{Z}}(-1)v_A(a)(x \otimes \xi) = \sum_{k=1}^{\infty} x(k)\delta_k \otimes \varphi^{k-1}(a_1 a_2) = c_A(a_1)v_A(a_2)(x \otimes \xi),$$

and therefore $u_{\mathbb{Z}}(-1)v_A(a) = c_A(a_1)v_A(a_2) \in \mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$. For a general $n \geq 1$, notice that $u_{\mathbb{Z}}(-n)v_A(a) = u_{\mathbb{Z}}(-n+1)u_{\mathbb{Z}}(-1)v_A(a) = u_{\mathbb{Z}}(-n+1)c_A(a_1)v_A(a_2)$ and we already proved above that $u_{\mathbb{Z}}(n-1)c_A(a_1) \in \mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$. Thus, we have shown that $u_{\mathbb{Z}}(\mathbb{Z}_{<0})v_A(a) \subseteq \mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$. We now check that $u_{\mathbb{Z}}(\mathbb{Z}_{>0})v_A(a) \subseteq \mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$. Indeed, for $n \geq 1$,

$$u_{\mathbb{Z}}(n)v_A(a)(x \otimes \xi) = \sum_{k=n}^{\infty}(tx)(k)\delta_{k-n} \otimes \varphi^{k-n}(\varphi^n(a))\xi.$$

Now choose a factorization

$$\varphi^n(a) = a_1\varphi(a_2)\ldots\varphi^n(a_{n+1}),$$

so that equation (6.3.12) in Lemma 6.3.40 gives

$$\begin{aligned} v_A(a_1)\cdots v_A(a_{n+1})(x\otimes\xi) &= \sum_{k=n}^{\infty}(tx)(k)\delta_{k-n}\otimes\varphi^{k-n}(a_1\varphi(a_2)\cdots\varphi^n(a_{n+1}))\xi \\ &= \sum_{k=n}^{\infty}(tx)(k)\delta_{k-n}\otimes\varphi^{k-n}(\varphi^n(a))\xi. \end{aligned}$$

This proves that $u_{\mathbb{Z}}(n)v_A(a) = v_A(a_1)\cdots v_A(a_{n+1}) \in \mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$ for any $n \geq 1$, as wanted. Next step is to verify $v_A(a)u_{\mathbb{Z}}(\mathbb{Z}_{<0}) \subseteq \mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$. If $n \geq 1$, we find

$$v_A(a)u_{\mathbb{Z}}(-n)(x\otimes\xi) = \sum_{k=1}^{\infty}(sx)(k)\delta_{k+n-2}\otimes\varphi^{k-1}(\varphi^{n-1}(a)).$$

Thus, for $n = 1$ we immediately see that $v_A(a)u_{\mathbb{Z}}(-1) = \varphi_A^\infty(a)$. For $n \geq 2$, we choose a factorization of A of the form

$$\varphi^{n-1}(a) = \varphi^{n-2}(a_1)\cdots\varphi(a_{n-2})a_{n-1},$$

whence using equation (6.3.11) from Lemma 6.3.40 gives $v_A(a)u_{\mathbb{Z}}(-n) = c_A(a_1)\cdots c_A(a_{n-1}) \in \mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$ for $n \geq 2$. This proves that $v_A(a)u_{\mathbb{Z}}(\mathbb{Z}_{<0}) \subseteq \mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$ and finishes the proof. ■

We are now ready to exhibit a covariant representation for $(\mathbb{Z}, A, \varphi_A)$ on $M(\mathcal{O}^p(\varphi_A^\infty, c_A, v_A))$.

Proposition 6.3.51. *Let $p \in (1,\infty)$, let $A \subseteq \mathcal{L}(L^p(\mu))$ be a nondegenerate L^p-operator algebra with a bicontractive approximate identity, let $\varphi_A^\infty\colon A \to \mathcal{L}(L^p(\nu\times\mu))$ be as defined by (6.3.8), let $u_{\mathbb{Z}}\colon \mathbb{Z} \to \mathcal{L}(L^p(\nu\times\mu))$ be as defined in (6.3.15), and let $q_0\colon \mathcal{T}^p(\varphi_A^\infty, c_A, v_A) \to \mathcal{O}^p(\varphi_A^\infty, c_A, v_A)$ be the quotient map (see Remark 6.3.46). Consider the map $\widetilde{q_0}\colon M(\mathcal{T}^p(\varphi_A^\infty, c_A, v_A)) \to M(\mathcal{O}^p(\varphi_A^\infty, c_A, v_A))$, from Proposition*

