Solutions: 1/24/18

Math 2300-007: §5.7 - Trig Integrals

(Thanks to Faan Tone Liu)

Key points:

- To integrate powers of $\sin(x)$ and $\cos(x)$ when at least one power is odd...

  If at least one power is odd, pull out one of these to play the role of du. (The other will be u). Convert the even power that remains into u using the Pythagorean Identity. See e.g. #1

- To integrate powers of $\sin(x)$ and $\cos(x)$ when at both powers are even...

  In this case, use the power reduction Fmlas $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$ and $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$. Sometimes you have to repeat this process. See e.g. #2, #3

- To integrate powers of $\sec(x)$ and $\tan(x)$...

  - Try to find $du = \sec^2 x\,dx \Rightarrow u = \tan x$

    OR

    $du = \sec x \tan x\, dx \Rightarrow u = \sec x$

  - Use Pythagorean Identities as needed.

  Guidelines/Tips:
  - If power on $\sec x$ even: $u = \tan x$
  - If power on $\tan x$ odd: $u = \sec x$
  - If power on $\tan x$ even: turn $\tan^2 x$'s into $\sec^2 x$'s
  - If power on $\sec x$ odd: good luck :)

- Helpful Trig Identities:

$$\left[\begin{aligned} \sin^2 x + \cos^2 x &= 1 \\ \tan^2 x + 1 &= \sec^2 x \\ 1 + \cot^2 x &= \csc^2 x \end{aligned}\right] \qquad \left[\begin{aligned} \sin^2\theta &= \frac{1-\cos(2\theta)}{2} \\ \cos^2\theta &= \frac{1+\cos(2\theta)}{2} \end{aligned}\right]$$

Pythagorean Identities   Power Reduction Fmlas

Examples:

1. $\int \sin^3 x \cos^4 x\, dx$

$$= \int \sin^2 x \cos^4 x \cdot \underbrace{\sin x\, dx}_{\text{Left over to be du}}$$

"du" = ± sin x looks good, so I'll pick

$$\left\{\begin{aligned} u &= \cos x \\ du &= -\sin x\, dx \\ -du &= \sin x\, dx \end{aligned}\right\}$$

$$= \int (1-\cos^2 x)\cos^4 x \cdot \sin x\, dx$$

$$= -\int (1-u^2)u \cdot du$$

$$= \int u^3 - u\, du$$

$$= \frac{1}{4}u^4 - \frac{1}{2}u^2 + C$$

$$= \frac{1}{4}\cos^4 x - \frac{1}{2}\cos^2 x + C$$

2. $\displaystyle\int_0^{\pi/4} \sin^2\theta\, d\theta$

⚠ Uh oh. Even powers of $\sin x$, $\cos x$. No "left over" $\sin x/\cos x$ for "du". Use $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$

$$= \int_0^{\pi/4} \frac{1-\cos(2\theta)}{2}\, d\theta = \frac{1}{2}\int_0^{\pi/4} 1-\cos(2\theta)\, d\theta = \left[\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\right]_0^{\pi/4}$$

$$= \left[\frac{1}{2}\left(\frac{\pi}{4}\right) - \frac{1}{4}\sin\left(\frac{\pi}{2}\right)\right] - \left[\frac{1}{2}(0) - \frac{1}{4}\sin(0)\right]$$

$$= \frac{\pi}{8} - \frac{1}{4} - 0$$

$$= \frac{\pi}{8} - \frac{1}{4}$$

3. $\displaystyle\int \sin^2\theta\cos^2\theta\, d\theta = \int \frac{1-\cos(2\theta)}{2}\cdot\frac{1+\cos(2\theta)}{2}\, d\theta$

$$= \frac{1}{4}\int 1-\cos^2(2\theta)\, d\theta$$

Thanks to Kevin!

$$= \frac{1}{4}\int \sin^2(2\theta)\, d\theta$$

use $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$ again but with "2θ" in for "θ"

$$= \frac{1}{4}\int \frac{1-\cos(4\theta)}{2}\, d\theta$$

$$= \frac{1}{8}\int 1-\cos(4\theta)\, d\theta$$

$$= \frac{1}{8}\left[\theta - \frac{1}{4}\sin(4\theta)\right] + C$$

4. $\displaystyle\int \tan^4 x\sec^2 x\, dx$

For this one, "left over" $\sec^2 x\,dx$ makes a good "du", so $u=\tan x$ is a good move:

$$\left\{\begin{aligned} u &= \tan x \\ du &= \sec^2 x\, dx \end{aligned}\right\}$$

