SPECTRA

TOPOLOGY SEMINAR, SPRING 2017

Here are some resources with links.

- M. Aguilar, S. Gitler, and C. Prieto, *Algebraic topology from a homotopical viewpoint* (Available online through the CU library)
- J. P. May, *A concise course in algebraic topology*
- A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Modern foundations for stable homotopy theory*
- L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure, *Equivariant stable homotopy theory*
- S. Schwede, Symmetric Spectra
- M. A. Mandell, J. P. May, S. Schwede, and B. Shipley, *Model categories of diagram spectra*
- J. Lurie, *Higher Topos Theory*
- J. Lurie, *Higher Algebra*
- K. Wickelgren, *Stable homotopy theory course notes*

And some surveys

- J. P. May, *Stable algebraic topology and stable topological algebra*
- J. P. C. Greenles, *Spectra for commutative algebraists*
- C. Malkieviech, *The stable homotopy category*
- J. Lurie, *Note on Chromatic Homotopy Theory*

1. Overview

This will be the first lecture and give an brief overview and motivation for the topics we will cover this semester, and recall some classical facts from homotopy theory that play a key role in the development of spectra.

2. Classical Spectra

2.1. **Cohomology Theories and Brown Representability.** Definition of a generalized cohomology/homology theory and the Brown Representability Theorem (see [AGP02, 12.2] and [May99, Chapter 18]) for cohomology theories.

2.2. **Prespectra and spectra.** Here, we will give the most elementary modern definition of the category of spectra.[1] The definition I have in mind are in references:

- [AGP02, Section 12.3]
- [Wei94, 10.9]

[1]I have decided to avoid the Spanier-Whitehead [SW53] and the Adams-Boardman [Ada95, Part III] categories. The second one gives the right homotopy category, but the treatment of maps is involved and I think we can avoid this with a more modern approach. We can come back to the history at the end if we feel like it.

- [EKrvzMM95, Chapter 1], ignoring the "coordinate free"
- [LMSM86, Chapter I] taking G trivial
- [May99, Chapter 22, 25.6, 25.7]

Discuss the relationship between prespectra and spectra and mention the spectrification functor L. Discuss the smash product $E \wedge X$ of a spectrum E and a space X and the function spectrum $F(X, E)$. Discuss Σ^∞, Ω^∞ and Q and relevant adjunctions. Define CW–prespectra and CW–spectra. Finally, make the connection with generalized cohomology theories and homology theories. Give HA as example.

2.3. **The stable homotopy category.** Here, we discuss homotopy groups of spectra, the concept of homotopy in spectra and weak homotopy equivalences. Discuss Whitehead's theorem for spectra ([Wei94, starting at 10.9.14]). Define the stable homotopy category of spectra. See [AGP02, p.414] and better, [Wei94, starting at 10.9.10]. Discuss the fact that this is a triangulated category. The handcrafted smash product and its properties.

Remark. Adams [Ada95, Part III] can be an excellent reference, but the terminology and methodology is not as modern as in the previous references. Be aware that what he calls spectra is what we have been calling pre-spectra, that his Ω–spectra are those for which $E_n \to \Omega E_n$ are weak equivalences, and that on his category, his maps are different (see , [Ada95, p.142]). However, his end result gives the same stable homotopy category.

2.4. **Examples.**

2.4.1. *Interlude: Vector bundle.* Something to get everyone up to speak, discuss Thom spaces and the Thom isomorphism [May99, 23.5] theorem.

2.4.2. *Thom Spectra.* Discuss Thom spectra ([AGP02, p.415], [May99, Chapter 25]) and the examples MO and MU (see for a first glance [AGP02, p.415]). Describe $MO_*(X)$, $MO^*(X)$, $MU_*(X)$, $MU^*(X)$ in terms of bordism and cobordism. Finally, if there is interest, I would like us to talk about framed cobordism and the Pontryagin-Thom theorem. Another reference is [Lura].

2.4.3. *Quillen's theorem: MU and formal group laws.* Introduce complex orientation and the connection between the spectrum MU and formal group laws. There are probably many good references for this, but a start is Adams [Ada95, Part II]. Another reference is [Lura].

2.4.4. *K–theory.* Here, we will give examples, among which we should have topology complex and real K-theories K and KO, and algebraic K–theory.

2.5. **Models for symmetric monoidal categories of spectra.** A good survey is [Gre].