4.1.8, let $\widetilde{\varphi_A^\infty} = \widetilde{q_0} \circ \varphi_A^\infty$, and let $\widetilde{u_{\mathbb{Z}}} = \widetilde{q_0} \circ u_{\mathbb{Z}}$. Then the pair $(\widetilde{\varphi_A^\infty}, \widetilde{u_{\mathbb{Z}}})$ is a covariant representation of $(\mathbb{Z}.A, \varphi_A)$ on $M(\mathcal{O}^p(\varphi_A^\infty, c_A, v_A))$.

Proof. For $a \in A$, let $\widetilde{\varphi_A^\infty}(a) = (L_0, R_0) \in M(\mathcal{O}^p(\varphi_A^\infty, c_A, v_A))$. By construction of $\widetilde{q_0}$ (see proof of Proposition 4.1.8), we have for any $w \in \mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$ that

$$L_0(q_0(w)) = q_0(\varphi_A^\infty(a)w)$$
$$R_0(q_0(w)) = q_0(w\varphi_A^\infty(a))$$

Hence, conditions 1-2 in Lemma 6.3.49 imply that the covariance condition will hold for $(\widetilde{\varphi_A^\infty}, \widetilde{u_{\mathbb{Z}}})$, provided that we show that for any $w \in \mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$, $n \geq 1$ and $a \in A$,

$$w\Big(\sum_{k=0}^{n-1} \theta_{\delta_k,\delta_k} \otimes \varphi_A^{k-n}(a)\Big), \Big(\sum_{k=0}^{n-1} \theta_{\delta_k,\delta_k} \otimes \varphi_A^{k-n}(a)\Big)w \in \mathcal{K}_A((\mathcal{F}^p(A), \mathcal{F}^q(A)),$$

but this follows at once from the fact that $\mathcal{K}_A((\mathcal{F}^p(A), \mathcal{F}^q(A))$ is a closed two sided ideal of $\mathcal{L}_A((\mathcal{F}^p(A), \mathcal{F}^q(A)))$ (see Proposition 5.5.1) which is equal to $\mathcal{K}(\ell^p(\mathbb{Z}_{\geq 0})) \otimes_p A$ (see Remark 6.3.43).

Similarly, for $n \in \mathbb{Z}$, let $\widetilde{u_{\mathbb{Z}}}(n) = (L_1, R_1) \in M(\mathcal{O}^p(\varphi_A^\infty, c_A, v_A))$. Thus, for each $w \in \mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$, the construction of $\widetilde{q_0}$ now implies that

$$L_1(q_0(w)) = q_0(u_{\mathbb{Z}}(n)w)$$
$$R_1(q_0(w)) = q_0(wu_{\mathbb{Z}}(n)).$$

Hence, thanks to conditions 3-5 in Lemma 6.3.49, we see that the map $\widetilde{u_{\mathbb{Z}}}(n) \colon \mathbb{Z} \to M(\mathcal{O}^p(\varphi_A^\infty, c_A, v_A))$ will be a group homomorphism if we prove that for any $w \in$

$\mathcal{T}^p(\varphi_A^\infty, c_A, v_A)$, if $n < 0$ and $m \geq 0$, we have

$$w\Big(\sum_{j=0}^{-n} \theta_{\delta_j, \delta_{j+n+m}} \otimes \mathrm{id}_{L^p(\mu)}\Big), \Big(\sum_{j=0}^{-n} \theta_{\delta_j, \delta_{j+n+m}} \otimes \mathrm{id}_{L^p(\mu)}\Big) w \in \mathcal{K}_A((\mathcal{F}^p(A), \mathcal{F}^q(A)),$$

when $m + n \geq 0$, and

$$w\Big(\sum_{j=0}^{m-1} \theta_{\delta_{j+1-n-m}, \delta_j} \otimes \mathrm{id}_{L^p(\mu)}\Big), \Big(\sum_{j=0}^{m-1} \theta_{\delta_{j+1-n-m}, \delta_j} \otimes \mathrm{id}_{L^p(\mu)}\Big) w \in \mathcal{K}_A((\mathcal{F}^p(A), \mathcal{F}^q(A)),$$

