Solutions: 3/20/18

Taylor Polynomials

(Thanks to Faan Tone Liu)

Key Points:

- The formula for $T_n(x)$, the nth degree taylor polynomial for $f(x)$ centered at $x = a$ is:

$$\boxed{\begin{aligned} T_n(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \end{aligned}}$$

- Other notes:

$T_n(x)$ has some of the same properties as $f(x)$ near a.
For example,

- $T_6(x)$ for $\cos(x)$ is even, just like $\cos(x)$
- We saw last week that $e^x \approx 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$. If you differentiate both sides, you get $e^x \approx 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!}$, so $T_n(x)$ is almost its own derivative (e^x is its own derivative)

Examples:

1. (a) Find $T_6(x)$ the 6th degree Taylor Polynomial for $f(x) = \cos(x)$ centered at $a = 0$.

$f(x) = \cos(x)$ $\quad f(0) = 1$

$f'(x) = -\sin(x)$ $\quad f'(0) = 0$

$f''(x) = -\cos(x)$ $\quad f''(0) = -1$

$f'''(x) = \sin(x)$ $\quad f'''(0) = 0$

$f^{(4)}(x) = \cos(x)$ $\quad f^{(4)}(0) = 1$

$f^{(5)}(x) = -\sin(x)$ $\quad f^{(5)}(0) = 0$

$f^{(6)}(x) = -\cos(x)$ $\quad f^{(6)}(0) = -1$

$$T_6(x) = f(0) + \frac{f'(0)}{1}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \frac{f^{(6)}(0)}{6!}x^6$$

$$= 1 + 0x + \frac{-1}{2}x^2 + 0x^3 + \frac{1}{24}x^4 + 0x^5 + \frac{-1}{6!}x^6$$

$$\boxed{T_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6}$$

(b) Use your $T_6(x)$ to estimate $\cos(5°)$.

$5°$ is $\frac{\pi}{36}$ radians

$$\cos(5°) = \cos\left(\frac{\pi}{36}\right) \approx 1 - \frac{1}{2}\left(\frac{\pi}{36}\right)^2 + \frac{1}{24}\left(\frac{\pi}{36}\right)^4 - \frac{1}{720}\left(\frac{\pi}{36}\right)^6 \approx 0.99619$$

2. What is $T_n(x)$, the nth degree Taylor polynomial for $f(x) = \ln(x)$ centered at $a = 1$?

n	$f^{(n)}(x)$	$f^{(n)}(1)$
$n=0$	$f(x) = \ln x$	$\ln(1) = 0$
1	$\frac{1}{x} = x^{-1}$	$\frac{1}{1} = 1$
2	$-x^{-2}$	$-\frac{1}{1^2} = -1$
3	$2x^{-3}$	$2!$
4	$-3!\,x^{-4}$	$-3!$
5	$+4!\,x^{-5}$	$4!$
$\vdots$		$\vdots$
n	$(-1)^{n-1}(n-1)!\,x^{-n}$	$(-1)^{n-1}(n-1)!$

$$T_n(x) = 0 + \frac{1}{1!}(x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2!}{3!}(x-1)^3 + \frac{-3!}{4!}(x-1)^4 + \cdots + \frac{(-1)^{n-1}(n-1)!}{n!}(x-1)^n$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots + \frac{(-1)^{n-1}}{n}(x-1)^n$$

$$= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}(x-1)^k$$

$$\boxed{T_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}(x-1)^k = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots + \frac{(-1)^{n-1}}{n}(x-1)^n}$$

3. How could you estimate $\ln(1.6)$? What could you to do improve your estimate?

To estimate $\ln(1.6)$, you could pick a Taylor polynomial, say $T_4(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$ and evaluate $T_4(x)$ at 1.6:

$$\ln(1.6) \approx T_4(1.6) = 0.4596.$$

(The actual value of $\ln(1.6)$ is ≈ 0.470004, so we are off by a bit)
To improve the estimate, use a larger n (i.e. make a polynomial of higher degree.) For example,

$$T_7(1.6) = 0.4714,$$

which is closer to $\ln(1.6)$.