2.5.1. *Symmetric, orthogonal and diagram Spectra.* For symmetric spectra [Sch], and more generally, for diagram spectra [MMSS01]. In particular, define the smash product and check that these categories are symmetric monoidal. Mention the difficulties in constructing the right homotopy category from this model. It would be good, if we cover this topic, to construct Eilenberg-MacLane spectra.

2.5.2. $\mathbb{S}$*–modules.* The category of $\mathbb{S}$–modules is a nice model of spectra in which to do algebra. The reference would be [EKrvzMM95], and we would first define the category of $\mathbb{L}$–spectra. This takes some work, but once we've done that, $\mathbb{S}$–modules fall out and the homotopy category is simple to define.

2.6. **Brave new algebra.** We pick our module of spectra and we can define $\mathbb{S}$–modules, $\mathbb{S}$–algebras, and modules over $\mathbb{S}$–algebras. Then cover some of the following topics.

2.6.1. *HR–modules.* Discuss [Gre, Theorem 5.2]

2.6.2. *The classical spectral sequences.* Discuss the Künneth, Universal Coefficient [EKrvzMM95, Section 9] and Eilenberg-Moore spectral sequences [EKMM97, Chapter IV.6].

2.6.3. *Quotients and localization and MU–module spectra.* Discuss [EKrvzMM95, Sections 11,12]. Define BP and the Morava K–theories.

2.6.4. *Miscellaneous.* Here, depending on people's interest, we could cover topics such

- Algebraic K–theory, A–theory, THH, TC
- E_∞–ring spectra

3. ∞–category of spectra

3.1. Higher categories.

3.1.1. *Interlude: Simplicial Sets.* Discuss simplicial sets, geometric realization, weak equivalence, Kan Complexes and Kan fibrations, Quillen equivalence between homotopy category of spaces and homotopy category of simplicial sets.

3.1.2. *∞–categories.* Give enough background on ∞–categories to move to the topic of spectra. This can be based on [Lur09, Chapter 1], making sure to cover the notion of homotopy category, limit and colimit, and the ∞–category of spaces $\mathcal{S}_*$. We may need to cover the topic of Ind-completion, but I suggest waiting until we need it. (There's also a very brief overview in Wickelgren's notes, L19.)

3.2. Spectra in the∞–categories context.

3.2.1. *Stable ∞–categories.* The definition of stable ∞–categories, covering [Lurb, Chapter 1.1].

3.2.2. *Spectrum objects and ∞–category of Spectra* $\mathrm{Sp}(\mathcal{S}_*)$. This should be based on [Lurb, Chapter 1.4].

3.2.3. *Symmetric monoidal structire on* $\mathrm{Sp}(\mathcal{S}_*)$ *(Maybe).* I think this should be based on [Lurb, Chapter 4.8.2], but I don't know if it can be made accessible without reading a lot more than that.

References

[Ada95] J. F. Adams, *Stable homotopy and generalised homology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1995, Reprint of the 1974 original. MR 1324104

[AGP02] Marcelo Aguilar, Samuel Gitler, and Carlos and Prieto, *Algebraic topology from a homotopical viewpoint*, Universitext, Springer-Verlag, New York, 2002, Translated from the Spanish by Stephen Bruce Sontz. MR 1908260

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47, American Mathematical Society, Providence, RI, 1997, With an appendix by M. Cole. MR 1417719

[EKrvzMM95] A. D. Elmendorf, I. Kří ž, M. A. Mandell, and J. P. May, *Modern foundations for stable homotopy theory*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 213–253. MR 1361891

[Gre] J. P. C. Greenlees, *Stable homotopy theory course notes.*

[LMSM86] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure, *Equivariant stable homotopy theory*, Lecture Notes in Mathematics, vol. 1213, Springer-Verlag, Berlin, 1986, With contributions by J. E. McClure. MR 866482

[Lura] Jacob Lurie, *Chromatic homotopy theory (252x).*

[Lurb] ———, *Higher algebra.*

[Lur09] ———, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659

[May99] J. P. May, *A concise course in algebraic topology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1999. MR 1702278

[MMSS01] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley, *Model categories of diagram spectra*, Proc. London Math. Soc. (3) **82** (2001), no. 2, 441–512. MR 1806878

[Sch] Stefan Schwede, *Symmetric spectra.*

[SW53] E. H. Spanier and J. H. C. Whitehead, *A first approximation to homotopy theory*, Proc. Nat. Acad. Sci. U. S. A. **39** (1953), 655–660. MR 0056290

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR 1269324