when $n + n < 0$. All instances will follow if we show that for any $x \in \ell^p(\mathbb{Z}_{\geq 0})$, $y \in \ell^q(\mathbb{Z}_{\geq 0})$ and $a \in A$,

$$(\theta_{x,y} \otimes \mathrm{id}_{L^p(\mu)}) c_A(a), (\theta_{x,y} \otimes \mathrm{id}_{L^p(\mu)}) v_A(a) \in \mathcal{K}_A((\mathcal{F}^p(A), \mathcal{F}^q(A)),$$
$$c_A(a)(\theta_{x,y} \otimes \mathrm{id}_{L^p(\mu)}), v_A(a)(\theta_{x,y} \otimes \mathrm{id}_{L^p(\mu)}) \in \mathcal{K}_A((\mathcal{F}^p(A), \mathcal{F}^q(A)).$$

To do so, we fix $x \in \ell^p(\mathbb{Z}_{\geq 0})$, $y \in \ell^q(\mathbb{Z}_{\geq 0})$ and $a \in A$. For each $z \in \ell^p(\mathbb{Z}_{\geq 0})$, $\xi \in L^p(\mu)$ we define

$$l_c(z \otimes \xi) = \sum_{k=1}^{\infty} y(k) \theta_{x, \delta_{k-1}}(z) \otimes \varphi^{k-1}(a)\xi,$$

and

$$l_v(z \otimes \xi) = \sum_{k=0}^{\infty} y(k) \theta_{x, \delta_{k+1}}(z) \otimes \varphi^{k}(a)\xi.$$

Notice that, for $0 \leq n \leq m$, Hölder inequality gives

$$\begin{aligned}\Big\| \sum_{k=n}^{m} y(k) \theta_{x, \delta_{k-1}}(z) \otimes \varphi^{k-1}(a)\xi \Big\|_p^p &= \sum_{j=0}^{\infty} \int_\Omega \Big| \sum_{k=n}^{m} y(k) x(j) z(k-1) (\varphi^{k-1}(a)\xi)(\omega) \Big|^p d\mu(\omega) \\ &\leq \|x\|_p^p \int_\Omega \Big(\sum_{k=n}^{m} |y(k)| |z(k-1)(\varphi^{k-1}(a)\xi)(\omega)| \Big)^p d\mu(\omega)\end{aligned}$$

$$\leq \|x\|_p^p\|y\|_q^p\Big(\sum_{k=n}^{m}|z(k-1)|^p\int_\Omega|(\varphi^{k-1}(a)\xi)(\omega)|^p d\mu(\omega)\Big)$$
$$\leq \|x\|_p^p\|y\|_q^p\|a\|^p\|\xi\|_p^p\sum_{k=n}^{m}|z(k-1)|^p,$$

and therefore $l_c(z\otimes\xi)$ is in $L^p(\nu\times\mu)$. A similar computation shows that $l_v(z\otimes\xi)$ is also an element of $L^p(\nu\times\mu)$. Furthermore, if $m\in\mathbb{Z}_{\geq 1}$, $z_1,\ldots,z_m\in\ell^p(\mathbb{Z}_{\geq 0})$, and $\xi_1,\ldots,\xi_m\in L^p(\mu)$, then Hölder inequality now gives