$$= \int u^4\, du$$

$$= \frac{1}{5}u^5 + C$$

$$= \frac{1}{5}\tan^5 x + C$$

5. $\int \tan^2 x \sec^4 x\, dx = \int \tan^2 x \sec^2 x \cdot \sec^2 x\, dx$

can make "left-over" $\sec^2 x\, dx$ into du, ~~even amount of s~~ so $u = \tan x$ is a good move. Even amount of $\sec x$'s remain which can turn into $\tan x$'s via $\sec^2 x = \tan^2 x + 1$

$$= \int \tan^2 x (1+\tan^2 x) \cdot \sec^2 x\, dx$$

$$= \int u^2 (1+u^2)\, du \qquad \left\{ \begin{array}{l} u = \tan x \\ du = \sec^2 x\, dx \end{array} \right\}$$

$$= \int u^2 + u^4\, du$$

$$= \tfrac{1}{3}u^3 + \tfrac{1}{5}u^5 + C$$

$$= \tfrac{1}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x + C$$

6. $\int \tan^3 x \sec^5 x\, dx = \int \tan^2 x \sec^4 x \cdot \tan x \sec x\, dx$

left-over for du. $u = \sec x$ looks good.

$$= \int (\sec^2 x - 1)\sec^4 x \cdot \tan x \sec x\, dx \qquad \left\{ \begin{array}{l} u = \sec x \\ du = \sec x \tan x\, dx \end{array} \right\}$$

$$= \int (u^2-1) u^4\, du$$

$$= \int u^6 - u^4\, du$$

$$= \tfrac{1}{7}u^7 - \tfrac{1}{5}u^5 + C$$

$$= \tfrac{1}{7}\sec^7 x - \tfrac{1}{5}\sec^5 x + C$$

7. $\int \tan^3 x\, dx = \int \tan x \cdot \tan^2 x\, dx = \int \tan x (\sec^2 x - 1)\, dx$

$$= \int \tan x \sec^2 x\, dx - \int \tan x\, dx$$

$$\left\{ \begin{array}{l} u = \tan x \\ du = \sec^2 x\, dx \end{array} \right\} = \int u\, du - \int \frac{\sin x}{\cos x}\, dx$$

$$= \tfrac{1}{2}u^2 - \int \frac{\sin x}{\cos x}\, dx \qquad \left\{ \begin{array}{l} u = \cos x \\ du = -\sin x\, dx \end{array} \right\}$$

$$= \tfrac{1}{2}\tan^2 x + \int \frac{1}{u}\, du$$

$$= \tfrac{1}{2}\tan^2 x + \ln|u| + C$$

$$= \tfrac{1}{2}\tan^2 x + \ln|\cos x| + C$$

These two are tricky!

need some secx's in there, so try to get some first

8. $\int \tan^4 x\, dx = \int (\sec^2 x - 1)\tan^2 x = \underbrace{\int \tan^2 x \sec^2 x\, dx}_{\text{can do u-sub}} - \int \tan^2 x\, dx$

write as $\sec^2 x - 1$

$\left\{ \begin{array}{l} u = \tan x \\ du = \sec^2 x\, dx \end{array} \right\}$

$$\int \tan^2 x \sec^2 x\, dx = \int u^2\, du = \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 x + C$$

$$\int \tan^2 x\, dx = \int \sec^2 x - 1\, dx = \tan x - x + C$$

Now, we have...

$$\left[\int \tan^4 x\, dx = \frac{1}{3}\tan^3 x - \tan x + x + C \right]$$

Odd powers of sec x are hard. The boomerang is helpful!

9. $\int \sec^5 x\, dx = \int \sec^3 x \cdot \sec^2 x\, dx = \sec^3 x \tan x - \int 3\sec^3 x \tan^2 x\, dx$

$\left\{ \begin{array}{ll} u = \sec^3 x & dv = \sec^2 x\, dx \\ du = 3\sec^2 x \cdot \sec x \tan x\, dx & v = \tan x \end{array} \right\}$

Int. By parts

$$= \sec^3 x \tan x - 3\int \sec^3 x(\sec^2 x - 1)\, dx$$

$$= \sec^3 x \tan x - 3\int \sec^5 x\, dx + 3\int \sec^3 x\, dx$$

We have by "boomerang"...

$$4\int \sec^5 x\, dx = \sec^3 x \tan x + 3\int \sec^3 x\, dx$$

$$\int \sec^5 x\, dx = \frac{1}{4}\sec^3 x \tan x + \underbrace{\frac{3}{4}\int \sec^3 x\, dx}$$

[did on Int. By parts WS #10]

$$\int \sec^5 x\, dx = \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\cdot\frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C$$

$$\boxed{\int \sec^5 x\, dx = \frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\ln|\sec x + \tan x| + C}$$