$$\begin{aligned}
\Big\|\sum_{j=1}^{m}l_c(z_j\otimes\xi_j)\Big\|^p &= \Big\|\sum_{k=1}^{\infty}\sum_{j=1}^{m}y(k)\theta_{x,\delta_{k-1}}(z_j)\otimes\varphi^{k-1}(a)\xi_j\Big\|_p^p\\
&= \|x\|_p^p\int_\Omega\Big|\sum_{k=1}^{\infty}y(k)\sum_{j=1}^{m}z_j(k-1)(\varphi^{k-1}(a)\xi_j)(\omega)\Big|^p d\mu(\omega)\\
&\leq \|x\|_p^p\|y\|_q^p\int_\Omega\sum_{k=1}^{\infty}\Big|\sum_{j=1}^{m}z_j(k-1)(\varphi^{k-1}(a)\xi_j)(\omega)\Big|^p d\mu(\omega)\\
&= \|x\|_p^p\|y\|_q^p\sum_{k=1}^{\infty}\int_\Omega\Big|\Big(\varphi^{k-1}(a)\big(\sum_{j=1}^{m}z_j(k-1)\xi_j\big)\Big)(\omega)\Big|^p d\mu(\omega)\\
&= \|x\|_p^p\|y\|_q^p\sum_{k=1}^{\infty}\Big\|\varphi^{k-1}(a)\big(\sum_{j=1}^{m}z_j(k-1)\xi_j\big)\Big\|_p^p\\
&\leq \|x\|_p^p\|y\|_q^p\|a\|^p\sum_{k=0}^{\infty}\Big\|\sum_{j=1}^{m}z_j(k)\xi_j\Big\|_p^p\\
&= \|x\|_p^p\|y\|_q^p\|a\|^p\Big\|\sum_{j=1}^{m}z_j\otimes\xi_j\Big\|_p^p.
\end{aligned}$$

This shows that l_c extends to well defined bounded linear map $l_c\in\mathcal{L}(L^p(\nu\times\mu))$ with $\|l_c\|\leq\|x\|_p\|y\|_q\|a\|$. An analogous computation shows that l_v extends to well defined bounded linear map $l_v\in\mathcal{L}(L^p(\nu\times\mu))$ with $\|l_v\|\leq\|x\|_p\|y\|_q\|a\|$. Furthermore, these calculations also show that

$$\lim_{n\to\infty}\Big\|l_c-\sum_{k=1}^{n}y(k)\theta_{x,\delta_{k-1}}\otimes\varphi^{k-1}(a)\Big\|=0$$

and

$$\lim_{n\to\infty}\left\|l_v-\sum_{k=0}^{n}y(k)\theta_{x,\delta_{k+1}}\otimes\varphi^k(a)\right\|=0.$$

Therefore, both l_c and l_v are norm limits of elements in $\mathcal{K}_A(\ell^p(\mathbb{Z}_{\geq 0}))\otimes_p A$, which is equal to $K_A((\mathcal{F}^p(A),\mathcal{F}^q(A)))$ as pointed out in Remark 6.3.43. Finally, a direct computation acting on elementary tensors in $L^p(\nu\times\mu)$ shows that $(\theta_{x,y}\otimes\mathrm{id}_{L^p(\mu)})c_A(a)=l_c\in K_A((\mathcal{F}^p(A),\mathcal{F}^q(A)))$ and that $(\theta_{x,y}\otimes\mathrm{id}_{L^p(\mu)})v_A(a)=l_v\in\mathcal{K}_A((\mathcal{F}^p(A),\mathcal{F}^q(A))$. Now, for each $z\in\ell^p(\mathbb{Z}_{\geq 0})$, $\xi\in L^p(\mu)$ we let

$$r_c(z\otimes\xi)=\sum_{k=1}^{\infty}x(k-1)\theta_{\delta_k,y}(z)\otimes\varphi^{k-1}(a)\xi,$$

and

$$r_v(z\otimes\xi)=\sum_{k=0}^{\infty}x(k+1)\theta_{\delta_k,y}(z)\otimes\varphi^{k}(a)\xi.$$

A similar analysis as to the one done for l_c and l_v now shows that r_c and r_v extend to bounded linear maps on all $L^p(\nu\times\mu)$, that $c_A(a)(\theta_{x,y}\otimes\mathrm{id}_{L^p(\mu)})=r_c\in K_A((\mathcal{F}^p(A),\mathcal{F}^q(A)))$, and that $v_A(a)(\theta_{x,y}\otimes\mathrm{id}_{L^p(\mu)})=r_v\in\mathcal{K}_A((\mathcal{F}^p(A),\mathcal{F}^q(A))$, finishing the proof. ■

Remark 6.3.52. As a final remark, we present an outline for the final step that would be needed in order to get the sought inverse map for γ from Definition 6.3.48. The main point is to be able to show that any nondegenerate covariant representation of $(\mathbb{Z},A,\varphi_A)$ on a L^p-space E factors through the covariant representation $(\widetilde{\varphi_A^\infty},\widetilde{u_{\mathbb{Z}}})$ from Proposition 6.3.51. That is, if $\pi\colon A\to\mathcal{L}(E)$ and $u\colon\mathbb{Z}\to\mathcal{L}(E)$ are such that (π,u) is a nondegenerate covariant representation of $(\mathbb{Z},A,\varphi_A)$ on E, then we want to construct a contractive nondegenerate representation $\rho\colon\mathcal{O}^p(\varphi_A^\infty,c_A,v_A)\to\mathcal{L}(E)$ such that $\pi=\widehat{\rho}\circ\widetilde{\varphi_A^\infty}$ and $u=\widehat{\rho}\circ\widetilde{u_{\mathbb{Z}}}$

where $\widehat{\rho}\colon M(\mathcal{O}^p(\varphi_A^\infty, c_A, v_A)) \to \mathcal{L}(E)$ is the map from Corollary 4.1.7. The importance of showing that ρ actually exists is that, by similar methods as those used in the proof of Raeburn's universal property for C*-crossed products (see Theorem 2.61 in [28]), we would be able to produce the desired contractive map $\mathcal{O}^p(\varphi_A^\infty, c_A, v_A) \to F^p(\mathbb{Z}, A, \varphi_A)$ that is an inverse for γ. The initial steps to define ρ are clear: For any $a \in A$ we are forced to set

$$\rho\big(q_0(v_A(a))\big) = \pi(a)u(1) = u(1)\pi(\varphi_A^{-1}(a)).$$
$$\rho\big(q_0(c_A(a))\big) = u(-1)\pi(a) = \pi(\varphi_A^{-1}(a))u(-1),$$

It s easy to check that $\big(\rho \circ q_0 \circ \varphi_A^\infty, \rho \circ q_0 \circ v_A, \rho \circ q_0 \circ c_A\big)$ satisfies the L^p-Fock covariant conditions and therefore ρ can be linearly extended to $q_0(\mathcal{T}_0^p(\varphi_A^\infty, c_A, v_A))$ (see Remark 6.3.38), which lies dense in $\mathcal{O}^p(\varphi_A^\infty, c_A, v_A)$. However, we have not been able to further extend ρ to all $\mathcal{O}^p(\varphi_A^\infty, c_A, v_A)$ due to not being able to show (without universality) that $\|\rho(q_0(w))\| \leq \|q_0(w)\|$ for any $w \in \mathcal{T}_0^p(\varphi_A^\infty, c_A, v_A)$. Given that universality has been established when $p = 2$ (see Chapter II), a Riesz-Thorin interpolation argument could be used to show that ρ actually extends to all values of $p \in [1, \infty)$, provided that we manage to also establish the extension for $p = 1$. This is more or less immediate to implement when $A = C_0(\Omega)$ for a locally compact Hausdorff space Ω. The advantage here is that $C_0(\Omega)$ is an L^p-operator algebra for each $p \in [1, \infty)$. In hopes of a more general result, this would require us to introduce (and precisely define) an extra assumption that A belongs to a "continuous" one-parameter family of L^p-operator algebras (parametrized by $p \in [1, \infty)$) such as for instance M_d^p and $\mathcal{O}_d^p$. The "constant" family $C_0(\Omega)$ should be a particular case of this potential scenario.

CHAPTER VII

FUTURE WORK

L^p-bimodules and Morita equivalence

We take advantage of Theorem 3.3.7 in the C*-case to define a notion of A-B L^p-bimodules via (A, B) L^p-correspondences:

Definition 7.1.1. Let A and B be L^p operator algebras. An L^p *A-B-bimodule* is an (A, B) L^p-correspondence $((\mathsf{X}, \mathsf{Y}), \varphi)$ for which $\mathcal{K}_B((\mathsf{X}, \mathsf{Y})) \subseteq \varphi(A)$.

As in the C*-case, any L^p-module over an L^p-operator algebra B is naturally an L^p *$\mathcal{K}_B((\mathsf{X}, \mathsf{Y}))$-$B$-bimodule* by letting φ be the inclusion $\mathcal{K}_B((\mathsf{X}, \mathsf{Y})) \hookrightarrow \mathcal{L}_B((\mathsf{X}, \mathsf{Y}))$.

Having a definition for L^p A-B-bimodule induces a working concept for Morita equivalent L^p-operator algebras.

Definition 7.1.2. Two nondegenerate L^p operator algebras $A \subseteq \mathcal{L}(L^p(\mu_1))$ and $A \subseteq \mathcal{L}(L^p(\mu_0))$ are said to be *Morita Equivalent* if there is an L^p A-B-bimodule $((\mathsf{X}, \mathsf{Y}), \varphi)$ such that

1. $\mathcal{K}_B((\mathsf{X}, \mathsf{Y})) = \varphi(A)$.
2. $\overline{(\mathsf{Y} \mid \mathsf{X})_B} = B$

The following immediate initial questions arise:

Q.1 Do we need to add that $\mathsf{X}L^p(\mu_0)$ dense in $L^p(\mu_1)$ and $\mathsf{Y}L^p(\mu_1)$ dense in $L^p(\mu_0)$ in Definition 7.1.2?

Q.2 Is Definition 7.1.2 the same as having (X, Y) a right L^p module over B which is also a left L^p-module over A, with dense compatible pairings:

$$_A(x \mid y)z = x(y \mid z)_B, \ (y \mid z)_B w = y_A(z \mid w)$$

for $x, z \in \mathsf{X}$ and $y, w \in \mathsf{Y}$?

Q.3 Is this a particular case of Morita equivalence of Banach algebras as defined in [19]?

Q.4 Can we translate the results in either [10] or [19] for computations of K-theory and KK-theory of L^p-operator alegebras?

C*-likeness of some L^p-modules

Explore in detail the C*-likeness for particular cases of A in Example 5.1.9.

A particular example is to take $A = M_2^p(\mathbb{C})$ and $d = 2$, so that the C*-like condition on $(\mathsf{X}, \mathsf{Y}) = (\ell_2^p \otimes_p A, \ell_2^q \otimes_p A)$ becomes the following problem.

Equip $M_2(\mathbb{C})^2$ with two different norms:

$$\|(a_1, a_2)\|_{\mathsf{X}} = \max_{\{\xi \in \ell_2^p : \|\xi\|_p = 1\}} \|(\|a_1\xi\|_p, \|a_2\xi\|_p)\|_p,$$

and

$$\|(b_1, b_2)\|_{\mathsf{Y}} = \max_{\{\xi_1, \xi_2 \in \ell_2^p : \|(\|\xi_1\|_p, \|\xi_2\|_p)\|_p = 1\}} \Big\| b_1\xi_1 + b_2\xi_2 \Big\|_p .$$

Conjecture 7.2.1.

$$\|(a_1, a_2)\|_{\mathsf{X}} = \max_{\|(b_1, b_2)\|_{\mathsf{Y}} = 1} \|b_1 a_1 + b_2 a_2\|_p$$

This conjecture is true when $p = 2$, but still is an open problem when $p \neq 2$. For $p = 1$, the Simplex method is certainly a good tool to look for a counterexample or increase the evidence that the conjecture holds. If It holds for both $p = 1$ and $p = 2$, some interpolation argument might give the result for any p.

L^p-Fock representations and Universality

There are two instances in the definitions of L^p-Fock representations that are, in some sense, automatic in the C*-case:

1. The first equalities in both conditions 1 and 2 of Definition 6.3.1 are redundant in the C*-case. See Remark 2.2.7 and Remark 2.4.2.

2. The map $\pi_{\mathcal{K}}$ from Definition 6.3.5 always exists in the C*-case. See Lemma 2.4.8.

We have not investigated whether this two instances still apply for the L^p-case, and it is likely that they will not hold in general. Furthermore, our current results from Chapter VI suggest the following lines of work

L.1 Answering both questions regarding universality posed in Question 6.3.29 and 6.3.47?

L.2 A more targeted project is to work on the universality of the representation from Proposition 6.3.51 for particular parametric families of L^p-operator algebras, as pointed out by the end of Remark 6.3.52.

L.3 Finally, a more general attempt (but also more technical) is to attempt a Fock space construction for any L^p-correspondence using in full generality Definition 6.2.1 to obtain the tensor correspondences and 5.3.2 for the direct sum of these.

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